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ON THE π-NILPOTENT LENGTH OF π-SOLVABLE GROUPS

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1. Introduction

In this paper, G is always a finite group. The *Fitting subgroup* $F(G)$ of G is the uniquely determined maximal normal subgroup. If G is solvable, we have the following normal series:

$$1 = F^0(G) \lhd F^1(G) \lhd \cdots \lhd F^s(G) = G\,,$$
$$F^{i+1}(G)/F^i(G) = F(G/F^i(G))\,.$$

The length s of this series is called the *nilpotent length* of G.

The purpose of this paper is to prove

Theorem 1. *The nilpotent length of a finite solvable group G is at most one plus the number of G-conjugate classes of the family of non-normal maximal subgroups of G.*

This result will be extended for π-solvable groups. Let π be a set of prime numbers and π' the complement of π in the set of all the prime numbers. We say that a number n belongs to π if n is divisible only by primes in π. A group G is called *π-group* if the order of G belongs to π. A group G is *π-separable* if every composition factor of G is either π-group or π'-group, and G is *π-solvable* if every composition factor is either π'-group or p-group for some prime p belonging to π. A group G is called *π-nilpotent* if G has a normal p-complement for all p in π. Let $F_\pi(G)$ be the uniquely determined maximal normal π-nilpotent subgroup of G. If G is π-solvable, then we have the following normal series of G:

$$1 = F_\pi^0(G) \lhd F_\pi^1(G) \lhd \cdots \lhd F_\pi^s(G)\,,$$
$$F_\pi^{i+1}(G)/F_\pi^i(G) = F_\pi(G/F_\pi^i(G))\,.$$

The length s of this normal series is called the *π-nilpotent length* of G. Then we have

Theorem 2. *The π-nilpotent length of a π-solvable group G is at most* *one plus the number of G-conjugate classes of the family of non-normal maximal subgroups of G whose indices belong to π.*

As a corollary we have

Corollary. *The π-length of a π-solvable group is at most one plus the number of G-conjugate classes of the family of non-normal maximal subgroups of G whose indices belong to π.*

At the last we shall show that these inequalities are best possible.

Notation. Let G be a finite group and X, Y the subsets of G. We use the following notation.

$Z(G)$	:	center of G
$N_G(X)$	:	normalizer of X in G
$\Phi(G)$	:	intersection of all the maximal subgroups of G
$\langle X, Y\rangle$	:	subgroup of G generated by X, Y
$C_G(X)$	:	centralizer of X in G
$[X, Y]$	:	comutator subgroup of X and Y
$GF(p)$	:	finite field of p elements
$\lvert X\rvert$	:	number of the elements of X

2. Proofs of the theorems

Proof of Theorem 1. We shall prove the theorem by induction on the number of G-conjugate classes of the family of non-normal maximal subgroups of G. For the case that every maximal subgroup of G is normal in G, Theorem 1 is trivial ([1]; p. 260). Suppose there exists a non-normal maximal subgroup of G. This means that G is not nilpotent ([1]; p. 260). We shall show that $\Delta(G)\not\geq F(G)$, where $\Delta(G)$ is the intersection of all the non-normal maximal subgroups of G. Since $\Delta(G/\Phi(G))=\Delta(G)/\Phi(G)$ and $F(G/\Phi(G))=F(G)/\Phi(G)$ ([1]; p. 270), we may suppose $\Phi(G)=1$. Therefore $\Delta(G)=Z(G)$ ([1]; p. 276). If $Z(G)=\Delta(G)\geq F(G)$, then $F(G)\geq C_G(F(G))=G$ ([1]; p. 277). It contradicts the assumption of non-nilpotency of G. Since $\Delta(G)\not\geq F(G)$, the number of $G/F(G)$-conjugate classes of the family of non-normal maximal subgroups of $G/F(G)$ is strictly less than that of G. Hence by induction, we can complete the proof.

