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SELF-DUAL GENERALIZED TAUB-NUT METRICS

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0. Introduction

The Taub-NUT metrics were found by Taub [10] in 1951, and extended by Newmann-Unti-Tamburino [8] in 1963. They are Ricci-flat self-dual metrics on $\boldsymbol{R}^4$ and were given by Hawking [4] as a non-trivial example of the gravitational instantons which he advocated in 1977. Iwai-Katayama [5] generalized the Taub-NUT metrics in the following way. Suppose that a metric $\check{g}$ on an open interval U in $(0, +\infty)$ and a family of Berger metrics $\hat{g}(r)$ on S^3 indexed by U are given, where a Berger metric is by definition a right invariant metric on $S^3 = Sp(1)$ which is further left $U(1)$-invariant. Then the twisted product $g = \check{g} + \hat{g}(r)$ on the annulus $U \times S^3 \subset \boldsymbol{R}^4 \setminus \{0\}$ is called a generalized Taub-NUT metric. In [5] they explicitly wrote out the conformally flat ones among those metrics and gave a condition for a generalized Taub-NUT metric to be self-dual in terms of a certain ordinary differential equation.

In this paper we will solve this differential equation and explicitly express all the self-dual generalized Taub-NUT metrics. As an application we get a family of complete Einstein self-dual generalized Taub-NUT metrics on 4-balls. This reproduces the Einstein self-dual metrics found by Pedersen [9] which have Berger spheres as conformal infinities.

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1. Calculations of the curvatures

Let $\boldsymbol{R}^4$ denote the Euclidean 4-space with the standard metric endowed with the standard orientation. Let (x, y, z, w) be the standard coordinates on $\boldsymbol{R}^4$ and r the square of the distance from the origin, that is, $r = x^2 + y^2 + z^2 + w^2$. Let us introduce a positively oriented orthonormal frame of the cotangent bundle of $\boldsymbol{R}^4 \setminus \{0\}$ as follows:

$$\left(\frac{dr}{2\sqrt{r}} \quad \sqrt{r}\sigma_1 \quad \sqrt{r}\sigma_2 \quad \sqrt{r}\sigma_3\right)=(dx \quad dy \quad dz \quad dw)\frac{1}{\sqrt{r}}\begin{pmatrix} x & -w & z & -y \\ y & -z & -w & x \\ z & y & -x & -w \\ w & x & y & z \end{pmatrix}$$

Then it holds that

$$d\sigma_1=2\sigma_2\wedge\sigma_3,$$
$$d\sigma_2=2\sigma_3\wedge\sigma_1,$$
$$d\sigma_3=2\sigma_1\wedge\sigma_2,$$

and σ_1, σ_2, σ_3 are the pullbacks of a basis of the space of the right invariant 1-forms on $Sp(1)=S^3$ by the standard projection of $\boldsymbol{R}^4\backslash\{0\}$ to S^3. Using these forms, the standard metric on $\boldsymbol{R}^4\backslash\{0\}$ is expressed as

$$\frac{1}{4r}(dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+r^2(2\sigma_3)^2),$$

which can be extended to the origin of $\boldsymbol{R}^4$.

DEFINITION. Let m be a positive constant. The metric on $\boldsymbol{R}^4\backslash\{0\}$

$$\left(1+\frac{4m}{r}\right)(dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2)+\frac{(4m)^2}{1+\frac{4m}{r}}(2\sigma_3)^2$$

is called the *Taub-NUT metric*. It is known that this metric can be extended to the origin and is a Ricci-flat self-dual metric on $\boldsymbol{R}^4$.

DEFINITION (Iwai-Katayama [5]). Let f, g be positive C^∞-functions on an open set U in the open half-line $(0, +\infty)$, and let D be an open set in the multi-annulus $B=\{p\in\boldsymbol{R}^4\backslash\{0\}\,;\ r(p)\in U\}$. Then the metric

$$f(r)(dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2)+g(r)(2\sigma_3)^2$$

on D is called a *generalized Taub-NUT metric*.

Let f, g, h be C^∞-functions on an open set U in $(0, +\infty)$. Now we will calculate the curvatures of the metric

$$dr^2+e^{2f(r)}(\sigma_1)^2+e^{2g(r)}(\sigma_2)^2+e^{2h(r)}(\sigma_3)^2$$

on an open set D in $B=\{p\in\boldsymbol{R}^4\backslash\{0\}\,;\ r(p)\in U\}$. We define positively oriented orthonormal 1-forms θ^i, $i=1, \cdots, 4$, by

$$\begin{aligned}\theta^1&=dr,\\ \theta^2&=e^{f(r)}\sigma_1,\\ \theta^3&=e^{g(r)}\sigma_2,\\ \theta^4&=e^{h(r)}\sigma_3.\end{aligned}$$

Let (ω^i_j) be the connection forms of the Levi-Civita connection, then

$$d\theta^i+\sum_{j=1}^{4}\omega^i_j\wedge\theta^j=0,\quad \text{for } i=1,\cdots,4.$$

From this equation we find that the connection forms are given as follows:

$$(\omega^i_j)=\begin{pmatrix}0 & -f'(r)\theta^2 & -g'(r)\theta^3 & -h'(r)\theta^4\\ f'(r)\theta^2 & 0 & C(r)\theta^4 & -B(r)\theta^3\\ g'(r)\theta^3 & -C(r)\theta^4 & 0 & A(r)\theta^2\\ h'(r)\theta^4 & B(r)\theta^3 & -A(r)\theta^2 & 0\end{pmatrix},$$

where functions $A(r)$, $B(r)$, $C(r)$ are defined by

$$\begin{aligned}A(r)&=-e^{f(r)-g(r)-h(r)}+e^{-f(r)+g(r)-h(r)}+e^{-f(r)-g(r)+h(r)},\\ B(r)&=e^{f(r)-g(r)-h(r)}-e^{-f(r)+g(r)-h(r)}+e^{-f(r)-g(r)+h(r)},\\ C(r)&=e^{f(r)-g(r)-h(r)}+e^{-f(r)+g(r)-h(r)}-e^{-f(r)-g(r)+h(r)},\end{aligned}$$

and the prime' denotes the derivation with respect to r.

