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<td><strong>Author(s)</strong></td>
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<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 18(1) P.1–P.11</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1981</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/12582">https://doi.org/10.18910/12582</a></td>
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<td>10.18910/12582</td>
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ON A PROJECTIVE PLANE CURVE WHOSE
COMPLEMENT HAS LOGARITHMIC
KODAIRA DIMENSION $-\infty$

MASAYOSHI MIYANISHI AND TOHRU SUGIE

(Received September 14, 1979)

Introduction. Let $k$ be an algebraically closed field of characteristic zero and let $D$ be a reduced curve on the projective plane $\mathbb{P}^2_k$ such that its complement $X := \mathbb{P}^2_k - D$ has logarithmic Kodaira dimension $-\infty$ (cf. Iitaka [3, 4] for the definition and relevant results). We call such a curve $D$ a projective plane curve with complementary logarithmic Kodaira dimension $-\infty$, a plane curve with $\text{CLKD} = -\infty$, for short. The purpose of the present article is to give a characterization of plane curves with $\text{CLKD} = -\infty$. Namely, we shall prove the following:

Theorem. Let $D$ be a reduced plane curve with $\text{CLKD} = -\infty$. Then the following assertions hold true:

(1) There exists an irreducible linear pencil $\Lambda$ on $\mathbb{P}^2_k$ such that $D$ is a union of (irreducible components of) members of $\Lambda$, where $\Lambda$ satisfies the following properties:

1° $\Lambda$ has only one base point $P_0$, and, for a general member $C$ of $\Lambda$, $C - \{P_0\}$ is isomorphic to the affine line $\mathbb{A}_1$; the point $P_0$ is, therefore, a one-place point of $C$.

2° All members of $\Lambda$ are irreducible, and $\Lambda$ has at most two multiple members.

3° Let $d$ be the degree of a general member $C$ of $\Lambda$; if $d = 1$ then $\Lambda$ is the pencil of lines through $P_0$; if $d > 1$, let $l_0$ be the tangent line of a general member $C$, i.e., a line of maximal contact with $C$ at $P_0$; if $dl_0 \in \Lambda$ then $dl_0$ is the unique multiple member of $\Lambda$; if $dl_0 \notin \Lambda$ then $\Lambda$ has two multiple members $aF$ and $bG$, where $F$ and $G$ are irreducible and $a, b$ are integers $\geq 2$ such that $a = \deg G$, $b = \deg F$, $d = ab$ and $\text{G.C.D.}(a, b) = 1$; $D$ is said to be of the first kind or of the second kind according as $dl_0 \in \Lambda$ or $dl_0 \notin \Lambda$, respectively.

(2) There exists a Cremona transformation $\varphi : \mathbb{P}^2_k \to \mathbb{P}^2_k$ of degree $d$ such that:

1° The pencil $\Lambda$ corresponds (under $\varphi$) to the pencil of lines through a point $Q_0$;

2° if $D$ is of the first kind, the line $l_0$ is the unique exceptional curve for $\varphi$; if
is the line through \( Q_0 \) corresponding to \( d_0 \), then \( L_0 \) is the unique exceptional curve for \( \varphi^{-1} \);

3° if \( D \) is of the second kind, the curves \( F \) and \( G \) exhaust the exceptional curves for \( \varphi \); if \( L_1 \) and \( L_2 \) are lines through \( Q_0 \) corresponding to \( aF \) and \( bG \) under \( \varphi \), then \( L_1 \) and \( L_2 \) exhaust the exceptional curves for \( \varphi^{-1} \).

Conversely, let \( \Lambda \) be an irreducible linear pencil on \( \mathbb{P}^2 \), satisfying the property 1° of the assertion (1), and let \( D \) be the union of (irreducible components of) members of \( \Lambda \). Then \( D \) is a plane curve with \( \text{CLKD} = \infty \).

If \( D \) is of the first kind, the theorem implies that the Cremona transformation \( \varphi \) induces a biregular automorphism of the affine plane \( \mathbb{P}^2 - l_0 \rightarrow \mathbb{P}^2 - L_0 \) and that \( X := \mathbb{P}^2 - D \) contains an open set which is isomorphic (under \( \varphi \)) to the affine plane with finitely many parallel lines deleted off. So far, we have only one example for \( D \) of the second kind. That is, a quintic rational curve with only one cusp of multiplicity 2, which was obtained by Yoshihara [8].

In the final section, we discuss Yoshihara's example together with some detailed observations on plane curves \( D \) of the second kind with \( \text{CLKD} = \infty \).

