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## ON THE SECTIONAL CURVATURES OF $R$ -SPACES

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### Introduction

Let  $\mathfrak{g}$  be a real semi-simple Lie algebra without compact factors,  $\mathfrak{k}$  a maximal compactly imbedded subalgebra of  $\mathfrak{g}$ , and  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  the Cartan decomposition of  $\mathfrak{g}$  relative to  $\mathfrak{k}$ . We denote by  $B$  the Killing form of  $\mathfrak{g}$ . We regard the subspace  $\mathfrak{p}$  as a Euclidean space with the inner product  $\langle \cdot, \cdot \rangle$  induced by the restriction of  $B$  to  $\mathfrak{p}$ . Let  $\text{Int}(\mathfrak{g})$  be the group of inner automorphisms of  $\mathfrak{g}$  and, the Lie algebra of  $\text{Int}(\mathfrak{g})$  being identified with  $\mathfrak{g}$ ,  $K$  the connected Lie subgroup of  $\text{Int}(\mathfrak{g})$  corresponding to the Lie subalgebra  $\mathfrak{k}$  of  $\mathfrak{g}$ . Then  $K$  leaves the subspace  $\mathfrak{p}$  invariant and acts on the Euclidean space  $\mathfrak{p}$  as an isometry group. Let  $S$  be the unit sphere of  $\mathfrak{p}$  and  $N$  an orbit of an element  $H_0$  in  $S$ . Denoting by  $K^*$  the stabilizer of  $H_0$  in  $K$ , the space  $N$  may be identified with the quotient space  $K/K^*$  and is called an  $R$ -space. We always assume that  $\dim N \geq 2$ , and the space  $N$  is *substantial*, i.e. there exist no proper subspaces of  $\mathfrak{p}$  containing  $N$ . The aim of this paper is to study the sectional curvatures of  $N$  with respect to the  $K$ -invariant Riemannian metric  $\langle \cdot, \cdot \rangle$  induced by the inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{p}$ .

It is known (Takeuchi-Kobayashi [8]) that if the pair  $(K, K^*)$  is a symmetric pair, the metric  $\langle \cdot, \cdot \rangle$  on  $N = K/K^*$  coincides with the  $K$ -invariant Riemannian metric defined by a  $K$ -invariant inner product on  $\mathfrak{k}$ , and so the sectional curvatures of  $N$  are always non-negative, and  $N$  has a positive sectional curvature along each plane section if and only if the pair  $(K, K^*)$  is of rank 1.

In this paper we shall show that in general cases the space  $N$  may have both positive and negative sectional curvatures. Indeed, the curvatures are related with the restricted root system  $\mathfrak{r}$  of  $\mathfrak{g}$ . Let  $\Delta$  be a fundamental root system of  $\mathfrak{r}$ . Then a subsystem  $\Delta_1$  of  $\Delta$  corresponds to the space  $N$  (See section 3), and we have:

(I) *If the restricted root system  $\mathfrak{r}$  is irreducible and the cardinality  $|\Delta - \Delta_1|$  of  $\Delta - \Delta_1$  is not less than 2, the space  $N$  has both positive and negative sectional curvatures.*

Furthermore we shall characterize the  $R$ -spaces with strictly positive sectional

curvatures:

(II) *The space  $N$  has a strictly positive sectional curvature along each plane section if and only if  $N$  is the unit sphere  $S$  or the pair  $(K, K^*)$  is a symmetric pair of rank 1.*

The class of  $R$ -spaces includes the spaces  $K/K^*=SO(3)/Q$ ,  $SU(3)/T$ ,  $Sp(3)/Sp(1) \times Sp(1) \times Sp(1)$  and  $F_4/Spin(8)$ , where  $Q$  (resp.  $T$ ) denotes the subgroup of all diagonal matrices in  $SO(3)$  (resp. in  $SU(3)$ ). Although none of these pairs is a symmetric pair of rank 1, Wallach [9] proved that each of these spaces has a  $K$ -invariant Riemannian metric with strictly positive sectional curvatures. The characterization (II) shows that the Riemannian metric of Wallach is not the same as ours.

