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Ultimate Strength of Stiffened Plates and Minimum Stiffness Ratio of Their Stiffeners (under Thrust)†

Yukio UEDA* and Tetsuya YAO**

Abstract

In this paper, a theoretical investigation into the effectiveness of a stiffener against the ultimate strength of a stiffened plate is performed. Series of buckling analyses, elastic large deflection analyses and elastic-plastic large deflection analyses are performed by the analytical method and the finite element method on the stiffened plate under thrust. Experiments are also carried out to confirm the theoretical results.

On buckling of a stiffened plate, it is well known that there exists a minimum stiffness ratio of a stiffener to the plate, $\gamma^B{}_{min}$, which gives the maximum limiting value of the buckling strength. Concerning the ultimate strength, it is found and confirmed that there exists a significant stiffness ratio of a stiffener to the plate, $\gamma^U{}_{min}$, similar to $\gamma^B{}_{min}$ for the buckling strength.

It is also found that there exist three typical collapse modes for the stiffened plate under thrust.

(1) MODE OO ; overall collapse after overall buckling
(2) MODE LO ; overall collapse after local buckling
(3) MODE LL ; local collapse after local buckling

Approximate methods are proposed to evaluate the ultimate strength and $\gamma^U{}_{min}$ of a multi-stiffened plate under thrust.

The effects of initial imperfections such as welding residual stresses and initial deflection on ultimate strength and $\gamma^U{}_{min}$ of a stiffened plate are also discussed.

KEY WORDS: (Compressive Primary Buckling) (Compressive Secondary Buckling) (Compressive Collapse) (Compressive Collapse Modes) (Stiffened Plates) (Minimum Stiffness Ratio) (El-Pl Large Deflection)

1. Introduction

In general, ship structures are constructed by using plates to reduce their weights and costs, and a number of stiffeners are attached to the plates to increase their rigidity and strength. The buckling strength of a stiffened plate increases with an increase of the flexual rigidity of the stiffener, but reaches to its maximum limiting value when the stiffness ratio of the stiffener to the plate, γ, reaches to $\gamma^B{}_{min}$, which is well known as the minimum stiffness ratio against the buckling strength of a stiffened plate.

As the load increases, a stiffened plate buckles in the overall mode when $\gamma < \gamma^B{}_{min}$, while the local buckling of the plate takes place in the case of $\gamma^B{}_{min} < \gamma$. When the thickness of the plate is thin, the stiffened plate can carry further load after buckling in both cases.

As to the buckling strength of a stiffened plate, the fundamental studies were systematically performed by S.P. Timoshenko and J.M. Gere[1)] and K. Klöpple and J. Sheer[2)]. The ultimate strength of a stiffened plate under thrust was also investigated by Y. Yoshiki *et al.*[3),4)]. However, it has not been clarified whether there exists the minimum stiffness ratio of a stiffener against the ultimate strength of a stiffened plate.

In this paper, the authors will summarize their recent research works on this subject[5)-9)]. First, the fundamental methods for theoretical analyses which the authors employed are described in Chap. 2. Then, in Chap. 3, the buckling and the ultimate strength of a stiffened plate under thrust are discussed together with the minimum stiffness ratio of a stiffener against the ultimate strength of a stiffened plate, which is newly defined in this investigation. The effects of initial imperfections due to welding on the compressive strengths are also discussed, and the theoretical results are confirmed by the experimental ones.

2. Methods of Theoretical Analyses

In this chapter, the outline of the methods for theoretical analyses which the authors employed in their re-

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search works will be described.

It is well known that an initially flat stiffened plate undergoes a primary buckling from an initially flat equilibrium state under external loads, if the loads are applied non-eccentrically. When the stiffened plate is accompanied by initial deflection or the load is eccentrically applied, lateral deflection increases from the begining of the loading, but the increase of deflection is small until the load reaches near the buckling load. Furthermore, the buckled stiffened plate sometimes undergoes the secondary buckling from the primary buckling mode. To analyse such behavior of a stiffened plate, the geometrical non-linearity must be taken into account.

For a further increase of the load, plastification gradually takes place in the plate and/or the stiffener, and the stiffened plate reaches to its ultimate strength. For the analysis of this stage, the material non-linearity must also be taken into account in addition to the geometrical one.

For the purpose of analysing such behaviors of a stiffened plate, two methods are employed. One is the analytical method for the elastic large deflection analysis, and the other is the numerical method for elastic-plastic large deflection analysis. In both methods, the stiffener of a flat bar type is considered, and its torsional rigidity is ignored.

2.1. Analytical method for elastic large deflection analysis

In the analytical method, Airy's stress function, F, is introduced using the coordinate systems shown in **Figs. 1 and 8.** The compatibility equation of the plate can be expressed as follows.

$$\nabla^4 F = E\,[(\partial^2 w/\partial x\partial y)^2 - (\partial^2 w/\partial x^2)(\partial^2 w/\partial y^2) - (\partial^2 w_0/\partial x\partial y)^2 + (\partial^2 w_0/\partial x^2)(\partial^2 w_0/\partial y^2)] \quad \cdots\cdots\cdots\cdot \quad (1)$$

where w_0 and w are the initial deflection and the total deflection under external load, respectively.

The equilibrium equation for the plate is

$$D\nabla^4 (w - w_0) = t\,[(\partial^2 F/\partial y^2)(\partial^2 w/\partial x^2) - 2\,(\partial^2 F/\partial x\partial y)(\partial^2 w/\partial x\partial y) + (\partial^2 F/\partial x^2)(\partial^2 w/\partial y^2)] \quad (2)$$

where

$D = Et^3/12\,(1 - \nu^2)$; flexual rigidity of the plate

t ; thickness of the plate

E ; Young's modulus

ν ; Poisson's ratio

In the analysis, strains are assumed to be continuous along the joint line of the plate and the stiffener, and the equilibrium equation for the stiffener becomes as follows.

$$EI_s d^4 (w - w_0)/dx^4 = A_s\,(\partial^2 F/\partial y^2 - \nu\partial^2 F/\partial x^2)\,d^2 w/dx^2 \quad (3)$$

where I_s and A_s are the moment of inertia and the sectional area of the stiffener, respectively.

In the analytical method, the initial and the total deflection, w_0 and w are first assumed so as to satisfy the boundary conditions. Then, the stress function of the plate, F, is derived substituting assumed w_0 and w into Eq. (1).

For one type of problems, the principle of virtual work is applied in the incremental form, and the fundamental equations to be solved are derived. For the other type of problems, Galerkin's method is applied to Eqs. (2) and (3) substituting w_0, w and F.

2.2. Numerical method for elastic-plastic large deflection analysis

Applying the finite element method as a numerical method, the fundamental equilibrium equations are derived in the incremental form based on the principle of virtual work.

According to the formulation which has been developed by one of the authors[10)], the final equilibrium equation can be expressed in the following form.

$$\{dF_g\} = ([K_{pl}]_g + [K_{st}]_g)\,\{dh_g\} - \{L_g\} \quad (4)$$

where $\{dF_g\}$ and $\{dh_g\}$ represent the nodal force and nodal displacement increments, respectively, and

$[K_{pl}]_g = \Sigma\,[\Lambda]^T\,[K_{pl}]^e\,[\Lambda]$;
summation of all the stiffness matrices for each individual element of the plate in the global coordinates

$[K_{st}]_g = \Sigma\,[\Lambda]^T [K_{st}]^e\,[\Lambda]$;
summation of all the stiffness matrices for each individual element of the stiffener in the global coordinates

$[\Lambda]$; transformation matrix of the coordinates

$\{L_g\}$; load correction vector

The detail of the formulation is presented in Ref. 10).