Our terminology and notation conform with those of Miyanishi [5] and Miyanishi-Sugie [7]. We only modify some of the definitions given in [5] so as to suit the present situation.

(1) Let \( V_0 \) be a nonsingular projective surface, let \( P_0 \) be a point on \( V_0 \) and let \( l_0 \) be an irreducible curve on \( V_0 \) such that \( P_0 \) is a simple point of \( l_0 \). Let \( d_0 \) and \( d_1 \) be positive integers such that \( d_1 < d_0 \). Define positive integers \( d_2, \ldots, d_a \) and \( q_1, \ldots, q_a \) by the following Euclidean algorithm:

\[
\begin{align*}
    d_0 &= d_1 q_1 + d_2 & 0 < d_2 < d_1 \\
    d_1 &= d_2 q_2 + d_3 & 0 < d_3 < d_2 \\
    &\vdots & \\
    d_{a-2} &= d_{a-1} q_{a-1} + d_a & 0 < d_a < d_{a-1} \\
    d_{a-1} &= d_a q_a & 1 < q_a.
\end{align*}
\]

Let \( N = q_a + \cdots + q_1 \). Define the infinitely near points \( P_i \)'s of \( P_0 \) (for \( 1 \leq i < N \)) and the quadratic transformation \( \sigma_i : V_{i-1} \rightarrow V_{i-1} \) with center \( P_{i-1} \) (for \( 1 \leq i \leq N \)) inductively in the following fashion:

(i) \( P_i \) is an infinitely near point of order one of \( P_{i-1} \) for \( 1 \leq i < N \);

(ii) let \( E_i = \sigma_i^{-1}(P_{i-1}) \) for \( 1 \leq i \leq N \) and let \( E(v,j) = E_i \) if \( i = q_0 + \cdots + q_{v-1} + j \) with \( 1 \leq v \leq \alpha \) and \( 1 \leq j \leq q_0 \), where we set \( q_0 := 0 \); \( P_i \) is the intersection point of the proper transform of \( E(v-1,q_{v-1}) \) on \( V_i \) and the curve \( E(v,j) \) if \( i = q_0 + \cdots + q_{v-1} + j \) with \( 1 \leq v \leq \alpha \) and \( 1 \leq j \leq q_v \) (\( 1 \leq j < q_\alpha \) if \( v = \alpha \)), where we set \( E(0,q_0) = l_0 \).

The composition \( \sigma = \sigma_1 \cdots \sigma_N \) is called the Euclidean transformation with re-
spect to a datum \((P_0, l_0, d_0, d_1)\), or the Euclidean transformation, for short, if the datum \((P_0, l_0, d_0, d_1)\) is clear by the context. The related definitions are given in Miyanishi [5; p. 92 and p. 214]. In particular, the weighted dual graph of \(\sigma^{-1}(l_0)\) is the same as given in [loc. cit.; p. 95]. If \(C\) is an irreducible curve on \(V_0\) such that \(P_0\) is a one-place point of \(C\) with \(d_0=\text{mult}_{P_0}C\) (the multiplicity of \(C\) at \(P_0\)), then the proper transform \(C_i:=(\sigma_1\cdots\sigma_{i-1})'(C)\) passes through \(P_i\) so that \(d_i:=(C_i\cdot E(\nu, j))\) and the intersection multiplicity of \(C_i\) with the proper transform of \(E(\nu-1, q_{i-1})\) on \(V_i\) is \(d_{i-1}-jd_{i-1}\), where \(1\leq i\leq \alpha\) and \(1\leq j\leq q_{i-1}\); the smaller one of \(d_i\) and \(d_{i-1}-jd_{i-1}\) is the multiplicity of \(C_i\) at \(P_i\). We note that \(\sigma'(C)\) meets only the last exceptional curve \(E(\alpha, q_{\alpha})\) and does not meet the other exceptional curves in the process \(\sigma\).

(2) Let \(V_0, P_0\) and \(l_0\) be as above. Let \(r\) be a positive integer. An equi-multiplicity transformation of length \(r\) with center \(P_0\), or an EM-transformation, for short, is the composition \(\sigma=\sigma_1\cdots\sigma_r\) of quadratic transformations defined as follows:

For \(1\leq i\leq r\), \(\sigma_i\) is the quadratic transformation with center \(P_{i-1}\) and \(P_i\) (for \(i<r\)) is a point on \(\sigma_i^{-1}(P_{i-1})\) other than the point \(\sigma_i(\sigma_{i-1}^{-1}(P_{i-2}))\cap\sigma_i^{-1}(P_{i-1})\) (\(\sigma_i(l_0)\cap\sigma_i^{-1}(P_0)\) if \(i=1\)). A related notion is the \((e, i)\)-transformation defined in [loc. cit.; p. 100]. If \(C\) is an irreducible curve on \(V_0\) such that \(P_0\) is a one-place point with \(d_0=i(C, l_0; P_0)\) equal to \(d_i:=(\sigma_i(C)\cap\sigma_i^{-1}(P_0))\) differs from \(\sigma_i(l_0)\cap\sigma_i^{-1}(P_0)\). If \(d_i:=(\sigma_i(C)\cap\sigma_i^{-1}(P_0))\) then \(P_2:=(\sigma_2(C)\cap\sigma_2^{-1}(P_0))\). Thus, this step can be repeated as long as the intersection multiplicity of the proper transform of \(C\) with the last exceptional curve equals the multiplicity of the proper transform of \(C\) at the intersection point, and the composition of performed quadratic transformations is an equi-multiplicity transformation with center \(P_0\).

1. The pencil \(\Lambda\)

Let \(D\) be a plane curve with \(CLKD=-\infty\) and let \(X:=\mathbb{P}^2-D\). Then \(X\) is a nonsingular rational affine surface with logarithmic Kodaira dimension \(\kappa(X)=-\infty\). By virtue of the analogue of Enriques' characterization theorem of ruled surfaces (cf. Miyanishi-Sugie [7] and Fujita [1]), \(X\) contains a cylinder-like open set \(U\cong U_0\times A^1\), where \(U_0\) is a nonsingular rational affine curve. Then there exists an irreducible linear pencil \(\Lambda\) on \(\mathbb{P}^2\) such that, for a general member \(C\) of \(\Lambda\), \(C\cap X\) is a general fiber of the canonical projection \(p_1:U\rightarrow U_0\). Hence \(C\cap X\) is isomorphic to the affine line \(A^1\). Let \(d\) be the degree of \(C\). Since \(d>0\), \(\Lambda\) has a unique base point \(P_0\), which is a one-place point for a general member \(C\) of \(\Lambda\). Let \(D_1\) be an irreducible component of the curve \(D\). Then \(D_1\) meets only at \(P_0\) with general members of \(\Lambda\). Hence \(D_1\) is an irreducible component
of a member of \( \Lambda \). Therefore, we know that \( D \) is a union of irreducible components of members of \( \Lambda \). If \( d = 1 \) the pencil \( \Lambda \) consists of lines through \( P_0 \), and we have nothing more to show in the theorem; take the identity automorphism of \( P^2_1 \) as the Cremona transformation \( \varphi \). Thus, we shall assume, henceforth, that \( d > 1 \).

Let \( C \) be a general member of \( \Lambda \) and let \( l_0 \) be the tangent line of \( C \) at \( P_0 \), which is the line of maximal contact with \( C \) at \( P_0 \). Then \( l_0 \) is the tangent line for almost all members of \( \Lambda \). Indeed, if \( d = 2 \) the pencil \( \Lambda \) is spanned by a nonsingular conic \( C \) through \( P_0 \) and \( 2l_0 \), where \( l_0 \) is the tangent line of \( C \) at \( P_0 \). Hence every irreducible member of \( \Lambda \) has \( l_0 \) as its tangent line at \( P_0 \). If \( d > 2 \) then \( P_0 \) is a singular point for almost all members of \( \Lambda \). If the assertion does not hold, after the quadratic transformation \( \sigma: V' \rightarrow V_0 := P^2_1 \), the exceptional curve \( E' := \sigma^{-1}(P_0) \) is a quasi-section of the pencil \( \sigma' \Lambda (= \text{the proper transform of} \ \Lambda \ \text{by} \ \sigma) \), (cf. [5; p. 190]). Since \( P_0 \) is a one-place point for almost all members \( C \) of \( \Lambda \) and \( (E' \cdot \sigma'(C)) > 1 \), this is a contradiction.

We shall prove the following:

**Lemma 1.** Let the notations and the assumptions be as above. Then the following assertions hold true:

1. Every member \( M \) of \( \Lambda \) is irreducible and \( M_{\text{red}} - \{ P_0 \} \) is isomorphic to the affine line \( A^1 \).
2. \( \Lambda \) has at most two multiple members.
3. If \( \Lambda \) has only one multiple member then it is \( dl_0 \).

