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ORTHOGONAL GROUPS AND SYMMETRIC SETS

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Orthogonal groups are considered as automorphism groups of some symmetric sets of vectors. From this point of view, we can prove the well-known theorem of simplicity on orthogonal groups. (The cases for the other classical groups are given in [5].) The proof consists of two steps. The first step which will be given in **1** is to show that a transitive symmetric set of non-isotropic lines (of a certain type) is simple. After a short review on simple symmetric set is given, we will show the above fact. A point here is that it is so when $\dim V$ is 3. The second step is to show that the group of displacements of the simple symmetric set is a simple group, which will be given in **2**. A useful supplement to the main theorem on simple symmetric sets will be found, and using it we can show the above fact when $\dim V \geq 5$.

1. A simple symmetric set of non-isotropic lines

Let V be a vector space over a field of characteristic $\neq 2$ with a non-singular orthogonal metric. Since the following results hold in a stronger sense for a finite field as was shown in [4], we assume in this note that the base field k is infinite. Suppose that $\dim V \geq 3$ and that V contains a hyperbolic plane. Then, there exists a vector v such that v is orthogonal to a hyperbolic plane and $(v, v) = \varepsilon \neq 0$. Throughout this note, we fix the element ε . Now we consider $A = \{\bar{u} \mid (u, u) = \varepsilon\}$, where $\bar{u} = \langle u \rangle =$ a subspace generated by u . On A , we define a binary operation: $\bar{u} \circ \bar{v} = \bar{w}$ with $w = u^{\tau_v}$, where τ_v is the symmetry with respect to the hyperplane orthogonal to v . A is then a symmetric set, i.e., satisfies $\bar{u} \circ \bar{u} = \bar{u}$, $(\bar{u} \circ \bar{v}) \circ \bar{v} = \bar{u}$ and $(\bar{u} \circ \bar{v}) \circ \bar{w} = (\bar{u} \circ \bar{w}) \circ (\bar{v} \circ \bar{w})$.

We summarize some definitions and properties on simple symmetric sets. Let $S = \{a, b, c, \dots\}$ be a symmetric set. The right multiplication by an element a is an automorphism of S , which we denote by σ_a . Let $G(S) = \langle \sigma_a \mid a \in S \rangle$ and $H(S) = \langle \sigma_a \sigma_b \mid a, b \in S \rangle$. The latter is called the group of displacements of S . Let T be another symmetric set. A homomorphism f of S onto T is called proper if it is not one to one and if T contains more than one element. When $G(S) = 1$, we say S is trivial. A non-trivial symmetric set is called simple if it has no proper homomorphism (to some symmetric set). It is important

to characterize a simple symmetric set in a different way as follows. Let f be a homomorphism of S to T . $f^{-1}(t)$, the set of all inverse images, for an element t in T is called a coset of f . Then, S is decomposed into disjoint cosets of f : $S = \cup S_i$ with $S_i = f^{-1}(t_i)$ with $S_i \cap S_j = \phi$ if $i \neq j$. In this case, as is easily seen, σ_a induces a permutation on $X = \{S_i\}$, the set of cosets S_i . It is also clear that σ_a and σ_b induce the same permutation on X if a and b belong to a same coset. Conversely, suppose that $S = \cup S'_i$ is a disjoint union of subsets S'_i satisfying the above two conditions on σ_a . Then we can define a symmetric structure on $X = \{S'_i\}$ in such a way that the restriction mapping of S to X is a homomorphism. Thus the concept of homomorphisms is equivalent with that of coset-decompositions. When a homomorphism f is proper, we say that the corresponding coset-decomposition is proper. The simplicity of a non-trivial symmetric set is now characterized by the fact that it has no proper coset-decomposition.

A symmetric set S is called transitive if $G(S)$ (or, equivalently $H(S)$) is a transitive permutation group on S . If S is simple, then S is transitive. For, otherwise, $S = \cup S_i$ with $S_i =$ orbits of elements in S by $G(S)$ would be a proper coset-decomposition. A symmetric set is called effective if $\sigma_a \neq \sigma_b$ whenever $a \neq b$. The following is the main theorem on simple symmetric set obtained in [2] and [3]. (See also [5].)

