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Supplement to "Note on Brauer’s Theorem of Simple Groups"

By Osamu NAGAI

Using the modular representation theory of groups, R. Brauer obtained very interesting results concerning a finite group satisfying the following conditions:

(*) The group $\mathcal{G}$ contains an element $P$ of prime order $p$ which commutes only with its own powers $P^i$.

(**) The commutator subgroup $\mathcal{G}'$ of $\mathcal{G}$ is equal to $\mathcal{G}$.

Namely:

Theorem. If $\mathcal{G}$ is a group of finite order $g$ satisfying the conditions (*) and (**), then $g = p(p-1)(1+np)/t$, where $n$ and $t$ are integers, and $t$ divides $p-1$. The group $\mathcal{G}$ contains exactly $1+np$ subgroups of order $p$ and $t$ classes of conjugate elements of order $p$. Moreover, if $n < (p+7)/3$, then either (1) $\mathcal{G} \cong LF(2,p)$ or (2) $p$ is a prime of the form $2^\mu+1$ and $\mathcal{G} \cong LF(2,2^\mu)$.

In a previous note, we considered the case $n \leq p+2$ and $t \equiv 0 \pmod{2}$, and proved that $p$ is of the form $2^\mu-1$ and $\mathcal{G} \cong LF(2,2^\mu)$. In this supplement we shall prove that, including the case $n = p+2$, the previous result is valid; that is,

Theorem. Let $\mathcal{G}$ be a group of finite order satisfying conditions (*) and (**). If $n \leq p+2$ and $t$ is odd, then $p$ is of the form $2^\mu-1$ and $\mathcal{G} \cong LF(2,2^\mu)$.

Before the proof, we shall mention Brauer’s results which is needed in the sequel. Under the condition (*), the order of $\mathcal{G}$ contains $p$ to the first power only. So the ordinary irreducible representations of $\mathcal{G}$ are of


3) [B] and R. Brauer, "On groups whose order contains a prime number to the first power I, II," Amer. J. of Math., vol. 54 (1942).
four different types: (I) Representations $\mathcal{A}_p$ of a degree $a_p = u_p p + 1$. (II) Representations $\mathcal{B}_\sigma$ of a degree $b_\sigma = v_\sigma p - 1$. (III) Representations $\mathcal{C}^{(\sigma)}$ of a degree $c = (w_\sigma + \delta)/\tau$, where $\delta = \pm 1$ and $w$ is a positive integer. There exist exactly $t$ such representations that are algebraically conjugate. (IV) Representations $\mathcal{D}_\tau$ of a degree $d_\tau = \mu_\tau r$. Denote by $A_p$, $B_\sigma$, $C^{(\sigma)}$ and $D_\tau$ the characters of $\mathcal{A}_p$, $\mathcal{B}_\sigma$, $\mathcal{C}^{(\sigma)}$ and $\mathcal{D}_\tau$ respectively.

If we have $x$ characters $A_p$, $\rho = 1, 2, \ldots, x$, and $y$ characters $B_\sigma$, $\sigma = 1, 2, \ldots, y$, then we have

$$x + y = (p-1)/t.$$  

Furthermore, for elements $G$ of order prime to $p$, we have

$$\sum A_p(G) + \delta C^{(\sigma)}(G) = \sum B_\sigma(G).$$

In particular, for $G = 1$, this gives

$$\sum a_p + \delta c = \sum b_\sigma, \text{ or } \sum u_\sigma + (\delta w + 1)/\tau = \sum v_\sigma.$$  

Since $g$ is equal to the sum of the squares of all the degrees, we have

$$\sum u_\sigma^2 + \sum v_\sigma^2 + w^2/\tau + \sum x_\sigma^2 = (pn - n + 1)/\tau,$$

$$\sum u_\sigma^2 + \sum v_\sigma^2 + w^2/\tau + \sum x_\sigma^2 = (p^2 + p - 1)/\tau \text{ (in the case } n = p + 2).$$

Since the first $p$-block $B(p)$ is of the only lowest kind of $G$, the full number of irreducible representations of $G$ whose degrees are prime to $p$ is $(p-1)/\tau + t$.

**Proof.**

It is sufficient to prove that such group does not exist in the case $n = p + 2$, for the case $n < p + 2$ was discussed in the previous note.

Let $n = p + 2$.