Proof. Let \( C_0 \) be a general member of \( \Lambda \). Then \( C_0 \) is a curve with \( CLKD = \infty \). In order to prove the above assertions, we may replace \( D \) by \( C_0 \) and assume that \( D \) consists of only one general member of \( \Lambda \). Then \( \text{Pic}(X) \) is a cyclic group of order \( d \), where \( X := P^2_1 - D \). Then we shall show that the projection \( \pi_1: U \rightarrow U_0 \), \( U \) being the cylinderlike open set of \( X \) which defines the pencil, extends to a surjective morphism \( \pi: X \rightarrow T \cong A^1_1 \), whose general fibers are isomorphic to \( A^1_1 \). Indeed, since \( P_0 \) is the single base point of \( \Lambda \), the rational map \( \Phi_\Lambda: P^2_1 \rightarrow P^1_1 \) defined by \( \Lambda \) is regular outside \( P_0 \). Hence the restriction \( \pi := \Phi_\Lambda|_X \) is a morphism. Since \( D \) is a member of \( \Lambda \), the image \( T := \pi(X) \) is isomorphic to the affine line. By construction, general fibers of \( \pi \) are isomorphic to \( A^1_1 \).

By virtue of Miyanishi [6; Lemma 1.1], we have

\[
\text{rank Pic}(X) \otimes \mathbb{Q} = \sum_{P \in T} (\mu_P - 1),
\]

where \( \mu_P \) is the number of irreducible components of the fiber \( \pi^{-1}(P) \) and where \( P \) ranges over all points of \( T \). Since \( \text{Pic}(X) \) is a cyclic group of order \( d \), we know that every fiber of \( \pi \) is irreducible; we know also by the same result (loc.
that every fiber of $\pi$ is isomorphic to $A_1$. This implies that every member of $\Lambda$ is irreducible and nonsingular outside the point $P_0$. This proves the assertion (1). We can strengthen, in effect, the cited result as follows: Let $P_1, \ldots, P_s$ exhaust the points of $T$ such that $\pi^*(P_i) = a_i \pi^{-1}(P_i)$ with $a_i \geq 2$ for $1 \leq i \leq s$; then $\text{Pic}(X)$ is a finite abelian group with generators $\xi_1, \ldots, \xi_s$ and relations $a_i \xi_1 = \cdots = a_s \xi_s = 0$. Hence $a_1, \ldots, a_s$ are pairwise coprime, and $d = a_1 \cdots a_s$.

If $s = 1$, we know thence that $a_i = d$. Let $F_1$ be the closure in $P_k^2$ of the irreducible curve $\pi^{-1}(P_i)$. Then $dF_1$ is linearly equivalent to $C$. Hence $F_1$ is a line. Since $F_1 \cap C = \{P_i\}$, we know that $F_1 = l_0$. Hence $dl_0$ is a member of $\Lambda$. This proves the assertion (3).

Let $\sigma : \tilde{V} \rightarrow V_0 := \mathbb{P}^k$ be the shortest succession of quadratic transformations with centers at base points (including infinitely near base points) of $\Lambda$ such that the proper transform $\tilde{\Lambda}$ of $\Lambda$ by $\sigma$ has no base points. Let $\tilde{E}$ be the exceptional curve obtained by the last quadratic transformation in the process $\sigma$. Then $\tilde{E}$ is a cross-section of $\tilde{\Lambda}$, i.e., a general member of $\tilde{\Lambda}$ meets $\tilde{E}$ transversally in a single point. Let $F_i$ be the closure in $P_k^2$ of $\pi^{-1}(P_i)$, let $\tilde{F}_i$ be the proper transform of $F_i$ by $\sigma$ and let $\Gamma_i$ be the member of $\tilde{\Lambda}$ for which $\tilde{F}_i$ is an irreducible component, where $1 \leq i \leq s$. Since $a_i \geq 2$, $\Gamma_i$ is a reducible member. Namely, $\Gamma_i$ contains other irreducible components by means of which the component $\tilde{F}_i$ is connected to the cross-section $\tilde{E}$. Those irreducible components of $\Gamma_i$'s other than $\tilde{F}_i$'s for $1 \leq i \leq s$ are exceptional curves obtained by the quadratic transformations in the process $\sigma$. The composition $\sigma$ of quadratic transformations is, in effect, a composition of Euclidean transformations and $\text{EM}$-transformations, which are uniquely determined by a general member $C$ of $\Lambda$ as explained in the Introduction. This is due to the fact that $P_0$ is a one-place point for a general member $C$ of $\Lambda$. By looking at the configurations of exceptional curves obtained by the Euclidean transformations or the $\text{EM}$-transformations (cf. [5; p. 95]), we know that the totality of exceptional curves (except $\tilde{E}$) obtained in the process $\sigma$ is divided into at most two connected components, and that each of (irreducible) exceptional curves (except $\tilde{E}$) is contained in a member of $\tilde{\Lambda}$. This observation implies that $\tilde{\Lambda}$ contains at most two reducible fibers. Therefore $\Lambda$ has at most two multiple members. This proves the assertion (2).

