Title	On a projective plane curve whose complement has logarithmic Kodaira dimension $-\infty$
Author(s)	Sugie, Tohru; Miyanishi, Masayoshi
Citation	Osaka Journal of Mathematics. 1981, 18(1), p. 1-11
Version Type	VoR
URL	https://doi.org/10.18910/12582
rights	
Note	

Miyanishi, M. and Sugie, T.
Osaka J. Math.
18 (1981), 1–11

ON A PROJECTIVE PLANE CURVE WHOSE COMPLEMENT HAS LOGARITHMIC KODAIRA DIMENSION $-\infty$

MASAYOSHI MIYANISHI AND TOHRU SUGIE

(Received September 14, 1979)

Introduction. Let k be an algebraically closed field of characteristic zero and let D be a reduced curve on the projective plane $\boldsymbol{P}_k^2$ such that its complement $X:=\boldsymbol{P}_k^2-D$ has logarithmic Kodaira dimension $-\infty$ (cf. Iitaka [3, 4] for the definition and relevant results). We call such a curve D *a projective plane curve with complementary logarithmic Kodaira dimension* $-\infty$, a plane curve with *CLKD* $-\infty$, for short. The purpose of the present article is to give a characterization of plane curves with *CLKD* $-\infty$. Namely, we shall prove the following:

Theorem. *Let D be a reduced plane curve with CLKD* $-\infty$. *Then the following assertions hold true*:

(1) *There exists an irreducible linear pencil* Λ *on* $\boldsymbol{P}_k^2$ *such that D is a union of* (*irreducible components of*) *members of* Λ, *where* Λ *satisfies the following properties*:

1° Λ *has only one base point* P_0, *and, for a general member* C *of* Λ, $C-\{P_0\}$ *is isomorphic to the affine line* $\boldsymbol{A}_k^1$; *the point* P_0 *is, therefore, a one-place point of* C.

2° *All members of* Λ *are irreducible, and* Λ *has at most two multiple members.*

3° *Let d be the degree of a general member* C *of* Λ; *if* $d=1$ *then* Λ *is the pencil of lines through* P_0; *if* $d>1$, *let* l_0 *be the tangent line of a general member* C, *i.e., a line of maximal contact with* C *at* P_0; *if* $dl_0\in\Lambda$ *then* dl_0 *is the unique multiple member of* Λ; *if* $dl_0\notin\Lambda$ *then* Λ *has two multiple members aF and bG, where F and G are irreducible and a, b are integers* $\geqq 2$ *such that* $a=\deg G$, $b=\deg F$, $d=ab$ *and* $G.C.D.(a,b)=1$; *D is said to be of the first kind or of the second kind according as* $dl_0\in\Lambda$ *or* $dl_0\notin\Lambda$, *respectively.*

(2) *There exists a Cremona transformation* φ: $\boldsymbol{P}_k^2\to\boldsymbol{P}_k^2$ *of degree d such that*:

1° *The pencil* Λ *corresponds* (*under* φ) *to the pencil of lines through a point* Q_0;

2° *if D is of the first kind, the line* l_0 *is the unique exceptional curve for* φ; *if*

L_0 is the line through Q_0 corresponding to dl_0 then L_0 is the unique exceptional curve for φ^{-1};

3° if D is of the second kind, the curves F and G exhaust the exceptional curves for φ; if L_1 and L_2 are lines through Q_0 corresponding to aF and bG under φ, then L_1 and L_2 exhaust the exceptional curves for φ^{-1}.

Conversely, let Λ be an irreducible linear pencil on $\boldsymbol{P}_k^2$ satisfying the property 1° of the assertion (1), *and let D be the union of* (*irreducible components of*) *members of Λ. Then D is a plane curve with CLKD $-\infty$.*

If D is of the first kind, the theorem implies that the Cremona transformation φ induces a biregular automorphism of the affine plane $\boldsymbol{P}_k^2-l_0 \xrightarrow{\sim} \boldsymbol{P}_k^2-L_0$ and that $X:=\boldsymbol{P}_k^2-D$ contains an open set which is isomorphic (under φ) to the affine plane with finitely many parallel lines deleted off. So far, we have only one example for D of the second kind. That is, a quintic rational curve with only one cusp of multiplicity 2, which was obtained by Yoshihara [8]. In the final section, we discuss Yoshihara's example together with some detailed observations on plane curves D of the second kind with $CLKD$ $-\infty$.

Our termiology and notation conform with those of Miyanishi [5] and Miyanishi-Sugie [7]. We only modify some of the definitions given in [5] so as to suit the present situation.