The curvature forms Ω^i_j are given by

$$\Omega^i_j=d\omega^i_j+\sum_{k=1}^{4}\omega^i_k\wedge\omega^k_j,\quad \text{for } i,j=1,\cdots,4.$$

By a straightforward calculation, we obtain

$$\begin{aligned}\Omega^2_1&=\{f''(r)+(f'(r))^2\}\theta^1\wedge\theta^2\\ &\quad-\{B(r)(h'(r)-f'(r))-C(r)(f'(r)-g'(r))\}\theta^3\wedge\theta^4,\\ \Omega^3_1&=\{g''(r)+(g'(r))^2\}\theta^1\wedge\theta^3\\ &\quad-\{C(r)(f'(r)-g'(r))-A(r)(g'(r)-h'(r))\}\theta^4\wedge\theta^2,\\ \Omega^4_1&=\{h''(r)+(h'(r))^2\}\theta^1\wedge\theta^4\\ &\quad-\{A(r)(g'(r)-h'(r))-B(r)(h'(r)-f'(r))\}\theta^2\wedge\theta^3,\end{aligned}$$

$$\begin{aligned}\Omega^3_4&=(A'(r)+A(r)f'(r))\theta^1\wedge\theta^2\\ &\quad+(-B(r)C(r)+C(r)A(r)+A(r)B(r)-g'(r)h'(r))\theta^3\wedge\theta^4,\\ \Omega^4_2&=(B'(r)+B(r)g'(r))\theta^1\wedge\theta^3\\ &\quad+(B(r)C(r)-C(r)A(r)+A(r)B(r)-h'(r)f'(r))\theta^4\wedge\theta^2,\\ \Omega^2_3&=(C'(r)+C(r)h'(r))\theta^1\wedge\theta^4\\ &\quad+(B(r)C(r)+C(r)A(r)-A(r)B(r)-f'(r)g'(r))\theta^2\wedge\theta^3,\end{aligned}$$

and the other curvature forms are known from $\Omega^i_j=-\Omega^j_i$.

The components R_{ijkl} of the Riemannian curvature tensor are given by

$$R_{ijkl}=\sum_{m=1}^{4}\delta_{im}R^{m}{}_{jkl}=R^{i}{}_{jkl}=\Omega_j^i(e_k,\ e_l)$$

for $i,\ j,\ k,\ l=1,\ \ldots,\ 4$, where δ_{im} denotes Kronecker's delta and $\{e_k\}$ denotes the dual frame of $\{\theta^i\}$. From the curvature forms above, one easily gets the explicit forms of R_{ijkl}.

The components R_{ij} of the Ricci tensor and the scalar curvature R are given by

$$R_{ij}=\sum_{k=1}^{4}R^{k}{}_{ikj},\quad \text{for } i,\ j=1,\ \cdots,\ 4,$$
$$R=\sum_{i=1}^{4}R_{ii}.$$

By an easy calculation, we get

$$\begin{aligned}
&R_{11}=-\{f''(r)+g''(r)+h''(r)+(f'(r))^2+(g'(r))^2+(h'(r))^2\},\\
&R_{22}=-f''(r)-f'(r)(f'(r)+g'(r)+h'(r))+2B(r)C(r),\\
&R_{33}=-g''(r)-g'(r)(f'(r)+g'(r)+h'(r))+2C(r)A(r),\\
&R_{44}=-h''(r)-h'(r)(f'(r)+g'(r)+h'(r))+2A(r)B(r),\\
&R_{ij}=0,\quad \text{for } i\neq j,
\end{aligned}$$

$$\begin{aligned}
R=&-2(f''(r)+g''(r)+h''(r))-\{(f'(r))^2+(g'(r))^2+(h'(r))^2\}\\
&-(f'(r)+g'(r)+h'(r))^2+2(B(r)C(r)+C(r)A(r)+A(r)B(r)).
\end{aligned}$$

The components C_{ijkl} of the Weyl tensor are given (cf. Eisenhart [3]) by

$$C_{ijkl}=R_{ijkl}-\frac{1}{2}(\delta_{ik}R_{jl}-\delta_{il}R_{jk}+\delta_{jl}R_{ik}-\delta_{jk}R_{il})+\frac{1}{6}R(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})$$

for $i,\ j,\ k,\ l=1,\ \cdots,\ 4$. We define a frame $\{\omega_+{}^i\}$ (resp., $\{\omega_-{}^i\}$) of the bundle of self-dual (resp., anti-self-dual) cotangent 2-vectors as follows:

$$\begin{aligned}
\omega_+{}^1&=\theta^1\wedge\theta^2+\theta^3\wedge\theta^4,\\
\omega_+{}^2&=\theta^1\wedge\theta^3+\theta^4\wedge\theta^2,\\
\omega_+{}^3&=\theta^1\wedge\theta^4+\theta^2\wedge\theta^3,\\
\omega_-{}^1&=\theta^1\wedge\theta^2-\theta^3\wedge\theta^4,\\
\omega_-{}^2&=\theta^1\wedge\theta^3-\theta^4\wedge\theta^2,\\
\omega_-{}^3&=\theta^1\wedge\theta^4-\theta^2\wedge\theta^3.
\end{aligned}$$

