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<th>Remarks on an exclusive extension generated by a super-primitive element</th>
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Throughout this paper, all fields, rings and algebras are assumed to be commutative with unity. Our special notations are indicated below, and our general reference for unexplained technical terms is [3].

Let \( R \) be a Noetherian domain and \( K \) its quotient field. Let \( \alpha \) be an algebraic element over \( K \) with the minimal polynomial \( \phi_\alpha(X) = X^d + \eta_1X^{d-1} + \cdots + \eta_d \). In [6], we have shown that if \( R[\alpha] \cap K = R \) then \( \bigcap_{i=1}^{d-1} I_{\eta_i} \subseteq I_{\eta_d} \). Our objective of this paper is to show the converse of this result under certain assumptions, which will be established in Theorem 5.

In what follows, we use the following notations unless otherwise specified:

- \( R \): a Noetherian integral domain,
- \( K:=K(R) \): the quotient field of \( R \),
- \( L \): an algebraic field extension of \( K \),
- \( \alpha \): a non-zero element of \( L \),
- \( d=[K(\alpha):K] \),
- \( \phi_\alpha(X) = X^d + \eta_1X^{d-1} + \cdots + \eta_d \), the minimal polynomial of \( \alpha \) over \( K \).
- \( I_{(\alpha)} := \bigcap_{i=1}^{d-1}(R;\eta_i)_R \), which is an ideal of \( R \).
- \( I_a := R:_{R} a R \) for \( a \in K \).

It is clear that for \( a \in K \), \( I_{(\alpha)}=I_a \) from definitions.

- \( J_{(\alpha)} := I_{(\alpha)}(1, \eta_1, \ldots, \eta_d) \),
- \( J_{(\alpha)} := I_{(\alpha)}(1, \eta_1, \ldots, \eta_{d-1}) \).

We also use the following standard notation:

\[ DP_1(R) := \{ p \in \text{Spec}(R) | \text{depth} R_p = 1 \}. \]

Let \( R \) be a Noetherian domain and \( K \) its quotient field. Take an element \( \alpha \) in an field extension of \( K \). When \( R[\alpha] \cap K = R \), we say that \( \alpha \) is an exclusive element over \( R \) and that \( R[\alpha] \) is an exclusive extension of \( R \). Let \( \pi: R[X] \rightarrow R[\alpha] \) be the \( R \)-algebra homomorphism sending \( X \) to \( \alpha \). The element \( \alpha \) is called an anti-integral element of degree \( d \) over \( R \) if \( \text{Ker} \pi = I_{(\alpha)}\phi_\alpha(X)R[X] \). When \( \alpha \) is an anti-integral element over \( R \), \( R[\alpha] \) is called an anti-integral extension of \( R \). For \( f(X) \in R[X] \),
let $C(f(X))$ denote the ideal generated by the coefficients of $f(X)$, that is, the content ideal of $f(X)$. Note that $J_{[a]} := I_{[a]} C(φ_a(X))$, which is an ideal of $R$ and contains $I_{[a]}$. The element $α$ is called a super-primitive element of degree $d$ over $R$ if $J_{[a]} \not\in p$ for all primes $p$ of depth one. It is known that a super-primitive element is an anti-integral element (See [5] for detail).

A non-zero element $α$ in $L$ is called co-monic if $1/α$ is integral over $R$ and a polynomial $f(X) ∈ R[X]$ is co-monic if the constant term of $f(X)$ is $1$.

**Proposition 1.** Assume that $α$ is anti-integral over $R$ of degree $d$. Then the following conditions are equivalent:

(i) $α$ is co-monic;
(ii) $η_d I_{[a]} = R$;
(iii) there exists $a ∈ R$ such that $η_d = 1/a$ and $I_{[a]} = aR$;
(iv) there exists a co-monic polynomial $g(X) ∈ R[X]$ of degree $d$ with $g(α) = 0$;
(v) $R[1/α]$ is a free $R$-module of rank $d$.

Proof. (i)⇒(ii): Take a co-monic polynomial $f(X)$ in $R[X]$ satisfying $f(α) = 0$. Then $f(X) ∈ I_{[a]} φ_a(X) R[X]$ and hence $f(0) = 1 ∈ η_d I_{[a]} ⊆ R$. So we have $η_d I_{[a]} = R$.

(ii)⇒(iii): There exists $a ∈ I_{[a]}$ such that $η_d a = 1$. Hence $η_d = 1/a$ and $I_{[a]} = aR$.

(iii)⇒(iv): Since $I_{[a]} = aR$, $g(X) := aφ_a(X) ∈ R[X]$ is a required polynomial.

(iv)⇒(v): The polynomial $f(Y) := X^{−d} g(X)$ in $R[Y]$ with $Y = 1/X$ is monic in $Y$ of degree $d$ and satisfies $f(1/α) = 0$. Hence $R[1/α]$ is a free $R$-module of rank $d$ because $1/α$ is degree $d$.