Theorem. *Suppose that S is a transitive and effective symmetric set. Then, S is simple if and only if $H(S)$ is a minimal normal subgroup of $G(S)$. The latter condition is equivalent with that $H(S)$ is either a simple group or a direct product of simple subgroups N_1 and N_2 such that $N_2 = N_1^{\sigma_a}$ for any element a in S . In the latter case, N_i are regular permutation groups on S .*

As the last part of Theorem is not explicitly given in the previous papers, we explain it here. It is known that N_i are transitive on S . Now suppose that $a^\tau = a$ for an element a in S and τ in N_i . Then, $\tau^{-1}\sigma_a\tau = \sigma_a$, or τ and σ_a are commutative, and so $\tau = \sigma_a^{-1}\tau\sigma_a$, which belongs to N_1 as well as to N_2 . Thus, $\tau = 1$.

We now return to V and A . Let $O(V)$ be the orthogonal group of V and $\Omega = \Omega(V)$ its commutator subgroup. Elements of $O(V)$ naturally induce automorphisms of A . So, there is a natural homomorphism h of $O(V)$ to the group of automorphisms of A . The kernel of h is $Z = \{\pm 1\}$, the center of $O(V)$, due to Lemma 5.5, p. 206 [1]. Thus, $PO(V) = O(V)/Z$ is considered as a group of automorphisms of A . In this respect, we want to show that $P\Omega = H(A)$. This is equivalent with $\Omega = h^{-1}(H(A))$. Since Ω is generated by $\tau_v\tau_w$ with $(v, v) = (w, w) \neq 0$, it is clear that $H(A) \subset P\Omega$. When $\dim V = 3$, $P\Omega$ is a simple group (Theorem 5.20, [1]). So, in this case, $H(A) = P\Omega$. To show it in a general case and also to show the simplicity of A , we use the following

Lemma. *If $(u, u)=(v, v)=\varepsilon$, then there exists a hyperbolic plane P such that $(u, P)=0$ and $v \in \langle u \rangle + P$.*

Proof. If $\langle u, v \rangle$ is singular, it is easy to find the above P . Suppose that $\langle u, v \rangle$ is non-singular. Then, $v = \alpha u + v'$, where $v' \in \langle u \rangle^\perp$ and $(v', v') \neq 0$. (Naturally we are assuming $\dim \langle u, v \rangle = 2$.) By the assumption on ε , $\langle u \rangle^\perp$ contains a hyperbolic plane. Since a hyperbolic plane contains an element w such that (w, w) is any element in the base field, i.e., is universal, we can find an isometry on $\langle u \rangle^\perp$ by Witt theorem which maps w to v' . We can let P be the image of the hyperbolic plane under the isometry.

From Lemma, we can conclude that $H(A) = P\Omega$, as we can always restrict our consideration to a 3-dimensional case. We can also conclude that A is transitive. For, Ω acts on A transitively. Here note that $\langle u \rangle^\perp$ is universal and we can insert σ_a with no effect on u where (a, a) is any prescribed value. Now we show that A is simple. Assume that A is not simple. Then there exists a proper coset-decomposition $A = \cup A_i$. Let \bar{a} and \bar{b} be two distinct elements in A_1 . Let $U = \langle a \rangle + P$, where $(a, P) = 0$ and $b \in U$ from Lemma. Then $A(U) = \{\bar{u} \mid u \in U, (u, u) = \varepsilon\}$ is simple by the main theorem. Restrict $A = \cup A_i$ to the elements in $A(U)$. We can conclude that $A(U) \subset A_1$. Now let Q be a hyperbolic plane such that $(a, Q) = 0$. Since $O(V) = O(P)\Omega$ as we know, there exists an element in Ω which fixes a and maps P to Q . This implies that A_1 contains every element \bar{w} such that $w \in \langle u \rangle + Q$, $(w, w) = \varepsilon$. Using Lemma, we can conclude that $A = A_1$, which is a contradiction. Thus, A is simple.