(1) Let V_0 be a nonsingular projective surface, let P_0 be a point on V_0 and let l_0 be an irreducible curve on V_0 such that P_0 is a simple point of l_0. Let d_0 and d_1 be positive integers such that $d_1<d_0$. Define positive integers $d_2, \cdots, d_\alpha$ and $q_1, \cdots, q_\alpha$ by the following Euclidean algorithm:

$$\begin{array}{ll} d_0 = d_1q_1+d_2 & 0<d_2<d_1 \\ d_1 = d_2q_2+d_3 & 0<d_3<d_2 \\ \cdots\cdots\cdots\cdots & \\ d_{\alpha-2} = d_{\alpha-1}q_{\alpha-1}+d_\alpha & 0<d_\alpha<d_{\alpha-1} \\ d_{\alpha-1} = d_\alpha q_\alpha & 1<q_\alpha . \end{array}$$

Let $N=q_1+\cdots+q_\alpha$. Define the infinitely near points P_i's of P_0 (for $1\leqq i<N$) and the quadratic transformation σ_i: $V_i\to V_{i-1}$ with center P_{i-1} (for $1\leqq i\leqq N$) inductively in the following fashion:

(i) P_i is an infinitely near point of order one of P_{i-1} for $1\leqq i<N$;

(ii) let $E_i=\sigma_i^{-1}(P_{i-1})$ for $1\leqq i\leqq N$ and let $E(\nu,j)=E_i$ if $i=q_0+\cdots+q_{\nu-1}+j$ with $1\leqq\nu\leqq\alpha$ and $1\leqq j\leqq q_\nu$, where we set $q_0:=0$; P_i is the intersection point of the proper transform of $E(\nu-1, q_{\nu-1})$ on V_i and the curve $E(\nu,j)$ if $i=q_0+\cdots+q_{\nu-1}+j$ with $1\leqq\nu\leqq\alpha$ and $1\leqq j\leqq q_\nu$ ($1\leqq j<q_\alpha$ if $\nu=\alpha$), where we set $E(0, q_0)=l_0$.

The composition $\sigma=\sigma_1\cdot\cdots\cdot\sigma_N$ is called *the Euclidean transformation with re-*

spect to a datum (P_0, l_0, d_0, d_1), or the Euclidean transformation, for short, if the datum (P_0, l_0, d_0, d_1) is clear by the context. The related definitions are given in Miyanishi [5; p. 92 and p. 214]. In particular, the weighted dual graph of $\sigma^{-1}(l_0)$ is the same as given in [*loc. cit.*; p. 95]. If C is an irreducible curve on V_0 such that P_0 is a one-place point of C with $d_0=i(C, l_0; P_0)$ and $d_1=\mathrm{mult}_{P_0}C$ (=the multiplicity of C at P_0), then the proper transform $C_i:=(\sigma_1\cdots\sigma_{i-1})'(C)$ passes through P_i so that $d_\nu=(C_i\cdot E(\nu, j))$ and the intersection multiplicity of C_i with the proper transform of $E(\nu-1, q_{\nu-1})$ on V_i is $d_{\nu-1}-jd_\nu$, where $1\leqq i\leqq N$ and $i=q_0+\cdots+q_{\nu-1}+j$ with $1\leqq\nu\leqq\alpha$ and $1\leqq j\leqq q_\nu$; the smaller one of d_ν and $d_{\nu-1}-jd_\nu$ is the multiplicity of C_i at P_i. We note that $\sigma'(C)$ meets only the last exceptional curve $E(\alpha, q_\alpha)$ and does not meet the other exceptional curves in the process σ.

(2) Let V_0, P_0 and l_0 be as above. Let r be a positive integer. *An equi-multiplicity transformation of length r with center* P_0, or an *EM*-transformation, for short, is the composition $\sigma=\sigma_1\cdot\cdots\cdot\sigma_r$ of quadratic transformations defined as follows:

For $1\leqq i\leqq r$, σ_i is the quadratic transformation with center P_{i-1} and P_i (for $i<r$) is a point on $\sigma_i^{-1}(P_{i-1})$ other than the point $\sigma_i'(\sigma_{i-1}^{-1}(P_{i-2}))\cap\sigma_i^{-1}(P_{i-1})$ $(\sigma_1'(l_0)\cap\sigma_1^{-1}(P_0)$ if $i=1)$. A related notion is the (e,i)-transformation defined in [*loc. cit.*; p. 100]. If C is an irreducible curve on V_0 such that P_0 is a one-place point with $d_0:=i(C, l_0; P_0)$ equal to $d_1:=\mathrm{mult}_{P_0}C$, $P_1:=\sigma_1'(C)\cap\sigma_1^{-1}(P_0)$ differs from $\sigma_1'(l_0)\cap\sigma_1^{-1}(P_0)$. If $d_1^{(1)}:=\mathrm{mult}_{P_1}\sigma_1'(C)$ equals $d_1=(\sigma_1'(C)\cdot\sigma_1^{-1}(P_0))$ then $P_2:=(\sigma_1\sigma_2)'(C)\cap\sigma_2^{-1}(P_1)$ differs from $\sigma_2'(\sigma_1^{-1}(P_0))\cap\sigma_2^{-1}(P_1)$. Thus, this step can be repeated as long as the intersection multiplicity of the proper transform of C with the last exceptional curve equals the multiplicity of the proper transform of C at the intersection point, and the composition of performed quadratic transformations is an equi-multiplicity transformation with center P_0.