Let W^+ and W^- be the self-dual part and the anti-self-dual part of the Weyl tensor, respectively. A tedious calculation results in

$$\begin{aligned}
W^+=&-2(C_{1313}+B'(r)+B(r)g'(r))(\omega_+{}^1\otimes\omega_+{}^1-\omega_+{}^2\otimes\omega_+{}^2)\\
&-2(C_{1414}+C'(r)+C(r)h'(r))(\omega_+{}^1\otimes\omega_+{}^1-\omega_+{}^3\otimes\omega_+{}^3),\\
W^-=&-2(C_{1313}-B'(r)-B(r)g'(r))(\omega_-{}^1\otimes\omega_-{}^1-\omega_-{}^2\otimes\omega_-{}^2)\\
&-2(C_{1414}-C'(r)-C(r)h'(r))(\omega_-{}^1\otimes\omega_-{}^1-\omega_-{}^3\otimes\omega_-{}^3),
\end{aligned}$$

where C_{1313}, C_{1414} are given by

$$C_{1313}=\frac{1}{6}(2B(r)C(r)-4C(r)A(r)+2A(r)B(r)+f''(r)-2g''(r)+h''(r)$$
$$+(f'(r))^2-2(g'(r))^2+(h'(r))^2+g'(r)h'(r)-2h'(r)f'(r)+f'(r)g'(r)),$$
$$C_{1414}=\frac{1}{6}(2B(r)C(r)+2C(r)A(r)-4A(r)B(r)+f''(r)+g''(r)-2h''(r)$$
$$+(f'(r))^2+(g'(r))^2-2(h'(r))^2+g'(r)h'(r)+h'(r)f'(r)-2f'(r)g'(r)).$$

From the Weyl tensor obtained above, we have

Proposition 1.1. *The metric*

$$g=dr^2+e^{2f(r)}(\sigma_1)^2+e^{2g(r)}(\sigma_2)^2+e^{2h(r)}(\sigma_3)^2$$

is self-dual if and only if

$$\begin{cases} C_{1313}-B'(r)-B(r)g'(r)=0, \\ C_{1414}-C'(r)-C(r)h'(r)=0 \end{cases}$$

on the open set $U'=\{r(p)\,;\ p\in D\}$ of U. The metric g is anti-self-dual if and only if

$$\begin{cases} C_{1313}+B'(r)+B(r)g'(r)=0, \\ C_{1414}+C'(r)+C(r)h'(r)=0 \end{cases}$$

on U'.

Applying the proposition to a generalized Taub-NUT metric, we easily see the following.

Corollary 1.2 (Iwai-Katayama [5]). *Let f, h be positive C^∞-functions on an open set U in $(0, +\infty)$ and g the generalized Taub-NUT metric*

$$f(r)(dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+(h(r))^2(2\sigma_3)^2)$$

on an open set D in $B=\{p\in \boldsymbol{R}^4\backslash\{0\}\,;\ r(p)\in U\}$. Then:

(1) *The metric g is self-dual if and only if $h(r)$ satisfies the ordinary differential equation*

$$rh'(r)-r^2h''(r)-\frac{1}{r^2}(h(r))^3-\frac{3}{r}(h(r)-rh'(r))h(r)=0 \tag{1.1}$$

on $U'=\{r(p)\,;\ p\in D\}$;

(2) *The metric g is anti-self-dual if and only if $h(r)$ satisfies the equation*

$$rh'(r)-r^2h''(r)-\frac{1}{r^2}(h(r))^3+\frac{3}{r}(h(r)-rh'(r))h(r)=0$$

on U' ;

(3) *The metric* g *is conformally flat if and only if*

$$h(r)=r$$

on U'.

REMARK. From conformal invariance of Weyl tensors (of type (1,3)), it follows that if the metric g in Corollary 1.2 is self-dual, then the self-dual part W^+ of the Weyl tensor of g is equal to the self-dual part multiplied by $f(r)$ of the Weyl tensor of the metric $\bar{g}=dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+(h(r))^2(2\sigma_3)^2$. It is given by

$$W^+=f(r)\frac{2}{r}\left(\frac{h(r)}{r}\right)'(\omega_+{}^1\otimes\omega_+{}^1+\omega_+{}^2\otimes\omega_+{}^2-2\omega_+{}^3\otimes\omega_+{}^3),$$

where $\{\omega_+{}^i\}$ is the frame for the metric $\bar{g}$.

Corollary 1.3. *Let* f, h *be positive* C^∞*-functions on an open set* U *in* $(0, +\infty)$ *and* g *the metric*

$$f(r)\{dr^2+h(r)((2\sigma_1)^2+(2\sigma_2)^2+(2\sigma_3)^2)\}$$

on an open set D *in* $B=\{p\in\boldsymbol{R}^4\backslash\{0\}\,;\ r(p)\in U\}$. *Then the metric* g *is always self-dual, and moreover it is conformally flat.*

2. Self-dual generalized Taub-NUT metrics

We will give explicit expression of self-dual generalized Taub-NUT metrics. From these explicit forms some properties of metrics will be obtained.

Theorem 2.1. (1) *Let* f *be a positive* C^∞*-function on an open set* V *in* $(0, +\infty)$ *and* a, b, c *constants. Suppose that the open set*

$$U=\{r\in V\,;\ (c-ar^2)(c+2br+ar^2)>0\}$$

in $(0, +\infty)$ *is not empty, in particular,* $(a, c)\neq(0, 0)$. *Then the metric*

$$f(r)\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+\left(\frac{r(c-ar^2)}{c+2br+ar^2}\right)^2(2\sigma_3)^2\}$$

on an open set D *in* $B=\{p\in\boldsymbol{R}^4\backslash\{0\}\,;\ r(p)\in U\}$ *is self-dual.*

(2) *Any self-dual generalized Taub-NUT metric* g *on a connected open set* D *in* $\boldsymbol{R}^4\backslash\{0\}$ *is expressed either as*

$$g=f(r)\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+\left(\frac{r}{1+br}\right)^2(2\sigma_3)^2\}$$

on D, *where* f *is a positive* C^∞*-function on* $U=\{r(p)\,;\ p\in D\}$ *and* b *is a constant*; *or as*

$$f(r)\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+\left(\frac{r(c-r^2)}{c+2br+r^2}\right)^2(2\sigma_3)^2\}$$

on D, where f is a positive C^∞-function on U and b, c are constants. In the first case $1+br$ is positive everywhere on D, and in the second case $(c-r^2)(c+2br+r^2)$ is positive everywhere on D.

Proof. (1) It is verified that the positive function $h(r)=r(c-ar^2)/(c+2br+ar^2)$ on U satisfies the equation (1.1) in Corollary 1.2.

