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# EXTENDIBILITY AND STABLE EXTENDIBILITY OF NORMAL BUNDLES ASSOCIATED TO IMMERSIONS OF REAL PROJECTIVE SPACES

Dedicated to the Memory of Professor Katsuo Kawakubo

TEIICHI KOBAYASHI, HARUO MAKI and TOSHIO YOSHIDA

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## 1. Introduction

The extension problem is one of the fundamental problems in topology. We consider the problem for vector bundles over real projective spaces.

Let $F$ be the real field $\boldsymbol{R}$, the complex field $\boldsymbol{C}$ or the quaternion field $\boldsymbol{H}$. Let $X$ be a space and $A$ be a subspace. A $t$-dimensional $F$-vector bundle $\zeta$ over $A$ is called *extendible* (respectively *stably extendible*) to $X$, if there is a $t$-dimensional $F$-vector bundle over $X$ whose restriction to $A$ is equivalent (respectively stably equivalent) to $\zeta$ as $F$-vector bundles, that is, if $\zeta$ is equivalent (respectively stably equivalent) to $i^*\alpha$ for a $t$-dimensional $F$-vector bundle $\alpha$ over $X$, where $i: A \to X$ is the inclusion (cf. [13] and [5]).

As is seen in [7, Theorem 6.4] and [11, Theorem 2.2], the extendibility (or the stable extendibility) is closely related to the span, i.e., the maximum number of linearly independent cross-sections of an $F$-vector bundle, and one can see in the proof of Theorem C of this paper how the stable extendibility is related to the immersion problem.

Let $\boldsymbol{R}^n$ be the $n$-dimensional Euclidean space and $F\boldsymbol{P}^n$ be the $n$-dimensional $F$-projective space. Concerning stably extendible $F$-vector bundles for $F = \boldsymbol{R}$ and $\boldsymbol{C}$, R.L.E. Schwarzenberger obtained the following results (cf. [2], [3], [7], [12] and [13]).

**Theorem** (Schwarzenberger). *Let $F = \boldsymbol{R}$ or $\boldsymbol{C}$. If a $k$-dimensional $F$-vector bundle $\zeta$ over $F\boldsymbol{P}^n$ is stably extendible to $F\boldsymbol{P}^m$ for every $m > n$, then $\zeta$ is stably equivalent to a sum of $k$ $F$-line bundles.*

In the original results of Schwarzenberger, the $F$-vector bundles are assumed to be extendible, but his results are also valid for the stably extendible $F$-vector bundles.

Recently, M. Imaoka and K. Kuwana have proved in [5] that if a $k$-dimensional

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$\boldsymbol{H}$-vector bundle $\zeta$ over $\boldsymbol{HP}^n$ is stably extendible to $\boldsymbol{HP}^m$ for every $m > n$ and its top non-zero Pontrjagin class is not zero mod 2, then $\zeta$ is stably equivalent to a sum of $k$ $\boldsymbol{H}$-line bundles provided $k \leq n$.

We study the question: Determine the necessary and sufficient condition that a $\boldsymbol{R}$-vector bundle over $\boldsymbol{RP}^n$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$. We have obtained the results for the tangent bundle $\tau = \tau(\boldsymbol{RP}^n)$ of $\boldsymbol{RP}^n$ (cf. [7] and [9]), for the normal bundle $\nu$ associated to an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^{2n+1}$ (cf. [10]) and for the complexification $c\nu$ of $\nu$ (cf. [10]) as follows:

1) $\tau$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$ if and only if $n = 1$, 3 or 7.
2) $\nu$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$ if and only if $1 \leq n \leq 8$.
3) $c\nu$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$ if and only if $1 \leq n \leq 9$.

The purpose of this paper is to improve 2) and 3) for the normal bundle $\nu$ associated to an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^{n+k}$ where $k$ is any positive integer and for the complexification $c\nu$ of $\nu$.

