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COMBINATORIAL PREBUNDLES PART I

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1. Introduction

In this paper we shall define the concept of combinatorial prebundles and prove the fundamental properties. Roughly speaking, a combinatorial prebundle is an object something like a PL bundle, but having only a trivialization over each simplex of the base complex. The advantage of weakening fiber structures is that the theory of regular neighborhoods can be fully applied for attacking normal prebundles.

The paper is organized as follows. In §2 the concepts are introduced. The structural groups and principal bundles for prebundles are defined as abstract simplicial (abbreviated by a. s.) groups and a. s. bundles respectively. In particular the structural group PR_n of combinatorial n cell prebundles contains the structural group ΠL_n of PL n cell bundles as a subgroup. In §3, by virtue of Zeeman's unknotting theorem [13], we show the stability theorem of the homotopy groups of PR_n which is quite similar to that of the orthogonal group O_n (see 3.3). In §4 we prove the existence of a normal prebundle for every locally flat PL embedding. It is shown that microequivalence classes or iso-neighboring classes in the sense of Hiroshi Noguchi [10] of locally flat PL embeddings of the m sphere of codimension n are one to one corresponding to elements of $\pi_{m-1}(PR_n)$ (see 4.6) and that isomorphism classes of PL tubes in the sense of M.W. Hirsch [4] for the standard $(m+n,m)$ sphere pair are one to one corresponding to elements of $\pi_m(PR_n, \Pi L_n)$ (see 4.7). Thus we obtain unified criteria for non existence and non uniqueness of normal PL cell bundles by means of the homomorphism i_k: $\pi_k(\Pi L_n)\rightarrow\pi_k(PR_n)$ (see 4.8). One of these criteria gives us an interpretation of Hirsch's example of a PL embedding of the 8 sphere of codimension 4 having no normal PL cell bundle [4], II (see 4.9). In view of the result of C.T.C. Wall and A. Haefliger [2] it is deduced that the stable homotopy groups of PR_n and those of ΠL_n coincide.

In the subsequent paper we shall show the existence of a collar neighborhood for a locally flatly embedded PL m sphere of codimension two for $m\geq 5$ with a number of interesting implications for PL locally flat embeddings of codimension two.

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Added in Proof. Rourke and Sanderson have also obtained an analogous theory, called the block-bundle theory, which is stronger than ours; Bull. A.M.S. vol. 72, 1966, pp. 1036–1039.

2. Prebundles and the a. s. principal bundles

In the following we shall work in the PL category consisting of polyhedra covered by rectilinear locally finite simplicial complexes and piecewise linear maps. Thus all maps, manifolds, and bundles are always understood to be piecewise linear.

Let P be a polyhedron and let p be a fixed point of P. *A* (P, p) *prebundle* is a triple $\{E, K, \sum\}$ consisting of

(1) a polyhedron E called the *total space,*

(2) a complex K called the *base complex* and

(3) a collection $\sum$ of pairs (A, f) satisfying the following four conditions:

(a) Each pair (A, f), called a *trivialization* of E over A, consists of a simplex A of K and an embedding $f\colon A\times P\to E$.

(b) For each simplex A of K there is a pair (A, f) in $\sum$ and $\bigcup f(A\times P)=E$, where the union is taken for all (A, f) in $\sum$.

(c) If (A, f) and (B, g) belong to $\sum$ and if $A\cap B$ is a non empty simplex C then $f(A\times P)\cap g(B\times P)=f(C\times P)=g(C\times P)$ and $f/C\times p=g/C\times p$.

(d) The collection $\sum$ is maximal with respect to the condition (c).

A second (P, p) prebundle $\{E', K, \sum'\}$ is *isomorphism* to $\{E, K, \sum\}$ if there is a homeomorphism $h\colon E\to E'$ called an isomorphism such that $hf(A\times P)=g(A\times P)$ and $hf/A\times p=g/A\times p$ for (A, f) in $\sum$ and (A, g) in $\sum'$. A product polyhedron $|K|\times P$ has the natural trivialization over each simplex A of K, that is the inclusion map $A\times P\subset |K|\times P$. The (P, p) prebundle so obtained is called the *product* (P, p) *prebundle over* K and simply denoted by $K\times(P, p)$. A (P, p) prebundle is called to be *trivial* if it is isomorphic to the product prebunlde $K\times(P, p)$. Let L be a subcomplex of K. Then the restricted prebundle $\{E/L, L, \sum/L\}$ is defined by setting $\sum/L=\{(A, g)\in\sum/A\in L\}$ and $E/L=\bigcup g(A\times P)$, where the union is taken for all (A, g) in $\sum/L$.

The concept of P prebundles is also defined in the same fashion as (P, p) prebundles deleting the conditions concerning the fixed point p.

REMARK. For a (P, p) bundle over the base space B, see [5], and any partition K of B, we have naturally a (P, p) prebundle called the *underlying* prebundle over K.

For a P prebundle $\{E, K, \sum\}$ *a cross section* $c:|K|\to E$ is an embedding such that $c(A)\subset f(A\times P)$ for each pair (A, f) in $\sum$. Every (P, p) prebundle has a cross section i: $|K|\to E$ called the *p section* which is defined by setting for each point x of $|K|$

$$i(x) = f(x, p) \text{ if } x \text{ in } A \text{ and if } (A, f) \text{ in } \textstyle\sum .$$

Let J^n denote the n fold cartesian product of the closed interval $[-1, 1]$ and let 0 denote the origin $(0, \cdots, 0)$ in J^n. Then a $(J^n, 0)$ prebundle is simply called an *n prebundle*, and the 0 section is called the *zero section*. For an n prebundle $\{E, K, \sum\}$ the associated $n-1$ sphere (∂J^n) prebundle $\{\partial E, K, \partial\sum\}$ is obtained by setting $\partial\sum=\{(A, h')/h'=h/A\times\partial J^n$ for (A, h) in $\sum\}$ and $\partial E=\bigcup h'(A\times\partial J^n)$, where the union is taken for all (A, h') in $\partial\sum$. A *non zero section* of an n prebundle is a cross section of the associated sphere prebundle.