3. Stiffened Plate under Thrust

Fundamentally, there are two ways to reinforce a plate by providing stiffeners. One is to furnish stiffeners on both sides of the plate, and the other is to put them on one side. The former is suitable to study the basic behaviors of a stiffened plate. The latter is the usual practice which is often used in actual structures.

3.1. Symmetrically both-sided stiffened plate

3.1.1. Analytical method for elastic large deflection analysis

First, as the most fundamental research model of a stiffened plate, a square plate with one flat stiffener is chosen. The stiffener is attached along one center line of the plate as shown in **Fig. 1**.

In general, the stiffener is welded to the plate, and consequently the plate and the stiffener are accompanied by welding residual stresses and initial deflection. In the analysis, the welding residual stresses shown in **Fig. 1 (a)** are assumed, which are based on the measured distribution in the test specimen[6]. As for the initial deflection, the following one is assumed.

$$w_0 = A_0 \sin\frac{\pi x}{b} \sin\frac{\pi y}{b} + B_0 \sin\frac{2\pi x}{b} \sin\frac{2\pi y}{b} \quad (5)$$

In the above equation, the first and the second terms represent the deflection modes of the overall and the local buckling, respectively, and A_0 and B_0 are the maximum deflections for the corresponding modes.

Although there exist an infinite number of primary buckling modes of stiffened plate, the subsequent buckling after the primary one shows a different character. In order to investigate this phenomenon, a large deflection analysis must be performed.

In the elastic large deflection analysis of the stiffened plate under thrust, the following total deflection is assumed.

$$w = A \sin\frac{\pi x}{b} \sin\frac{\pi y}{b} + B \sin\frac{2\pi x}{b} \sin\frac{2\pi y}{b} \quad (6)$$

where A and B are the maximum deflections of the corresponding modes.

Substituting Eqs. (5) and (6) into Eq. (1), the stress function, F, is obtained. Substituting this stress function and Eqs. (5) and (6) into Eqs. (2) and (3), the following two equations are derived applying Galerkin's method.

$$\alpha_1 (A^2 - A_0{}^2)A + \alpha_2 (AB - A_0B_0)B + (\alpha_3 + \alpha_5)(A - A_0) - \alpha_4(B^2 - B_0{}^2)A - (\alpha_6\sigma - \alpha_7)A = 0 \quad (7)$$

$$\beta_1 (B^2 - B_0{}^2)B + \beta_2 (AB - A_0B_0)A + \beta_3 (B - B_0) - (\beta_4\sigma - \beta_5)B = 0 \quad (8)$$

where σ is the applied mean compressive stress, and

$$\alpha_1 = \pi^4 tE/32b^2 + \pi^4 (2 - \nu)A_sE/32b^3,$$
$$\alpha_2 = 2\pi^4 tE/25b^2, \quad \alpha_3 = \pi^4 D/b^2, \quad \alpha_4 = \pi^4 A_sE/4b^3,$$
$$\alpha_5 = \pi^4 I_sE/2b^3, \quad \alpha_6 = \pi^2 t(1 + 2A_s/bt)/4,$$
$$\alpha_7 = \pi^2 [t_s \int_{-h/2}^{h/2} h(z)dz + t\int_0^b g(y)\sin^2\frac{\pi y}{b}dy]/2b,$$

$$\beta_1 = \pi^4 tE/2b^2, \quad \beta_2 = 2\pi^4 tE/25b^2, \quad \beta_3 = 4\pi^2 D/b^2,$$
$$\beta_4 = \pi^2 t, \quad \beta_5 = 2\pi^2 t[\int_0^b g(y)\sin^2\frac{2\pi y}{b}dy]/b,$$
$$I_s = t(h^3 - t_s{}^3), \quad A_s = t_s(h - t)$$

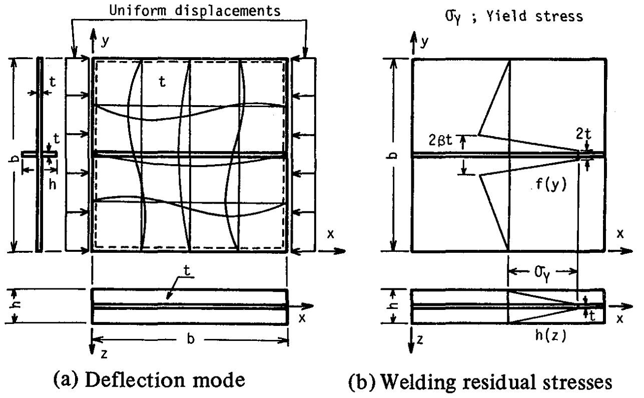
The solutions of the simultaneous equations, (7) and (8), describe the elastic large deflection behavior of a stiffened plate with initial imperfections due to welding.

To calculate the buckling strength of a stiffened plate, the coefficients of initial deflection, A_0 and B_0, in Eqs. (7) and (8) must be zero, and the resulting equations become

$$\alpha_1 A^3 + (\alpha_2 - \alpha_4)AB^2 + (\alpha_3 + \alpha_5 + \alpha_7 - \alpha_6\sigma)A = 0 \quad (10)$$

$$\beta_1 B^3 + \beta_2 A^2 B + (\beta_3 + \beta_5 - \beta_4\sigma)B = 0 \quad (11)$$

The buckling strength of a stiffened plate can be calculated by Eqs. (10) and (11). First, assuming A (or B) to be zero, the primary buckling strength is obtained as the stress at the bifurcation point. Above this stress, B (or A)



(a) Deflection mode (b) Welding residual stresses

Fig. 1 Square plate with a symmetrically both-sided stiffener under thrust is no longer zero. After the primary buckling has occured, the deflection of mode B (or A) increases, and the subsequent buckling strength is again obtained at the bifurcation point. Above this stress, A (or B) again becomes non-zero. This buckling will be called the secondary buckling hereafter. The deflection modes of the primary and the secondary buckling are determined by the magnitude of the stiffness ratio, γ, whether this value is less than or greater than $\gamma^B{}_{min}$.

(a) $\gamma < \gamma^B_{min}$

In this case, the stiffened plate buckles first in the overall mode at the critical stress given below.

$$\sigma_{cr}{}^1 = (\alpha_3 + \alpha_5 + \alpha_7)/\alpha_6 \tag{12}$$

After the overall buckling, the secondary buckling in the local mode occurs, of which critical stress is given in the following form.

$$\sigma_{cr}{}^2 = \{\alpha_1(\beta_3 + \beta_5) - (\alpha_3 + \alpha_5 + \alpha_7)\beta_2\}/(\alpha_1\beta_4 - \alpha_6\beta_2) \tag{13}$$

(b) $\gamma^B_{min} \leqq \gamma$

Contrary to the case (a), the primary and the secondary bucklings are in the local and the overall modes, respectively, and the corresponding critical stresses become as follows

$$\sigma_{cr}{}^3 = (\beta_3 + \beta_5)/B_4 \tag{14}$$

$$\sigma_{cr}{}^4 = \{(\alpha_3 + \alpha_5 + \alpha_7)\beta_1 - (\alpha_2 - \alpha_4)(\beta_3 + \beta_5)\}/\{\alpha_6\beta_1 - (\alpha_2 - \alpha_4)\beta_4\} \tag{15}$$

3.1.2. Buckling strength

Taking b and t as 500 mm and 3.13 mm, respectively, the primary and the secondary buckling strengths are calculated changing the welding residual stresses and the height of the stiffener. The calculated buckling strengths are plotted in **Fig. 2**. It is observed that the welding they residual stresses reduce the local buckling strength, while increase the overall buckling strength. Consequently, the minimum stiffness ratio, γ^B_{min}, against the buckling strength decreases when there exist welding residual stresses. It should be noted that the maximum limiting value of the buckling strength also decreases.