Q.E.D.

We have also the following:

**Lemma 2.** Let the notations and the assumptions be as above. Then the following conditions are equivalent to each other:

1. For a general member $C$ of $\Lambda$, we have

$$(C \cdot l_0) = i(C, l_0; P_0) = d.$$
(2) \( d\ell_0 \in \Lambda \), i.e., \( D \) is of the first kind.

(3) \( \Lambda \) has only one multiple member.

Proof. (1) \( \Rightarrow \) (2): By assumption, \( C - \{ P_0 \} \) is contained in the affine plane \( X_0 := P^1_k - l_0 \). Since \( C - \{ P_0 \} \) is isomorphic to the affine line, we may choose coordinates \( x, y \) on \( X_0 \) so that the curve \( C - \{ P_0 \} \) is defined by \( x = 0 \) (cf. Abhyankar-Moh's Embedding theorem [5; p. 90]). Let \( C' \) be another general member of \( \Lambda \). Then \( C' - \{ P_0 \} \) is defined by \( f(x, y) = 0 \), where \( f(x, y) \in k[x, y] \).

Since \( C \cap C' = \{ P_0 \} \), we have \( f(0, y) \neq 0 \), i.e., \( f(x, y) = c + xg(x, y) \) with \( c \in k^* := k - (0) \) and \( g(x, y) \in k[x, y] \). Since \( k[x, y]/(f(x, y)) \) is a polynomial ring in one variable over \( k \), there exists an element \( c' \in k^* \) such that \( x - c' \) is divisible by \( f(x, y) \). Hence we may assume that \( f(x, y) = x - c' \).

Since \( \Lambda \) is spanned by \( C \) and \( C' \), almost all members of \( \Lambda \) restricted on \( X \) are defined by equations of the form \( x = c \) with \( c \in k^* \). This implies that \( \Lambda \) is spanned by \( C \) and \( d\ell_0 \), and that every member of \( \Lambda \) except \( d\ell_0 \) is irreducible and reduced.

(2) \( \Rightarrow \) (1): Since \( P_0 \) is the unique base point of \( \Lambda \), we have
\[
i(C, l_0; P_0) = (C \cdot l_0) = d.
\]

(2) \( \Rightarrow \) (3): We have shown, in the course of the proof (1) \( \Rightarrow \) (2), that \( d\ell_0 \) is the unique multiple member of \( \Lambda \). The implication (3) \( \Rightarrow \) (2) is proved in Lemma 1. Q.E.D.

Assume that \( \Lambda \) has two multiple members \( aF \) and \( bG \). Assuming, as in the proof of Lemma 1, that \( D \) consists of only one general member of \( \Lambda \), \( \text{Pic}(X) \) (with \( X := P^1_k - D \)) is a cyclic group of order \( d \) generated by \([F]\) and \([G]\) with relations \( a[F] = b[G] = 0 \), where \([F]\) and \([G]\) are the divisor classes represented by \( F \) and \( G \), respectively. Then it is clear that \( b = \deg F \), \( a = \deg G \), \( d = ab \) and \( \text{G.C.D.}(a, b) = 1 \). Therefore, we proved the assertion (1) of the theorem.

2. The Cremona transformation \( \varphi \)

We retain the notations in the previous section. Let \( \sigma : V \to V_0 := P^1_k \) be the shortest succession of quadratic transformations with centers at base points (including infinitely near base points) of \( \Lambda \) such that the proper transform \( \tilde{\Lambda} \) of \( \Lambda \) by \( \sigma \) has no base points. As already remarked, \( \sigma \) is the composition of Euclidean transformations and \( EM \)-transformations. Since the process \( \sigma \) is uniquely determined, the Euclidean transformations and the \( EM \)-transformations which constitute \( \sigma \) are uniquely determined. Let \( C \) be a general member of \( \Lambda \), let \( d_0 = i(C, l_0; P_0) \) and let \( d_1 = \text{mult}_{P_0} C \) (= the multiplicity of \( C \) at \( P_0 \)). Then we have \( d_0 \leq d_1 \leq d \), hence the process \( \sigma \) starts with the Euclidean transformation \( \sigma_1 : V_1 \to V_0 \) with respect to a datum \((P_0, l_0, d_0, d_1)\). Write
\[
\sigma = \sigma_1 \tau_1 \sigma_2 \tau_2 \cdots \sigma_n \tau_n,
\]
where $\sigma_i$'s ($1 \leq i \leq n$) are Euclidean transformations and $\tau_j$'s ($1 \leq j \leq n$) are the $EM$-transformations; if the $EM$-transformation, say $\tau_j$, is not necessary in the process $\sigma$, we understand $\tau_j = id$.