Let A and B be polyhedra and let f: $A\to B$ be a map. For a polyhedron P maps $f\times P$: $A\times P\to B\times P$ and $P\times f$: $P\times A\to P\times B$ are defined by setting for each x in A and for each y in P

$$f\times P(x, y) = (f(x), y) \text{ and } P\times f(y, x) = (y, f(x)) .$$

We shall mean by a simplex both the polyhedron and the complex consisting of the faces. A complex K is called to be *ordered* if the vertices are totally ordered. For ordered complexes K and L a monotone map $F: K\to L$ is a simplicial embedding preserving order of the vertices. Let us consider the unit simplex Δ_q in the euclidean $q+1$ space R^{q+1} with cooridinates $(x_0, \cdots, x_q)$. The vertices $e^0, \cdots, e^q$ of Δ_q are the unit points on the coordinate axes of R^{q+1}. If we regard R^q as the subspace of R^{q+1} given by $x_q=0$ then Δ_{q-1} is a face of Δ_q and has vertices $e^0, \cdots, e^{q-1}$. Let Ω_n denote the category consisting of objects Δ_q, $q=0, \cdots, n$ (possibly $n=\infty$) and monotone maps d: $\Delta_p\to\Delta_q$, $p\leq q\leq n$. Let S denote the category of sets and maps and let G denote the category of groups and homomorphisms. An n dimensional *abstract simplicial* (abbreviated by *a.s.*) *complex* K^* is a contravariant functor K^*: $\Omega_n\to S$. A *simplicial map* between a.s. complexes K^* and L^* is a natural transformation f: $K^*\to L^*$. A *q simplex* of K^* is an element of $K^*(\Delta_q)=K^*_q$ and a face map of K^* is an image $K^*(d)=d^*$. In the above replacing S by G, we may also define the concept of a.s. groups. Following A. Heller [3], p.p. 303–304, we may define the concept of product a.s. complexes and a.s. bundles.

Now we define the a.s. group $PR(P, p)$ as follows; A q simplex of $PR(P, p)$ is an isomorphism of the product (P, p) prebundle $\Delta_q\times(P, p)$ onto itself. The operation of composing isomorphisms makes the set $PR(P, p)^q$ of q simplexes into a group. The monotone maps $d\colon \Delta_p \to \Delta_q$ induce homomorphisms $d^*\colon PR(P, p)^p \to PR(P, p)^q$ given by $d^*f=F$ for f in $PR(P, p)^q$ in such a way that a diagram

$$\begin{array}{ccc} \Delta_q \times (P, p) & \xrightarrow{f} & \Delta_q \times (P, p) \\ \uparrow{\scriptstyle dxP} & & \uparrow{\scriptstyle dxP} \\ \Delta_p \times (P, p) & \xrightarrow{F} & \Delta_p \times (P, p) \end{array}$$

commutes. Thus $PR(P, p)=\{PR(P, p)^q, d^*\}$ is an a.s. group.

Following Milnor, §5 in [9], we define the associated principal $PR(P, p)$ bundle of a (P, p) prebundle $\{E, K, \sum\}$ as follows. Choose some ordering for the vertices of K. The base complex K is the a.s. complex consisting of all monotone simplicial maps $F\colon \Delta_q \to K$. A q simplex of the total space E^* consists of

(1) a q simplex F of K^{*q} together with

(2) a map $f\colon \Delta_q \times P \to E$ which is factored as follows:

$f=h(F\times P)$ for $(F(\Delta_q), h)$ in $\sum$.

The functions $d^*\colon E^{*q} \to E^{*q}$ are defined by the formulas $d^*\,(F, f)=(Fd, f(d\times P))$. The right translation function $E^*\times PR(P, p)\to E^*$ is given by $(F, f)g=(F, fg)$. Since the group $PR(P, p)$ operates freely on E^*, it follows that E^* is an a.s. principal $PR(P, p)$ bundle with the orbit complex K^*.

The following Propositions are easily verified, see pp. 25–26 in [9].

Proposition 2.1. *Two (P, p) prebundles $\{E, K, \sum\}$ and $\{E', K', \sum'\}$ are isomorphic if and only if E^* and E'^* are isomorphic.*

Proposition 2.2. *Let K be a complex. A principal $PR(P, p)$ bundle E^* over K^* is isomorphic to the associated principal bundle of a (P, p) prebundle $\{E, K, \sum\}$.*

In the rest of the section we shall define the homotopy groups of the a.s. structural groups of prebundles.

For each integer $k>0$, we specify the face maps $d_i^k\colon \Delta_{k-1}\to\Delta_k$, $i=0, \cdots, k$ given by the vertex assignments:

$$d_i^k(e_j) = e_j \text{ if } 0\leq j<i \text{ and}$$
$$d_i^k(e_j) = e_{j+1} \text{ if } i\leq j\leq k-1\,.$$

An a. s. complex K is said to be an *a. s. Kan complex* if for every pair of integers (i, k) such that $0\leq i\leq k$ and for every $k-1$ simplexes $f_0, \cdots, f_{i-1}, f_{i+1}, \cdots, f_k$ in K such that $d_{j-1}^{(k-1)*} f_e = d_e^{(k-1)*} f_j$ for $e<j$ and $e\neq i\neq j$, there exists a k simplex f in K such that

$$d_e^{k*} f = f_e \text{ for } e\neq i\,.$$

Let (G, H) be a pair of a.s. groups such that H is an a.s. subgroup of G. Then the group pair (G, H) is said to be a *Kan group pair*, if G and H are a.s. Kan complexes.

For a Kan group pair (G, H) we define the relative homotopy groups $\pi_k(G, H)$ $(k\geq 0)$ as follows;

Let $C(G, H)$ be the a. s. subgroup of G of which k simplexes $(k\geq 1)$ f satisfy that

$$d_i^{k*}f=\text{id.}, \text{ for } i=1, \cdots, k-1, \text{ and } d_k^{k*}f \text{ belongs to } H.$$

We put $B^k(G, H)=d_0^{k+1*}(C^{k+1}(G, H))$, $Z^k(G, H)=C^k(G, H)\cap \text{Kernel } d_0^{k*}$, and $Z^0(G, H)=G^0$ for $k\geq 0$.

Then we have:

Lemma 2.3. *The subgroup $B^k(G, H)$ is a normal subgroup of $Z^k(G, H)$.*

Proof. Let f be a k simplex of $B^k(G, H)$ and let g be a k simplex of $Z^k(G, H)$. We must show that $g^{-1}fg$ belongs to $B^k(G, H)$. Let F be a $k+1$ simplex of $C(G, H)$ such that $d_0^{k+1*}F=f$. Since (G, H) is a Kan group pair, we have a $k+1$ simplex E of G such that $d_0^{k+1*}E=g$, $d_{k+1}^{k+1*}E$ belongs to H, $d_i^{k+1*}E=\text{id.}$, for i=2, $\cdots$, k and $d_1^{k+1*}E$ belongs to G. Let $D=E^{-1}FE$. Then $d_{k+1}^{k+1*}D$ belongs to H, $d_i^{k+1*}D=\text{id.}$, for $i=1, \cdots, k$. Hence D belongs to $C^{k+1}(G, H)$. Since $g^{-1}fg=d_0^{k+1*}D$, it follows that $g^{-1}fg$ belongs to $B^k(G, H)$, completing the proof.