1. The pencil Λ

Let D be a plane curve with *CLKD* $-\infty$ and let $X:=\boldsymbol{P}_k^2-D$. Then X is a nonsingular rational affine surface with logarithmic Kodaira dimension $\bar{\kappa}(X)=-\infty$. By virtue of the analogue of Enriques' characterization theorem of ruled surfaces (cf. Miyanishi-Sugie [7] and Fujita [1]), X contains a cylinder-like open set $U\cong U_0\times\boldsymbol{A}^1$, where U_0 is a nonsingular rational affine curve. Then there exists an irreducible linear pencil Λ on $\boldsymbol{P}_k^2$ such that, for a general member C of Λ, $C\cap X$ is a general fiber of the canonical projection $p_1: U\to U_0$. Hence $C\cap X$ is isomorphic to the affine line $\boldsymbol{A}_k^1$. Let d be the degree of C. Since $d>0$, Λ has a unique base point P_0, which is a one-place point for a general member C of Λ. Let D_1 be an irreducible component of the curve D. Then D_1 meets only at P_0 with general members of Λ. Hence D_1 is an irreducible component of a member of Λ. Therefore, we know that D is a union of irreducible components of members of Λ. If $d=1$ the pencil Λ consists of lines through P_0, and we have nothing more to show in the theorem; take the identity automorphism of $\boldsymbol{P}_k^2$ as the Cremona transformation φ. Thus, we shall assume, henceforth, that $d>1$.

Let C be a general member of Λ and let l_0 be the tangent line of C at P_0, which is the line of maximal contact with C at P_0. Then l_0 is the tangent line for almost all members of Λ. Indeed, if $d=2$ the pencil Λ is spanned by a nonsingular conic C through P_0 and $2l_0$, where l_0 is the tangent line of C at P_0. Hence every irreducible member of Λ has l_0 as its tangent line at P_0. If $d>2$ then P_0 is a singular point for almost all members of Λ. If the assertion does not hold, after the quadratic transformation $\sigma\colon V'\to V_0:=\boldsymbol{P}_k^2$, the exceptional curve $E':=\sigma^{-1}(P_0)$ is a quasi-section of the pencil $\sigma'\Lambda$ (=the proper transform of Λ by σ), (cf. [5; p. 190]). Since P_0 is a one-place point for almost all members C of Λ and $(E'\cdot\sigma'(C))>1$, this is a contradiction.

We shall prove the following:

Lemma 1. *Let the notations and the assumptions be as above. Then the following assertions hold true*:

(1) *Every member M of Λ is irreducible and $M_{\text{red}}-\{P_0\}$ is isomorphic to the affine line $\boldsymbol{A}_k^1$.*

(2) *Λ has at most two multiple members.*

(3) *If Λ has only one multiple member then it is dl_0.*

Proof. Let C_0 be a general member of Λ. Then C_0 is a curve with $CLKD$ $-\infty$. In order to prove the above assertions, we may replace D by C_0 and assume that D consists of only one general member of Λ. Then $\mathrm{Pic}(X)$ is a cyclic group of order d, where $X=\boldsymbol{P}_k^2-D$. Then we shall show that the projection $p_1\colon U\to U_0$, U being the cylinderlike open set of X which defines the pencil, extends to a surjective morphism $\pi\colon X\to T\cong\boldsymbol{A}_k^1$, whose general fibers are isomorphic to $\boldsymbol{A}_k^1$. Indeed, since P_0 is the single base point of Λ, the rational map $\Phi_\Lambda\colon\boldsymbol{P}_k^2\to\boldsymbol{P}_k^1$ defined by Λ is regular outside P_0. Hence the restriction $\pi:=\Phi_\Lambda|_X$ is a morphism. Since D is a member of Λ, the image $T:=\pi(X)$ is isomorphic to the affine line. By construction, general fibers of π are isomorphic to $\boldsymbol{A}_k^1$.