First of all, we shall prove that such group $G$ must be simple. Let $G$ have a proper normal subgroup $H$ of order $h$. From [B], Theorem 3 and Theorem 4, $G/H$ also satisfies condition (*) and at the same time $h \equiv 1$ (mod $p$) and $(1 + np) \equiv 0$ (mod $h$). Since $n = p + 2$, we have $(p + 1)^2 \equiv 0$ (mod $h$), $h \equiv 1$ (mod $p$) and $g = p(p-1)(p+1)^2/\tau$. We put $h = 1 + \alpha p$ and $(p+1)^2 = \beta h$, then $(p+1)^2 = \beta(1+\alpha p)$. So $\beta \equiv 1$ (mod $p$). We put $\beta = 1 + \gamma p$. Then $(p+1)^2 = (1 + \alpha p)(1 + \gamma p)$. This gives $p + 2 = \alpha \gamma p + \alpha + \gamma$. If $\gamma = 0$, then $\alpha = p + 2$. We have $h = (p+1)^2$. Since $G/H$ also satisfies condition (**), $G/H$ can not be a metacyclic group of order $p(p-1)/\tau$. If $\gamma \neq 0$, then, since $\alpha = 0$, we have $\alpha = 1$ and $\gamma = 1$. So we have $h = p + 1$. This means $g/h = p(p-1)(p+1)/\tau$.  


From [B], Theorem 10, \( t \) must be even\(^4\). This is a contradiction.

Then, we shall examine the degrees of the irreducible representations of \( \mathcal{G} \). In the case \( n = p + 2 \), \( n \) is represented as \( n = F(p, u^{(\nu)}, h^{(\nu)}) \) in two kinds such that \( \nu^{(\nu)} = \frac{u}{h} \) and \( h^{(\nu)} = \frac{1}{h} \). So from [B], Theorem 7, the degrees of the irreducible representations of \( \mathcal{G} \), as far as they are prime to \( p \), can only have some of the values

\[
\begin{align*}
\alpha_\sigma &= 1, np + 1, wp + 1, p + 1, \\
b_\sigma &= p - 1, ((n-1)/u)p - 1, (n-2)p - 1, \\
c &= (np + 1)/t, (wp + 1)/t, (p + 1)/t, (p - 1)/t, \\
&= (\frac{n-1}{u}p - 1)/t, ((n-2)p - 1)/t.
\end{align*}
\]

Since \( n = p + 2 \) is represented as \( n = \frac{wp + w^2 + u + 1}{u + 1} \), we have \( p = u^2 - u - 1 \) (this means \( u \geq 3 \)). Using these relations of \( n \) and \( p \), we can simplify some of above values such that

\[
\begin{align*}
\alpha_\sigma &= 1, np + 1, wp + 1, p + 1, \\
b_\sigma &= p - 1, ((n-1)/u)p - 1 = (p + u)p/(u + 1) - 1 = (u - 1)p - 1, \\
&= ((n-2)p - 1 = p^2 - 1, \\
c &= (np + 1)/t, (wp + 1)/t, (p + 1)/t, (p - 1)/t, \\
&= (\frac{n-1}{u}p - 1)/t = ((u - 1)p - 1)/t, \\
&= ((n-2)p - 1)/t = (p^2 - 1)/t.
\end{align*}
\]

Now we shall eliminate the above values of degrees one by one.

If \( \mathcal{G} \) possesses the irreducible representations \( \mathcal{I} \) of degree \( p + 1 \), then we can decompose the character \( \zeta_\mathcal{I} \) of \( \mathcal{I} \) in the normalizer \( \mathfrak{N}(\mathfrak{P}) = \{P, Q\} \) of \( p \)-Sylow subgroup \( \mathfrak{P} \) into its irreducible constituents. But it is easy to find all irreducible characters of the group \( \mathfrak{N}(\mathfrak{P}) \) of order \( p(p - 1)/t = pq \). Let \( \omega \) be a primitive \( q \)-th root of unity. We then have \( q \) linear characters \( \omega_\mu, (\mu = 0, 1, 2, \ldots, q - 1) \) defined by

\[
\omega_\mu(Q^j) = \omega^{\mu j}, \quad \omega_\mu(P^j) = 1.
\]

\(^4\) Furthermore, by considering the automorphism of \( \mathcal{G} \) induced by the element of \( \mathfrak{Q} \), we can find \( p + 1 = 2^a \) and \( \mathcal{G} \) must be an abelian group of type \( (2,2,\ldots,2) \). Thus in the case \( t \equiv 0 \pmod{2} \), the structure of the non-simple group \( \mathcal{G} \) is determined: that is, \( \mathcal{G} \) contains an abelian normal subgroup of type \( (2,2,\ldots,2) \) and the factor-group \( \mathcal{G}/\mathcal{G} \cong LF(2, p) \) and \( p = 2^{a - 1} \). This remark is due to Mr. N. Itô.

\(^5\) Cf. [B], Theorem 7.
Besides, we have \( t \) conjugate characters \( Y^{(\gamma)} \) of degree \( q \) such that \( Y^{(\gamma)}(Q') = 0 \) for \( j \equiv 0 \pmod{q} \).