Let $\phi(n)$ be the number of integers $s$ such that $0 < s \leq n$ and $s \equiv 0$, 1, 2 or 4 mod 8. Then we have

**Theorem A.** *Let $\nu$ be the normal bundle associated to an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^{n+k}$, where $k > 0$. Then $\nu$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$ if and only if $k \geq 2^{\phi(n)} - n - 1$.*

**Theorem B.** *Let $\nu$ be the normal bundle associated to an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^{n+k}$, and let $n + 1 \leq k \leq n + 12$. Then the following three conditions are equivalent*:

(1) *$\nu$ is extendible to $\boldsymbol{RP}^m$ for every $m > n$.*
(2) *$\nu$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$.*
(3) $1 \leq n \leq 8$.

These are improvements of Theorem A in [10].

Let $[x]$ denote the integral part of a real number $x$. Then for the complexification of the normal bundle, we have

**Theorem C.** *Let $c\nu$ be the complexification of the normal bundle $\nu$ associated to an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^{n+k}$, where $k > 0$. Then the following hold.*

(i) *For $n \geq 6$, $c\nu$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$ if and only if $k \geq 2^{[n/2]} - n - 1$.*
(ii) *For $1 \leq n \leq 5$, $c\nu$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$.*

The following is an improvement of Theorem 4.4 in [10].

**Theorem D.** *Let $c\nu$ be the complexification of the normal bundle $\nu$ associated to an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^{n+k}$, and let $n \leq k \leq n + 8$. Then the following three* *conditions are equivalent:*

(1) *$c\nu$ is extendible to $\boldsymbol{RP}^m$ for every $m > n$.*

(2) *$c\nu$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$.*

(3) $1 \leq n \leq 9$.

This note is arranged as follows. In Section 2 we study relations between extendibility and stable extendibility. In Section 3 we prove Theorem A. We prove Theorem B and give some examples in Section 4. In Section 5 we prove Theorem C. We prove Theorem D and give some examples in Section 6.

## 2. Extendibility and stable extendibility

In the following, we use the same letter for a vector bundle and its equivalence class, and use an integer $k$ for a $k$-dimensional trivial bundle.

Let $d$ denote $\dim_{\boldsymbol{R}} F$, where $F = \boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$. The following fact is known (cf. [4, Theorem 1.5, p.100]).

(2.1). If $\alpha$ and $\beta$ are two $t$-dimensional $F$-vector bundles over an $n$-dimensional CW-complex $X$ such that $\langle (n+2)/d - 1 \rangle \leq t$ and $\alpha \oplus k = \beta \oplus k$ for some $k$-dimensional trivial $F$-bundle $k$ over $X$, then $\alpha = \beta$, where $\oplus$ denotes the Whitney sum and $\langle x \rangle$ denotes the smallest integer $m$ with $x \leq m$.

**Theorem 2.2.** *Let $X$ be a subcomplex of a finite dimensional CW-complex $Y$ and let $\zeta$ be an $\boldsymbol{R}$-vector bundle over $X$ such that* $\dim \zeta > \dim X$. *Then $\zeta$ is extendible to $Y$ if and only if $\zeta$ is stably extendible to $Y$.*

*In case* $\dim \zeta = \dim X$, *this does not hold in general.*

Proof. The "only if" part is clear. Suppose that $\zeta$ is stably equivalent to $i^*(\alpha)$ for some $\boldsymbol{R}$-vector bundle $\alpha$ over $Y$, where $i : X \to Y$ is the inclusion. In case $\dim \zeta > \dim X$, $\zeta$ is equivalent to $i^*(\alpha)$ by (2.1).

A counter example is given by the $n$-sphere $S^n$ in the $(n+1)$-sphere $S^{n+1}$ and the tangent bundle $\tau = \tau(S^n)$ of $S^n$ for $n \neq 1, 3, 7$. In fact, $\tau \oplus 1$ is the $(n+1)$-dimensional trivial bundle over $S^n$ and so $\tau \oplus 1 = i^*(n) \oplus 1$, where $i : S^n \to S^{n+1}$ is the inclusion and $n$ denotes the $n$-dimensional trivial $\boldsymbol{R}$-vector bundle over $S^{n+1}$. Hence $\tau$ is stably extendible to $S^{n+1}$. On the other hand, if there is an $n$-dimensional $\boldsymbol{R}$-vector bundle $\alpha$ over $S^{n+1}$ such that $\tau = i^*(\alpha)$, $\tau$ is trivial, since $i : S^n \to S^{n+1}$ is homotopic to a constant map. Hence $n = 1$, 3 or 7. So $\tau$ is not extendible to $S^{n+1}$ for $n \neq 1, 3, 7$. $\square$

The following is proved in the way similar to the former part of the proof of Theorem 2.2.