Now we define the k-th homotopy group of (G, H) by $\pi_k(G, H)=Z^k(G, H)/B^k(G, H)$. In case $H=\{\text{id.}\}$, we shall denote the group $\pi_k(G, H)$ by $\pi_k(G)$. Then we have a homomorphism

$$\partial_k\colon \pi_k(G, H)\to \pi_{k-1}(G)$$

induced from the homomorphism $d_k^{k*}\colon Z^k(G, H)\to Z^{k-1}(H, \{\text{id.}\})$.

By the usual manner we have the following exact sequence, which will be called the *homotopy exact sequence* for the Kan group pair (G, H):

$$\cdots\to\pi_{k+1}(G, H)\xrightarrow{\partial_{k+1}}\pi_k(H)\xrightarrow{i_k}\pi_k(G)\xrightarrow{j_k}\pi_k(G, H)\xrightarrow{\partial_k}\pi_{k-1}(H)\to$$
$$\cdots\to\pi_1(G, H)\xrightarrow{\partial_1}\pi_0(H)\xrightarrow{i_0}\pi_0(G)\xrightarrow{j_0}\pi_0(G, H).$$

Proposition 2.4. *Let P be a polyhedron and let p be a fixed point of P. The a. s. structural group $PR(P, p)$ $(PR(P))$ of (P, p) prebundles (P prebundles) is an a. s. Kan group.*

Proof. Given $k-1$ simplexes $f_0, \cdots, f_{i-1}, f_{i+1}, \cdots, f_k$ in $PR(P, p)$ such that $d_{j-1}^{k-1*}f_e=d_e^{k-1*}f_j$ for $e<j$ and $e\neq i\neq j$, then they define a (P, p) prebundle isomorphism $g\colon V\times(P, p)\to V\times(P, p)$ such that $d_e^{k*}g=f_e$ for $e=0,\cdots,i-1, i+1,\cdots,k$,

where $V=\partial\Delta_k-d_i^k(\Delta_{k-1})$. Let h: $I\times|V|\to\Delta_k$ be a homeomorphism such that

$$h(0, x) = x \text{ for all points } x \text{ in } |V|.$$

Then a (P, p) prebundle isomorphism

f: $\Delta_k\times(P,p)\to\Delta_k\times(P,p)$ is given by $f=(h\times P)(I\times g)(h^{-1}\times P)$.

Since $f|V\times(P,p)=g$, the k simplex f in $PR(P,p)$ is the required one. In the same way, we may prove that $PR(P)$ is an a. s. Kan group, completing the proof.

All a. s. subgroups of the structural groups of prebundles which will appear in the rest of the paper will be a. s. Kan groups. For example, the c. s. s. structural groups of bundles are Kan groups as a. s. groups, since c. s. s. groups are always c. s. s. Kan complexes.

Proposition 2.5. *Every (P, p) prebundle $\{E, K, \sum\}$ is trivial, if K is collapsible.*

Proof. Since the restricted prebundle over a vetex is trivial, it suffices to show that if K_0 elementary collapses to K_1, and if f: $K_1\times(P,p)\to E|K_1$ is an isomorphism, then there exists an isomorphism

F: $K_0\times(P,p)\to E|K_0$ such that $F|K_1\times(P,p)=f$.

Let K_0-K_1 consist of a principal simplex A of K_0 and its free face B and let V be the complex $\partial A-B$. Let h: $A\times(P,p)\to E|A$ be a trivialization. Then $h^{-1}f|V\times(P,p)$ is an isomorphism of $V\times(P,p)$ onto itself. By Proposition 2.4, we have an isomorphism g: $A\times(P,p)\to A\times(P,p)$ such that $g|V\times(P,p)=h^{-1}f|V\times(P,p)$.

Then the required isomorphism F is obtained by setting

$F|K_1\times(P,p)=f$ and $F|A\times(P,p)=hg$, completing the proof.

3. The stability theorem

The structural groups of prebundles are written as follows: $PR_n=PR(J^n, 0)$, $PR'_n=PR(J^n)$, $\partial PR_n=PR(\partial J^n)$, and $\partial_0 PR_n=PR(\partial J^n, e)$, where e denotes the point $(0^{n-1}, 1)$ in ∂J^n.

The structural groups of $(J^n, 0)$ and $(R^n, 0)$ bundles are written ΠL_n and PL_n respectively. Thus PR'_n contains PR_n and ΠL_n as subgroups.

Moreover the following injections are obtained:

i^{n-m}: $PR_m\to PR_n$ $(m<n)$ is defined by the formula $i^{n-m}(f)=f\times J^{n-m}$ for all f in PR_m, and j: $\partial PR_n\to PR_n$ is defined as follows; For each f in ∂PR_n^q assuming inductively that $j(f)/$(the k skeleton of $\Delta_q)\times J^n\bigcup\Delta_q\times\partial J^n$ is already obtained, set for each $k+1$ face A of Δ_q with the barycenter a, $j(f)/A\times J^n$ to be the join extension of $j(f)/\partial A\times J^n\bigcup A\times\partial J^n$ from $(a, 0)$. Then the homeomerphism $j(f)$ so defined is uniquely determined by f. Thus j: $\partial PR_n\to PR_n$ is an

a. s. injection. Let ∂: $PR_n \to \partial PR_n$ denote the homomrophism defined by the restriction $\partial h = h/\Delta_q \times \partial J^n$ for all h in PR_n^q.

Then the composition ∂i^1: $PR_n \to \partial_0 PR_{n+1}$ is also an injection.

Proposition 3.1. *The following three injections are homotopy equivalences;*

(i) *The inclusion map* $PR_n \subset PR_n'$,
(ii) *the injection* $j: \partial PR_n \to PR_n$ *and*
(iii) *the injection* ∂i^1: $PR_n \to \partial_0 PR_{n+1}$.