By \([B]\), Lemma 3, \( \mathcal{S}(N) \) (\( N \) in \( \mathfrak{R}(\mathfrak{F}) \)) contains only two linear characters: \( \mathcal{S}(N) = \omega_p(N) + \omega_q(N) + \sum Y^{(\gamma)}(N) \). So the determinant of \( \mathfrak{H}(Q') \) (\( j \equiv 0 \pmod{q} \)) has the value

\[
\omega^{(p+1)}(N) = \omega^{(p+1)}(N) = (-1)^{(p+1)}t = \omega^{(p+1)}(N) = (-1)^t
\]

Since \( t \) is odd, we have \( |\mathfrak{H}(Q')| = -\omega^{(p+1)} \). But since the determinant of \( \mathfrak{H}(G) \) (\( G \) in \( \mathfrak{G} \)) forms a representation of degree 1 of \( \mathfrak{G} \), this value must be equal to 1 for all \( j \equiv 0 \pmod{q} \). This is obviously impossible, except the case \( q = (p-1)/t = 2 \). But in this excluded case, if \( \mathfrak{G} \) possesses the irreducible representation of degree \( p+1 \), then by (2)

\[
e = ((u-1)p-1)/t \text{ or } (p^2-1)/t.
\]

If \( e = ((u-1)p-1)/t \), then by (2), \( 1 = (u-2)/t \). But since \( p-1 = u^2-u-2 \) and \( (p-1)/t = 2 \), we have \( u+1 = 2 \). This is impossible. If \( e = (p^2-1)/2 \), then by (2), \( 1 = (p-1)/t \). This is impossible.

Thus \( \mathfrak{G} \) does not possess the irreducible representation of degree \( p+1 \).

Since \( t \) is odd, \( \mathfrak{G} \) does not possess the representations of degree \( p-1, (p-1)/t \) and \( (p+1)/t \). Furthermore, according to the relation (3), \( \mathfrak{G} \) does not possess the irreducible representations of degree \( np+1 \) and \( (np+1)/t \).

If \( \mathfrak{G} \) possesses the representations of degree \( p^2-1 \), then we can assume that the first \( p \)-block \( B_p(p) \) contains one character of degree 1, \( x \) characters of degree \( wp+1 \), \( y_1 \) characters of degree \( (u-1)p-1 \), \( y_2 \) characters of degree \( p^2-1 \) and \( t \) conjugate characters of degree \( (wp+\delta)/t \). From (3), we have

\[
u^2 x + (u-1)^2 y_1 + p^2 y_2 + w^2/t \leq (p^2+p-1)/t.
\]

Now it is sufficient to draw a contradiction only in the case \( t = 1 \). For, if \( t \geq 3 \), then above inequality shows \( p^2 y_2 \leq (p^2+p-1)/3 \). This is impossible.

Let \( t = 1 \). In this case the character \( C^{(\gamma)}(G) \) is considered as one of those \( A_p(G) \) (\( p \neq 1 \)) or \( B_p(G) \). So we again assume that \( B_p(p) \) consists of one character of degree 1, \( x \) characters of degree \( wp+1 \), \( y \) characters of degree \( (u-1)p-1 \) and \( y_2 \) characters of degree \( p^2-1 \), where \( 1+x+y_1+y_2=p \). From (3), we have

6) Cf. The relation (2)'.

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From (2), \( x \neq 0 \). Then we have
\[
\begin{align*}
    u^2 + y_1 + p^2 &\leq p^2 + p - 1, \\
    u^2 + y_1 &\leq p - 1 = u^2 - u - 2.
\end{align*}
\]
This is impossible.

Thus \( \emptyset \) does not possess the irreducible representations of degree \( p^2 - 1 \). Furthermore, above calculations show that \( \emptyset \) does not possess the representations of degree \( (p^2 - 1)/t \).

Then, it remains only the following cases to be considered; \( B_1(p) \) consists of one character of degree \( 1 \), \( x \) characters of degree \( wp + 1 \), \( y \) characters of degree \( (u-1)p-1 \) and either \( t \) characters of degree \( (wp+1)/t \) or those of degree \( ((u-1)p-1)/t \).

Case A: \( B_1(p) \) contains the characters of degree \( (wp+1)/t \).