I wish to express my sincere gratitude to Professor M. Takeuchi for his kind guidance and encouragements.

## 1. Preliminaries

1.1. The assumptions and the notation are the same as those in Introduction. Let  $\mathfrak{a}$  be a maximal abelian subspace in  $\mathfrak{p}$ . We shall identify  $\mathfrak{a}$  with the dual space  $\mathfrak{a}^*$  of  $\mathfrak{a}$  by means of the duality defined by the Killing form  $B$  of  $\mathfrak{g}$ .

For an element  $\lambda \in \mathfrak{a}$ , we define subspaces  $\mathfrak{k}_\lambda$  and  $\mathfrak{p}_\lambda$  of  $\mathfrak{g}$  as follows:

$$(1.1) \quad \begin{cases} \mathfrak{k}_\lambda = \{X \in \mathfrak{k}; ad(H)^2 X = B(\lambda, H)^2 X, \text{ for all } H \in \mathfrak{a}\}, \\ \mathfrak{p}_\lambda = \{X \in \mathfrak{p}; ad(H)^2 X = B(\lambda, H)^2 X, \text{ for all } H \in \mathfrak{a}\} \end{cases}$$

Then  $\mathfrak{k}_{-\lambda} = \mathfrak{k}_\lambda$ ,  $\mathfrak{p}_{-\lambda} = \mathfrak{p}_\lambda$  and  $\mathfrak{p}_0 = \mathfrak{a}$ . It is known (Satake [4]) that if we put

$$\mathfrak{r} = \{\lambda \in \mathfrak{a}; \lambda \neq 0, \mathfrak{p}_\lambda \neq \{0\}\},$$

$\mathfrak{r}$  is a root system in  $\mathfrak{a}$ . The root system  $\mathfrak{r}$  is called the *restricted root system* of  $\mathfrak{g}$ . We denote by  $\mathfrak{r}^+$  the set of positive roots of  $\mathfrak{r}$  with respect to a linear order in the subspace  $\mathfrak{a}$ . Then we have the following orthogonal decompositions of  $\mathfrak{k}$  and  $\mathfrak{p}$  with respect to the Killing form  $B$  (cf. Helgason [2]).

$$(1.2) \quad \mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in \mathfrak{r}^+} \mathfrak{k}_\lambda, \quad \mathfrak{p} = \mathfrak{a} + \sum_{\lambda \in \mathfrak{r}^+} \mathfrak{p}_\lambda$$

## 2. Second fundamental forms of $R$ -spaces

2.1. In Introduction we assume that the point  $H_0 \in N$  is contained in the unit sphere  $S$ . Moreover we may assume  $H_0 \in S \cap \mathfrak{a}$  by virtue of the following lemma (cf. Helgason [2]).

**Lemma 1.** *For each element  $X \in \mathfrak{p}$ , there exists an element  $k \in K$  such that  $kX$  is contained in the subspace  $\mathfrak{a}$ .*

Let  $T_{H_0}(N)$  be the tangent space of  $N$  at  $H_0$ . We can identify the tangent space  $T_{H_0}(N)$  with a subspace of  $\mathfrak{p}$  in a canonical manner, and we have

$$T_{H_0}(N) = [\mathfrak{k}, H_0].$$

Choose a linear order in the subspace  $\mathfrak{a}$  such that  $\langle \lambda, H_0 \rangle \geq 0$  for each positive root  $\lambda \in \mathfrak{r}$  with respect to this order, and fix this order once for all. Put

$$\begin{aligned} \mathfrak{r}_1^+ &= \{\lambda \in \mathfrak{r}^+; \langle \lambda, H_0 \rangle = 0\}, \\ \mathfrak{r}_2^+ &= \{\lambda \in \mathfrak{r}^+; \langle \lambda, H_0 \rangle > 0\}. \end{aligned}$$