3.1.3. Post-buckling behavior and ultimate strength

A series of elastic-plastic large deflection analyses is carried out for the stiffened plate shown in Fig. 1 changing the height of the stiffener. The initial deflection of the form expressed by Eq. (5) is assumed, but the welding residual stresses are not taken into account.

For some typical cases, the load-deflection curves are shown in **Fig. 3**, and the deflection modes and stress distributions at collapse in **Fig. 4**. The evaluated ultimate strengths are plotted against the height of the stiffener in **Fig. 5**, where U1-U7 represents the ultimate strength of a stiffened plate with small initial deflection ($A_0/t = B_0/t = 0.01$), and UD1-UD4 represents that with large initial deflection ($A_0/t = B_0/t = 0.5$).

Here, the behavior of the stiffened plate with small initial deflection will be described.

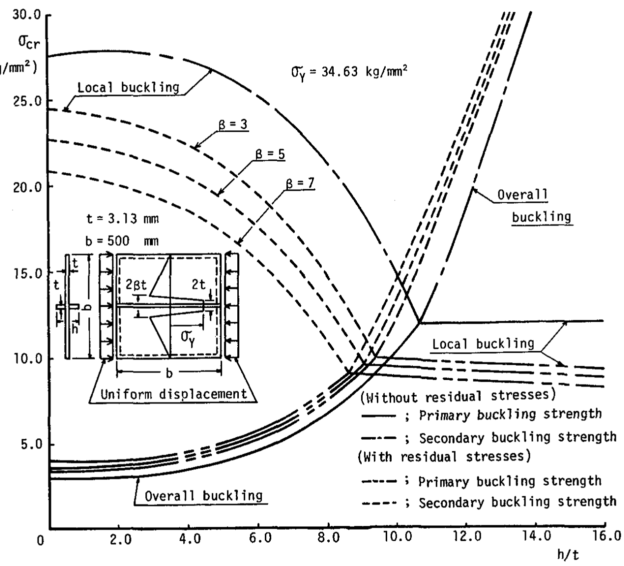


Fig. 2 Buckling strength of a symmetrically both-sided stiffened plate under thrust

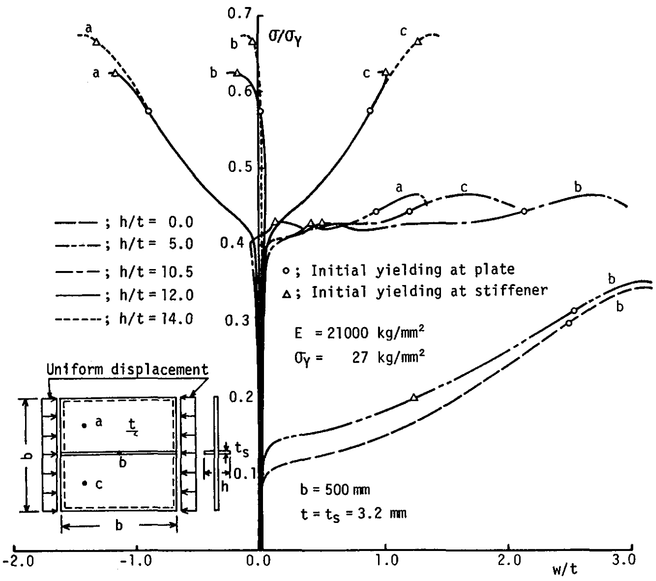


Fig. 3 Load deflection curves of a symmetrically both-sided stiffened plate under thrust

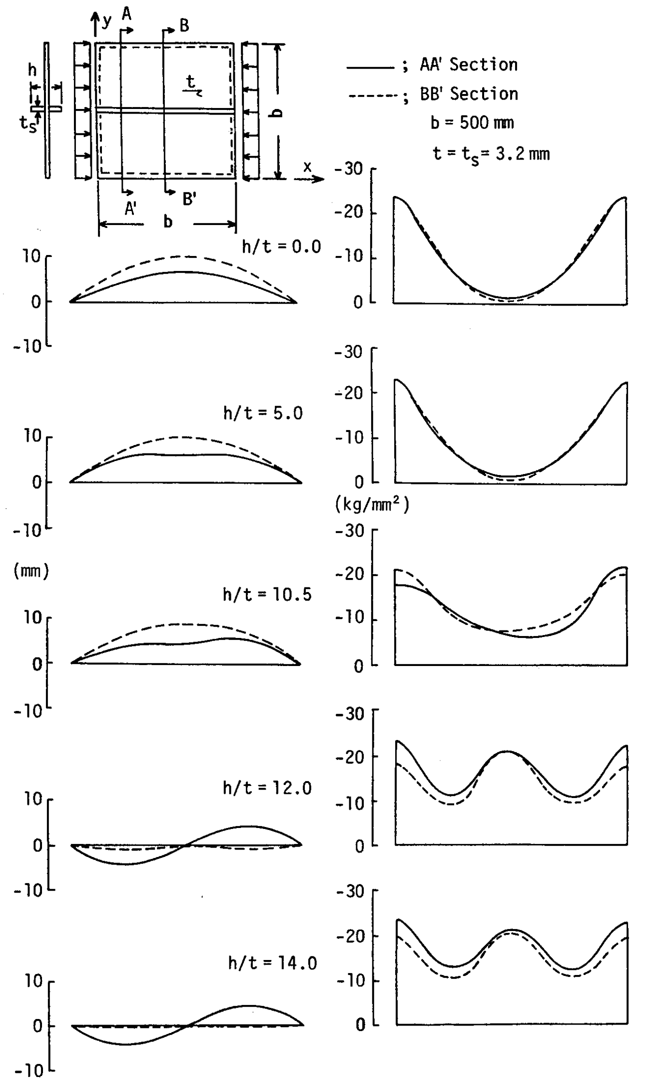


Fig. 4 Deflection mode and stress distribution of a symmetrically both-sided stiffened plate at collapse

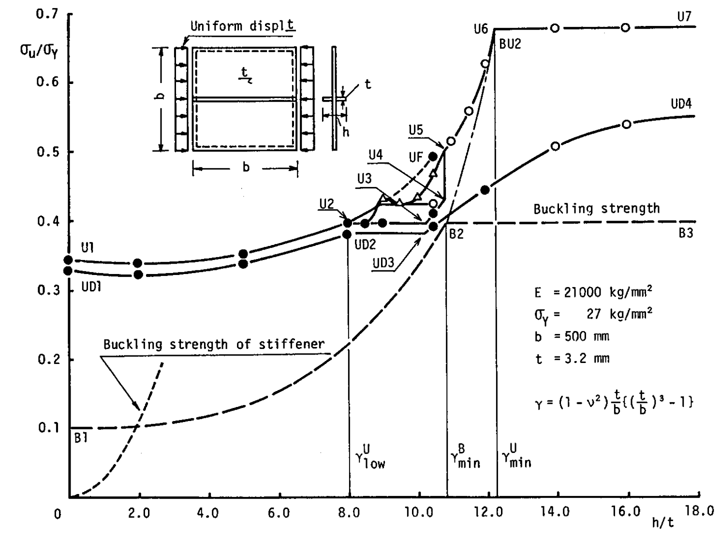


Fig. 5 Ultimate strength and minimum stiffness ratios of a symmetrically both-sided stiffened plate under thrust

(a) U1 – U2

The ultimate strength for a small ratio of h/t ($h/t < 4$) is approximately constant and it gradually increases as h/t increases ($h/t > 4$). It stops increasing and shows the local maximum at point U2. For a smaller ratio of h/t in this range, the initial yielding occurs first in the plate, and the yielding of the stiffener follows. Contrary to this, the order of the initial yielding is reversed for larger value of h/t. However, regardless of the order of the initial yielding, the stiffened plate collapses in the overall mode in this range of h/t.