**Theorem 2.3.** *Let $X$ be a subcomplex of a finite dimensional CW-complex $Y$ and let $\zeta$ be a $\boldsymbol{C}$-vector bundle over $X$ such that* $\dim\zeta \geq \langle(\dim X)/2\rangle$. *Then $\zeta$ is extendible to $Y$ if and only if $\zeta$ is stably extendible to $Y$.*

**Corollary 2.4.** *Let $M$ be a submanifold of a finite dimensional differentiable manifold $N$ and $c\tau(M)$ be the complexification of the tangent bundle $\tau(M)$ of $M$. Then $c\tau(M)$ is extendible to $N$ if and only if $c\tau(M)$ is stably extendible to $N$.*

## 3. Proof of Theorem A

Let $\xi_n$ be the canonical line bundle over $\boldsymbol{RP}^n$.

**Lemma 3.1.** *Let $\nu$ be the normal bundle associated to an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^{n+k}$, where $k > 0$. Then the equality*

$$\nu = (a2^{\phi(n)} - n - 1)\xi_n + n + k + 1 - a2^{\phi(n)}$$

*holds in $KO(\boldsymbol{RP}^n)$, where $a$ is any integer.*

Proof. Let $\tau = \tau(\boldsymbol{RP}^n)$ be the tangent bundle of $\boldsymbol{RP}^n$. Then we have $\tau \oplus \nu = n+k$ and $\tau \oplus 1 = (n+1)\xi_n$. Hence

$$\nu = n + k + 1 - (n+1)\xi_n = (a2^{\phi(n)} - n - 1)\xi_n + n + k + 1 - a2^{\phi(n)}$$

in $KO(\boldsymbol{RP}^n)$ for any integer $a$, since $\xi_n - 1$ is of order $2^{\phi(n)}$ (cf. [1, Theorem 7.4]). $\square$

**Theorem 3.2.** *Let $\nu$ be the normal bundle associated to an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^{n+k}$, where $k > 0$. Then $\nu$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$ if $k \geq 2^{\phi(n)} - n - 1$, and if $k > n$, in addition, $\nu$ is extendible to $\boldsymbol{RP}^m$ for every $m > n$.*

Proof. By Lemma 3.1, we have $\nu = A\xi_n + B$, where $A = 2^{\phi(n)} - n - 1$ and $B = n + k + 1 - 2^{\phi(n)}$. Clearly $A \geq 0$, and $B \geq 0$ by the assumption. For $m > n$, $i^*(A\xi_m \oplus B) = A\xi_n \oplus B$, where $i\colon \boldsymbol{RP}^n \to \boldsymbol{RP}^m$ is the standard inclusion. Hence $\nu$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$, since $\nu$ is stably equivalent to $A\xi_n \oplus B$. If $k > n$, in addition, $\dim \boldsymbol{RP}^n = n < k = \dim\nu = A + B$, and so we obtain $\nu = A\xi_n \oplus B$ by (2.1). Thus $\nu$ is extendible to $\boldsymbol{RP}^m$ for every $m > n$. $\square$

The following result ([9, Theorem 4.1]) is the "stably extendible version" of Theorem 6.2 in [7].

(3.3). Let $\zeta$ be a $t$-dimensional $\boldsymbol{R}$-vector bundle over $\boldsymbol{RP}^n$. Assume that there is a positive integer $l$ such that $\zeta$ is stably equivalent to $(t+l)\xi_n$ and $t + l < 2^{\phi(n)}$. Then

$n < t + l$ and $\zeta$ is not stably extendible to $\boldsymbol{RP}^{t+l}$.

Using (3.3), we have obtained the following in [10, Theorem 2.4] (cf. [11, Proposition 6.4(iii)(b)]).

(3.4). The normal bundle associated to an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^{n+k}$ is not stably extendible to $\boldsymbol{RP}^{n+k+1}$, if $0 < k < 2^{\phi(n)} - n - 1$.