(v)⇒(i) is obvious. □

Recall the following result shown in [1, Theorem 7]:

**Lemma 2.** Assume that $α$ is anti-integral over $R$ of degree $d$. Let $Δ_{R[α]/R} := \{ p ∈ \text{Spec}(R) | p R[α] = R[α] \}$ and $Γ_{J_{[a]} } := \{ p ∈ \text{Spec}(R) | p + J_{[a]} = R \}$. Then $Δ_{R[α]/R} = V(J_{[a]} ) ∩ Γ_{J_{[a]}^d}$, where $V(J_{[a]} )$ denotes $\{ p ∈ \text{Spec}(R) | J_{[a]} \subseteq p \}$.

**Remark 1.** Under the assumptions in Lemma 2, if $J_{[a]} = R$ or $\text{grade } J_{[a]} > 1$ then $p R[α] ≠ R[α]$ for all $p ∈ D_{p_d}(R)$.

**Lemma 3.** Assume that $α$ is super-primitive over $R$. The following statements are equivalent:

(i) $J_{[a]} = R$ or $\text{grade } J_{[a]} > 1$ (i.e., $J_{[a]}$ contains an $R$-regular sequence of length $>1$; see [3] for the definition);
(ii) $\bigcap_{l=1}^{d} I_{η_l} ⊆ I_{η_d}$.
Proof. (i) $\Rightarrow$ (ii): First consider the case $J_{[\alpha]} \neq R$. Suppose that there exists $a \in \bigcap_{i=1}^{d-1} I_{[\alpha]}$, but $a \notin I_{\eta \alpha}$. Then $\beta := an_d \notin R$. Since $\beta I_{[\alpha]} = (an_I)\eta_d \subseteq R$ and $\beta \eta [\alpha] = (an\eta_d) \subseteq R$, we have $\beta I_{[\alpha]}(1, \eta_1, \cdots, \eta_{d-1}) \subseteq R$. Since grade $J_{[\alpha]}$ = grade $I_{[\alpha]}(1, \eta_1, \cdots, \eta_{d-1}) > 1$, $\beta \eta \subseteq \beta I_{[\alpha]}(1, \eta_1, \cdots, \eta_{d-1}) \subseteq R$ for all $p \in Dp(R)$. Hence $\beta \in \bigcap_{i=1}^{d-1} I_{[\alpha]}$, which is a contradiction. Thus $\bigcap_{i=1}^{d-1} I_{[\alpha]} \subseteq I_{\eta \alpha}$.

Second, suppose that $J_{[\alpha]} = R$. By the assumption, there is an equation $1 = a_0 + a_1 \eta_1 + \cdots + a_{d-1} \eta_{d-1}$ with $a_i \in I_{[\alpha]}$. So $\eta_d = a_0 \eta_d + (a_1 \eta_d) \eta_1 + \cdots + (a_{d-1} \eta_d) \eta_{d-1}$. Take $x \in \bigcap_{i=1}^{d-1} I_{[\alpha]}$. Then $x \eta_d = (a_1 \eta_d) x \eta_1 + \cdots + (a_{d-1} \eta_d) x \eta_{d-1} \in R$, and hence $x \in I_{\eta \alpha}$. Hence $\bigcap_{i=1}^{d-1} I_{[\alpha]} \subseteq I_{\eta \alpha}$.

(ii) $\Rightarrow$ (i): We may assume that $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \cdots, \eta_{d-1}) = R$. Suppose that grade $J_{[\alpha]} = \text{grade } I_{[\alpha]}(1, \eta_1, \cdots, \eta_{d-1}) = 1$. Then there exists $p \in Dp_1(R)$ such that $I_{[\alpha]}(1, \eta_1, \cdots, \eta_{d-1}) \subseteq p$. Since $p \in Dp(R)$, there exists $\beta \in K$ such that $\beta \notin R$ and $I_p = p$. Then $\beta I_{[\alpha]}(1, \eta_1, \cdots, \eta_{d-1}) \subseteq I_p$. Thus $I_{[\alpha]} \beta \eta \subseteq I_{[\alpha]} \beta I_{[\alpha]}(1, \eta_1, \cdots, \eta_{d-1}) \subseteq I_p$. Hence $I_{[\alpha]} \beta \eta \subseteq R$, which shows that $J_{[\alpha]} \beta \subseteq R$. But since $\alpha$ is super-primitive over $R$, $J_{[\alpha]} \beta \neq p$ for all $p \in Dp_1(R)$. Thus $\beta \in R$, which is a contradiction. So we conclude that grade $J_{[\alpha]} > 1$.