Then the tangent space  $T_{H_0}(N)$  and the orthogonal complement  $T_{H_0}^\perp(N)$  in  $\mathfrak{p}$  are given by

$$(2.1) \quad \begin{cases} T_{H_0}(N) = \sum_{\lambda \in \mathfrak{r}_2^+} \mathfrak{p}_\lambda, \\ T_{H_0}^\perp(N) = \mathfrak{a} + \sum_{\lambda \in \mathfrak{r}_1^+} \mathfrak{p}_\lambda. \end{cases}$$

Let

$$\alpha: T_{H_0}(N) \times T_{H_0}(N) \rightarrow T_{H_0}^\perp(N)$$

be the second fundamental form at  $H_0$  of the submanifold  $N$  of  $\mathfrak{p}$ . In the same way as in Takagi-Takahashi [6] we get the following:

**Proposition 2.** For  $X_\lambda \in \mathfrak{p}_\lambda$ ,  $Y_{\lambda'} \in \mathfrak{p}_{\lambda'}$  ( $\lambda, \lambda' \in \mathfrak{r}_2^+$ ,  $\lambda' \geq \lambda$ ), the second fundamental form  $\alpha$  is given by

$$(2.2) \quad \alpha(X_\lambda, Y_{\lambda'}) = \begin{cases} \text{(i)} & -\frac{\langle X_\lambda, Y_{\lambda'} \rangle}{\langle \lambda, H_0 \rangle} \lambda & \text{if } \lambda' = \lambda, \\ \text{(ii)} & \mathfrak{p}_\mu\text{-component of } -\frac{1}{\langle \lambda, H_0 \rangle^2} [[H_0, X_\lambda], Y_{\lambda'}] & \text{if } \lambda' > \lambda \text{ and } \mu = \lambda' - \lambda \in \mathfrak{r}_1^+, \\ \text{(iii)} & 0 & \text{otherwise.} \end{cases}$$

### 3. Sectional curvatures of $R$ -spaces

3.1. Let  $R$  be the curvature tensor at  $H_0$  of the space  $N$ . By the equation of Gauss and the flatness of the space  $\mathfrak{p}$  we have

$$(3.1) \quad \langle R(X, Y)Y, X \rangle = \langle \alpha(X, X), \alpha(Y, Y) \rangle - \langle \alpha(X, Y), \alpha(X, Y) \rangle$$

for any  $X, Y \in T_{H_0}(N)$  (cf. Kobayashi-Nomizu [3]). For  $X, Y \in T_{H_0}(N)$  we denote by  $K_{X,Y}$  the sectional curvature along the plane section spanned by  $X$  and  $Y$ ,  $X$  and  $Y$  being assumed to be linearly independent.

**Proposition 3.** If there exist roots  $\lambda, \lambda' \in \mathfrak{r}_2^+$  such that  $\langle \lambda, \lambda' \rangle < 0$ , then the space  $N$  has both positive and negative sectional curvatures.

Proof. Since  $\lambda \neq \lambda'$ , from (2.2) and (3.1) follows that

$$(3.2) \quad K_{X_\lambda, Y_{\lambda'}} = \langle \alpha(X_\lambda, X_\lambda), \alpha(Y_{\lambda'}, Y_{\lambda'}) \rangle - \langle \alpha(X_\lambda, Y_{\lambda'}), \alpha(X_\lambda, Y_{\lambda'}) \rangle \\ \leq \frac{\langle \lambda, \lambda' \rangle}{\langle \lambda, H_0 \rangle \langle \lambda', H_0 \rangle}$$

for  $X_\lambda \in \mathfrak{p}_\lambda$ ,  $Y_{\lambda'} \in \mathfrak{p}_{\lambda'}$  with  $\langle X_\lambda, X_\lambda \rangle = \langle Y_{\lambda'}, Y_{\lambda'} \rangle = 1$ . Since  $\langle \lambda, \lambda' \rangle < 0$ , we have  $K_{X_\lambda, Y_{\lambda'}} < 0$ .