That is, the following relative homotopy groups vanish for all k;

$$\pi_k(PR_n', PR_n),\ \pi_k(PR_n, j(\partial PR_n)) \quad \text{and} \quad \pi_k(\partial_0 PR_{n+1}, \partial i_1(PR_n))\,.$$

Proof of (i). For any element of $\pi_k(PR_n', PR_n)$ we may take a representation f in $PR_n'^k$, such that $f/\partial\Delta_k \times 0 = \text{id.}$. Since $(\Delta_k \times J^n, \Delta_k \times 0)$ is a flat cell pair it follows from Corollary 1 to theorem 9 in [12] which is valid for any flat embedding that there is an ambient isotopy g of $\Delta_k \times J^n$ keeping $\partial(\Delta_k \times J^n)$ fixed such that $gf/\Delta_k \times 0 = \text{id.}$, or gf in PR_n^k. Then g represents the trivial element of $\pi_k(PR_n', PR_n)$, and gf represents also the trivial element of (PR_n', PR_n). Hence f represents always the trivial element. Thus the relative homotopy group $\pi_k(PR_n', PR_n)$ consists of only the trivial element. This completes the proof of (i).

Proof of (ii). For any element of $\pi_k(PR_n, j(\partial PR_n))$ we may take a representation f in PR_n^k such that $f/\partial\Delta_k \times J^n = j\partial(f)/\partial\Delta_k \times J^n$. Since $\partial(f) = \partial j\partial(f)$, or $f^{-1}(j\partial(f))/\partial(\Delta_k \times J^n) = \text{id.}$, it follows from the join extension argument in the Lemma 8 in [11] that $f^{-1}(j\partial(f))$ is isotopic to the identity keeping $\partial(\Delta_k \times J^n)$ and $\Delta_k \times 0$ fixed. Thus f and $j\partial(f)$ represent the same element of $\pi_k(PR_n, j(\partial PR_n))$. However, $j\partial(f)$ belongs to $j(\partial PR_n)$ and hence represents the trivial element in $\pi_k(PR_n, j(\partial PR_n))$. Therefore f represents the trivial element, completing the proof.

Proof of (iii). For any element of $\pi_k(\partial_0 PR_{n+1}, \partial i^1(PR_n))$ we may take a representation f in $\partial_0 PR_{n+1}^k$ such that $f/\partial\Delta_k \times \partial J^{n+1} = g \times J/\partial\Delta_k \times \partial J^{n+1}$ for some isomorphism g of the product $(J^n, 0)$ prebundle $\partial\Delta_k \times (J^n, 0)$.

Let e denote the point $(0^n, 1)$ in ∂J^{n+1}. Since $f/\Delta_k \times e = \text{id.}$, and since $f(\Delta_k \times J^n \times 1)$ and $\Delta_k \times J^n \times 1$ are regular neighborhoods of $\Delta_k \times e \cup \Delta_k \times J^n \times 1$ mod $(\partial\Delta_k \times (\partial J^{n+1} - Int J^n \times 1))$ in $\Delta_k \times \partial J^{n+1}$, it follows from the uniqueness of relative regular neighborhoods [6] that there is an ambient isotopy g: $\Delta_k \times \partial J^{n+1} \to \Delta_k \times \partial J^{n+1}$ keeping $\Delta_k \times e$ and $\partial\Delta_k \times \partial J^{n+1}$ fixed so that $gf(\Delta_k \times J^n \times 1) = \Delta_k \times J^n \times 1$. Since g represents the trivial element of $\pi_k(\partial_0 PR_{n+1}, \partial i^1(PR_n))$, f and gf represent the same element. Now we define an element h in PR_n^k by setting

$$(h(x, u), 1) = gf(x, u, 1) \text{ for all } (x, u) \text{ in } \Delta_k \times J^n.$$

Then $\partial i^1(h)$ and gf coincide on $\Delta_k \times J^n \times 1 \cup \partial\Delta_k \times \partial J^{n+1}$, and $\Delta_k \times (\partial J^{n+1} - IntJ^n \times 1)$ is a $k+n$ cell. It follows from the Alexander trick that $(gf)^{-1}(\partial i^1)(h)$ is isotopic to the identity keeping $\Delta_k \times J^n \times 1$ and $\partial\Delta_k \times \partial J^{n+1}$ fixed. Hence f and $\partial i^1(h)$ represent the same element.

Since $\partial i^1(h)$ belongs to $\partial i^1(PR_n)$, f represents the trivial element, completing the proof.

Proposition 3.2. $\pi_k(\partial PR_{n+1}, \partial_0 PR_{n+1}) \cong 0$ *for* $k+1 \leq n$.

Proof. For any element of $\pi_k(\partial PR_{n+1}, \partial_0 PR_{n+1})$ we may take a representation f in ∂PR^k_{n+1} such that $f/\partial\Delta_k \times e = \text{id.}$. Let e' denote the point $(0^n, -1)$ in ∂J^{n+1}. Consider the intersection of $f(\Delta_k \times e)$ and $\Delta_k \times e'$. Since $k+k-(k+n) = k-n \leq -1$, it follows from the general position argument (see Chapter 6 in [12]) that there is an abmient isotopy g: $\Delta_k \times \partial J^{n+1} \to \Delta_k \times \partial J^{n+1}$ keeping $\Delta_k \times \partial J^{n+1}$ fixed such that $gf(\Delta_k \times e)$ is disjoint from $\Delta_k \times e'$.Let ε be a positive number and let εJ^n denote the n fold cartesian product of the closed interval $[-\varepsilon, \varepsilon]$. Choosing sufficiently small number ε, we may assume that $gf(\Delta_k \times e)$ is disjoint from $\Delta_k \times \varepsilon J^n \times (-1)$. Then $\Delta_k \times (\partial J^{n+1} - IntJ^n \times (-1))$ is a $k+n$ cell, and $gf(\Delta_k \times e)$ and $\Delta_k \times e$ are two k cells which coincide on the boundary $\partial\Delta_k \times \partial J^{n+1}$. Since $gf/\partial\Delta_k \times e = \text{id.}$, it follows from Corollary to Theorem 9 in [12] that if $n \geq 3$, then there is an abmient isotopy h: $\Delta_k \times \partial J^{n+1} \to \Delta_k \times \partial J^{n+1}$ keeping $\partial\Delta_k \times \partial J^{n+1}$ and $\Delta_k \times \varepsilon J^n \times (-1)$ fixed such that $hgf/\Delta_k \times e = \text{id.}$, or hgf in $\partial_0 PR_{n+1}$. In case $(n, k) = (2, 1)$, by Lemma 9.1 in [1], we may also obtain such an ambient isotopy h. Since h, g and hgf represent the trivial element, it follows that f represents the trivial element, completing the proof.

For $m<n$ identify PR_m with the subgroup $i^{n-m}(PR_m)$ in PR_n.