**Proposition 4.** Assume that $\alpha$ is super-primitive and co-monic over $R$. If $\bigcap_{i=1}^{d-1} I_{[\alpha]} \subseteq I_{\eta \alpha}$ then $R[\alpha] \cap K = R$, i.e., $\alpha$ is exclusive.

Proof. Suppose that $R[\alpha] \cap K \neq R$. Then there exists $\beta \in R[\alpha] \cap K \setminus R$. Write $\beta = c_0 + c_1 \alpha + \cdots + c_d \alpha^d$ with $c_i \in R$. We shall show that $\alpha \in I_p R[\alpha]$. Since $\beta, c_0, \cdots, c_d \in K$ and $\alpha$ is an algebraic element of degree $d$ over $K$, we have $n \geq d$. From this we have:

(1) \[(1/\alpha)^n \beta = c_0 (1/\alpha)^n + c_1 (1/\alpha)^{n-1} + \cdots + c_d \in R[1/\alpha].\]

Since $\alpha$ is co-monic, $R[1/\alpha]$ is a free $R$-module of rank $d$ by Proposition 1. So we rewrite (1) and have:

\[(1/\alpha)^n \beta = b_0 (1/\alpha)^d + b_1 (1/\alpha)^{d-2} + \cdots + b_{d-1}\]

with $b_i \in R$. Hence we obtain:

(2) \[(1/\alpha)^n = \beta^{-1} b_0 (1/\alpha)^d + \beta^{-1} b_1 (1/\alpha)^{d-2} + \cdots + \beta^{-1} b_{d-1}.\]

On the other hand, since $(1/\alpha)^n \in R[1/\alpha]$, we have:

(3) \[(1/\alpha)^n = a_0 (1/\alpha)^d + a_1 (1/\alpha)^{d-2} + \cdots + a_{d-1} \text{ with } a_i \in R.\]

Comparing the coefficients of (2) with those of (3), we have $a_i = \beta^{-1} b_i$ ($0 \leq i \leq d-1$), and hence $\beta a_i = b_i \in R$. Thus $(a_0, a_1, \cdots, a_{d-1}) \in \beta$. It is obvious that $\alpha \in (a_0, a_1, \cdots, a_{d-1}) R[\alpha] \subseteq I_p R[\alpha]$ by (3). Take $p \in Dp_1(R)$ such that $I_p \subseteq p$. By (3), $1/\alpha = a_0 \alpha^{n-d} + a_1 \alpha^{n-d+1} + \cdots + a_{d-1} \alpha^{-1}$. So we get $1/\alpha \in p R[\alpha]$. Thus $p R[\alpha]$
contains $\alpha, 1/\alpha$. So we have $pR[\alpha] = R[\alpha]$, which is a contradiction because of Remark 1 and Lemma 3. Therefore $R[\alpha] \cap K = R$. □

**Remark 2.** In [6], we showed that if $\alpha$ is super-primitive over $R$ with $\eta_\alpha \in R$ then $\alpha$ is exclusive.

In the next theorem, the implication (i) $\Rightarrow$ (ii) has been shown in [6, Proposition 1] without any assumptions.

**Theorem 5.** Assume that $R$ contains an infinite field $k$ and that $\alpha$ is super-primitive over $R$. Then the following statements are equivalent:

(i) $\alpha$ is exclusive over $R$;
(ii) $\bigcap_{1}^{d-1} I_{\eta_\alpha} \subseteq I_{\eta_\alpha}$;
(iii) $\text{grade } \overline{J}_{[\alpha]} > 1$ or $\overline{J}_{[\alpha]} = R$.

**Proof.** The implication (i) $\Rightarrow$ (ii) was shown in [6, Proposition 1]. (ii) $\Leftrightarrow$ (iii) follows from Lemma 3. So we have only to prove the implication (ii) $\Rightarrow$ (i). Take $p \in D_{p_1}(R)$. If $I_{[\alpha]} \not\subseteq p$, then $\alpha$ is integral over $R_p$ and hence $R[\alpha] \cap K \subseteq R_p[\alpha] \cap K = R_p$. Consider the case $I_{[\alpha]} \subseteq p$. Since $\alpha$ is super-primitive, $J_{[\alpha]} \not\subseteq p$. Since $k$ is infinite, there exists $\lambda \in k$ such that $\alpha - \lambda$ is co-monic over $R_p$. Moreover $\alpha - \lambda$ is super-primitive by [5, (1.14)]. Hence by Proposition 4, we have $R[\alpha] \cap K = R[\alpha - \lambda] \cap K \subseteq R_p[\alpha - \lambda] \cap K = R_p[\alpha] \cap K = R_p$. Thus $R[\alpha] \cap K \subseteq \bigcap_{p \in D_{p_1}(R)} R_p = R$, and hence $R = R[\alpha] \cap K$, that is, $R[\alpha]$ is exclusive over $R$. □

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**References**


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