For each root  $\mu \in \mathfrak{r}$  we denote by  $S_\mu$  the reflection of  $\alpha$  with respect to  $\mu$ . Then  $\lambda'' = S_\lambda(\lambda')$  is a root in  $\mathfrak{r}$ . Since  $\langle \lambda, \lambda' \rangle < 0$ , we find that

$$\lambda'' = \lambda' - \frac{2\langle \lambda, \lambda' \rangle}{\langle \lambda, \lambda \rangle} \lambda \in \mathfrak{r}_2^+$$

and

$$\langle \lambda, \lambda'' \rangle = \langle \lambda, S_\lambda(\lambda') \rangle = \langle S_\lambda(\lambda), \lambda' \rangle \\ = -\langle \lambda, \lambda' \rangle > 0.$$

For  $Z_{\lambda''} \in \mathfrak{p}_{\lambda''}$  with  $\langle Z_{\lambda''}, Z_{\lambda''} \rangle = 1$ , since  $\langle \lambda'' - \lambda, H_0 \rangle > 0$ , we have  $\alpha(X_\lambda, Z_{\lambda''}) = 0$  by (2.2) (iii). So in the same way as above we have  $K_{X_\lambda, Z_{\lambda''}} > 0$ .

3.2. In the following we assume that the root system  $\mathfrak{r}$  is irreducible. Let  $\Delta$  be the fundamental root system of  $\mathfrak{r}$  consisting of simple roots with respect to the order chosen in section 2, and put

$$\Delta_1 = \{\lambda \in \Delta; \langle \lambda, H_0 \rangle = 0\}.$$

Let  $\mathfrak{h}$  be a Cartan subalgebra containing  $\alpha$ . Let  $\mathfrak{g}^c$  be the complexification of  $\mathfrak{g}$ ,  $\mathfrak{h}^c$  the subspace of  $\mathfrak{g}^c$  spanned by  $\mathfrak{h}$ , and  $\mathfrak{h}_0$  the real part of  $\mathfrak{h}^c$ . Let  $\sigma$  be the conjugation of  $\mathfrak{g}^c$  with respect to  $\mathfrak{g}$ , and choose a  $\sigma$ -order in the sense of Satake [4] on  $\mathfrak{h}_0$ , extending our order on  $\alpha$ . In the root system of  $\mathfrak{g}^c$  relative to  $\mathfrak{h}^c$ , let  $\hat{\Delta}$  be the fundamental root system consisting of simple roots with respect to this order, and denote the Satake diagram of  $\hat{\Delta}$  by the same symbol  $\hat{\Delta}$ . Then  $\hat{\Delta}$  defines our fundamental root system  $\Delta$  by the projection  $p$  of  $\mathfrak{h}_0$  onto  $\alpha$ :  $\Delta = p(\hat{\Delta}) - \{0\}$ . Let  $\hat{\Delta}_1 = p^{-1}(\Delta_1)$ . It is known (Takeuchi [7]) that isomorphic pairs  $(\hat{\Delta}, \hat{\Delta}_1)$  of Satake diagrams give rise to isomorphic pairs  $(K, K^*)$ : We say that the pair  $(\hat{\Delta}, \hat{\Delta}_1)$  is isomorphic to the pair  $(\hat{\Delta}', \hat{\Delta}'_1)$  if there exists an isomorphism  $\varphi$  of  $\hat{\Delta}$  onto  $\hat{\Delta}'$  such that  $\varphi$  maps  $\hat{\Delta}_1$  onto  $\hat{\Delta}'_1$ , and the pair  $(K, K^*)$  is said to be isomorphic to the pair  $(K', K'^*)$  if there exists an isomorphism  $f$  of  $K$  onto  $K'$  such that  $f$  maps  $K^*$  onto  $K'^*$ .

For a set  $A$  we will denote by  $|A|$  the cardinality of  $A$ .