To prove Theorem 2, we need the following lemmas

Lemma 1. *G is π-nilpotent if and only if every maximal subgroup of G, whose index belongs to π, is normal in G and G is π-separable.*

Proof of Lemma 1. Let G be π-nilpotent. Then from the definition there exists a π'-Hall subgroup R of G which is normal in G and G/R is nilpotent. The order of G/R belongs to π. Thus G is π-separable. Any maximal subgroup M of G, whose index belongs to π, contains R. The image of M under the natural homomorphism $G \to G/R$ is maximal in G/R. Since G/R is nilpotent, it must be normal in G/R, and M is also normal in G. Let us prove the "if" part. Since G is π-separable, G possesses a π'-Hall subgroup R ([2]; p. 229). If the normalizer $N_G(R)$ of R in G is smaller than G, then there exists a maximal subgroup M such that $N_G(R) \leq M < G$ and $N_G(M) = M$ ([2]; p. 230). It contradicts the assumption M is normal in G. Thus $N_G(R) = G$, namely R is normal in G. Let L/R be a maximal subgroup of G/R. Then L is maximal in G whose index in G belongs to π. From the assumption, L is normal in G, and L/R is also normal in G/R. Thus G/R is nilpotent and is π-nilpotent.

Lemma 2. *Let G be π-separable, then the index of any maximal subgroup of G belongs to π or π'.*

Proof. Let M be a maximal subgroup of G. Regard $G/\bigcap_{g \in G} M^g$ as the permutation group on the left cosets of G by M. Then this permutation group is primitive and its degree is equal to the index of M in G. Since the minimal normal subgroup of $G/\bigcap_{g \in G} M^g$ is transitive, the index of M divides the order of the minimal normal subgroup of $G/\bigcap_{g \in G} M^g$, which belongs to π or π'. Thus Lemma 2 is proved.

Proof of Theorem 2. We shall prove by induction on the number of G-conjugate classes of the family of non-normal maximal subgroups of G whose indices in G belong to π. For the case that every maximal subgroup of G, whose index belongs to π, is normal in G, it is trivial from Lemma 1. Suppose there exists a non-normal maximal subgroup of G whose index belongs to π. We shall show that $\Delta_\pi(G) \ngeq F_\pi(G)$, where $\Delta_\pi(G)$ is the intersection of all non-normal maximal subgroups whose indices belong to π. First of all, let R be a π'-Hall subgroup of $F_\pi(G)$, then R is a characteristic subgroup of $F_\pi(G)$, therefore R is normal in G. Since G is π-separable, Sylow-theorem holds for the π'-Hall subgroups of G. Therefore $\Delta_\pi(G) \geq R$. Since $F_\pi(G/R) = F_\pi(G)/R$ and $\Delta_\pi(G/R) = \Delta_\pi(G)/R$, we may suppose that $R=1$. In addition, $\Delta_\pi(G/\Phi(G)) = \Delta_\pi(G)/\Phi(G)$ and $F_\pi(G/\Phi(G)) = F_\pi(G)/\Phi(G)$ ([1]; p. 689). Thus we may suppose that $\Phi(G) = 1$. Then $F_\pi(G)$ is a π-group, therefore $F_\pi(G) \leq \Delta_{\pi'}(G)$. On the other hand $\Delta(G) = \Delta_\pi(G) \cap \Delta_{\pi'}(G)$ from Lemma 2. Therefore $\Delta_\pi(G) \geq F_\pi(G)$ if and only if $\Delta(G) \geq F_\pi(G)$. If $\Delta(G) \geq F_\pi(G)$, then $Z(G) = \Delta(G) \geq F_\pi(G)$. Set $M/F_\pi(G)$ a minimal normal subgroup of $G/F_\pi(G)$. We shall show that M is a normal π-nilpotent subgroup of G. At first, let $M/F_\pi(G)$ be a π-group. Since $G/F_\pi(G)$ is π-solvable, $M/F_\pi(G)$ is abelian. Therefore, $[M, M] \leq F_\pi(G)$ and $1 = [F_\pi(G), M] \geq [[M, M], M]$. Thus M is nilpotent, especially, M is π-nilpotent. On the other hand, let $M/F_\pi(G)$ be a π'-group. Then there exists a π'-Hall subgroup S of M such that $M = F_\pi(G)S$ and $F_\pi(G) \cap S = 1$. From $F_\pi(G) \leq Z(G)$, we have