Let i^{n-m}_k: $\pi_k(PR_m) \to \pi_k(PR_n)$ denote the homomorphism induced from the injection i^{n-m}: $PR_m \to PR_n$. From Propositions 3.1 and 3.2 we immediately derive the following.

Theorem 3.3. *The relative homotopy groups* $\pi_k(PR_n, PR_m)$ *vanish for all* $k<m<n$. *That is, the homomorphisms* i^{n-m}_k: $\pi_k(PR_m) \to \pi_k(PR_n)$ *are surjective for all* $k<m<n$ *and injective for all* $k<m-1<n$.

4. Normal prebundles

Let f: $M \to W$ be an embedding of an m manofild M into an $m+n$ manifold W and let K be a partition of M. An n prebundle $\{N, K, \sum\}$ is a *normal prebundle for f over K, if*

(1) N is a closed neighborhood of $f(M)$ in W and

(2) f: $M \to W$ coincides with the zero section.

Then it is not hard to see that N is a regulra neighborhood of $f(M)$ in W and that f is locally flat. For simplicity the normal prebundle $\{N, K, \sum\}$ is denoted by N or $N(f)$.

Let $\{E, K, \sum\}$ and $\{E, K_1, \sum_1\}$ be (P, p) prebundles. We say that $\{E, K_1, \sum_1\}$ is a *subdivision* of $\{E, K, \sum\}$, if K_1 is a subdivision of K and if for each simplex A of K there is (A, f) in $\sum$ such that

f: $A_1 \times (P, p) \to E \mid A_1$ is an isomorphism from $A_1 \times (P, p)$ to $\{E \mid A_1, A_1, \sum_1 \mid A_1\}$, where A_1 is a subcomplex of K_1 covering A.

Theorem 4.1 (m). *Let K be a k $(\leq m)$ dimensional complex. Given an n prebundle $\{E, K, \sum\}$ and a subdivision K_1 of K, then there exists a subdivision $\{E, K_1, \sum_1\}$.*

Corollary 4.2 (m). *Let $\{E, K, \sum\}$ be an n prebundle such that $|K|$ is an m cell. Then $\{E, K, \sum\}$ is trivial.*

Proof of Corollary 4.2 (m). Since $|K|$ is a cell, $|K|$ is collapsible. Hence there exists a subdivision K_1 of K such that K_1 is collapsible. By Theorem 4.1 (m) there is a subdivision $\{E, K_1, \sum_1\}$ of $\{E, K, \sum\}$. By Proposition 2.5, $\{E, K_1, \sum_1\}$ is trivial. Therefore $\{E, K, \sum\}$ is clearly trivial, completing the proof.

To prove Theorem 4.1 (m) we need:

Theorem 4.3 (m). *Let S be an m sphere with a partition K and let W be an $m+n$ manifold. Let f: $S \to W$ be an embedding. If N_1 and N_2 are normal prebundles for f: $S \to W$ over K, then they are isomorphic. Moreover, if N_1 and N_2 are contained in Int W, then there is an ambient isotopy F of W keeping $f(S)$ fixed such that $F \mid N$: $N_1 \to N_2$ is a prebundle isomorphism.*

Proof of Theorems 4.1 (m) and 4.2 (m). Let us prove 4.1 (m) and 4.2 (m) by induction on the dimension m.

(i) *Theorem* 4.1 (0) *is obvious.*

(ii) *Theorem* 4.1 (m) (*Corollary* 4.2 (m)) *implies Theorem* 4.3 (m).

Proof of (ii). Since N_1 and N_2 are regular neighborhoods of $f(S)$ in W, replacing N_1 and N_2 by smaller regular neighborhoods, if necessary, we may assume that N_1 and N_2 are contained in Int W. Let A be a principal simplex of K and let B denote the cell $S - \text{Int } A$ and also the partition $K - A$.

Since N_i/B, $i=1, 2$ are regular neighborhoods of $f(B)$ mod $f(A)$ in W, it follows from the uniqueness of relative regular neighborhoods [6] that there is an ambient isotopy H: $W \to W$ keeping $f(S)$ fixed such that $H(N_2/B) = N_1/B$. Since B is a cell, by Corollary 4.2 (m) N_i/B $i=1, 2$ are trivial n prebundles. Choosing trivializations h_i: $B \times (J^n, 0) \to N_i/B$ $i=1, 2$, suitably, we may assume that $h_1^{-1} H h_2$: $B \times J^n \to B \times J^n$ is orientation preserving. By the Lemma 8 in [11], $h_1^{-1} H h_2 / \partial(B \times J^n)$ is isotopic to the identity keeping $\partial B \times 0$ fixed. By embedding the isotopy on a compatible collar of $(\partial(N_1/B), \partial f(B))$ in $(W - \text{Int}(N_1/B), f(A))$, see [12], we may extend $H h_2 h_1^{-1}$ to an ambient isotopy G: $W \to W$ keeping $f(S)$

fixed. Since $H^{-1}G(N_1/A)$ and N_2/A are regular neighborhoods of $f(A)\bigcup N_2/\partial A$ mod $\partial(N_1/B)-\mathrm{Int}(N_2/\partial A)$ in $W-\mathrm{Int}(N_1/B)$, we may assume that $H^{-1}G(N_1/A)=N_2/A$. Thus $H^{-1}G$ is the required ambient isotopy, completing the proof.

(iii) *Theorems* 4.1 $(m-1)$ *and* 4.3 $(m-1)$ *imply Theorem* 4.1 (m) $(m\geq 1)$.

Proof of (iii). Let L be the subcomplex of K_1 covering $|K^{m-1}|$. By Theorem 4.1 $(m-1)$, we have a subdivision $\{E|K^{m-1}, L, \sum_L\}$ of $\{E|K^{m-1}, K^{m-1}, \sum|K^{m-1}\}$. Let A be an m simplex of K and let A_1 be the subcomplex of K_1 covering A. Let (A, f) belong to $\sum$. Then $f|\partial A\times(J^n, 0)$ gives a trivial normal prebundle of $i|\partial A\colon \partial A\to\partial(E|A)$ over ∂A_1, where $i\colon |K|\to E$ is the zero-section of the n prebundle $\{E, K, \sum\}$. While $\{E|\partial A, \partial A_1, \sum_L|\partial A_1\}$ is a normal prebundle of $i|\partial A\colon \partial A_1\to\partial(E|A)$. Since $E|\partial A$ is contained in $\mathrm{Int}\,\partial(E|A)=\partial(E|A)$ and since ∂A is an $m-1$ sphere, it follows from Theorem 4.3 $(m-1)$, there exists a homeomorphism $g\colon \partial(E|A)\to\partial(E|A)$ such that for each simplex B of ∂A_1, $gf|B\times(J^n, 0)$ belongs to $\sum_L|\partial A_1$.