**Proposition 4.** *If the root system  $\mathfrak{r}$  is irreducible and  $|\Delta - \Delta_1| \geq 2$ , the space  $N$  has both positive and negative sectional curvatures.*

Proof. Assume that  $\lambda, \mu \in \Delta - \Delta_1$  and  $\lambda \neq \mu$ . Then there exist simple

roots  $\lambda_0, \lambda_1, \dots, \lambda_s \in \Delta$  such that the following conditions are satisfied:

$$(3.3) \quad \begin{cases} \text{(i)} & \lambda_0 = \lambda, \quad \lambda_s = \mu \\ \text{(ii)} & \langle \lambda_i, \lambda_j \rangle = \begin{cases} < 0 & \text{if } j = i+1 \\ = 0 & \text{otherwise} \end{cases} \end{cases}$$

where  $0 \leq i < j \leq s$

Put

$$\lambda'' = S_{\lambda_{s-1}} S_{\lambda_{s-2}} \cdots S_{\lambda_1}(\lambda_0) \text{ and } \lambda'' = \lambda_0 + \sum_{i=1}^{s-1} m_i \lambda_i.$$

Then by (3.3) we have  $m_i > 0$  and  $\lambda'' \in \mathfrak{r}_2^+$ . Since

$$\langle \lambda'', \mu \rangle = m_{s-1} \langle \lambda_{s-1}, \lambda_s \rangle < 0,$$

the proposition follows from Proposition 3.

An element  $H$  of  $\mathfrak{a}$  is called regular if  $\langle \lambda, H \rangle \neq 0$  for any  $\lambda \in \mathfrak{r}$ . Then the following Corollary is an immediate consequence of Proposition 4.

**Corollary.** *If the root system  $\mathfrak{r}$  is irreducible and the rank of  $\mathfrak{r}$  is not less than 2, then the  $K$ -orbit  $N$  through a regular element of  $\mathfrak{a}$  has both positive and negative sectional curvatures.*

REMARK. If  $\mathfrak{g}$  is the direct sum of  $r$ -copies of  $\mathfrak{sl}(2, \mathbf{R})$  ( $r \geq 2$ ), the space  $N$  is the  $r$ -dimensional flat torus. According to the following result (A) and (B), it follows that, except for the above case, the space  $N$  has always a plane section along which the sectional curvature is strictly positive.

(A) *Let  $M$  be an  $n$ -dimensional manifold and  $G$  a compact connected transitive Lie transformation group of  $M$ . If the universal covering manifold  $\tilde{M}$  of  $M$  satisfies*

$$H_i(\tilde{M}, \mathbf{Z}_2) = \{0\} \quad \text{for any } i > 0,$$

*then  $G$  is the  $n$ -dimensional toral group,  $M$  is the  $n$ -dimensional torus and  $G$  acts on  $M$  as translations (M. Takeuchi).*

(B) *A simply connected complete Riemannian manifold with non-positive sectional curvatures is diffeomorphic with a Euclidean space (Theorem of Cartan-Hadamard, cf. [3]).*

3.3. Next we shall see under which conditions  $N$  has a strictly positive sectional curvature along each plane section.

Assume that the space  $N$  has strictly positive sectional curvatures. Decompose the root system  $\mathfrak{r}$  into the sum of irreducible components  $\mathfrak{r}^{(i)}$ :

$$\mathfrak{r} = \mathfrak{r}^{(1)} \cup \dots \cup \mathfrak{r}^{(m)}$$

If we put  $(\mathfrak{r}^{(i)})_2^+ = \mathfrak{r}^{(i)} \cap \mathfrak{r}_2^+$  for  $1 \leq i \leq m$ , then the space  $N$  is substantial if and only if  $(\mathfrak{r}^{(i)})_2^+ \neq \phi$  for each  $i$ . Thus the root system  $\mathfrak{r}$  should be irreducible by (3.2). If  $|\Delta| = 1$ , then  $\Delta_1 = \phi$  and the space  $N$  coincides with the unit sphere  $S$  itself. So we assume that  $|\Delta| \geq 2$  in the following. Then we have by Proposition 4 and (3.2):

$$(3.4) \quad |\Delta - \Delta_1| = 1$$

$$(3.5) \quad \langle \lambda, \lambda' \rangle > 0 \quad \text{for all } \lambda, \lambda' \in \mathfrak{r}_2^+$$