(2) We will solve the equation (1.1) in Corollary 1.2. Let the metric g be of the form in Corollary 1.2 and set

$$U=\{r(p)\,;\ p\in D\},$$

which is an open interval, possibly unbounded, in $(0, +\infty)$. Then the positive C^∞-function $h(r)$ on U satisfies the equation (1.1). Let

$$y(r)=\frac{r-h(r)}{rh(r)} \quad \text{on } U. \tag{2.1}$$

Then $y(r)$ satisfies the following differential equation:

$$-y(r)y'(r)+y''(r)+ry(r)y''(r)-2r(y'(r))^2=0 \quad \text{on } U. \tag{2.2}$$

In the case that $y'(r)\equiv 0$ on U, $y(r)$ satisfies the equation (2.2) and

$$y(r)=b \quad \text{on } U,$$

where b is a constant. Then we have

$$h(r)=\frac{r}{1+br} \quad \text{on } U.$$

In the case that $y'(r)\not\equiv 0$ on U, let U' be a connected component of the set $\{r\in U\,;\ y'(r)\neq 0\}$. Let

$$z(r)=\frac{1+ry(r)}{y'(r)} \quad \text{on } U', \tag{2.3}$$

then from (2.2) $z(r)$ satisfies

$$z'(r)+r=0 \quad \text{on } U'.$$

Thus,

$$z(r)=-\frac{1}{2}(r^2-c) \quad \text{on } U',$$

where c is a constant. Then from (2.3) $y(r)$ satisfies

$$\frac{1+ry(r)}{y'(r)}=-\frac{1}{2}(r^2-c) \quad \text{on } U'. \tag{2.4}$$

Let

$$w(r)=-\frac{1}{2}(r^2-c)y(r) \quad \text{on } U'. \tag{2.5}$$

Then from (2.4) $w(r)$ satisfies

$$w'(r)=1 \quad \text{on } U'.$$

Thus

$$w(r)=r+b \quad \text{on } U',$$

where b is a constant. From (2.5) we have

$$y(r)=\frac{2b+2r}{c-r^2} \quad \text{on } U',$$

and hence, from (2.1) we have

$$h(r)=\frac{r(c-r^2)}{c+2br+r^2} \quad \text{on } U'.$$

Since $h(r)$ is a positive C^∞-function on U and we have

$$y'(r)=\frac{2(c+2br+r^2)}{(c-r^2)^2} \quad \text{on } U',$$

there is no point r_0 in U such that $y'(r_0)=0$. Thus $U'=U$ and hence,

$$h(r)=\frac{r(c-r^2)}{c+2br+r^2} \quad \text{on } U.$$

This completes the proof for the second case. Note that both in the first and the second case $h(r)$ is written in the unified form as in (1). Q.E.D.

In the theorem above we have explicitly expressed all the self-dual generalized Taub-NUT metrics. We next will consider the Einstein property, the extendability to the origin of $\boldsymbol{R}^4$ and the completeness of our metrics. And we will classify the non-conformally flat complete Einstein self-dual generalized Taub-NUT metrics on 4-balls or on $\boldsymbol{R}^4$.

In the following proposition, we will not specify the domains of definition of metrics to avoid complexity.

Proposition 2.2. *Let k be a C^∞-function on an open interval in* $(0, +\infty)$.
(1) *Let b, c be constants such that $(b, c)\neq(0, 0)$. Then the metric*

$$\bar{g}=e^{2k(r)}\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+\left(\frac{r(c-r^2)}{c+2br+r^2}\right)^2(2\sigma_3)^2\}$$

on a connected open set in $\boldsymbol{R}^4\backslash\{0\}$ *is Einstein if and only if*

$$e^{2k(r)}=\alpha\frac{c+2br+r^2}{r(bc+2cr+br^2)^2},\quad \alpha \text{ a non-zero constant.} \tag{2.6}$$

In this case the metric $\bar{g}$ *has the scalar curvature* $24bc/\alpha$.

(2) *Let* b *be a non-zero constant. Then the metric*

$$\bar{g}=e^{2k(r)}\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+\left(\frac{r}{1+br}\right)^2(2\sigma_3)^2\}$$

on a connected open set in $\boldsymbol{R}^4\backslash\{0\}$ *is Einstein if and only if*

$$e^{2k(r)}=\alpha\frac{1+br}{r},\quad \alpha \text{ a non-zero constant.} \tag{2.7}$$

In this case the metric $\bar{g}$ *is Ricci-flat.*

(3) (Iwai-Katayama [5]) *The metric*

$$\bar{g}=e^{2k(r)}\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+r^2(2\sigma_3)^2\}$$

on a connected open set in $\boldsymbol{R}^4\backslash\{0\}$ (*which is conformally flat from* Corollary 1.2) *is Einstein if and only if*

$$e^{2k(r)}=\frac{1}{r(\beta+\gamma r)^2},$$

where β, γ *are constants such that* $(\beta,\gamma)\neq(0,0)$. *In this case the metric* $\bar{g}$ *has the scalar curvature* $12\beta\gamma$.