The stiffness ratio, γ, calculated from h/t at point U2 is defined as γ^{U}_{low}, which gives the local maximum of the compressive ultimate strength. The presence of this local maximum is due to the sudden expansion of the yielded zone in the stiffener.

(b) U2 – U5

In this range, the stiffened plate shows a very complicated behavior. When $h/t > 9.5$, there exist two peak values in the load-deflection curves as shown in Fig. 3. These two peak values are plotted by ○ and Δ in Fig. 5. The first peak is due to the yielding of the stiffener, and the second peak that of the plate. In this range of h/t, the collapse mode is still in overall.

(c) U5 – U6

Between U5 and U6, the stiffness ratio, γ, is greater than γ^{B}_{min}. Therefore, the local buckling of the plate occurs first, and the overall buckling of the stiffened plate follows, when there is no initial deflection. When the initial defleciton is small, the deflection of the stiffener increases rapidly as the load approaches to the secondary buckling strength, and the plastification takes place in the stiffener. This plastification accelarates an increase of the deflection in the overall mode, and the stiffened plate can carry no more loads. For this reason, the ultimate strength of a stiffened plate in this range of h/t is very close to the secondary buckling strength when the initial deflection is zero or very small.

(d) U6 – U7

In this range, the plate collapses locally after local buckling before the secondary buckling in the overall mode takes place. This local failure of the plate leads to the simultaneous collapse of the stiffened plate. The stiffened plate shows almost the same ultimate strength in this range of h/t regardless of the stiffness of the stiffener, which may be regarded as the maximum limiting value of the compressive ultimate strength of a stiffened plate. The stiffness ratio, γ, which is calculated from h/t at point U6 is defined as the minimum stiffness ratio, γ^{U}_{min}, against the ultimate strength of a stiffened plate, which guarantees the maximum ultimate strength.

As is known from the above discussion, there exist three typical types of collapse mode of a stiffened plate under thrust depending on the stiffness of the stiffener, which can be classified as follows.

(a) MODE OO ($\gamma < \gamma^{B}_{min}$)
; overall collapse after overall buckling (U1 – U5)

(b) MODE LO ($\gamma^{B}_{min} \leqq \gamma < \gamma^{U}_{min}$)
; overall collapse after local buckling (U5 – U6)

(c) MODE LL ($\gamma^{U}_{min} \leqq \gamma$)
; local collapse after local buckling (U6 – U7)

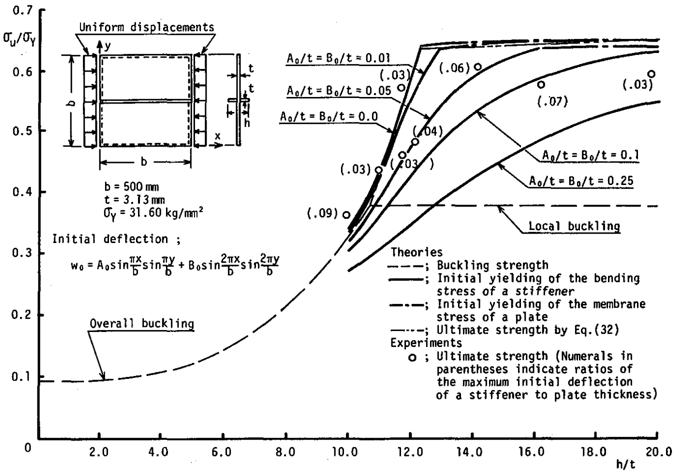
3.1.4. Effects of initial imperfections due to welding on ultimate strength and γ^{U}_{min}

A series of experiments is carried out to confirm the theoretical prediction on the behavior of a stiffened plate including the effects of initial imperfections due to welding. The test specimen shown in Fig. 1 is used taking b and t as 500 mm and 3.13 mm, respectively, of which stiffener has a stiffness ratio greater than γ^{B}_{min}. The observed ultimate strengths of the annealed specimens and the as-weld specimens are plotted by O in **Figs. 6 (a) and (b)**, respectively, against the height of the stiffener.

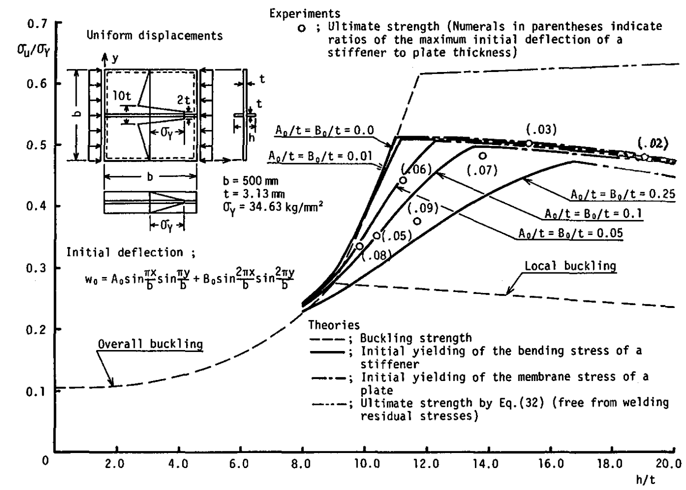
The solid lines and the chain lines with one dot in Fig. 6 represent the predicted ultimate strengths by an analytical method using Eqs. (7) and (8), in which the load at the initial yielding of a stiffened plate is regarded as its ultimate strength. The solid lines represent the ultimate strength in MODE LO, which is determined by the initial yielding of the stiffener. Contrary, the chain lines with one dot represent the ultimate strength in MODE LL, which is determined by the initial yielding at the middle plane of the plate. The validity of the above mentioned method of calculation on the ultimate strength of a symmetrically both-sided stiffened plate is confirmed by the finite element method in Refs. 6) and 9). In Fig. 6, the predicted ultimate strengths correspond well with those by the experiments.

As is known from Figs. 5 and 6, the initial deflection reduces the compressive ultimate strength of a stiffened plate especially when the collapse takes place in MODE LO. However, if the value of γ becomes greater than a certain value, the collapse occurs in MODE LL, and the ultimate strength shows a nearly constant value which depends on the local collapse strength of a plate regardless of the stiffness of the stiffener. The authors have named this specific value of the stiffness ratio as the "effective minimum stiffness ratio, ${}_{eff}\gamma^{U}_{min}$" against the ultimate strength of a stiffened plate with initial deflection.

When the welding residual stresses exist, the ultimate strength is greatly reduced as observed in Fig. 6. γ^{U}_{min} or ${}_{eff}\gamma^{U}_{min}$ also becomes small as in the case of γ^{B}_{min}.



(a) Stiffened plates free from welding residual stresses



(b) Stiffened plate with welding residual stresses

Fig. 6 Effects of initial deflection and welding residual stresses on ultimate strength of stiffened plates under thrust

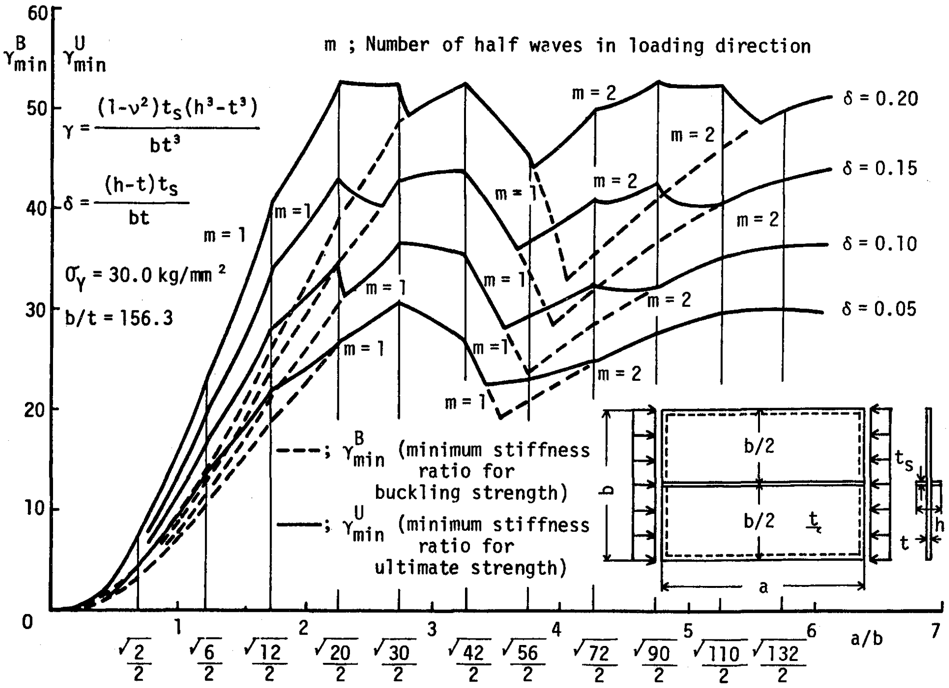
3.1.5. Minimum stiffness ratios, γ^{B}_{min} and γ^{U}_{min}