We remark that the properties (3.4) and (3.5) depend only on the root system  $\mathfrak{r}$ . So we shall consider the following situation:

Let  $V$  be a real vector space and  $\mathfrak{s}$  a *reduced* irreducible root system in  $V$ . Choosing a linear order in the space  $V$ , let  $\mathfrak{s}^+$  denote the set of positive roots and  $\Delta$  the fundamental root system of  $\mathfrak{s}$  consisting of simple roots. The Dynkin diagram of  $\Delta$  will be also denoted by the same symbol  $\Delta$ . Let  $\Delta_1$  be a subset  $\Delta$  of such that  $|\Delta - \Delta_1| = 1$ ,

$$\Delta = \{\lambda_1, \lambda_2, \dots, \lambda_l\} \quad \text{and} \quad \Delta - \Delta_1 = \{\lambda_1\}.$$

We put

$$\mathfrak{s}_2^+ = \{\lambda \in \mathfrak{s}^+; \lambda = \sum_{i=1}^l m_i \lambda_i, m_1 > 0\}.$$

We denote by  $\lambda_0$  the highest root in  $\mathfrak{s}$  and  $S_\lambda$  the reflection of  $V$  with respect to the root  $\lambda \in \mathfrak{s}$ . We fix an inner product  $(\ , \ )$  on  $V$  invariant under the Weyl group of  $\mathfrak{s}$ . Since the root system  $\mathfrak{s}$  is irreducible, such an inner product is unique up to a multiple of positive constant. The diagram obtained from the Dynkin diagram  $\Delta$  by adding  $-\lambda_0$  is called the *extended Dynkin diagram* of  $\mathfrak{s}$ . The table of extended Dynkin diagrams is seen, for instance, in Borel-de Siebenthal [1]. We shall consider the condition

$$(3.6) \quad (\lambda, \lambda') > 0 \quad \text{for all } \lambda, \lambda' \in \mathfrak{s}_2^+$$

and prove the following:

**Proposition 5.** *The pair  $(\Delta, \Delta_1)$  satisfies the condition (3.6) if and only if the Dynkin diagram  $\Delta$  is of type  $A_l$  and  $\lambda_1$  is one of the terminals of  $\Delta$ , i.e. the vertex  $\lambda_1$  is connected with only one vertex in  $\Delta$ .*

**Proof.** Let  $\Delta$  be of type  $A_l$  and  $\lambda_1$  one of the terminals of  $\Delta$ . We may assume that there exist an  $(l+1)$ -dimensional vector space  $W \supset V$  with an inner product  $(\ , \ )$  and an orthonormal basis  $e_1, \dots, e_{l+1}$  of  $W$  such that the following conditions are satisfied:

(i) The restriction to  $V$  of the inner product  $(\ , \ )$  on  $W$  is invariant under the Weyl group of  $\mathfrak{s}$ .

- (ii)  $\mathfrak{s} = \{e_i - e_j; 1 \leq i, j \leq l+1, i \neq j\}$
- (iii)  $\Delta = \{e_i - e_{i+1}; 1 \leq i \leq l\}, \lambda_1 = e_1 - e_2$

Then we have

$$\mathfrak{s}_2^+ = \{e_i - e_i; 2 \leq i \leq l+1\}.$$

Hence the pair  $(\Delta, \Delta_1)$  satisfies the condition (3.6).

It remains to prove that if the pair  $(\Delta, \Delta_1)$  satisfies (3.6), then the Dynkin diagram  $\Delta$  is of type  $A_l$  and  $\lambda_1$  is one of the terminals of  $\Delta$ .

**Lemma 6.** *If the pair  $(\Delta, \Delta_1)$  satisfies (3.6), the inner product  $(\lambda_0, \lambda_1) \neq 0$ .*

Proof. Since the highest root  $\lambda_0$  is contained in  $\mathfrak{s}_2^+$ , we have  $(\lambda_0, \lambda_1) > 0$ .