By virtue of Miyanishi [6; Lemma 1.1], we have

$$\text{rank Pic}(X)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=\sum_{P\in T}(\mu_P-1),$$

where μ_P is the number of irreducible components of the fiber $\pi^{-1}(P)$ and where P ranges over all points of T. Since $\mathrm{Pic}(X)$ is a cyclic group of order d, we know that every fiber of π is irreducible; we know also by the same result (*loc.*

cit.) that every fiber of π is isomorphic to $\boldsymbol{A}_k^1$. This implies that every member of Λ is irreducible and nonsingular outside the point P_0. This proves the assertion (1). We can strengthen, in effect, the cited result as follows: Let $P_1, \cdots, P_s$ exhaust the points of T such that $\pi^*(P_i)=a_i\pi^{-1}(P_i)$ with $a_i \geqq 2$ for $1 \leqq i \leqq s$; then $\mathrm{Pic}(X)$ is a finite abelian group with generators $\xi_1, \cdots, \xi_s$ and relations $a_1\xi_1=\cdots=a_s\xi_s=0$. Hence $a_1, \cdots, a_s$ are pairwise coprime, and $d=a_1\cdots a_s$.

If $s=1$, we know thence that $a_1=d$. Let F_1 be the closure in $\boldsymbol{P}_k^2$ of the irreducible curve $\pi^{-1}(P_1)$. Then dF_1 is linearly equivalent to C. Hence F_1 is a line. Since $F_1 \cap C=\{P_0\}$, we know that $F_1=l_0$. Hence dl_0 is a member of Λ. This proves the assertion (3).

Let $\tilde{\sigma}: \tilde{V} \rightarrow V_0:=\boldsymbol{P}_k^2$ be the shortest succession of quadratic transformations with centers at base points (including infinitely near base points) of Λ such that the proper transform $\tilde{\Lambda}$ of Λ by $\tilde{\sigma}$ has no base points. Let $\tilde{E}$ be the exceptional curve obtained by the last quadratic transformation in the process $\tilde{\sigma}$. Then $\tilde{E}$ is a cross-section of $\tilde{\Lambda}$, i.e., a general member of $\tilde{\Lambda}$ meets $\tilde{E}$ transversally in a single point. Let F_i be the closure in $\boldsymbol{P}_k^2$ of $\pi^{-1}(P_i)$, let $\tilde{F}_i$ be the proper transform of F_i by $\tilde{\sigma}$ and let Γ_i be the member of $\tilde{\Lambda}$ for which $\tilde{F}_i$ is an irreducible component, where $1 \leqq i \leqq s$. Since $a_i \geqq 2$, Γ_i is a reducible member. Namely, Γ_i contains other irreducible components by means of which the component $\tilde{F}_i$ is connected to the cross-section $\tilde{E}$. Those irreducible components of Γ_i's other than $\tilde{F}_i$'s for $1 \leqq i \leqq s$ are exceptional curves obtained by the quadratic transformations in the process $\tilde{\sigma}$. The composition $\tilde{\sigma}$ of quadratic transformations is, in effect, a composition of Euclidean transformations and EM-transformations, which are uniquely determined by a general member C of Λ as explained in the Introduction. This is due to the fact that P_0 is a one-place point for a general member C of Λ. By looking at the configurations of exceptional curves obtained by the Euclidean transformations or the EM-transformations (cf. [5; p. 95]), we know that the totality of exceptional curves (except $\tilde{E}$) obtained in the process $\tilde{\sigma}$ is divided into at most two connected components, and that each of (irreducible) exceptional curves (except $\tilde{E}$) is contained in a member of $\tilde{\Lambda}$. This observation implies that $\tilde{\Lambda}$ contains at most two reducible fibers. Therefore Λ has at most two multiple members. This proves the assertion (2). Q.E.D.

We have also the following:

Lemma 2. *Let the notations and the assumptions be as above. Then the following conditions are equivalent to each other*:

(1) *For a general member C of Λ, we have*

$$(C \cdot l_0)=i(C, l_0; P_0)=d\,.$$

(2) $dl_0 \in \Lambda$, i.e., *D is of the first kind.*
(3) Λ *has only one multiple member.*

Proof. (1)$\Rightarrow$(2): By assumption, $C-\{P_0\}$ is contained in the affine plane $X_0:=\boldsymbol{P}_k^2-l_0$. Since $C-\{P_0\}$ is isomorphic to the affine line, we may choose coordinates x, y on X_0 so that the curve $C-\{P_0\}$ is defined by $x=0$ (cf. Abhyankar-Moh's Embedding theorem [5; p. 90]). Let C' be another general member of Λ. Then $C'-\{P_0\}$ is defined by $f(x, y)=0$, where $f(x, y)\in k[x, y]$. Since $C\cap C'=\{P_0\}$, we have $f(0, y)\neq 0$, i.e., $f(x, y)=c+xg(x, y)$ with $c\in k^*:=k-(0)$ and $g(x, y)\in k[x, y]$. Since $k[x, y]/(f(x, y))$ is a polynomial ring in one variable over k, there exists an element $c'\in k^*$ such that $x-c'$ is divisible by $f(x, y)$. Hence we may assume that $f(x, y)=x-c'$. Since Λ is spanned by C and C', almost all members of Λ restricted on X are defined by equations of the form $x=c$ with $c\in k^*$. This implies that Λ is spanned by C and dl_0, and that every member of Λ except dl_0 is irreducible and reduced.
(2)$\Rightarrow$(1): Since P_0 is the unique base point of Λ, we have