2. Simplicity of the group $H(A)$

Let $S = \{a, b, \dots\}$ be an effective simple symmetric set. Suppose that $H(S)$ is not a simple group. Then, $H(S) = N_1 \times N_2$ with simple subgroups N_i such that $N_2 = N_1^{\sigma_a}$ for any element a in S . Moreover, N_i are regular permutation groups on S . Let τ be an element in N_1 and express it as a product of disjoint cyclic permutations: $\tau = (\dots, a, b, c, \dots)$. $(\dots) \dots$ We show that $a^{\sigma_b} = c$. Since N_1 and N_2 are commutative, we have $\tau\tau^{\sigma_b} = \tau^{\sigma_b}\tau$, or $\tau\sigma_b\tau\sigma_b = \sigma_b\tau\sigma_b\tau$. Therefore, $\sigma_b(\tau\sigma_b\tau\sigma_b) = \tau\sigma_b\tau = (\tau\sigma_b\tau\sigma_b)\sigma_b$. So, $\rho^{-1}\sigma_b\rho = \sigma_b$, where $\rho = \tau\sigma_b\tau\sigma_b = \tau\tau^{\sigma_b}$. Since $\rho^{-1}\sigma_b\rho = \sigma_b^\rho$ and S is effective, we have $b^\rho = b$, or $b^{\tau\tau^{\sigma_b}} = b$. So, $b^{(\tau\sigma_b)^{-1}} = b^\tau = c$. On the other hand, $\tau^{\sigma_b} = (\dots, a^{\sigma_b}, b, c^{\sigma_b}, \dots) \dots$. So, $b^{(\tau\sigma_b)^{-1}} = a^{\sigma_b}$. Therefore, $a^{\sigma_b} = c$, as required. In the above, (\dots, a, b, c, \dots) coincides with a cycle defined in the theory of symmetric set. Note also that τ is a product of cycles of the same length and every element of S must appear in a cycle. This is a supplement to the main theorem on simple symmetric sets. In the above, especially, $\tau = (a, b)(c, d) \dots$ if $a^{\sigma_b} = a$, $c^{\sigma_d} = c$, etc. Since N_1 is regular, such an element τ exists if there exist a and b such that $a^{\sigma_b} = a$. In this case, we can show that if

an element σ in H fixes a , b and c , then σ must fix d as well. For, $\tau^\sigma = (a, b)(c, d^\sigma)\cdots$ must coincide with τ as both are elements in a regular permutation group and move a to b .

Now we return to A . Assume that $\dim V \geq 5$ and that $H(A)$ is not simple. Thus, $H(A) = N_1 \times N_2$ as above. Let u_1 be an element in V such that $(u_1, u_1) = \varepsilon$ and let P be a hyperbolic plane orthogonal to u_1 . Select u_2 in P such that $(u_2, u_2) = \varepsilon$. Since N_1 is transitive, there exists an element τ in N_1 such that $\bar{u}_1 = u_2$. Then, $\tau = (\bar{u}_1, u_2)(\vartheta, \bar{w})\cdots$, where we assume that $\vartheta \in P$. Since $\vartheta^{\sigma\bar{w}} = \vartheta$, we have $(\vartheta, w) = 0$. Also we have that $(u_2, w) = 0$, because τ maps \bar{u}_1 and ϑ to u_2 and \bar{w} respectively and $(u_1, \vartheta) = 0$. Thus, $(w, P) = 0$ as $P = \langle u_2, \vartheta \rangle$. Let $W = P^\perp$. W contains u_1 and w and $\dim W \geq 3$. There exists an element ρ in $\Omega(W)$ such that $\bar{u}_1^\rho = u_1$ and $\bar{w}^\rho \neq \bar{w}$. For example, let ρ be a rotation around u_1 in some non-singular subspace of dim 3 containing u_1 and w . Then, ρ (extended as an element in $\Omega(V)$ in a natural way) fixes u_1, \bar{u}_2 and ϑ but moves \bar{w} , which contradicts the above argument. Thus, we have shown that $H(A)$ must be a simple group if $\dim V \geq 5$.

References

- [1] E. Artin: *Geometric algebra*, Interscience, New York, 1957.
- [2] H. Nagao: *A remark on simple symmetric sets*, Osaka J. Math. **16** (1979), 349–352.
- [3] N. Nobusawa: *Simple symmetric sets and simple groups*, Osaka J. Math. **14** (1977), 411–415.
- [4] ———: *Primitive symmetric sets in finite orthogonal geometry*, Osaka J. Math. **17** (1980), 407–410.
- [5] ———: *A remark on conjugacy classes in a simple group*, Osaka J. Math. **18** (1981), 749–754.

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