For the analysis of a stiffened plate with an arbitrary number of stiffeners and an arbitrary aspect ratio, which is subjected to thrust, the following deflection is assumed.

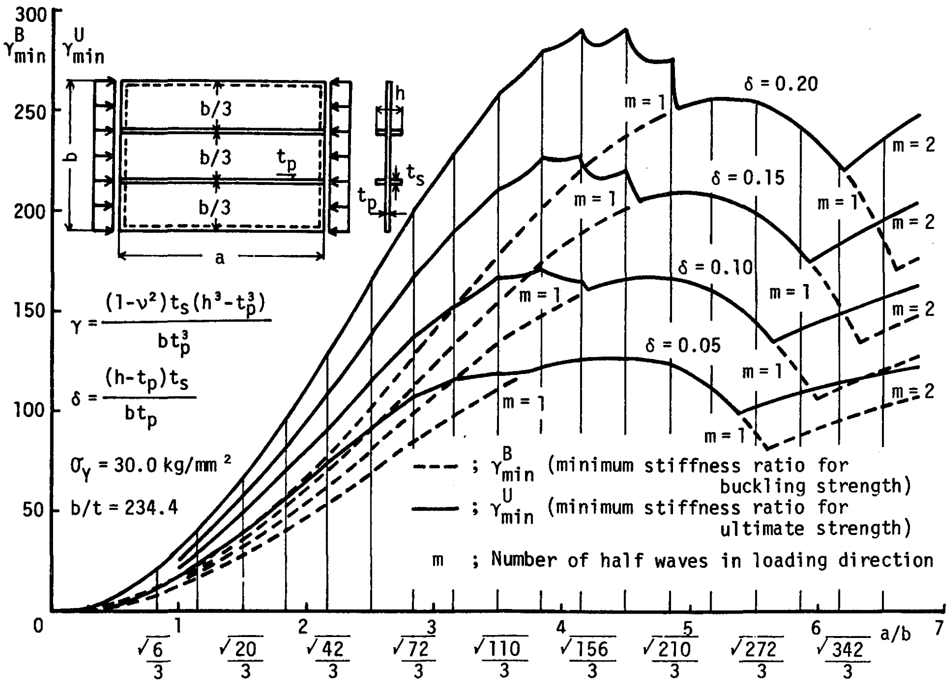
$$w = A \sin\frac{m\pi x}{a} \sin\frac{\pi y}{b} + B \sin\frac{k\pi x}{a} \sin\frac{l\pi y}{b} \qquad (16)$$

where a and b are the length and the breadth of the plate as indicated in **Fig. 7**. In the above equation, the first and the second terms represent the overall and the local buckling modes. Here, the minimum stiffness ratios, γ^{B}_{min} and γ^{U}_{min}, are calculated for the plates with one stiffener and those with two stiffeners. l in Eq. (16) is taken as 2 and 3 for the former and the latter stiffened plates, respectively.

The calculated results are shown in **Figs. 7 (a) and (b).** The broken lines and the solid lines in Fig. 7 represent γ^{B}_{min} and γ^{U}_{min}, respectively. For certain combinations of the aspect ratio of the plate and the number of the stiffeners, there exists no secondary buckling. In this case, the stiffened plate always collapses in MODE LL when the local buckling of the plate takes place, and the collapse in MODE LO does not exist. In this sense, γ^{U}_{min} becomes equal to γ^{B}_{min} for these stiffened plates.



(a) Stiffened plates with one longitudinal stiffener



(b) Stiffened plates with two longitudinal stiffeners

Fig. 7 Minimum stiffness ratios, γ^{B}_{min} and γ^{U}_{min}, of a both-sided stiffener against buckling and ultimate strengths

3.2. One-sided stiffened plate

3.2.1. Analytical method for elastic large deflection analysis

In the above discussion, the stiffeners are assumed to be symmetric with respect to the middle plane of the plate. However, in the actual structures, stiffeners are usually put on one side of the plate as shown in **Fig. 8**. For a wide use of the analytical method, it is assumed that N stiffeners of the equal dimensions are furnished to the plate at equal distances. The deflection of a stiffened plate under thrust may be represented in the following form.

$$w = A \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + B \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (17)$$

where n is equal to $N+1$. The same mode is assumed for the initial deflection, that is,

$$w_0 = A_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + B_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (18)$$

In Eqs. (17) and (18), the first and the second terms represent the overall and the local buckling modes, respectively.

Substituting Eqs. (17) and (18) into Eq. (1), the stress function, F, is derived. Using this stress function, the stress components of the plate can be expressed in the following forms.

$$\begin{aligned}\sigma_x &= \partial^2 F/\partial y^2 - Ez\,\{\partial^2 (w-w_0)/\partial x^2 \\ &\quad + \nu\partial^2 (w-w_0)/\partial y^2\}/(1-\nu^2) \\ \sigma_y &= \partial^2 F/\partial x^2 - Ez\,\{\partial^2 (w-w_0)/\partial y^2 \\ &\quad + \nu\partial^2 (w-w_0)/\partial x^2\}/(1-\nu^2) \\ \tau_{xy} &= -\partial^2 F/\partial x\partial y - Ez\partial^2 (w-w_0) \\ &\quad /\partial x\partial y/(1+\nu)\end{aligned} \quad (19)$$

From the elastic stress-strain relation in the plane stress state, the strain components can be expressed in the following forms.

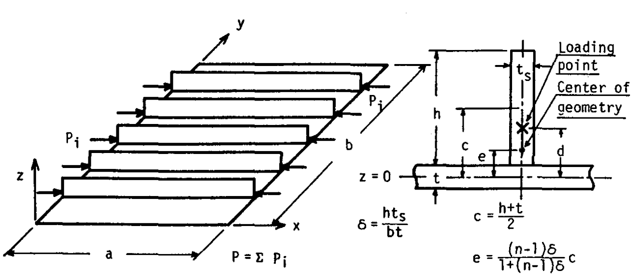


Fig. 8 One-sided stiffened plate and its coordinate system

$$\begin{aligned}\epsilon_x &= (\sigma_x - \nu\sigma_y)/E \\ \epsilon_y &= (\sigma_y - \nu\sigma_x)/E \\ \gamma_{xy} &= 2(1+\nu)\tau_{xy}/E\end{aligned} \quad (20)$$

It is assumed that the strains are continuous along the joint line of the plate and the stiffener, and the strain in the i-th stiffener can be represented as follows.

$$\epsilon_{si} = \{(\partial^2 F/\partial y^2 - \nu\partial^2 F/\partial x^2)/E - zd^2(w-w_0)/dx^2\}\,\Big|_{y=ib/n} \quad (21)$$