Proof. (1) We know the components R_{ij} of the Ricci tensor of the metric $g=e^{-2k(r)}\bar{g}$ from Section 1. We give orthonormal 1-forms $\bar{\theta}^i$, $i=1,\dots,4$, with respect to the metric $\bar{g}$ by

$$\bar{\theta}^i=e^{k(r)}\theta^i \quad \text{for } i=1,\dots,4.$$

Let $\bar{R}_{ij}$ be the components of Ricci tensor of $\bar{g}$ with respect to $\bar{\theta}^i$. Then (cf. Eisenhart [3])

$$\bar{R}_{ij}=e^{-2k(r)}\{R_{ij}-2(k_{i,j}-k_ik_j)-2(\sum_{l=1}^4 k_lk_l)\delta_{ij}-(\sum_{l=1}^4 k_{l,l})\delta_{ij}\}$$

for $i,j=1,\dots,4$, where k_i are defined by

$$dk=\sum_{i=1}^4 k_i\theta^i,$$

and $k_{i,j}$ are defined by

$$dk_i-\sum_{l=1}^{4}k_l\omega_i^l=\sum_{j=1}^{4}k_{i,j}\theta^j.$$

Let $\bar{R}^0_{ij}$ be the components of the traceless part of the Ricci tensor of $\bar{g}$. It is easily shown that $\bar{R}^0_{ij}=0$ for $i\neq j$. Now we will solve the simultaneous ordinary differential equations: $\bar{R}^0_{ii}=0$, $i=1,\ldots,4$. By a computation we have

$$\begin{aligned}(2.9)\qquad &\bar{R}^0_{11}+3\bar{R}^0_{22}\\ &=2(bcr+6c^2r^2+12bcr^3+8b^2r^4+2cr^4+3br^5+2bc^2r^2k'(r)\\ &\quad+4b^2cr^3k'(r)+4c^2r^3k'(r)+12bcr^4k'(r)+4b^2r^5k'(r)\\ &\quad+4cr^5k'(r)+2br^6k'(r))/\{r^2(r^2-c)(c+2br+r^2)^2e^{2k(r)}\}.\end{aligned}$$

By solving the equation: $(2.9)=0$ we see that $k(r)$ must be

$$(2.10)\qquad -\frac{1}{2}\log r+\frac{1}{2}\log(|c+2br+r^2|)-\log(|bc+2cr+br^2|)+const.$$

Thus we get (2.6). Conversely,if $k(r)$ is of the form (2.10), then it is verified that $\bar{R}^0_{ii}=0$ for $i=1,\ldots,4$.

The proof of (2) is similar to that of (1). See Iwai-Katayama [5] for the proof of (3). Q.E.D.

REMARK. It follows from (3) of Corollary 1.2 that any non-conformally flat Einstein self-dual generalized Taub-NUT metric is of the form in (1) or (2) of Proposition 2.2.

From the following lemma, we can tell whether our metric can be extended to the origin of $\mathbf{R}^4$ or not.

Lemma 2.3. *Let ψ, φ_1, φ_2, φ_3 be C^∞-functions on a neighborhood U of the origin of $\boldsymbol{R}$ which are all positive on $U\cap(0,+\infty)$. Then the metric*

$$g=\frac{\psi(r)}{4r}\{dr^2+\varphi_1(r)(2\sigma_1)^2+\varphi_2(r)(2\sigma_2)^2+\varphi_3(r)(2\sigma_3)^2\}$$

on $B=\{p\in\boldsymbol{R}^4\backslash\{0\};\ r(p)\in U\cap(0,+\infty)\}$ can be extended to the origin of $\boldsymbol{R}^4$ in class C^∞, if and only if

$$\begin{cases}\lim_{r\to0}\psi(r)=\alpha, & \alpha \text{ a positive constant,}\\ \lim_{r\to0}\dfrac{\varphi_i(r)}{r^2}=1 & \text{for } i=1,2,3.\end{cases}$$

In this case, the extended metric g at the origin 0 is α-times the standerd metric on $T_0\boldsymbol{R}^4$.

For examining the completeness of our metrics, we use the following lemma.

Lemma 2.4. *Let ψ, φ_1, φ_2, φ_3 be positive C^∞-functions on an open set U of $(0, +\infty)$ and let $B=\{p\in \boldsymbol{R}^4\backslash\{0\}\,;\ r(p)\in U\}$. If a C^∞-curve $\tau(t)$ in B lies in a straight line of $\boldsymbol{R}^4$ through the origin, and has a constant speed with respect to the metric*

$$g=\frac{\psi(r)}{4r}\{dr^2+\varphi_1(r)(2\sigma_1)^2+\varphi_2(r)(2\sigma_2)^2+\varphi_3(r)(2\sigma_3)^2\}$$

on B, then $\tau(t)$ is a geodesic of g.

For $0<\rho\leq+\infty$ we set $B(\rho)=\{p\in\boldsymbol{R}^4\,;\ r(p)<\rho\}$, which is an open 4-ball or $\boldsymbol{R}^4$.

Proposition 2.5. (1) *Let b, c be constants such that* (i) $c>0$, $-\sqrt{c}\leq b<0$, *or* (ii) $c<0$, $b<0$. *Then the self-dual generalized Taub-NUT metric*

$$\frac{-24bc(c+2br+r^2)}{r(bc+2cr+br^2)^2}\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+\left(\frac{r(c-r^2)}{c+2br+r^2}\right)^2(2\sigma_3)^2\}$$

is defined on $B((-c+c\sqrt{1-b^2/c})/b)\backslash\{0\}$ and can be extended to the origin. The extended metric on $B((-c+c\sqrt{1-b^2/c})/b)$, which will be denoted by $g(1, b, c)$, is complete and Einstein with the scalar curvature -1.

(2) *Let b be a positive constant. Then the sefl-dual generalized Taub-NUT metric*

$$\frac{b(1+br)}{4r}\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+\left(\frac{r}{1+br}\right)^2(2\sigma_3)^2\}$$

is defined on $\boldsymbol{R}^4\backslash\{0\}$ and can be extended to the origin. The extended metric on $\boldsymbol{R}^4$, which will be denoted by $g(0, b, 1)$, is complete and Ricci-flat.

(3) *Let $0<\rho\leq+\infty$. If a non-conformally flat Einstein self-dual generalized Taub-NUT metric on $B(\rho)\backslash\{0\}$ can be extended to a complete metric $\tilde{g}$ on $B(\rho)$, then either we have*

$$\tilde{g}=ag(1, b, c)$$

for some constants a, b, c such that (i)' $a>0$, $c>0$, $-\sqrt{c}\leq b<0$, *or* (ii)' $a>0$, $c<0$, $b<0$; *or we have*

$$\tilde{g}=ag(0, b, 1)$$

for some positive constants a, b.