$$i(C, l_0; P_0) = (C\cdot l_0) = d\,.$$

(2)$\Rightarrow$(3): We have shown, in the course of the proof (1)$\Rightarrow$(2), that dl_0 is the unique multiple member of Λ. The implication (3)$\Rightarrow$(2) is proved in Lemma 1. Q.E.D.

Assume that Λ has two multiple members aF and bG. Assuming, as in the proof of Lemma 1, that D consists of only one general member of Λ, $\mathrm{Pic}(X)$ (with $X:=\boldsymbol{P}_k^2-D$) is a cyclic group of order d generated by $[F]$ and $[G]$ with relations $a[F]=b[G]=0$, where $[F]$ and $[G]$ are the divisor classes represented by F and G, respectively. Then it is clear that $b=\deg F$, $a=\deg G$, $d=ab$ and $G.C.D.(a,b)=1$. Therefore, we proved the assertion (1) of the theorem.

2. The Cremona transformation φ

We retain the notations in the previous section. Let $\tilde{\sigma}$: $\tilde{V}\to V_0:=\boldsymbol{P}_k^2$ be the shortest succession of quadratic transformations with centers at base points (including infinitely near base points) of Λ such that the proper transform $\tilde{\Lambda}$ of Λ by $\tilde{\sigma}$ has no base points. As already remarked, $\tilde{\sigma}$ is the composition of Euclidean transformations and EM-transformations. Since the process $\tilde{\sigma}$ is uniquely determined, the Euclidean trasnformations and the EM-transformations which constitute $\tilde{\sigma}$ are uniquely determined. Let C be a general member of Λ, let $d_0=i(C, l_0; P_0)$ and let $d_1=\mathrm{mult}_{P_0}C$ (=the multiplicity of C at P_0). Then we have $d_1<d_0\leqq d$. Hence the process $\tilde{\sigma}$ starts with the Euclidean transformation $\sigma_1: V_1\to V_0$ with respect to a datum (P_0, l_0, d_0, d_1). Write

$$\tilde{\sigma} = \sigma_1\tau_1\sigma_2\tau_2\cdots\sigma_n\tau_n\,,$$

where σ_i's $(1\leqq i\leqq n)$ are Euclidean transformations and τ_j's $(1\leqq j\leqq n)$ are the EM-transformations; if the EM-transformation, say τ_j, is not necessary in the process $\tilde{\sigma}$, we understand $\tau_j=id.$.

Let $\rho_m=\sigma_1\tau_1\cdots\sigma_m$ with $m<n$. Then the totality of (irreducible) exceptional curves obtained in the process ρ_m is contained in one and the same member of the pencil $\rho'_m\Lambda$. Indeed, the pencil $\rho'_m\Lambda$ has the unique base point P_m on the last exceptional curve E_m which does not lie on any other exceptional curves appeared in the process ρ_m. Since each exceptional curve, which is an irreducible component of $\rho_m^{-1}(P_0)$, is contained in a member of $\rho'_m\Lambda$ and since $\sigma_m^{-1}(P_0)$, is connected, E_m and all other exceptional curves are contained in one and the same member of $\rho'_m\Lambda$. A similar assertion holds true for $\rho_m\tau_m$. Since the last exceptional curve $\tilde{E}$ obtained in the process $\tilde{\sigma}$ is a cross-section of the pencil $\tilde{\Lambda}=\tilde{\sigma}'\Lambda$, we know easily that $\tau_n\neq id.$ if D is of the first kind and $\tau_n=id.$ if D is of the second kind. Moreover, if D is of the second kind, one of two multiple members aF and bG of Λ, say aF, has the corresponding irreducible member $a\psi'(F)$ in the pencil $\psi'\Lambda$, where $\psi=\rho_m$ or $\rho_m\tau_m$ with $m<n$; $\psi'(G)$ then belongs to the member of $\psi'\Lambda$ containing all the exceptional curves; in the final step, $\tilde{\sigma}'(F)$ becomes an irreducible component of a reducible member of $\tilde{\sigma}'\Lambda$, whose irreducible components (except $\tilde{\sigma}'F$) are exceptional curves arising from the Euclidean transformation σ_n (cf. the configuration in [5; p. 95]).