**Lemma 7.** *If the pair  $(\Delta, \Delta_1)$  satisfies (3.6),  $\lambda_1$  is one of the terminals of  $\Delta$ .*

Proof. Suppose that  $\lambda_1$  is not a terminal. Then there exist two simple roots  $\lambda, \mu \in \Delta$  such that  $(\lambda, \lambda_1) \neq 0$  and  $(\mu, \lambda_1) \neq 0$ . Since the Dynkin diagram  $\Delta$  contains no cycles, we have  $(\lambda, \mu) = 0$ . It is known that  $(\lambda, \lambda_1) < 0$  and  $(\mu, \lambda_1) < 0$ . Then it follows that  $\lambda + \lambda_1$  is a root (cf. Serre [5]). Since  $(\lambda + \lambda_1, \mu) < 0$ , it follows also that  $\lambda + \lambda_1 + \mu$  is a root. On the other hand  $\lambda_1 + \lambda + \mu$  belongs to  $\mathfrak{s}_2^+$  by definition, so that  $(\lambda_1 + \lambda + \mu, \lambda_1) > 0$ . Thus we see that  $\lambda + \mu = (\lambda + \lambda_1 + \mu) - \lambda_1$  is a root.

Now let

$$\lambda - p\mu, \dots, \lambda - \mu, \lambda, \lambda + \mu, \dots, \lambda + q\mu$$

be a  $\mu$ -series of roots such that neither  $\lambda - (p+1)\mu$  nor  $\lambda + (q+1)\mu$  is a root. Then we have (cf. Serre [5]).

$$q - p = -\frac{2(\lambda, \mu)}{(\mu, \mu)}.$$

Since  $(\lambda, \mu) = 0$  and  $p = 0$ , we get  $q = 0$ , which contradicts to  $\lambda + \mu \in \gamma$ .

**Lemma 8.** *Suppose that the pair  $(\Delta, \Delta_1)$  satisfies the condition (3.6). If the Dynkin diagram  $\Delta$  is one of the classical types, then  $\Delta$  is of type  $A_l$  and  $\lambda_1$  is one of the terminals of  $\Delta$ .*

Proof. It suffices to show that the Dynkin diagram  $\Delta$  is of type  $A_l$ . Suppose that  $\Delta$  is of type  $B_l (l \geq 3)$  or of type  $D_l (l \geq 4)$ . Then, looking at their extended Dynkin diagrams, we see from Lemma 6 and Lemma 7 that there exist no pairs  $(\Delta, \Delta_1)$  satisfying the condition (3.6).

Suppose that the pair  $(\Delta, \Delta_1)$  satisfies (3.6) and the Dynkin diagram  $\Delta$  is of type  $C_l (l \geq 2)$ . Then we may assume that there exists an orthonormal basis  $\{e_1, \dots, e_l\}$  of  $V$  such that the following conditions are satisfied:



$$(i) \quad \mathfrak{s} = \{\pm e_i \pm e_j, \pm 2e_i; 1 \leq i, j \leq l, i \neq j\}$$

$$(ii) \quad \Delta = \{e_1 - e_2, e_2 - e_3, \dots, e_{l-1} - e_l, 2e_l\}$$

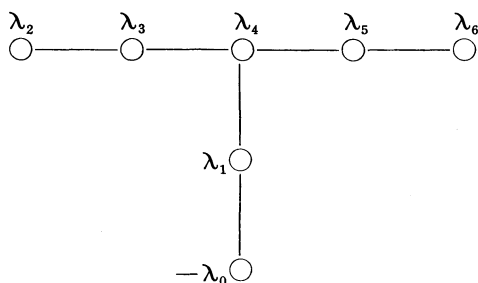
By Lemma 6 and Lemma 7 we should have  $\lambda_1 = e_1 - e_2$ . So we have

$$\mathfrak{s}_2^+ = \{e_1 \pm e_i, 2e_i; 2 \leq i \leq l\}.$$

For  $e_1 - e_2, e_1 + e_2 \in \mathfrak{s}_2^+$  we have  $(e_1 - e_2, e_1 + e_2) = 0$ , which is a contradiction.