Proof. (3) Recall the remark following Proposition 2.2. Consider first the metric

$$g=\frac{a(c+2br+r^2)}{r(bc+2cr+br^2)^2}\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+\left(\frac{r(c-r^2)}{c+2br+r^2}\right)^2(2\sigma_3)^2\}$$

in (1) of Proposition 2.2. From Lemma 2.3, it is defined on a punctured ball around the origin and can be extended to the origin if and only if $bc\neq 0$ and $ac>0$, which will be assumed in the following. Then, from Lemma 2.4 any geodesic of the extended metric $\tilde{g}$ starting from the origin is a part of a straight line through the origin. Suppose that $bc+2cr+br^2$ has positive zeros and the smaller positive zero r_0 is the smallest among the positive zeros of three functions $bc+2cr+br^2$, $c+2br+r^2$, $c-r^2$. This is just the cases (i) $c>0$, $-\sqrt{c}\leq b<0$, or (ii) $c<0$, $b<0$, and r_0 is given by $r_0=(-c+c\sqrt{1-b^2/c})/b$. Then $\tilde{g}$ is defined on $B(r_0)$ and

$$\frac{a(c+2br+r^2)}{r(bc+2cr+br^2)^2}\geq\frac{K}{4r(r_0-r)^2}$$

on $(0, r_0)$ for some positive constant K. It follows that any geodesic of $\tilde{g}$ starting from the origin can be extended infinitely, and therefore $\tilde{g}$ is complete on $B(r_0)$ (cf. Kobayashi-Nomizu [6]). Suppose that $bc+2cr+br^2$ has no positive zero or the smaller positive zero r_0 of $bc+2cr+br^2$ is larger than the smallest positive zeros of two functions $c+2br+r^2$, $c-r^2$, then $\tilde{g}$ is not complete on any ball where $\tilde{g}$ is defined. Thus we get the first case in (3).

By considering the metric in (2) of Proposition 2.2, we similarly get the second case in (3).

The assertions (1), (2) will be clear from the above arguments and Proposition 2.2. Q.E.D.

Now, we will classify our metrics in Proposition 2.5 by isometries.

Theorem 2.6. *Let $\mathcal{S}$ denote the family of the following two kinds of non-conformally flat complete Einstein self-dual generalized Taub-NUT metrics:*

(1) $$\frac{24c(c-2r+r^2)}{ar(c-2cr+r^2)^2}\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+\left(\frac{r(c-r^2)}{c-2r+r^2}\right)^2(2\sigma_3)^2\},$$

where $a>0$, and $c\geq 1$ or $c<0$, which is defined on $B(c-c\sqrt{1-1/c})$ and has the scalar curvature $-a$.

(2) $$\frac{(1+r)}{4ar}\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+\left(\frac{r}{1+r}\right)^2(2\sigma_3)^2\},\quad a>0,$$

which is defined on $\boldsymbol{R}^4$ and Ricci-flat. Then $\mathcal{S}$ is a set of complete representatives of the isometry classes of non-conformally flat complete Einstein self-dual generalized Taub-NUT metrics on 4-balls or on $\boldsymbol{R}^4$.

Proof. Let $0<\alpha_1$, $\alpha_2\leq+\infty$, and g_1, g_2 non-conformally flat complete Einstein self-dual generalized Taub-NUT metrics on $B(\alpha_1)$, $B(\alpha_2)$, respectively. Suppose that there exists an isometry φ of $(B(\alpha_1), g_1)$ onto $(B(\alpha_2), g_2)$. We can write

$$g_i=f_i(r)\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+(rk_i(r))^2(2\sigma_3)^2\}$$

by some positive C^∞-functions f_i, k_i on $(0, \alpha_i)$, for $i=1, 2$. Let W_1^+, W_2^+ be the self-dual parts of the Weyl tensors W_1, W_2 of g_1, g_2, respectively. Let $\| \ \|_1$, $\| \ \|_2$ be the norms induced by g_1, g_2, respectively. Here note that φ is orientation-preserving, and hence $\varphi^* W_2^+=W_1^+$ and $\|W_2^+(\varphi(p))\|_2=\|W_1^+(p)\|_1$ for any $p\in B(\alpha_1)$. In fact, since $\varphi^* W_2=W_1$ we have $\varphi^* W_2^+=W_1^+$ or $\varphi^* W_2^+=W_1^-$ according as φ is orientation-preserving or not. In the latter case, the self-duality of g_1 implies $W_2^+=0$, and thus $W_2=0$, which contradicts non-conformally flatness of g_2. Therefore, by Remark following Corollary 1.2 we have

$$\left|\frac{k'_2(r\circ\varphi(p))}{r\circ\varphi(p)f_2(r\circ\varphi(p))}\right|=\left|\frac{k'_1(r(p))}{r(p)f_1(r(p))}\right|$$

for any $p\in B(\alpha_1)\backslash\{0\}$. Thus, for each $\rho\in(0, \alpha_1)$, φ sends the sphere $S_\rho=\{p\in \boldsymbol{R}^4;\ r(p)=\rho\}$ into the subset

$$S'_\rho=\left\{q\in B(\alpha_2);\ \left|\frac{k'_2(r(q))}{r(q)f_2(r(q))}\right|=\left|\frac{k'_1(\rho)}{\rho f_1(\rho)}\right|\right\}$$

of $B(\alpha_2)$. We here assume that $(B(\alpha_2), g_2)$ is not of the form in (1) of (i) of Proposition 2.5 with $b=-\sqrt{c}$. Then since $k'_2(r)/rf_2(r)$ is a non-constant rational function on $[0, \alpha_2)$, S'_ρ is a finite union of spheres in $B(\alpha_2)$. The image $\varphi(S_\rho)$ is, however, actually identical with a sphere in $B(\alpha_2)\backslash\{0\}$ because φ is a diffeomorphism. Since this holds for an arbitrary $\rho\in(0, \alpha_1)$, it follows that φ sends the origin of $B(\alpha_1)$ to that of $B(\alpha_2)$, and that the map $\psi:\ (0, \alpha_1)\to(0, \alpha_2)$ determined by $\varphi(S_\rho)=S_{\psi(\rho)}$ is a strictly increasing diffeomorphism.