**Theorem 3.5.** *Let $\nu$ be the normal bundle associated to an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^{n+k}$. Then $\nu$ is not stably extendible to $\boldsymbol{RP}^m$ for $m = \min\{2^{\phi(n)} - n - 1, n + k + 1\}$, if $0 < k < 2^{\phi(n)} - n - 1$.*

Proof. Put $\zeta = \nu$, $t = k$ and $l = 2^{\phi(n)} - n - k - 1$ in (3.3). Then clearly $t + l < 2^{\phi(n)}$, and $l > 0$ by the assumption. So $\nu$ is not stably extendible to $\boldsymbol{RP}^m$ for $m = 2^{\phi(n)} - n - 1$. By (3.4), $\nu$ is not stably extendible to $\boldsymbol{RP}^m$ for $m = n + k + 1$. □

Putting $n = 9$ in Theorem 3.5, we have

**Corollary 3.6.** *If $1 \leq k \leq 21$, the normal bundle associated to an immersion of $\boldsymbol{RP}^9$ in $\boldsymbol{R}^{9+k}$ is not stably extendible to $\boldsymbol{RP}^m$ for $m = \min\{22, k + 10\}$.*

Proof of Theorem A. The "if" part follows from Theorem 3.2 and the "only if" part follows from Theorem 3.5. □

## 4. Proof of Theorem B

Let $\xi_n$ be the canonical line bundle over $\boldsymbol{RP}^n$.

**Theorem 4.1.** *Let $\nu = \nu(f_n)$ be the normal bundle associated to an immersion $f_n \colon \boldsymbol{RP}^n \to \boldsymbol{R}^{n+k}$, where $k > 0$. Then, for $1 \leq n \leq 10$, we have the equalities*

$$\begin{aligned} &\nu(f_1) = k, && \nu(f_2) = \xi_2 + k - 1, && \nu(f_3) = k, \\ &\nu(f_4) = 3\xi_4 + k - 3, && \nu(f_5) = 2\xi_5 + k - 2, && \nu(f_6) = \xi_6 + k - 1, \\ &\nu(f_7) = k, && \nu(f_8) = 7\xi_8 + k - 7, && \nu(f_9) = 22\xi_9 + k - 22 \\ &\textit{and} \quad \nu(f_{10}) = 53\xi_{10} + k - 53 \end{aligned}$$

*in $KO(\boldsymbol{RP}^n)$.*

*If $1 \leq n \leq 8$ and $k > n$ or if $n \geq 9$ and $k \geq 2^{\phi(n)} - n - 1$, the equalities hold in the set of equivalence classes of $\boldsymbol{R}$-vector bundles over $\boldsymbol{RP}^n$.*

Proof. By Lemma 3.1, we have

$$\nu = n + k + 1 - (n + 1)\xi_n = (a2^{\phi(n)} - n - 1)\xi_n + n + k + 1 - a2^{\phi(n)}$$

in $KO(\boldsymbol{RP}^n)$ for any integer $a$. So we have the former part by putting $a = 1$.

The latter part is a consequence of the former part by (2.1), since $\nu = A\xi_n + B$ for non-negative integers $A$ and $B$ such that $\dim \boldsymbol{RP}^n = n < k = \dim \nu = A + B$, if $1 \leq n \leq 8$ and $k > n$ or if $n \geq 9$ and $k \geq 2^{\phi(n)} - n - 1$. $\square$

**Corollary 4.2.** *If $1 \leq n \leq 8$ and $k > n$ or if $n \geq 9$ and $k \geq 2^{\phi(n)} - n - 1$, $\nu(f_n)$ is extendible to $\boldsymbol{RP}^m$ for every $m > n$.*

Proof. Since $\xi_n$ and the trivial $\boldsymbol{R}$-bundles over $\boldsymbol{RP}^n$ are extendible to $\boldsymbol{RP}^m$ for every $m > n$, the result follows from the latter part of Theorem 4.1. $\square$

Theorem B is a consequence of the following

**Theorem 4.3.** *Let $\nu$ be the normal bundle associated to an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^{n+k}$. Then we have*
(i) *$\nu$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$ if $1 \leq n \leq 8$ and $k \geq n$, and $\nu$ is extendible to $\boldsymbol{RP}^m$ for every $m > n$ if $1 \leq n \leq 8$ and $k > n$.*
(ii) *$\nu$ is not stably extendible to $\boldsymbol{RP}^{n+k+1}$ if $n \geq 9$ and $1 \leq k \leq n + 12$.*