In the analysis, the plastification of the stiffener is taken into account, and the stress in the stiffener is put equal to the yield stress, σ_Y, at the region where $E\cdot\epsilon_{si}$ reaches σ_Y, that is,

$$\sigma_{si} = \begin{cases} E\cdot\epsilon_{si} & (E\cdot\epsilon_{si} < \sigma_Y) \\ \sigma_Y & (E\cdot\epsilon_{si} \geqq \sigma_Y)\end{cases} \quad (22)$$

Now, it is assumed that the stiffened plate is in the equilibrium state under the external load, $P\,(=(n-1)P_i)$. In this state, under the load increment, ΔP, the equilibrium equation is derived applying the principle of virtual work,

$$\delta W = \delta U_{pl} + \sum_{i=1}^{N} \delta U_{si} \quad (23)$$

where

$$\begin{aligned}\delta W &= \sum_{i=1}^{N} (P_i + \Delta P_i)\,\delta\Delta u_i \\ \Delta u_i &= [\textstyle\int_0^a\{(\partial^2\Delta F/\partial y^2 - \nu\partial^2\Delta F/\partial x^2)/E - \\ &\quad (\partial\Delta w/\partial x)^2/2 + (\partial w/\partial x)(\partial\Delta w/\partial x)\}dx + \\ &\quad 2d\partial\Delta w/\partial x\,|_{x=0}]\,|_{y=ib/n} \\ \delta U_{pl} &= \textstyle\int_V \{(\sigma_x+\Delta\sigma_x)\delta\Delta\epsilon_x + (\sigma_y+\Delta\sigma_y)\delta\Delta\epsilon_y + \\ &\quad (\tau_{xy}+\Delta\tau_{xy})\delta\Delta\gamma_{xy}\}dV \\ \delta U_{si} &= \textstyle\int_V (\sigma_{si}+\Delta\sigma_{si})\delta\Delta\epsilon_{si}dV\end{aligned} \quad (24)$$

In the above equation, Δ represents an increment. Substituting Eq. (24) into Eq. (23), the following equation can be derived.

$$[K]\,\{\Delta A, \Delta B\}^T = \{Q\}\Delta\sigma + \{R_w\} + \{L\} \quad (25)$$

where

$[K]$; function of A, B, A_0, B_0, σ and welding residual stresses

$\{Q\}$; function of A, B, A_0, B_0 and $(d-e)$
$\{R_w\}$; function of A, B, A_0, B_0 and welding residual stresses
$\{L\}$; function of A, B, A_0, B_0 and σ

3.2.2. Buckling strength

When the load is applied to the center of geometry of the section of a stiffened plate free from welding residual stresses and initial deflection, Eq. (25) becomes as follows.

$$\begin{bmatrix} K_{AA} & K_{AB} \\ K_{BA} & K_{BB} \end{bmatrix} \begin{Bmatrix} \Delta A \\ \Delta B \end{Bmatrix} = 0 \qquad (26)$$

The conditions for bifurcation of the overall buckling and the local buckling can be expressed in the following forms, respectively.

$$|\,[K_{AA}]\,| = 0 \qquad (27)$$
$$|\,[K_{BB}]\,| = 0 \qquad (28)$$

Solving Eqs. (27) and (28) with respect to the average applied stress, σ, the following buckling stresses are obtained.

Overall buckling ;

$$\sigma_{cr}^{\ 1} = [n\gamma_e + (1+\alpha^2)^2]\,\pi^2 D/a^2 t\,(1+n\delta) \qquad (29)$$

Local buckling ;

$$\sigma_{cr}^{\ 2} = \pi^2 D(m^2 + \alpha^2 n^2)^2 / a^2 t m^2 \qquad (30)$$

where

$$\gamma_e = E\,[I_s + ht_s c^2\,\{1 - 8N\delta/\pi^2\,(1+N\delta)\}]/bD$$
$$c = (h+t)/2,\; I_s = t_s h^3/12,\; \delta = ht_s/bt,\; \alpha = a/b$$

If the primary buckling expressed by Eq. (29) or Eq. (30) takes place, $\{Q\}$ in Eq. (25) is no longer zero, and consequently, both ΔA and ΔB become non-zero. For this reason, there exists no secondary buckling in the case of one-sided stiffened plates.

When the stiffened plate is accompanied by welding residual stresses, $\{R_w\}$ in Eq. (25) is not zero, and both ΔA and ΔB become non-zero even when the external load is not applied. Therefore, the buckling in the strict sense does not occur.

3.2.3. Post-buckling behavior and ultimate strength

First, a square stiffened plate with one stiffener is considered. The stiffener is furnished along one center line of a plate. **Figure 9** shows the load-deflection curves of such stiffened plates which are subjected to thrust at the center of geometry of the section. In this figure, the solid and the broken lines represent the results obtained by the analytical method and the finite element method, respectively.

As in the case of a symmetrically both-sided stiffened plate discussed in 3.1, there exist three typical collapse modes depending on the stiffness of the stiffener, which are MODE OO, MODE LO and MODE LL. Here, the large deflection behavior of a one-sided stiffened plate will be described for the three typical collapse modes based on the results by the analytical method.

(a) MODE OO ($h/t = 5.0$)

When h/t is small, plastification of the stiffener (P_s) is observed first. The stiffened plate can carry further load, and plastification of the plate takes place at point a along the supporting side of the plate as indicated in Fig. 9 (P_p). Then, the stiffened plate will show its ultimate strength.

(b) MODE LO ($h/t = 6.25$ and 6.5)

For the case of h/t being intermidiate, the local buckling of the plate proceeds. Consequently, the rigidity of the plate decreases and the load applied originally at the center of geometry of the section becomes eccentrical. Then, the deflection of the stiffener increases rapidly, and plastification takes place in the stiffener. Soon after

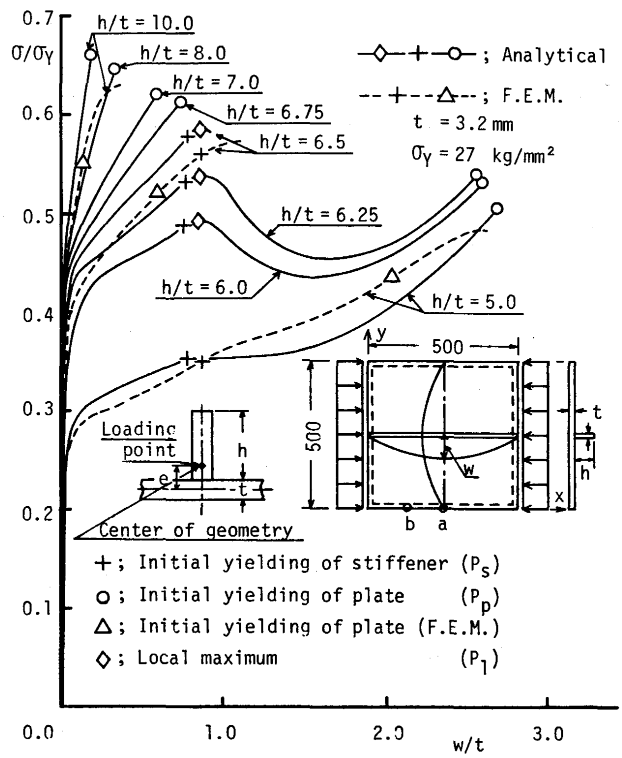


Fig. 9 Load-deflection curves of one-sided stiffened plates under thrust

plastification of the stiffener, the load shows a local maximum (P_l) and decreases as the deflection increases. However, the stiffened plate again sustains the increasing load. Then, plastification of the plate takes place between points a and b indicated in Fig. 9 (Pp), and the plate may reach its ultimate strength.