Let $\rho_m = \sigma_1 \tau_1 \cdots \sigma_m$ with $m < n$. Then the totality of (irreducible) exceptional curves obtained in the process $\rho_m$ is contained in one and the same member of the pencil $\rho_m \Lambda$. Indeed, the pencil $\rho_m \Lambda$ has the unique base point $P_m$ on the last exceptional curve $E_m$ which does not lie on any other exceptional curves appeared in the process $\rho_m$. Since each exceptional curve, which is an irreducible component of $\rho_m^{-1}(P_0)$, is contained in a member of $\rho_m \Lambda$ and since $\sigma_m^{-1}(P_0)$ is connected, $E_m$ and all other exceptional curves are contained in one and the same member of $\rho_m \Lambda$. A similar assertion holds true for $\rho_m \tau_m$. Since the last exceptional curve $E$ obtained in the process $\sigma$ is a cross-section of the pencil $\Lambda = \sigma' \Lambda$, we know easily that $\tau_n = id$. if $D$ is of the first kind and $\tau_n = id$. if $D$ is of the second kind. Moreover, if $D$ is of the second kind, one of two multiple members $aF$ and $bG$ of $\Lambda$, say $aF$, has the corresponding irreducible member $a\psi'(F)$ in the pencil $\psi' \Lambda$, where $\psi = \rho_m$ or $\rho_m \tau_m$ with $m < n$; $\psi'(G)$ then belongs to the member of $\psi' \Lambda$ containing all the exceptional curves; in the final step, $\sigma'(F)$ becomes an irreducible component of a reducible member of $\sigma' \Lambda$, whose irreducible components (except $\sigma'F$) are exceptional curves arising from the Euclidean transformation $\sigma_n$ (cf. the configuration in [5; p. 95]).

Since $\Lambda = \sigma' \Lambda$ has no base points, $\Lambda$ defines a surjective morphism $p = \Phi_\Lambda: \tilde{V} \to \mathbf{P}^1$ such that, if $\mathbf{P}^1 - \{P_0\}$ is naturally identified with an open set of $\tilde{V}$, $p$ coincides with the rational map $\Phi_\Lambda$ on $\mathbf{P}^1 - \{P_0\}$, and that the general fibers of $p$ are non-singular rational curves. The curve $\tilde{E}$ is a cross-section of $\tilde{p}$. We shall construct a $\mathbf{P}^1$-bundle from $\tilde{V}$ by contracting, one by one, all possible exceptional curves of the first kind contained in fibers of $p$. This contraction process is possible by virtue of the following:

**Lemma 3** [5; Lemma 2.2, p. 115]. Let $f: V \to B$ be a surjective morphism from a nonsingular projective surface $V$ onto a nonsingular complete curve $B$ such that almost all fibers are isomorphic to $\mathbf{P}^1$. Let $F = n_1 C_1 + \cdots + n_r C_r$ be a singular fiber of $f$, where $C_i$ is an irreducible curve, $C_i \neq C_j$ if $i \neq j$, and $n_i > 0$. Then we have:

1. $G.C.D.(n_1, \ldots, n_r) = 1$ and $\text{Supp}(F) := \bigcup_{i=1} C_i$ is connected.
2. For $1 \leq i \leq r$, $C_i$ is isomorphic to $\mathbf{P}^1$ and $\langle C_i \rangle < 0$.
3. For $i \neq j$, $(C_i \cdot C_j) = 0$ or $1$.
4. For three distinct indices $i, j$ and $l$, $C_i \cap C_j \cap C_l = \phi$.
5. One of $C_i$'s, say $C_{i_1}$, is an exceptional curve of the first kind. If $\tau: V \to W$ is the contraction of $C_{i_1}$, then $f$ factors as $f: V \to W G B$, where $g: W \to B$ is a fibration by $\mathbf{P}^1$.
6. If one of $n_i$'s, say $n_i$, equals $1$ then there is an exceptional curve of the first
kind among $C^i$'s with $2 \leq i \leq r$.