Proof. The former part of Theorem 4.1 implies the former part of (i). In fact, if $k \geq n$, the $\boldsymbol{R}$-vector bundles $k$, $\xi_2 \oplus (k-1)$, $k$, $3\xi_4 \oplus (k-3)$, $2\xi_5 \oplus (k-2)$, $\xi_6 \oplus (k-1)$, $k$ and $7\xi_8 \oplus (k-7)$ over $\boldsymbol{RP}^n$, where $1 \leq n \leq 8$ respectively, are extendible to $\boldsymbol{RP}^m$ for every $m > n$, and they are stably equivalent to $\nu(f_n)$ respectively.

The latter part of (i) follows from the former part of (i) by Theorem 2.2.

(ii) is a consequence of (3.4), because $0 < k < 2^{\phi(n)} - n - 1$ if $n \geq 9$ and $1 \leq k \leq n + 12$. $\square$

In [6, Theorem 1], the following (4.4) is proved (cf. [11, Corollary 2.3 (2)]).

(4.4). Let $\zeta$ be a $t$-dimensional $\boldsymbol{R}$-vector bundle over $\boldsymbol{RP}^n$. If $n < t$, $\zeta$ is extendible to $\boldsymbol{RP}^m$ for every $m$ with $n < m \leq t$.

The next example is due to (4.4) and Corollary 3.6.

EXAMPLE 4.5. The normal bundle associated to an immersion of $\boldsymbol{RP}^9$ in $\boldsymbol{R}^{30}$ is extendible to $\boldsymbol{RP}^{21}$, but is not stably extendible to $\boldsymbol{RP}^{22}$.

## 5. Proof of Theorem C

**Lemma 5.1.** *Let $c\nu$ be the complexification of the normal bundle $\nu$ associated to an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^{n+k}$, where $k > 0$. Then the equality*

$$c\nu = (b2^{[n/2]} - n - 1)c\xi_n + n + k + 1 - b2^{[n/2]}$$

*holds in* $K(\boldsymbol{RP}^n)$, *where* $b$ *is any integer.*

Proof. Complexifying the equality in Lemma 3.1 and considering that $c\xi_n - 1$ is of order $2^{[n/2]}$, we have the equality above, since $[n/2] \leq \phi(n)$. □

**Theorem 5.2.** *Let* $c\nu$ *be the complexification of the normal bundle* $\nu$ *associated to an immersion of* $\boldsymbol{RP}^n$ *in* $\boldsymbol{R}^{n+k}$, *where* $k > 0$. *Then* $c\nu$ *is stably extendible to* $\boldsymbol{RP}^m$ *for every* $m > n$ *if* $k \geq 2^{\phi(n)} - n - 1$, *or if* $k \geq 2^{[n/2]} - n - 1 \geq 0$. *And if* $2k \geq n$, *in addition,* $c\nu$ *is extendible to* $\boldsymbol{RP}^m$ *for every* $m > n$.

Proof. To prove the first part, by Lemma 5.1, we have $c\nu = Ac\xi_n + B$, where $A = 2^{\phi(n)} - n - 1$ and $B = n + k + 1 - 2^{\phi(n)}$, since we may take $b = 1$ if $n \equiv 6$, 7 or 0 mod 8 and $b = 2$ otherwise. Clearly $A \geq 0$, and $B \geq 0$ by the assumption. For $m > n$, $i^*(Ac\xi_m \oplus B) = Ac\xi_n \oplus B$, where $i\colon \boldsymbol{RP}^n \to \boldsymbol{RP}^m$ is the standard inclusion. Hence $c\nu$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$, since $c\nu$ is stably equivalent to $Ac\xi_n \oplus B$.

To prove the second part, taking $b = 1$ in Lemma 5.1, we have $c\nu = Ac\xi_n + B$, where $A = 2^{[n/2]} - n - 1$ and $B = n + k + 1 - 2^{[n/2]}$. By the assumption $A \geq 0$ and $B \geq 0$. So $c\nu$ is stably extendible to $\boldsymbol{RP}^m$ for every $m > n$, in the way similar to the proof above.