$M=F_\pi(G)\times S$. Therefore M is π-nilpotent by the nilpotency of $F_\pi(G)$. This is a contradiction. Thus we can show that $\Delta_\pi(G)\supsetneqq F_\pi(G)$. From $\Delta_\pi(G)\supsetneqq F_\pi(G)$, the number of $G/F_\pi(G)$-conjugate classes of the family of non-normal maximal subgroups of $G/F_\pi(G)$, whose indices in $G/F_\pi(G)$ belong to π, is strictly less than that of G. Thus by induction Theorem 2 is proved.

3. Example

Theorem 3. *For any n there exists a finite solvable group G such that the nilpotent length of G is equal to n and the number of G-conjugate classes of the family of non-normal maximal subgroups of G is equal to $n-1$.*

Proof. The existence of the finite solvable groups satisfying the conditions of Theorem 3 is easily proved from the construction of π-solvable groups satisfying the conditions of Theorem 4.

Theorem 4. *For any n and any π, there exists a π-solvable group G such that the π-nilpotent length of G is equal to n and the number of G-conjugate classes of the family of non-normal maximal subgroups of G, whose indices belong to π, is equal to $n-1$.*

Construction

G_1: $|G_1|=p_1$, $p_1\in\pi$

T_1: G_1 possesses a faithful irreducible representation on $GF(q_1)$, $q_1\in\pi'$. Therefore we can construct $T_1=G_1Q_1$, the semidirect product of G_1 and Q_1 which is the elementary abelian group of the order $q_1^{l_1}$, for some l_1, and then Q_1 is a unique minimal normal subgroup of T_1.
For $n\geqslant 2$, we construct G_n and T_n as below.

G_n: T_{n-1} possesses a faithful irreducible representation on $GF(p_n)$, $p_n\in\pi$, for the sake of the uniqueness of the minimal normal subgroup of T_{n-1}. Therefore, we can construct $G_n=T_{n-1}P_n$, the semidirect product of T_n and P_n which is the elementary abelian group of the order $p_n^{m_n}$, for some m_n, and then P_n is a unique minimal normal subgroup of G_n.

T_n: G_n possesses a faithful irreducible representation on $GF(q_n)$, $q_n\in\pi'$, for the sake of the uniqueness of the minimal normal subgroup of G_n. Therefore, we can construct $T_n=G_nQ_n$, the semidirect product of G_n and Q_n which is the elementary abelian group of the order $q_n^{l_n}$, for some l_n, and then Q_n is a unique minimal normal subgroup of T_n.

Proof. We shall show by induction on n that G_n satisfies the conditions of Theorem 4. It is trivial that the π-nilpotent length of G_n is equal to n.

Since a maximal subgroup of G_n containing P_n, whose index belongs to π, contains $Q_{n-1}P_n$ and $G_n/Q_{n-1}P_n$ is isomorphic to G_{n-1}, the number of G_n-conjugate classes of the family of non-normal maximal subgroups, which have indices belonging to π and contain P_n, is equal to $n-1$ by induction hypothesis. Now T_{n-1} is maximal in G_n and not normal in G_n. We shall show that any maximal subgroup M of G_n which does not contain P_n is conjugate to T_{n-1}. Now $G_n=MP_n$ and $M\cap P_n=1$. M is isomorphic to G_n/P_n. Let N be a minimal normal subgroup of M. Then $NP_n=Q_{n-1}P_n$. By Zassenhaus theorem $N^g=Q_{n-1}$, for some $g\in Q_{n-1}P_n$. Therefore $M^g\triangleright N^g=Q_{n-1}$, then $M^g=T_{n-1}$. For if $M^g\neq T_{n-1}$, then $Q_{n-1}\triangleright\langle M^g, T_{n-1}\rangle=G$. This is the contradiction. Thus, the number of G_n-conjugate classes of the family of non-normal maximal subgroups of G_n, whose indices belong to π, is equal to $n-1$.

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References

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