Let $\phi:\ T_0B(\alpha_1)=\boldsymbol{R}^4\to T_0B(\alpha_2)=\boldsymbol{R}^4$ be the differential of φ at 0. Since the metrics g_1, g_2 at 0 are the multiples of the standard metric of $T_0\boldsymbol{R}^4=\boldsymbol{R}^4$, we have $\phi\in SO(4)$. Furthermore, any geodesic of g_i, $i=1, 2$, starting from 0 lies in a straight line of $\boldsymbol{R}^4$ through 0. It follows that under the identifications $B(\alpha_i)\backslash\{0\}=(0,\alpha_i)\times S^3$, $i=1, 2$, φ is identical with the product map $\psi\times\phi$. Now by elementary linear algebra we can prove: Let σ_1, σ_2, σ_3 be an orthonormal basis of the space of right-invariant 1-forms on $S^3=Sp(1)$, and let $g_i=a_i(\sigma_1)^2+a_i(\sigma_2)^2+b_i(\sigma_3)^2$, a_i, $b_i>0$ for $i=1, 2$ be two Berger metrics on S^3; if there is an element $\phi\in O(4)$ such that $\phi^*g_1=g_2$ then $g_1=g_2$, i.e., $a_1=a_2$, $b_1=b_2$. So we may assume that

$$\varphi=\psi\times id.:\ (0, \alpha_1)\times S^3\to(0, \alpha_2)\times S^3.$$

Thus

$$\begin{aligned}&f_1(r)\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+(rk_1(r))^2(2\sigma_3)^2\}\\&=g_1=\varphi^*(g_2)\end{aligned}$$

$$=f_2\circ\psi(r)\{(\psi'(r))^2dr^2+(\psi(r))^2(2\sigma_1)^2+(\psi(r))^2(2\sigma_2)^2$$
$$+(\psi(r)k_2\circ\psi(r))^2(2\sigma_3)^2\}.$$

Comparing the coefficients, we have

$$f_1(r)=f_2\circ\psi(r)(\psi'(r))^2,$$
$$f_1(r)r^2=f_2\circ\psi(r)(\psi(r))^2.$$

Thus

$$\frac{1}{r^2}=\left(\frac{\psi'(r)}{\psi(r)}\right)^2.$$

Therefore

$$\psi(r)=Cr \quad \text{or} \quad Cr^{-1},$$

where C is a constant. Since ψ is a strictly increasing diffeomorphism, we have

$$\psi(r)=Cr, \tag{2.11}$$

where C is a positive constant.

Let us consider the metrics $ag(1, b, c)$, $ag(0, b, 1)$ in (3) of Proposition 2.5. We first consider a metric $ag(1, b, c)$ with $a>0, c>0, b=-\sqrt{c}$, which is the complex hyperbolic metric on $B(\sqrt{c})$ with the scalar curvature $-a^{-1}$. Then $ag(1, -\sqrt{c}, c)$ and $a'g(1, -\sqrt{c'}, c')$ is isometric if and only if $a=a'$, $-\sqrt{c}=-C\sqrt{c'}$, $c=C^2c'$ for some positive constant C. If $a'g(1, b', c')$ is not of the type above, it follow from (2.11) that $ag(1, b, c)$ is isometric to $a'g(1, b', c')$ if and only if $a=a'$, $b=Cb'$, $c=C^2c'$ for some positive constant C. It also follows from (2.11) that $ag(0, b, 1)$ is isometric to $a'g(0, b', 1)$ if and only if $a=a'$. Thus we can choose as our representatives the metrics $a^{-1}g(1, -1, c)$ with $a>0$, and $c\geq 1$ or $c<0$, and $a^{-1}g(0, 1, 1)$ with $a>0$. These are just the family $\mathcal{S}$. Q.E.D.

We will last classify the metrics in the family $\mathcal{S}$ in Theorem 2.6 by conformally equivalence.

Theorem 2.7. *Under the notation in* Proposition 2.5, *let* $\mathcal{T}$ *denote the subfamily of* $\mathcal{S}$ *consisiting of the metrics* $g(1, -1, c)$ *with* $c\geq 1$ *or* $c<0$ *and the metric* $g(0, 1, 1)$. *Then* $\mathcal{T}$ *is a set of complete representatives of the conformally equivalence classes of metrics in* $\mathcal{S}$.

Proof. Let $0<\alpha_1, \alpha_2\leq +\infty$, and g_1, g_2 metrics on $B(\alpha_1), B(\alpha_2)$, respectively, in the family $\mathcal{S}$ in Theorem 2.6. Let g_1 and g_2 be conformally equivalent, then there exist a positive C^∞-function β on $B(\alpha_2)$ and an isometry φ of $(B(\alpha_1), g_1)$ onto $(B(\alpha_2), \beta g_2)$. Let

$$g_i=f_i(r)\{dr^2+r^2(2\sigma_1)^2+r^2(2\sigma_2)^2+(rk_i(r))^2(2\sigma_3)^2\},$$

where f_i and k_i are positive C^∞-functions on $(0, \alpha_i)$ for $i=1, 2$. We define

$$F(r)=\frac{|k_1'(r)|}{rf_1(r)}$$

for $r\in(0, \alpha_1)$. We easily see from explicit forms of $f_1(r)$, $k_1(r)$ that $F(r)$ has no zero in $(0, \alpha_1)$ and extends to 0 in class C^∞ with $F(0)>0$, which also holds for $|k_2'(r)|/rf_2(r)$. Let $\bar{g}_1=(F\circ r)g_1$ and $\bar{g}_2=(F\circ r\circ\varphi^{-1})\beta g_2$, which are metrics on $B(\alpha_1)$ and $B(\alpha_2)$, respectively, and let $\bar{W}_i^+$ be the self-dual part of the Weyl tensor of $\bar{g}_i$ for $i=1, 2$ Then φ induces an isometry of $(B(\alpha_1), \bar{g}_1)$ onto $(B(\alpha_2), \bar{g}_2)$, and thus $\|\bar{W}_2^+(\varphi(p))\|_2=\|\bar{W}_1^+(p)\|_1$ for each $p\in B(\alpha_1)$. It follows from Remark following Corollary 1.2 that $\|\bar{W}_1^+\|_1$ is constant on $B(\alpha_1)$, and therefore