If $2k \geq n$, in addition, $\langle(\dim \boldsymbol{RP}^n)/2\rangle = \langle n/2\rangle \leq k = \dim c\nu = A + B$, and so we obtain $c\nu = Ac\xi_n \oplus B$ by (2.1). Thus $c\nu$ is extendible to $\boldsymbol{RP}^m$ for every $m > n$. □

We recall the following result ([9, Theorem 2.1]) which is the "stably extendible version" of Theorem 4.2 for $d = 1$ in [8].

(5.3). Let $\zeta$ be a $t$-dimensional $\boldsymbol{C}$-vector bundle over $\boldsymbol{RP}^n$. Assume that there is a positive integer $l$ such that $\zeta$ is stably equivalent to $(t+l)c\xi_n$ and $t+l < 2^{[n/2]}$. Then $[n/2] < t+l$ and $\zeta$ is not stably extendible to $\boldsymbol{RP}^m$ for every $m$ with $t+l \leq [m/2]$.

**Theorem 5.4.** *Let* $c\nu$ *be the complexification of the normal bundle* $\nu$ *associated to an immersion of* $\boldsymbol{RP}^n$ *in* $\boldsymbol{R}^{n+k}$, *where* $k > 0$. *Then* $c\nu$ *is not stably extendible to* $\boldsymbol{RP}^m$ *for every* $m$ *with* $2^{[n/2]+1} - 2n - 2 \leq m$, *if* $k < 2^{[n/2]} - n - 1$.

Proof. Put $\zeta = c\nu$, $t = k$ and $l = 2^{[n/2]} - n - k - 1$ in (5.3). Then clearly $t + l < 2^{[n/2]}$, and $l > 0$ by the assumption. So $c\nu$ is not stably extendible to $\boldsymbol{RP}^m$ for every $m$ with $2^{[n/2]} - n - 1 \leq [m/2]$. □

Proof of Theorem C. (i) For $n \geq 6$, the "only if" part follows from Theorem 5.4, and the "if" part follows from Theorem 5.2, since $2^{[n/2]} - n - 1 \geq 0$ if $n \geq 6$.

(ii) As is well-known, $\boldsymbol{RP}^1 \subseteq \boldsymbol{R}^2$, $\boldsymbol{RP}^2 \subseteq \boldsymbol{R}^3$, $\boldsymbol{RP}^3 \subseteq \boldsymbol{R}^4$, $\boldsymbol{RP}^4 \subseteq \boldsymbol{R}^7$ and $\boldsymbol{RP}^5 \subseteq \boldsymbol{R}^7$, where we denote by $\boldsymbol{RP}^n \subseteq \boldsymbol{R}^N$ the existence of an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^N$, and these immersions are best possible, that is, there do not exist immersions of $\boldsymbol{RP}^n$ in

$\boldsymbol{R}^{N-1}$. Hence we have $k \geq 2^{\phi(n)} - n - 1$ for $1 \leq n \leq 5$. So the result follows also from Theorem 5.2. □

## 6. Proof of Theorem D

**Theorem 6.1.** *Let $c\nu = c\nu(f_n)$ be the complexification of the normal bundle $\nu = \nu(f_n)$ associated to an immersion $f_n \colon \boldsymbol{RP}^n \to \boldsymbol{R}^{n+k}$, where $k \geq n$. Then we have the Whitney sum decompositions as follows:*

$$c\nu(f_1) = k, \qquad c\nu(f_2) = c\xi_2 \oplus (k-1), \qquad c\nu(f_3) = k,$$
$$c\nu(f_4) = 3c\xi_4 \oplus (k-3), \qquad c\nu(f_5) = 2c\xi_5 \oplus (k-2), \qquad c\nu(f_6) = c\xi_6 \oplus (k-1),$$
$$c\nu(f_7) = k, \qquad c\nu(f_8) = 7c\xi_8 \oplus (k-7) \quad \text{and} \quad c\nu(f_9) = 6c\xi_9 \oplus (k-6).$$