If \( x = 0 \), then from (2)',
\[
(u+1)/t = y(u-1).
\]
But from the relations \( 1 + x + y = (p-1)/t \) and \( p = u^2 - u - 1 \), we have
\[
\begin{align*}
    (u+1)/t &= (u^2 - u - 2)(u-1)/t - (u-1), \\
    t(u-1) &= (u+1)(u^2 - 3u + 1), \\
    (u+1)/t &= (u-1)/(u^2 - 3u + 1).
\end{align*}
\]
Since \( (u+1)/t \geq 1 \), \( u-1 \geq u^2 - 3u + 1 \). Hence \( u = 3 \). But \( u+1 \equiv 0 \pmod{t} \). This contradicts \( t \equiv 0 \pmod{2}\). Thus we can assume \( x \geq 1 \).

The degree \( a_p \) must divide the order \( g \) of \( \emptyset \). But using \( p = u^2 - u - 1 \), \( a_p \) and \( g \) are decomposed into the forms:
\[
\begin{align*}
    a_p &= wp + 1 = (u-1)^2(u+1), \\
    g &= (u+1)(p-1)(p+1)/t = (u^2 - u - 1)(u-2)(u+1)u^2(u-1)^2/t.
\end{align*}
\]
This gives \( (u-2)u^2 \equiv 0 \pmod{t} \). But since \( u+1 \equiv 0 \pmod{t} \), \( t = 3 \) or \( t = 1 \).

If \( t = 1 \), then by (2)' and by (1), we have
\[
ux + u + 1 = y(u-1) \text{ and } 1 + x + y = p - 1.
\]
So
\[
\begin{align*}
    u(u^2 - u - 3 - y) + u + 1 &= y(u-1), \\
    y(2u-1) &= u^3 - u^2 - 2u + 1.
\end{align*}
\]

7) If \( t=1 \), then the character of type \( C \) may be considered as one of those of type \( A_p \) or of type \( B_\sigma \). So even in this case \( B_1(p) \) contains the character of degree \( wp+1 \).
Since \((u^3-u^2-2u+1), (2u-1)) = 1\), such \(y\) cannot be a rational integer.

If \(t = 3\), then we have
\[
ux + (u+1)/3 = y(u-1) \quad \text{and} \quad 1 + x + y = (p-1)/3.
\]

So
\[
u((u^2-u-2)/3) - y - 1 + (u+1)/3 = y(u-1),
\]
\[
3y(2u-1) = u^3-u^2-4u+1.
\]

But \(8(u^3-u^2-4u+1) = (2u-1) (4u^2-2u-17)-9\). This means \(9 \equiv 0 \pmod{2u-1}\). Hence \(u = 5\). So \(y = 3\), \(p = 19\) and \(x = 2\). Thus \(g = 2^3 \cdot 3 \cdot 5^2 \cdot 19\), \(a_1 = 1\), \(a_2 = 2^3 \cdot 3\), \(b_\sigma = 5^2 \cdot 3\) and \(c = 2^5\). Since the characters \(A_\sigma(G)\) and \(C^{(\psi)}(G)\) are of highest kind for 2 and since \(B_\sigma(G)\) is of highest kind for 3 and furthermore since the normalizer \(N(\Psi)\) of \(p\)-Sylow subgroup \(\Psi\) contains an element \(Q\) of order \((p-1)/t = 6\), we have
\[
A_\psi(Q) = 1, \quad A_\psi(Q) = 0, \quad B_\psi(Q) = 0 \quad \text{and} \quad C^{(\psi)}(Q) = 0.
\]

This contradicts (2).

Case B: \(B(p)\) contains the characters of degree \(((u-1)p-1)/t\).

If \(y = 0\), then from (1) and (2)'
\[
u((p-1)/t-1) = (u-2)/t,
\]
\[
u^3-u^2-3u+2 = ut.
\]

So \(2 \equiv 0 \pmod{u}\). This is impossible. Thus we can assume \(y \geq 1\).

As in the case \(A\), since \(b_\sigma = (u-1)p-1 = u(u-2)\) must divide the order \(g\) of \(\Psi\), we have \(t = 3\) or \(t = 1\).

If \(t = 1\), then from (2)' and (1), we have
\[
ux = y(u-1)+u-2 \quad \text{and} \quad 1 + x + y = p-1.
\]

So
\[
u(u^3-u^2-3-y) = y(u-1)+u-2,
\]
\[
y(2u-1) = u^3-u^2-4u+2.
\]

But such \(y\) cannot be a rational integer.

If \(t = 3\), then we have
\[
ux = y(u-1)+(u-2)/3 \quad \text{and} \quad 1 + x + y = (p-1)/3.
\]

So
\[
3y((p-1)/3-y-1) = 3y(u-1)+u-2,
\]
\[
3y(2u-1) = u^3-u^2-6u+2.
\]

But such \(y\) cannot be a rational integer.

Thus, in the case \(n = p+2\), such group can not exist.

Combining this with the previous result, we get the Theorem.

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