Let $\theta_1: \tilde{V} \to W$ be the contraction of all possible exceptional curves of the first kind contained in the reducible fibers of $p$. Then $p$ factors as $p: \tilde{V}^\theta \to \tilde{W}^\theta \to P^i$ (cf. Lemma 3, (5)) and $W$ is a $P^i$-bundle over $P^i$. If $\Delta$ is a reducible fiber of $p$, the irreducible component $\Delta_i$ of $\Delta$ meeting the cross-section $E$ has multiplicity 1. Hence we may assume that $\Delta$ is not contracted in the process $\theta_1$, i.e., $\theta_1(\Delta_i)$ is the fiber of $q$ lying over the point $p(A)$, (cf. Lemma 3, (6)); we assume that this assumption is valid for all reducible fibers of $p$. Then $\theta_1(E)$ is a cross-section of $q$ with $(\theta_1(E))^{\alpha} = -1$. This implies that $W$ is the Hirzebruch surface of degree 1, i.e., $W = \text{Proj}(O_{P^i} \oplus O_{P^i}(1))$. Then, by contracting $\theta_1(E)$, we obtain the projective plane $P^i$ and all fibers of $q$ become lines through the point $Q_0 := \theta_2(\theta_1(A))$ where $\theta_2: W \to P^i$ is the contraction of $\theta_1(E)$. Let $\varphi := \theta \circ \theta^{-1}: \tilde{V} \to \tilde{P}^i$. Then $\varphi$ is the required Cremona transformation. By construction of $\varphi$, it is now straightforward to see that the assertion (2) of the theorem is verified by $\varphi$.

The converse of the theorem follows from the analogue of Enriques' characterization theorem of ruled surfaces (cf. [7] and [11]).

3. Further properties of $\Lambda$

In this section, we consider the case where $D$ is of the second kind. We retain the notations and the assumptions in the previous section. Let $C$ be a general member of $\Lambda$, and let $d_0$ and $d_1$ be the same as defined as before. The pencil $\Lambda$ has two multiple members $aF$ and $bG$; we assume that $a\psi'(F)$ is an irreducible member of $\psi'\Lambda$ for $\psi = \rho_m$ or $\rho_m \tau_m$ with $m < n$. Let $b_0 = i(F, l_0; P_0)$ and let $b_1 = \text{mult}_{P_0} F$; then $b_0 < b_1 = \deg F$. Define positive integers $d_2, \ldots, d_n$ and $q_1, \ldots, q_n$ by the Euclidean algorithm with respect to $d_0$ and $d_1$ (cf. Introduction). Similarly, define positive integers $b_2, \ldots, b_n$ and $p_1, \ldots, p_n$ by the Euclidean algorithm with respect to $b_0$ and $b_1$.

Let $\sigma_1: V_1 \to V_0 := P^i_k$ be the first Euclidean transformation with respect to a datum $(P_0, l_0, d_0, d_1)$. Then, since

$$ (\sigma_1(C))^2 = (C^2) - \sum_{i=1}^{n} q_i d_i^2 = d^2 - d_d > 0, $$

the pencil $\sigma_1(\Lambda)$ has still a base point $P_1'$ on the last exceptional curve $E_1'$ obtained in the process $\sigma_1$. This implies that $\alpha = \beta$, $p_i = q_i$ for $1 \leq i \leq \alpha$ and $d_\alpha := (\sigma_1(C) \cdot E_1') = (a\sigma_1(F) \cdot E_1') = ab$. Therefore we have $d_i = ab_i$ for $0 \leq i \leq \alpha$.

For $1 \leq m < n$, let $\eta_m = \rho_m \tau_m$, let $E_m$ be the last exceptional curve obtained in the process $\eta_m$, and let $P_m$ be the base point of the pencil $\eta_m \Lambda$, i.e., $P_m = \eta_m(C) \cap E_m$. Let $d_m = i(\eta_m C, E_m; P_m) = (\eta_m C \cdot E_m)$, $d_\alpha = \text{mult}_{P_m} \eta_m C$, $b_m = i(\eta_m F, E_m; P_m)$, $b_0 = i(\eta_0 F, E_m; P_m)$, $b_1 = \text{mult}_{P_m} \eta_1 F$. Set $d_0 = d_0$, $d_\alpha = d_i$, $b_0 = b_0$ and $b_1 = b_1$. 