Proof. Complexifying the equalities in the former part of Theorem 4.1, we have the equalities above for $1 \leq n \leq 8$ using (2.1). So it suffices to prove the equality for $n = 9$. By the former part of Theorem 4.1, $\nu(f_9) = 22\xi_9 + k - 22$, and so $c\nu(f_9) = 22c\xi_9 + k - 22$. According to [1, Theorem 7.3], $c\xi_9 - 1$ is of order 16. Hence $16c\xi_9 - 16 = 0$ in $K(\boldsymbol{RP}^9)$, and so $c\nu(f_9) = 6c\xi_9 + k - 6$. Therefore, $c\nu(f_9) = 6c\xi_9 \oplus (k-6)$ by (2.1). □

**Corollary 6.2.** *If $1 \leq k \leq 20$, the complexification $c\nu(f_{10})$ of the normal bundle $\nu(f_{10})$ associated to an immersion $f_{10} \colon \boldsymbol{RP}^{10} \to \boldsymbol{R}^{10+k}$ is not stably extendible to $\boldsymbol{RP}^{42}$.*

Proof. By the former part of Theorem 4.1, $\nu(f_{10}) = 53\xi_{10} + k - 53$, and so $c\nu(f_{10}) = 53c\xi_{10} + k - 53 = 21c\xi_{10} + k - 21$, since $c\xi_{10} - 1$ is of order 32. Hence we have the result from (5.3) by putting $n = 10$, $\zeta = c\nu(f_{10})$, $t = k$ and $l = 21 - k$, since $l > 0$ for $1 \leq k \leq 20$ and since $t + l = 21 < 2^{[10/2]} = 32$. □

Define $l(n) = 2^{[n/2]} - n - k - 1$. Then we have

**Lemma 6.3.** *$l(n) > 0$ for any $k$ and $n$ such that $10 \leq n \leq k \leq n + 8$, and $k + l(n) < 2^{[n/2]}$, for any $k$ and $n$.*

Proof. For $10 \leq n \leq 17$, the inequalities hold clearly. For $n \geq 18$, we prove the inequalities by induction. □

Theorem D is a consequence of the following

**Theorem 6.4.** *Let $c\nu$ be the complexification of the normal bundle $\nu$ associated to an immersion of $\boldsymbol{RP}^n$ in $\boldsymbol{R}^{n+k}$. Then we have*

(i) *$c\nu$ is extendible to $\boldsymbol{RP}^m$ for every $m > n$ if $1 \leq n \leq 9$ and $k \geq n$.*

(ii) *$c\nu$ is not stably extendible to $\boldsymbol{RP}^m$ for every $m$ with $2^{[n/2]+1}-2n-2 \leq m$ if $n \geq 10$ and $n \leq k \leq n+8$.*

Proof. Since $c\xi_n$ and the trivial $\boldsymbol{C}$-vector bundles over $\boldsymbol{RP}^n$ are extendible to $\boldsymbol{RP}^m$ for every $m > n$, Theorem 6.1 implies (i).

By Lemma 5.1, we have

$$c\nu = \{b2^{[n/2]} - (n+1)\}c\xi_n + n + k + 1 - b2^{[n/2]}$$

for any integer $b$. (ii) follows from (5.3), Lemma 6.3 and the equality above by putting $\zeta = c\nu$, $t = k$ and $l = 2^{[n/2]} - n - k - 1$. □

In [6, Theorem 2], the following (6.5) is proved (cf. [11, Corollary 2.3 (2)]).

(6.5). Let $\zeta$ be a $t$-dimensional $\boldsymbol{C}$-vector bundle over $\boldsymbol{RP}^n$. If $n < 2t+1$, $\zeta$ is extendible to $\boldsymbol{RP}^m$ for every $m$ with $n < m \leq 2t+1$.

The next example is due to (6.5) and Corollary 6.2 for $k = 20$.

EXAMPLE 6.6. The complexification of the normal bundle associated to an immersion of $\boldsymbol{RP}^{10}$ in $\boldsymbol{R}^{30}$ is extendible to $\boldsymbol{RP}^{41}$, but is not stably extendible to $\boldsymbol{RP}^{42}$.

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T. Kobayashi
Kochi Asakura-ki 292-21
Kochi 780-8066, Japan

H. Maki
Department of Mathematics
Faculty of Culture and Education
Saga University
Saga 840-8502, Japan

T. Yoshida
Department of Mathematics
Faculty of Integrated Arts and Sciences
Hiroshima University
Higashi-Hiroshima 739-8521, Japan