Since $\tilde{\Lambda}=\tilde{\sigma}'\Lambda$ has no base points, $\tilde{\Lambda}$ defines a surjective morphism $p:=\Phi_{\tilde{\Lambda}}\colon \tilde{V}\to \boldsymbol{P}_k^1$ such that, if $\boldsymbol{P}_k^2-\{P_0\}$ is naturally identified with an open set of $\tilde{V}$, p coincides with the rational map Φ_Λ on $\boldsymbol{P}_k^2-\{P_0\}$, and that the general fibers of p are non-singular rational curves. The curve $\tilde{E}$ is a cross-section of p. We shall construct a $\boldsymbol{P}^1$-bundle from $\tilde{V}$ by contracting, one by one, all possible exceptional curves of the first kind contained in fibers of p. This contraction process is possible by virtue of the following:

Lemma 3 [5; Lemma 2.2, p. 115]. *Let $f\colon V\to B$ be a surjective morphism from a nonsingular projective surface V onto a nonsingular complete curve B such that almost all fibers are isomorphic to $\boldsymbol{P}_k^1$. Let $F=n_1C_1+\cdots+n_rC_r$ be a singular fiber of f, where C_i is an irreducible curve, $C_i\neq C_j$ if $i\neq j$, and $n_i>0$. Then we have:*

(1) *$G.C.D.(n_1,\cdots,n_r)=1$ and* $\mathrm{Supp}(F):=\bigcup_{i=1}^{r}C_i$ *is connected.*

(2) *For $1\leqq i\leqq r$, C_i is isomorphic to $\boldsymbol{P}_k^1$ and $(C_i^2)<0$.*

(3) *For $i\neq j$, $(C_i\cdot C_j)=0$ or 1.*

(4) *For three distinct indices i, j and l, $C_i\cap C_j\cap C_l=\phi$.*

(5) *One of C_i's, say C_1, is an exceptional curve of the first kind. If $\tau\colon V\to W$ is the contraction of C_1, then f factors as $f\colon V\xrightarrow{\tau}W\xrightarrow{g}B$, where $g\colon W\to B$ is a fibration by $\boldsymbol{P}_k^1$.*

(6) *If one of n_i's, say n_1, equals 1 then there is an exceptional curve of the first*

kind among C_i's with $2 \leqq i \leqq r$.

Let θ_1: $\tilde{V} \to W$ be the contraction of all possible exceptional curves of the first kind contained in the reducible fibers of p. Then p factors as $p: \tilde{V} \xrightarrow{\theta_1} W \xrightarrow{q} \boldsymbol{P}_k^1$ (cf. Lemma 3, (5)) and W is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}_k^1$. If Δ is a reducible fiber of p, the irreducible component Δ_1 of Δ meeting the cross-section $\tilde{E}$ has multiplicity 1. Hence we may assume that Δ_1 is not contracted in the process θ_1, i.e., $\theta_1(\Delta_1)$ is the fiber of q lying over the point $p(\Delta)$, (cf. Lemma 3, (6)); we assume that this assumption is valid for all reducible fibers of p. Then $\theta_1(\tilde{E})$ is a cross-section of q with $(\theta_1(\tilde{E})^2) = -1$. This implies that W is the Hirzebruch surface of degree 1, i.e., $W = \mathrm{Proj}(\boldsymbol{O}_{P^1} \oplus \boldsymbol{O}_{P^1}(1))$. Then, by contracting $\theta_1(\tilde{E})$, we obtain the projective plane $\boldsymbol{P}_k^2$ and all fibers of q become lines through the point $Q_0 := \theta_2(\theta_1(\tilde{E}))$, where θ_2: $W \to \boldsymbol{P}_k^2$ is the contraction of $\theta_1(\tilde{E})$. Let $\theta = \theta_2 \cdot \theta_1$ and let $\varphi := \theta \cdot \tilde{\sigma}^{-1}: \boldsymbol{P}_k^2 \to \tilde{V} \to \boldsymbol{P}_k^2$. Then φ is the required Cremona transformation. By construction of φ, it is now straightforward to see that the assertion (2) of the theorem is verified by φ.

The converse of the theorem follows from the analogue of Enriques' characterization theorem of ruled surfaces (cf. [7] and [11]).

3. Further properties of Λ

In this section, we consider the case where D is of the second kind. We retain the notations and the assumptions in the previous section. Let C be a general member of Λ, and let d_0 and d_1 be the same as defined as before. The pencil Λ has two multiple members aF and bG; we assume that $a\psi'(F)$ is an irreducible member of $\psi'\Lambda$ for $\psi = \rho_m$ or $\rho_m \tau_m$ with $m < n$. Let $b_0 = i(F, l_0; P_0)$ and let $b_1 = \mathrm{mult}_{P_0}, F$; then $b_1 < b_0 \leqq b = \deg F$. Define positive integers $d_2, \cdots, d_\alpha$ and $q_1, \cdots, q_\alpha$ by the Euclidean algorithm with respect to d_0 and d_1 (cf. Introduction). Similarly, define positive integers $b_2, \cdots, b_\beta$ and $p_1, \cdots, p_\beta$ by the Euclidean algorithm with respect to b_0 and b_1.