Then it is easy to obtain the following relations:

1. \( d^m = \text{G.C.D.}(d^{(m-1)}, d_{1}^{(m-1)}) \) and \( b^m = \text{G.C.D.}(b^{(m-1)}, b_{1}^{(m-1)}) \) for \( 1 \leq m < n \);
2. \( \text{G.C.D.}(d^{(s-1)}, d_{1}^{(s-1)}) = \text{G.C.D.}(b^{(s-1)}, b_{1}^{(s-1)}) = 1 \);
3. \( d_0^m = ab^m \) and \( d_1^m = ab^m \) for \( 0 \leq m < n - 1 \).

Note that \( \bar{A} = \sigma' \Lambda \) has no base points and that \( \sigma' F \) is the unique exceptional curve of the first kind in the reducible member of \( \bar{A} \) of which \( \sigma' F \) is an irreducible component. Hence we obtain:

4. \( ((\eta_{s-1} C)^2) = d_{0}^{(s-1)}d_1^{(s-1)} \) and \( ((\eta_{s-1} F)^2) = b_{0}^{(s-1)}b_1^{(s-1)} - 1 \).

Since \( \eta_{s-1} C \) is linearly equivalent to \( a\eta_{s-1} F \) and since \( d_0^{(s-1)} = ab_0^{(s-1)} \), we obtain by a straightforward computation with the relation (4) taken into account:

5. \( d_0^{(s-1)} = a^2, d_1^{(s-1)} = ab_1^{(s-1)} - 1 \) and \( b_0^{(s-1)} = a \).

Let \( r_m \) be the length of the EM-transformation \( \tau_m \) for \( 1 \leq m < n \). Then we have:

6. \( d^2 = \sum_{m=0}^{d_0} d_0^m d_1^m + \sum_{m=0}^{d_1} (d_0^m)^2 r_m, \) and \( b^2 = \sum_{m=0}^{b_0} b_0^m b_1^m + \sum_{m=0}^{b_1} (b_0^m)^2 r_m - 1 \).

Since \( p_\sigma(\sigma' C) = p_\sigma(\sigma' F) = 0 \), we obtain:

7. \( 3d = d_0 + \sum_{m=0}^{d_0} d_0^m + \sum_{m=0}^{d_1} d_0^m r_m + 1, \) and \( 3b = b_0 + \sum_{m=0}^{b_0} b_0^m + \sum_{m=0}^{b_1} b_0^m r_m \).

Since \( d_0 < d = ab \) and \( a^2 | d_0 \) by (5), we have:

8. \( a < b \).

Since \( b_0^{(s-1)} = a \) by (5) and \( b_0^{(s-1)} | b_0^{(m)} \) for \( 0 \leq m < n \), we obtain from (6):

9. \( a | (b^2 + 1) \).

On the other hand, by virtue of Noether's inequality (cf. Hudson [2; p. 9]), the sum of three highest (or three equally highest) multiplicities of the singular points (including infinitely near singular points) of \( C \) centered at \( P_0 \) is larger than \( d \). Hence we have:

10. If \( \alpha \geq 2 \) then \( q_1 = 1 \); if \( \alpha = 1 \) then \( q_1 \leq 2 \). Hence we have \( d < 3d_1 \) and \( b < 3b_1 \).

Finally, we consider Yoshihara's quintic rational curve \( F \) which is defined by the following equation:

\[ F: (YZ - X^2)(YZ^2 - X^2Z - 2XY^2) + Y^5 = 0. \]
Then $F$ has only one singular point $P_0:=(X=0, Y=0, Z=1)$, which is a cuspidal point (i.e., a one-place point) of multiplicity 2. The tangent line of $F$ at $P_0$ is given by

$$l_0: Y = 0.$$ 

New consider the conic $G$ defined by

$$G: YZ - X^2 = 0.$$ 

Let $\Lambda$ be the linear pencil on $P^2$ spanned by $2F$ and $5G$. Then, with the foregoing notations, we have:

$$d=10, \ a=2, \ b=5, \ d_0=8, \ d_1=4, \ b_0=4, \ b_1=2.$$ 

Then $\sigma^{-1}(l_0 \cup F \cup G)$ has the following configuration on $\tilde{V}$:

![Diagram](attachment:diagram.png)

where $I_0:=\sigma'(l_0), \ F:=\sigma'(F)$ and $G:=\sigma'(G)$. Let $\theta$ be the contraction of $G, \ E_5, E_4, E_3, E_2, E_1, \ F, E_8, E_7$ and $\tilde{E}$ in this order. Then the resulting configuration on $P^2$ is given by:

![Diagram](attachment:diagram2.png)

where $\theta(I_0)$ is a nonsingular conic. Moreover, $\varphi=\theta \cdot \sigma^{-1}$ is a Cremona transformation of degree 10.
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