By the join extension argument, we may extend the homeomorphism g to a homeomorphism h of the pair $(E|A, i(A))$ onto itself such that $h|i(A)=$identity. Then for each simplex C of A_1, $hf|C\times(J^n, 0)$ gives a trivialization compatible with $\sum_L|\partial A_1$. Thus the subdivision $\{E|K^{m-1}, L, \sum_L\}$ may be extended over A_1. Since for each m simplex A of K we may obtain such an extension independently, we have the required subdivision $\{E, K_1, \sum_1\}$, completing the proof.

By (i), (ii) and (iii), Theorems 4.1 (m) and 4.3 (m) are now complete.

Theorem 4.4. *Let $f\colon M\to W$ be a locally flat embedding of an m manifold M into an $m+n$ manifold W. For any partition K of M there is a normal prebundle N for f over K.*

Proof. Let K' and L be a subdivision of K and a partition of W respectively such that $f\colon K'\to L$ is simplicial and that $f(K')$ is full in L.

Let K'_k denote a subcomplex of the barycentric subdivision of K' covering the k skeleton of the dual cell complex of K'. For each $m-k$ simplex A of K' let C and D denote the dual k and $k+n$ cells of A and $f(A)$ in K' and L respectively, and let P denote a subcomplex of K'_k covering C. We shall prove the following Proposition for $k=m$ by induction.

[k]: *There is an n prebundle $N_k=\bigcup D_\alpha$ over K'_k such that the zero section coincides with the restriction $f/|K'_k|$ and that $N_k/P_\alpha=D_\alpha$, where α ranges over all indices of $m-k$ simplexes A_α of K'.*

[0]: Obvious.

[k]$\Rightarrow$[$k+1$]: Let A be an arbitrary $m-k-1$ simplex of K'. Since by the Lemma 1 of [10] $f/C\colon C\to D$ is flat, there is a homeomorphism $h\colon C\times J^n\to D$ such that $h(x, 0)=f(x)$ for all x in C.

Then $N_k/\partial P$ and $h(\partial C\times J^n)$ are normal prebundles for $f/\partial C$: $\partial C\to\partial D$ over ∂P. By Theorem 4.3 and by a join extension, we have a homeomorphism g: $D\to D$ such that $g/f(C)=\text{id.}$, and $gh/\partial C\times J^n$: $\partial C\times J^n\to N_k/\partial P$ is a trivialization. Then gh yields the required trivializations over simplexes B of P, which extend those of $N_k/\partial P$, by setting $gh/B\times J^n$: $B\times J^n\to N_{k+1}$. It follows from the induction that f has a normal prebundle N over the barycentric subdivision of K' such that N is a derived neighborhood of $f(K')$ in L. By Corollary 4.2, we may reduce the prebundle N over K' to over K. This completes the proof of Theorem 4.4.

An embedding f: $M\to W$ is called to be *proper*, if $f(\partial M)\subset\partial W$ and $f(\text{Int}\, M)\subset\text{Int}\, W$. By Zeeman's unknotting theorem, every proper embedding f of M into W of codimension ≥ 3 is always locally flat. Thus we have:

Corollary 4.5. *Every proper embedding of codimension ≥ 3 has a normal prebundle.*

The normal prebundle constructed by the above Propositions [k] for $k\leq m$ is called to be *compatible* with the dual cell structures of K' and L.

Let M be an oriented manifold. For an oriented manifold W, an embedding f: $M\to W$ is called to be *oriented.* Two oriented embeddings f: $M\to W$ and g: $M\to W'$ are *microequivalent* if there are neighborhoods U and U' of $f(M)$ and $g(M)$ in W and W' respectively and a homeomorphism h: $U\to U'$ such that h preserves orientations of U and U' induced from those of W and W' respectively and $hf=g$. The microequivalence relation of embeddings is clearly an equivalence relation.

REMARK. The original concept of microequivalence of embeddings is isoneighboring due to H. Noguchi, [10]. For locally flat embeddings of a sphere by the uniqueness of regular neighborhoods the two concepts of microequivalence and isoneighboring are equivalent.

Let $\mathcal{E}^n(M)$ denote the set of all microequivalence classes of oriented locally flat embeddings of M of codimension n. Let S_k denote the standard oriented k sphere $\partial\Delta_{k+1}$.

Theorem 4.6. *There is a set identification $\mathcal{E}^n(S_k)=\pi_{k-1}(PR_n)$.*

Proof. By Theorems 4.1 and 4.4 every oriented locally flat embedding f of S_k of codimension n has uniquely oriented normal prebundles $N(f)$ over S_k with orientations induced from those of S_k and the ambient manifold.

Thus by the classification theorem of oriented prebundles over S_k (see §5, Theorem 5.2), we may associate to each class $\{f\}$ in $\mathcal{E}^n(S_k)$ the class $\{N(f)\}$ in $\pi_{k-1}(PR_n)$. We define a correspondence N: $\mathcal{E}^n(S_k)\to\pi_{k-1}(PR_n)$ by setting $N\{f\}=\{N(f)\}$. If $\{E, S_k, \sum\}$ is an oriented n prebundle over S_k with the zero section i: $S_k\to E$, then E is an oriented manifold having the orientation from which the orientation of the normal prebundle $E(i)=\{E, S_k, \sum\}$ is induced. Thus N is surjective. Conversely if $N(f)$ and $N(g)$ are isomorphic oriented normal prebundles for f and g over S_k respectively, then f and g are obviously microequivalent. This completes the proof of Theorem 4.6.

The following notion of tubes is due to M.W. Hirsch [4]. Suppose that a manifold pair (W, M) has an oriented normal cell bundle v. A *tube* for (W, M) is the triple (W, M, v). A second tube (W, M, u) is *isomorphic* to (W, M, v) if there is a homeomorphism $h\colon W\to W$ called an *isomorphism* such that h/v is an isomorphism onto u. The isomorphism relation of tubes is clearly an equivalence relation. Let (T, S) denote the standard oriented $(k+n, k)$ sphere pair $(\partial(\Delta_{k+1}\times J^n), \partial\Delta_{k+1}\times 0)$. Let $\tau(k, n)$ denote the set of all isomorphism classes of tubes for (T, S).