Let σ_1: $V_1 \to V_0 := \boldsymbol{P}_k^2$ be the first Euclidean transformation with respect to a datum (P_0, l_0, d_0, d_1). Then, since

$$(\sigma_1'(C)^2) = (C^2) - \sum_{i=1}^{\alpha} q_i d_i^2 = d^2 - d_0 d_1 > 0\,,$$

the pencil $\sigma_1'\Lambda$ has still a base point P_1' on the last exceptional curve E_1' obtained in the process σ_1. This implies that $\alpha = \beta$, $p_i = q_i$ for $1 \leqq i \leqq \alpha$ and $d_\alpha = (\sigma_1'(C) \cdot E_1') = (a\sigma_1'(F) \cdot E_1') = ab_\beta$. Therefore we have $d_i = ab_i$ for $0 \leqq i \leqq \alpha$.

For $1 \leqq m < n$, let $\eta_m = \rho_m \tau_m$, let E_m be the last exceptional curve obtained in the process η_m, and let P_m be the base point of the pencil $\eta_m'\Lambda$, i.e., $P_m = \eta_m'(C) \cap E_m$. Let $d_0^{(m)} = i(\eta_m'C, E_m; P_m) = (\eta_m'C \cdot E_m)$, $d_1^{(m)} = \mathrm{mult}_{P_m} \eta_m'C$, $b_0^{(m)} = i(\eta_m'F, E_m; P_m) = (\eta_m'F \cdot E_m)$ and $b_1^{(m)} = \mathrm{mult}_{P_m} \eta_m'F$. Set $d_0^{(0)} = d_0$, $d_1^{(0)} = d_1$, $b_0^{(0)} = b_0$ and $b_1^{(0)} = b_1$.

Then it is easy to obtain the following relations:

(1) $d_0^{(m)}=G.C.D.(d_0^{(m-1)}, d_1^{(m-1)})$ *and*
$b_0^{(m)}=G.C.D.(b_0^{(m-1)}, b_1^{(m-1)})$ *for* $1\leqq m<n$;

(2) $G.C.D.(d_0^{(n-1)}, d_1^{(n-1)})=G.C.D.(b_0^{(n-1)}, b_1^{(n-1)})=1$;

(3) $d_0^{(m)}=ab_0^{(m)}$ *and* $d_1^{(m)}=ab_1^{(m)}$ *for* $0\leqq m<n-1$.

Note that $\tilde{\Lambda}=\sigma'\Lambda$ has no base points and that $\sigma'F$ is the unique exceptional curve of the first kind in the reducible member of $\tilde{\Lambda}$ of which $\sigma'F$ is an irreducible component. Hence we obtain:

(4) $((\eta'_{n-1}C)^2)=d_0^{(n-1)}d_1^{(n-1)}$ *and* $((\eta'_{n-1}F)^2)=b_0^{(n-1)}b_1^{(n-1)}-1$.

Since $\eta'_{n-1}C$ is linearly equivalent to $a\eta'_{n-1}F$ and since $d_0^{(n-1)}=ab_0^{(n-1)}$, we obtain by a straightforward computation with the relation (4) taken into account:

(5) $d_0^{(n-1)}=a^2$, $d_1^{(n-1)}=ab_1^{(n-1)}-1$ *and* $b_0^{(n-1)}=a$.

Let r_m be the length of the *EM*-transformation τ_m for $1\leqq m<n$. Then we have:

(6) $d^2=\sum_{m=0}^{n-1} d_0^{(m)}d_1^{(m)}+\sum_{m=1}^{n-1}(d_0^{(m)})^2r_m$, *and* $b^2=\sum_{m=0}^{n-1} b_0^{(m)}b_1^{(m)}+\sum_{m=1}^{n-1}(b_0^{(m)})^2r_m-1$.

Since $p_a(\sigma'C)=p_a(\sigma'F)=0$, we obtain:

(7) $3d=d_0+\sum_{m=0}^{n-1} d_1^{(m)}+\sum_{m=1}^{n-1} d_0^{(m)}r_m+1$, *and* $3b=b_0+\sum_{m=0}^{n-1} b_1^{(m)}+\sum_{m=1}^{n-1} b_0^{(m)}r_m$.

Since $d_0<d=ab$ and $a^2|d_0$ by (5), we have:

(8) $a<b$.