(c) MODE LL ($h/t > 6.75$)

When h/t becomes larger, the local buckling of the plate occurs. Then plastification of the plate takes place at point b indicated in Fig. 9 (P_p) before the load shows a local maximum due to plastification of the stiffener.

The ultimate strength of a one-sided stiffened plate may be approximately predicted as the initial yielding load, P_p, of the plate in the case of the collapse in MODE OO or LL. In contrast with this, the ultimate strength may be approximately estimated by the higher load among the initial yielding load of the plate, P_p, and the local maximum load, P_l, when the collapse takes place in MODE LO.

Such ultimate strengths predicted by this approximate method are plotted against the height of the stiffener in **Fig. 10**, together with those by the finite element method. They show good correlations. In Fig. 10, the solid lines between two points indicated by x represent the ultimate strength when MODE LO collapse takes place. In this range, the load shows the local maximum. When the height of the stiffener is higher than that at the point of higher x, the collapse occurs in MODE LL. In contrast with these, collapse in MODE OO takes place when the height of the stiffener is lower than that at the point of lower x. At points indicated by x, P_l is equal to P_p.

Hereafter, the approximate method described above will be employed to evaluate the compressive ultimate strength of a one-sided stiffened plate.

3.2.4. Minimum stiffness ratios, γ^U_{min} and $_{eff}\gamma^U_{min}$, of a stiffener against ultimate strength

Figure 11 shows the ultimate strength of a one-sided stiffened plate with initial deflection. When initial deflection is small, the minimum stiffness ratio, γ^U_{min}, can be determined as the stiffness ratio which terminates the collapse mode from MODE LO to MODE LL. This stiffness ratio corresponds to that at the point of higher x in Fig. 10. When initial deflection is large or the load is applied eccentrically, the effective minimum stiffness ratio, $_{eff}\gamma^U_{min}$, can be determined in the similar way as γ^U_{min}. These points are represented by O and Δ in Fig. 11, respectively.

In the case of a symmetrically both-sided stiffened plate, the maximum ultimate strength is attained if the stiffener of which stiffness ratio is γ^U_{min} is furnished. On the other hand, in the case of a one-sided stiffened plate, the ultimate strength increases as the stiffness of the stiffener increases even when γ is greater than γ^U_{min}. This is due to the following reason: After local buckling of the plate, the rigidity of the plate decreases, and the load which is first applied non-eccentrically becomes eccentrical as mentioned in 3.2.3. For this reason, the load carrying capacity of the stiffener changes depending on its stiffness when it is not so large, and so the ultimate strength

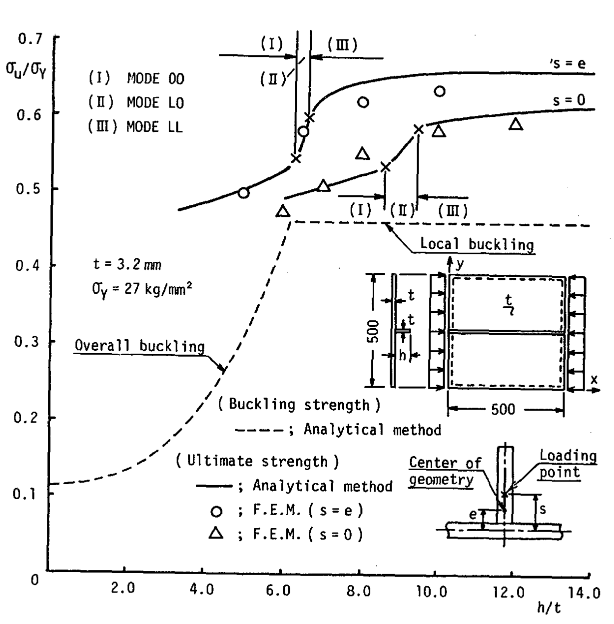


Fig. 10 Ultimate strength of a one-sided stiffened plate under thrust

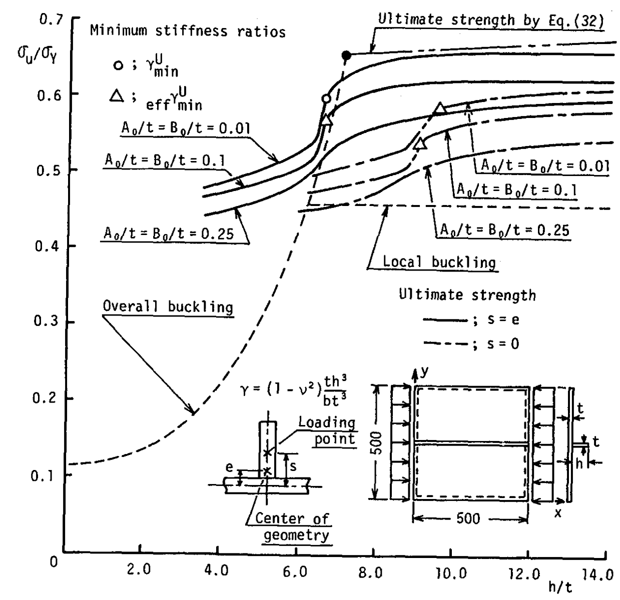


Fig. 11 Minimum stiffness ratios, γ^U_{min} and $_{eff}\gamma^U_{min}$ against ultimate strength of a one-sided stiffened plate under thrust

of a stiffened plate in MODE LL. Therefore, the stiffener of which stiffness ratio is about 3.5 times of γ^U_{min} is necessary to attain the maximum limiting value of the ultimate strength of a one-sided stiffened plate even the initial deflection is small.

The value of γ^U_{min} can be approximately determined as the stiffness ratio at the intersecting point of the overall buckling strength curve and the ultimate strength curve of MODE LL, which are represented by the dashed line and the chain line in Fig. 11, respectively. This point is indicated by ● in Fig. 11. The ultimate strength of MODE LL collapse can be approximately estimated by Eq. (32), which will be described later in 3.2.6.

When initial deflection is pretty large, there exists no significant point which terminates the collapse mode from MODE LO to MODE LL. In this case, the ultimate strength gradually increases as the stiffness increases, and it is no use to define the effective minimum stiffness ratio.

3.2.5. Effect of yield strength of a stiffener on ultimate strength and γ^U_{min}

When the primary buckling of a stiffened plate takes place in the elastic range, the value of γ^B_{min} is not affected by the yield strength of the stiffener. Contrary to this, the value of γ^U_{min} is greatly affected by it, because the ultimate strength in the overall collapse mode after local buckling depends on the plastification of the stiffener. Here, such effect on the value of γ^U_{min} and the ultimate strength of a stiffened plate will be discussed.

The ultimate strength is calculated by the analytical method described in 3.2.3, and the results are shown in **Fig. 12** together with the initial yielding strength of a stiffener. In the analysis, the yield stress of a stiffener is taken as 1.0, 1.5 and 2.0 times that of the plate. It is observed that the initial yielding strength of a stiffener increases as its yield stress increases. Consequently, the ultimate strength of a stiffened plate in overall collapse mode increases with an increase of the yield strength of a stiffener. In contrast with this, the ultimate strength in MODE LL is not affected by the yield strength of a stiffener. As the value of γ^U_{min} can be determined as the stiffness ratio which terminates the collapse mode from MODE LO to MODE LL, γ^U_{min} decreases as the yield strength of the stiffener increases.

Concerning the effect of the yield strength of a stiffener on γ^U_{min} and the ultimate strength of a stiffened plate, the effect of the shape of the cross section of the stiffener is also discussed in Ref. 7) in addition to that of the yield stress of the stiffener.