Theorem 4.7. *There is a set identification* $\tau(k, n)=\pi_k(PR_n, \Pi L_n)$.

Proof. The proof of Theorem 4.3 ensures that every tube for (T, S) is isomorphic to a tube (T, S, v) such that the underlying prebundle of v over S is isomorphic to the product prebundle $S\times(J^n, 0)$ by the identity isomorphism. Thus we may associate to each tube a relative $(PR_n, \Pi L_n)$ bundle over $(\Delta_{k+1}, \partial\Delta_{k+1})$ which consists of the product n prebundle $\Delta_{k+1}\times J^n$ over Δ_{k+1} and the n cell bundle v over $\partial\Delta_{k+1}$. (The concept of relative a. s. bundles are defined in the same way as in [S], p.p. 43–44) It is clear by the join extension argument that two tubes are isomorphic if and only if the associated relative bundles are isomorphic. Since the set of all isomorphism classes of relative $(PR_n, \Pi L_n)$ bundles over $(\Delta_{k+1}, \partial\Delta_{k+1})$ are one to one corresponding to elements of $\pi_k(PR_n, \Pi L_n)$, (see §5, Theorem 5.1), it follows that the required set identification is obtained, completing the proof.

Observing the homotopy exact sequence for $(PR_n, \Pi L_n)$ together with the above set identifications 4.6 and 4.7, we immediately obtain the following.

Theorem 4.8. (1) *Every locally flat embedding of S_k of codimension n has a normal cell bundle if and only if the homomorphism $i_{k-1}\colon \pi_{k-1}(\Pi L_n)\to\pi_{k-1}(PR_n)$ is surjective.*

(2) *Every normal cell bundle for the standard $(k+n, k)$ sphere pair (T, S) is trivial if and only if i_{k-1} is injective.*

Example 4.9 (M. W. Hirsch). In [4], I, Hirsch has found a tube t for $(k, n)=(7, 4)$ such that the class $\{t\}\neq 0$ in $\pi_7(PR_4, \Pi L_4)$, but $\partial_7\{t\}=0$ in $\pi_6(\Pi L_4)$.

Hence ∂_7 has non trivial kernel. Therefore i_7 has non trivial cokernel. It follows that there is a locally flat embedding of S_8 of codimension 4 having no normal cell bundle, compare [4], II.

Example 4.10 (N. H. Kuiper and R. K. Lashof). In [8], Kuiper and

Lashof have proven that the homomorphism $\pi_k(0_4)\to\pi_k(\Pi L_4)$ is injective for all k, and deduced that there are non trivial normal 4 cell bundles for the $(k+4, k)$ sphere pair, provided that $7\leq k\leq 9$ and $k=11$. It follows from Theorem 4.8 that i_k: $\pi_k(\Pi L_4)\to\pi_k(PR_4)$ is not injective for $6\leq k\leq 8$ and $k=10$.

Corollary 4.11. *The relative homotopy groups $\pi_k(PR_n, \Pi L_n)$ consist of only the trivial elements for $k+2\leq n$. That is, the homomorphisms i_k: $\pi_k(\Pi L_n)\to\pi_k(PR_n)$ are surjective for $k+2\leq n$ and injective for $k+3\leq n$.*

Prooof. By the Corollary 4.2 of [2], if $k+2\leq n$, then every embedding of S_{k+1} of codimension n has a normal cell bundle, and if $k+3\leq n$, then normal cell bundles for the standard $(k+1+n, k+1)$ sphere pair are unique, that is, trivial. Thus the conclusion follows from Theorem 4.8. This completes the proof of Corollary 4.11.

By the obstructuion theory we may deduce the following.

Corollary 4.12. *Every n prebundle over a complex K has an n cell bundle reduction, provided that dim. $K+1\leq n$.*

Applying Theorem 4.4 and the above, we may sharpen the Corollary 4.2 in [2] as follows.

Corollary 4.13. *Let f: $M\to W$ be a proper embedding of an m manifold M into an $m+n$ manifold W. Let K and L be partitions of M and W respectively such that f: $K\to L$ is simplicial and $f(K)$ is full in L.*
If $n\geq m+1$ and $m\geq 2$, then there is a normal cell bundle for f which is compatible with the dual cell structures of K and L.

Let PL denote the structural group of stable microbundles.

Since $\pi_k(PR_n)\cong\pi_k(\Pi L_n)\cong\pi_k(PL_n)$ for $k+3\leq n$, $\pi_k(PL)\cong\pi_k(PL_n)$ for $k+2\leq n$, and $\pi_k(PR_{k+2})\cong\pi_k(PR_{k+3})$, we may deduce the following.

Theorem 4.14. *By the isomorphism $\pi_k(PL_n)\cong\pi_k(PR_n)$ for $k+2\leq n$, the Hirsch-Mazur's exact sequence is rewritten as follows;*

$$0\to\pi_k(0_n)\to\pi_k(PR_n)\to\Gamma_k\to 0 \text{ for } k+2\leq n\,.$$

5. Appendix

The classification theorem for relative $(PR_n, \Pi L_n)$ bundles over $(\Delta_{k+1}, \partial\Delta_{k+1})$.

Let ξ be an element of $\pi_k(PR_n, \Pi L_n)$. Then ξ is represented by a k simplex f of $Z^k(PR_n, \Pi L_n)$. Pasting $\Delta_k\times J^n$ to $\Delta_{k+1}\times J^n$ by the embedding $(d_0^{k+1}\times J^n)f$: $\Delta_k\times J^n\to\Delta_{k+1}\times J^n$, we have an n prebundle over Δ_{k+1}. Moreover, since $(d_0^{k+1}\times J^n)f/\partial\Delta_k\times J^n$: $\partial\Delta_k\times J^n\to\partial d_0^{k+1}(\Delta_k)\times J^n$ is an n cell bundle isomorphism, we have a $(PR_n, \Pi L_n)$ bundle $\rho(f)$ over $(\Delta_{k+1}, \partial\Delta_{k+1})$. If a second k simplex g of $Z^k(PR_n, \Pi L_n)$ belongs to ξ, we have a second $(PR_n, \Pi L_n)$ bundle

$\rho(g)$ over $(\Delta_{k+1}, \partial\Delta_{k+1})$. However, $g^{-1}f$ is extendable to a $k+1$ simplex F of $C^{k+1}(PR_n, \Pi L_n)$ such that $d_0^{k+1*}F=g^{-1}f$, $d_{k+1}^{k+1*}F$ belongs to ΠL_n, and $d_i^{k+1*}F$ $=$id., for $i=1, \cdots, k$. This implies that F is an isomorphism between two $(PR_n, \Pi L_n)$ bundles $\rho(f)$ and $\rho(g)$ over $(\Delta_{k+1}, \partial\Delta_{k+1})$. Thus we obtain a correspondence ρ_*: $\pi_k(PR_n, \Pi L_n)\to$\{isomorphism classes of $(PR_n, \Pi L_n)$ bundles over $(\Delta_{k+1}, \partial\Delta_{k+1})$\} by $\rho_*(\xi)=$the isomorphism class of $\rho(f)$.