Since $b_0^{(n-1)}=a$ by (5) and $b_0^{(n-1)}|b_0^{(m)}$ for $0\leqq m<n$, we obtain from (6):

(9) $a|(b^2+1)$.

On the other hand, by viitue of Noether's inequality (cf. Hudson [2; p. 9]), the sum of three highest (or three equally highest) multiplicities of the singular points (including infinitely near singular points) of C centered at P_0 is larger than d. Hence we have:

(10) *If* $\alpha\geqq 2$ *then* $q_1=1$; *if* $\alpha=1$ *then* $q_1\leqq 2$. *Hence we have* $d<3d_1$ *and* $b<3b_1$.

Finally, we consider Yoshihara's quintic rational curve F which is defined by the following equation:

$$F: (YZ-X^2)(YZ^2-X^2Z-2XY^2)+Y^5=0\,.$$

Then F has only one singular point $P_0 := (X=0, Y=0, Z=1)$, which is a cuspidal point (i.e., a one-place point) of multiplicity 2. The tangent line of F at P_0 is given by

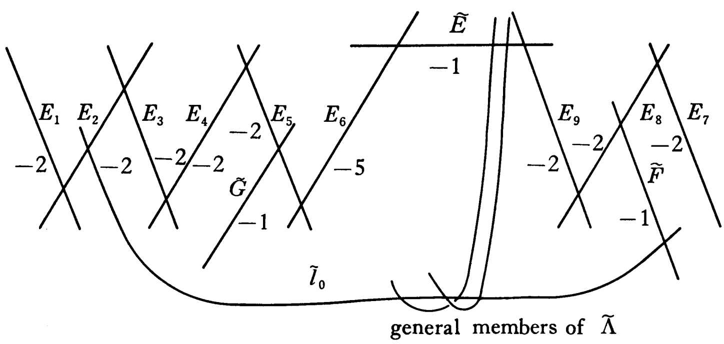
$$l_0 \colon \; Y = 0 \, .$$

New consider the conic G defined by

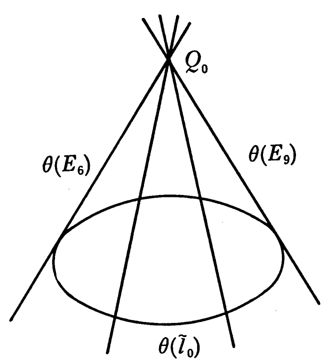
$$G \colon \; YZ - X^2 = 0 \, .$$

Let Λ be the linear pencil on $\boldsymbol{P}_k^2$ spanned by $2F$ and $5G$. Then, with the foregoing notations, we have:

$d=10$, $a=2$, $b=5$, $d_0=8$, $d_1=4$, $b_0=4$, $b_1=2$. Then $\tilde{\sigma}^{-1}(l_0 \cup F \cup G)$ has the following configuration on $\tilde{V}$:



where $\tilde{l}_0 := \tilde{\sigma}'(l_0)$, $\tilde{F} := \tilde{\sigma}'(F)$ and $\tilde{G} := \tilde{\sigma}'(G)$. Let θ be the contraction of $\tilde{G}$, $E_5, E_4, E_3, E_2, E_1, \tilde{F}, E_8, E_7$ and $\tilde{E}$ in this order. Then the resulting configuration on $\boldsymbol{P}_k^2$ is given by:



where $\theta(\tilde{l}_0)$ is a nonsingular conic. Moreover, $\varphi = \theta \cdot \tilde{\sigma}^{-1}$ is a Cremona transformation of degree 10.

References

[1] T. Fujita: *On Zariski Problem,* Proc. Japan Acad. **55** (1979), 106–110.

[2] H.P. Hudson: Cremona transformations in plane and space, Cambridge Univ. Press, 1927.

[3] S. Iitaka: *On logarithmic Kodaira dimension of algebraic varieties,* Complex analysis and algebraic geometry, 1977, Iwanami, Tokyo.

[4] S. Iitaka: *Geometry on complements of lines in* $\boldsymbol{P}^2$, Tokyo J. Math. **1** (1978), 1–19.

[5] M. Miyanishi: Lectures on curves on rational and unirational surfaces, Tata Institute of Fundamental Research, 1978, Springer, Berlin-Heidelberg-New York.

[6] M. Miyanishi: *Regular subrings of a polynomial ring,* Osaka J. Math. **17** (1980), 329–338.

[7] M. Miyanishi and T. Sugie: *Affine surfaces containing cylinderlike open sets,* J. Math. Kyoto Univ., **20** (1980), 11–42.

[8] H. Yoshihara: *A problem on plane rational curves,* Sugaku **31** (1979), 256–261, (in Japanese).

Masayoshi Miyanishi
Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan

Tohru Sugie
Department of Mathematics
Kyoto University
Kyoto 606
Japan