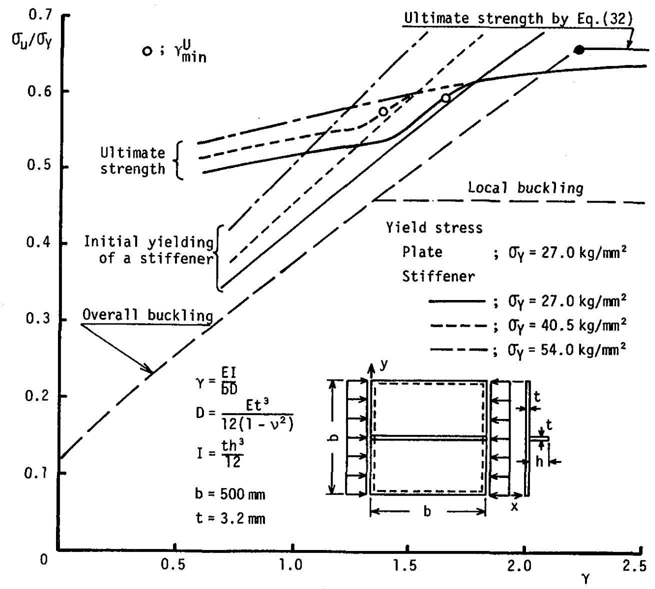


Fig. 12 Effect of the yield strength of a stiffener on the ultimate strength and U min of a one-sided stiffened plate under thrust

3.2.6. Compressive ultimate strength of a multi-stiffened plate

For the plate of which aspect ratio is 0.5, the ultimate strength is calculated by the analytical method described in 3.2.3, and the results are represented by solid lines in **Fig. 13** changing the number of stiffeners from 3 to 6.

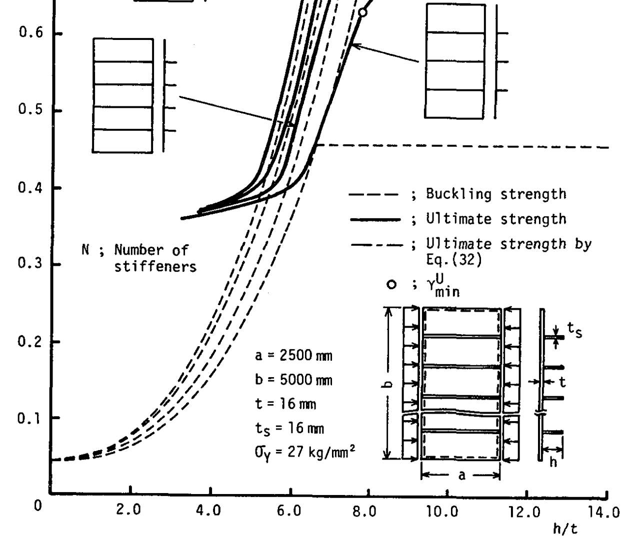


Fig. 13 Ultimate strength of one-sided stiffened plate under thrust

The broken lines in Fig. 13 represent the buckling strength. When MODE LO collapse occurs, the ultimate strength may be approximately estimated by the overall buckling strength, assuming that the local buckling of the plate does not take place, which can be expressed in the following form.

$$\sigma_u/\sigma_Y = [(N+1)\gamma_e + \{1+(a/b)^2\}^2]\,\pi^2 D/ [a^2 t\{1+(N+1)\delta\}\sigma_Y] \quad (31)$$

where N is the number of stiffeners.

The chain lines in Fig. 13 represent the approximate ultimate strength when MODE LL collapse occurs. It is evaluated as the mean value of the ultimate strengths of the plate and the stiffeners, which can be represented as follows.

$$\sigma_u/\sigma_Y = [\sigma_{up}/\sigma_Y + N\delta\sigma_{us}/\sigma_Y]\,/\,(1+N\delta) \quad (32)$$

where σ_{up}/σ_Y and σ_{us}/σ_Y are the ultimate strengths of the plate and the stiffener under thrust, respectively.

First, the ultimate strength of a plate will be considered. A rectangular plate shows a various deflection modes under thrust depending on the magnitude and the mode of initial deflection, and so its ultimate strength depending on the following collapse mode. The minimum limiting value of the compressive ultimate strength of a rectangular plate for all possible collapse mode can be expressed in the following form when the initial deflection is very small and the welding residual stresses do not exist[12].

i) $0.8 \leqq \xi < 2.0$

$$\sigma_{up}/\sigma_Y = -0.2961\,(\xi^2 - 4.0) + 0.6709\,(\xi - 2.0) + 0.8935 \quad (33)$$

ii) $2.0 \leqq \xi < 3.5$

$$\sigma_{up}/\sigma_Y = 0.8598/(\xi - 0.6678) + 0.2481 \quad (34)$$

where

$$\xi = b/(N+1)t \cdot \sqrt{\sigma_Y/E}$$

The compressive ultimate strength of a stiffener, σ_{us}/σ_Y, can be evaluated by the following equation employing Johnson's formula.

i) $\sigma_E \leqq \sigma_Y/2$

$$\sigma_{us}/\sigma_Y = \sigma_E/\sigma_Y$$

ii) $\sigma_Y/2 < \sigma_E$

$$\sigma_{us}/\sigma_Y = 1 - \sigma_Y/4\sigma_E$$

where

$$\sigma_E = \pi^2 E h^2/12a^2$$

The ultimate strengths calculated by Eq. (32) show good correlations with those by the analytical method.

In Fig. 13, ○ represents the accurate values of γ^U_{min} which are calculated by the analytical method. In contrast with this, ● represents the approximate value of γ^U_{min} which is determined as the intersecting point of the ultimate strength curves by Eqs. (31) and (32). The approximate values of γ^U_{min} coincide well with the accurate ones.

4. Conclusions

In this paper, the buckling and the ultimate strengths of a stiffened plate under thrust are investigated both theoretically and experimentally. In parallel with this, the minimum stiffness ratios of a stiffener against the buckling and the ultimate strengths are also studied. From the results of the investigation, the following conclusions are deduced.

(1) When the stiffener is symmetric with respect to the middle plane of the plate, the secondary buckling takes place after the primary buckling.

(2) For the symmetrically both-sided stiffened plate, it is found and confirmed that there exist two significant stiffness ratios, γ^U_{low} and γ^U_{min}, of a stiffener to the plate against the ultimate strength. Among these two stiffness ratios, γ^U_{min} is especially important since γ^U_{min} is the minimum required value of the stiffness ratio which guarantees to attain the maximum limiting value of the ultimate strength as the minimum stiffness ratio, γ^B_{min}, against the buckling strength.

(3) There exist three typical collapse modes of a stiffened plate according to the stiffness of a stiffener. They are

(a) MODE OO ; overall collapse after overall buckling

(b) MODE LO ; overall collapse after local buckling

(c) MODE LL ; local collapse after local buckling

The collapse mode is terminated from MODE LO to MODE LL at the minimum stiffness ratio, γ^U_{min}.

(4) For the one-sided stiffened plate, the minimum stiffness ratio, γ^U_{min}, can also be determined as defined above (3). However, about 3.5 times of the stiffness ratio of γ^U_{min} is necessary to attain the maximum limiting value of the ultimate strength in this case.

(5) When the stiffened plate is accompanied by initial deflection or the load is applied eccentrically, the effective minimum stiffness ratio, ${}_{eff}\gamma^U_{min}$, against

the ultimate strength may be defined, which has the same physical meaning as γ^U_{min} and greater than γ^U_{min}. The initial deflection or the eccentricity of the load reduces the ultimate strength.

(6) When welding residual stresses exist, the minimum stiffness ratios, γ^B_{min} and γ^U_{min}, become low, and they reduce the attained maximum limiting value of the buckling and the ultimate strengths.

(7) The approximate evaluating method of the compressive ultimate strength of a multi-stiffened plate is proposed when MODE LO and MODE LL collapses take place. The minimum stiffness ratio, γ^U_{min}, can be approximately determined as the stiffness ratio at the intersecting point of these approximate ultimate strength curves of MODE LO and MODE LL.

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