Theorem 5.1. *The correspondence ρ_* is bijective.*

Proof. Let $\{E, \Delta_{k+1}, \sum\}$ be an n prebundle over Δ_{k+1} such that $\{E\,|\,\partial\Delta_{k+1}, \partial\Delta_{k+1}, \sum|\partial\Delta_{k+1}\}$ has a distinguished n cell bundle reduction. Let h: $(\partial\Delta_{k+1}-\Delta_k)\times J^n\to E/(\partial\Delta_{k+1}-\Delta_k)$ be an n cell bundle isomorphism, where $\Delta_k=d_{k+1}(\Delta_k)$. Let g: $\Delta_{k+1}\times J^n\to E$ be a trivialization of the n prebundle. Since $g^{-1}h$: $(\partial\Delta_{k+1}-\Delta_k)\times J^n\to(\partial\Delta_{k+1}-\Delta_k)\times J^n$ is an n prebundle isomorphism and since PR_n is a Kan group, we may extend $g^{-1}h$ to a $k+1$ simplex f of PR_n. Replacing g by gf, if necessary, we may assume that $g/(\partial\Delta_{k+1}-\Delta_k)\times J^n=h$. Let h': $\Delta_k\times J^n\to E/\Delta_k$ be an n cell bundle isomorphism. Then $g^{-1}h'/\partial\Delta_k\times J^n$ is an n cell bundle isomorphism of the product n cell bundle $\partial\Delta_k\times J^n$. Since ΠL_n is a Kan group, we may extend $g^{-1}h'/(\partial\Delta_k-\Delta_{k-1})\times J^n$ to a k simplex f' of ΠL_n, where $\Delta_{k-1}=d_k(\Delta_{k-1})$. Replacing h' by $h'(f')^{-1}$, if necessary, we may assume that $g/(\partial\Delta_k-\Delta_{k-1})\times J^n=h'/(\partial\Delta_k-\Delta_{k-1})\times J^n$.

Thus $g^{-1}h'$: $\Delta_k\times J^n\to\Delta_k\times J^n$ belongs to $Z^k(PR_n, \Pi L_n)$, and $\rho(g^{-1}h')$ is just isomorphic to the given relative $(PR_n, \Pi L_n)$ bundle over $(\Delta_{k+1}, \partial\Delta_{k+1})$. Hence ρ_* is surjective. Let f and g belong to $Z^k(PR_n, \Pi L_n)$. Suppose that $\rho(f)$ is isomorphic to $\rho(g)$. Then there is an n prebundle isomorphism h: $\Delta_{k+1}\times J^n$ $\to\Delta_{k+1}\times J^n$ such that $(d_0^{k+1*}h)(fg)$, $d_i^{k+1*}h$, $i=1, \cdots, k+1$ are k simplexes in ΠL_n. Since f and g belong to $Z^k(PR_n, \Pi L_n)$, it follows that $d_i^{k*}((d_0^{k+1*}h)fg^{-1})$ $=d_i^{k*}d_0^{k+1*}h$, for $i\neq k$. We have, therefore, a $k+1$ simplex h_1 of ΠL_n such that $d_0^{k+1*}h_1=(d_0^{k+1*}h)fg^{-1}$, and $d_i^{k+1*}h_1=d_i^{k+1*}h$ for $i=1, \cdots, k$. Put $h_2=h_1^{-1}h$. Then $d_0^{k+1*}h_2=d_0^{k+1*}h_1^{-1}d_0^{k+1*}h=gf^{-1}(d_0^{k+1*}h)^{-1}(d_0^{k+1*}h)=gf^{-1}$, $d_i^{k+1*}h_2=$id., for $i=1, \cdots, k$, and $d_{k+1}^{k+1*}h_2=$belongs to ΠL_n. Hence gf^{-1} belongs to $B^k(PR_n, \Pi L_n)$.

Thus f and g belong to the same class in $\pi_k(PR_n, \Pi L_n)$, completing the proof. In the same way we may show the following:

Theorem 5.2. *There is a one to one correspondence between the set of all isomorphism classes of oriented n prebundles over $\partial\Delta_{k+1}$ and the set $\pi_k(PR_n)$.*

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References

[1] H. Gluck: *The embedding of two spheres in the four sphere,* Trans. Amer. Math. Soc. **104** (1962), 303–333.

[2] A. Haefliger and C. T. C. Wall: *Piecewise linear bundles in the stable range,* Topology **4** (1965), 209–214.

[3] A. Heller: *Homotopy resolutions of semi-simplicial complexes,* Trans. Amer. Math. Soc. **80** (1955), 299–344.

[4] M. W. Hirsch: *On tubular neighbourhoods of manifolds* I, II, Proc. Cambridge Philos. Soc. **62** (1966), 177–181, 183–185.

[5] M. W. Hirsch: *On non-linear cell bundles,* Ann. of Math. **84** (1966), 373–385.

[6] J. F. P. Hudson and E. C. Zeeman: *On regular neighbourhoods,* Proc. London Math. Soc. (3) **14** (1964), 719–745. (Correction, to appear).

[7] D. M. Kan: *A combinatorial definition of homotopy groups,* Ann. of Math. **67** (1958), 282–312.

[8] N. H. Kuiper and R. K. Lashof: Microbundles and Bundles Part C, (mimeographed).

[9] J. Milnor: Microbundles and Differentiable Structures, (mimeographed) Princeton University, 1961.

[10] H. Noguchi: *One flat 3-manifolds in 5-space,* Osaka J. Math. **1** (1964), 117–125.

[11] H. Noguchi: *Obstructions to locally flat embedding of combinatorial manifolds,* Topology **5** (1966), 203–213.

[12] E. C. Zeeman: Seminar on Combinatorial Topology, (mimeographed) Inst. Hautes Études Sci. Paris, 1963.

[13] E. C. Zeeman: *Unknotting combinatorial balls,* Ann. of Math. **78** (1963), 501–526.

[S] N. E. Steenrod: The Topology of Fibre Bundles, Princeton University Press, 1951.