$$F(r(p))\beta(\varphi(p))=\frac{K|k_2'(r\circ\varphi(p))|}{r\circ\varphi(p)\cdot f_2(r\circ\varphi(p))}$$

on $B(\alpha_1)$ for some positive constant K. If g_2 is not a complex hyperbolic metric $a^{-1}g(1, -1, 1)$, then we find that the scalar curvature of metric $\bar{g}_2$ is a non-constant rational function of r on $[0, \alpha_2)$. Therefore, doing the similar way to the proof of Theorem 2.6 on the scalar curvatures instead of the norm of the self-dual part of the Weyl tensors, we get the following: metrics $a^{-1}g(1, -1, c)$ and $a'^{-1}g(1, -1, c')$ in $\mathcal{S}$ are conformally equivalent if and only if $c=c'$, while $a^{-1}g(1, -1, c)$ and $a'^{-1}g(0, 1, 1)$ are not conformally equivalent. This implies the theorem. Q.E.D.

3. Examples

EXAMPLE 3.1. Suppose that there are given a metric $\breve{g}$ on an open interval U in $(0, +\infty)$ and a C^∞-family of biinvariant metrics $\hat{g}(r)$ on $S^3=Sp(1)$ indexed by U. Let us consider the twisted product $g=\breve{g}+\hat{g}(r)$ on the annulus $U\times S^3\subset \boldsymbol{R}^4\setminus\{0\}$, that is, $g_{(r,s)}=p_1^*(\breve{g}_r)+p_2^*(\hat{g}(r)_s)$ for $r\in U$, $s\in S^3$, where p_1, p_2 denote the canonical projections of $U\times S^3$ to U, S^3, respectively. Then is can be written as

$$g=f(r)dr^2+g(r)(\sigma_1^2+\sigma_2^2+\sigma_3^2) \tag{3.1}$$

by some positive C^∞-functions $f(r)$, $g(r)$ on U. From Corollary 1.3, g is self-dual and further conformally flat. We note that an $Sp(1)$-biinvariant metric g on the annulus $U\times S^3=U\times Sp(1)\subset\boldsymbol{R}^4\setminus\{0\}$ is of the form (3.1).

EXAMPLE 3.2. In Example 3.1, consider Berger metrics $\hat{g}(r)$ instead of biinvariant metrics. Then the twisted product $g=\breve{g}+\hat{g}(r)$ can be written as

$$g=f(r)dr^2+g(r)(\sigma_1^2+\sigma_2^2)+h(r)\sigma_3^2 \tag{3.2}$$

by some positive C^∞-functions $f(r)$, $g(r)$, $h(r)$ on U. We make a change of variable

$$R=\exp\int^r 2\sqrt{\frac{f(t)}{g(t)}}dt,$$

and set $r=\psi(R)$. Then g becomes a generalized Taub-NUT metric

$$f\circ\psi(R)\cdot(\psi'(R))^2\{dR^2+R^2(2\sigma_1)^2+R^2(2\sigma_2)^2\}+\frac{1}{4}h\circ\psi(R)(2\sigma_3)^2.$$

Thus, our definition of a generalized Taub-NUT metric is identical with that in Introduction. If g is self-dual then g can be obtained by rewriting a self-dual generalized Taub-NUT metric, which has been written out explicitly in Theorem 2.1, through a change of variable r. We note that a right $Sp(1)$-invariant and left $U(1)$-invariant metric g on $U\times S^3=U\times Sp(1)\subset \boldsymbol{R}^4\backslash\{0\}$ with respect to which U and S^3 are orthogonal to each other is of the form(3.2).

EXAMPLE 3.3. The metrics found by Pedersen [9]

$$\frac{1}{(1-r^2)^2}\left\{\frac{1+mr^2}{1+mr_4}dr^2+r^2(1+mr^2)(\sigma_2{}^2+\sigma_1{}^2)+\frac{r^2(1+mr^4)}{1+mr^2}\sigma_3{}^2\right\},\quad m\geq -1$$

are complete self-dual Einstein metrics on $B(1)$ which has the conformal infinity (cf. LeBrun [7]) of the Berger sphere $(S^3,\sigma_1{}^2+\sigma_2{}^2+1/(1+m)\sigma_3{}^2)$ if $m>-1$. They are isometric to the metrics

$$\frac{c(c-2R+R^2)}{2R(c-2cR+R^2)^2}\left\{dR^2+R^2(2\sigma_1)^2+R^2(2\sigma_2)^2+\left(\frac{R(c-R^2)}{c-2R+R^2}\right)^2(2\sigma_3)^2\right\},$$
$$c=-\frac{1}{m},$$

by the change of variable

$$r=\sqrt{\frac{2R}{1-mR^2}}.$$

These are the metrics with the scalar curvature -48 in the family $\mathcal{S}$ in Theorem 2.6.

EXAMPIE 3.4. The Eguchi-Hanson metrics [1], [2]

$$\frac{1}{1-\left(\frac{a}{r}\right)^2}dr^2+r^2\sigma_1{}^2+r^2\sigma_2{}^2+r^2\left(1-\left(\frac{a}{r}\right)^4\right)\sigma_3{}^2,\quad a>0$$

are isometric to the self-dual Ricci-flat metrics

$$\frac{a^2}{8}\frac{1+R^2}{R^3}\left\{dR^2+R^2(2\sigma_1)^2+R^2(2\sigma_2)^2+\left(\frac{R(1-R^2)}{1+R^2}\right)^2(2\sigma_3)^2\right\},$$

which are not isometric to any metrics in the family $\mathcal{S}$ in Theorem 2.6.

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