**Lemma 9.** *If the Dynkin diagram  $\Delta$  is one of the exceptional types, there exist no pairs  $(\Delta, \Delta_1)$  satisfying the condition (3.6).*

*Proof.* Suppose that the pair  $(\Delta, \Delta_1)$  satisfies the condition (3.6) and the Dynkin diagram  $\Delta$  is of type  $E_6$ . By Lemma 6 and the extended Dynkin diagram we should have



We put

$$\mathfrak{s}' = \{\lambda \in \mathfrak{s}; \lambda = \sum_{i=1}^6 m_i \lambda_i, m_2 = m_6 = 0\},$$

$$\Delta' = \{\lambda_1, \lambda_3, \lambda_4, \lambda_5\} \text{ and } \Delta_1' = \{\lambda_3, \lambda_4, \lambda_5\}.$$

Then  $\mathfrak{s}'$  is a root system of type  $D_4$  with the fundamental root system  $\Delta'$ . The subset  $(\mathfrak{s}')_2^+$  of  $\mathfrak{s}'$  corresponding to the pair  $(\Delta', \Delta_1')$  is given by  $(\mathfrak{s}')_2^+ = \mathfrak{s}_2^+ \cap \mathfrak{s}'$ . It follows from Lemma 8 that there exist two roots  $\lambda, \mu \in (\mathfrak{s}')_2^+$  such that  $(\lambda, \mu) \leq 0$ . This is a contradiction.

In the cases of type  $E_7$  and  $E_8$  we can prove the lemma in the same way, making use of Lemma 8 for  $\Delta$  of type  $D_5$  and  $D_7$  respectively.

In the case of type  $F_4$  we can prove in the same way, making use of Lemma 8 for  $\Delta$  of type  $B_3$ .

In the case of type  $G_2$  the lemma is easily proved.

By Lemma 8 and Lemma 9 we have the complete proof of Proposition 5.

Now we give the table of pairs  $(\hat{\Delta}, \hat{\Delta}_1)$  of Satake diagrams (up to an isomorphism) with the following properties.

- 1) The Satake diagram  $\hat{\Delta}$  has no compact factors.
- 2) The pair  $(\Delta, \Delta_1)$  of fundamental root systems obtained from  $(\hat{\Delta}, \hat{\Delta}_1)$

by the projection has the properties:

- (i) The Dynkin diagram  $\Delta$  is of type  $A_l$  with  $l \geq 2$ .
- (ii) The set  $\Delta - \Delta_1$  consists of one of the terminals of  $\Delta$ .

Here the vertexes in  $\hat{\Delta} - \hat{\Delta}_1$  are represented by  $\odot$ .  $p'(\mathbf{F})$  denotes the  $l$ -dimensional projective space over  $\mathbf{F}$  where  $\mathbf{F}$  denotes  $\mathbf{R}$ ,  $\mathbf{C}$ ,  $\mathbf{H}$  (the algebra of real quaternions), or  $\mathbf{K}$  (the algebra of real Cayley numbers).

	$(\hat{\Delta}, \hat{\Delta}_1)$	$N$
A		$SU(l+1) / S(U(1) \times U(l)) = P'(\mathbf{C})$
AI		$SO(l+1) / S(O(1) \times O(l)) = P'(\mathbf{R})$
AII		$Sp(l+1) / Sp(1) \times Sp(l) = P'(\mathbf{H})$
EIV		$F_4 / Spin(9) = P^2(\mathbf{K})$

Each of the pairs  $(K, K^*)$  appeared in this table is a symmetric pair of rank 1 and the space  $N$  has strictly positive sectional curvatures as we have noted in Introduction. Thus we have proved the following:

**Proposition 10.** *The space  $N$  has strictly positive sectional curvatures if and only if  $N$  is the unit sphere  $S$  or the pair  $(K, K^*)$  is a symmetric pair of rank 1.*

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