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Osaka University
ON THE LARGE TIME BEHAVIOR OF SOLUTIONS
FOR SOME DEGENERATE QUASILINEAR
PARABOLIC SYSTEMS

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(Received May 20, 1991)

1. Introduction

We consider the large time behavior of the solutions for the following
Cauchy problem:

\begin{align*}
    u_t &= (u^m)_{xx} - v^n u^n \quad \text{in } \mathbb{R} \times (0, \infty) \\
    v_t &= (v^m)_{xx} - u^n v^n \quad \text{in } \mathbb{R} \times (0, \infty)
\end{align*}

with initial conditions

\begin{align*}
    u(\cdot, 0) &= u_0 \quad \text{and} \quad v(\cdot, 0) = v_0 \quad \text{on } \mathbb{R}.
\end{align*}

Here, $m > 1$ and $n \geq 1$ are real numbers. Throughout this paper, we assume
that $m > 1$ and $n \geq 1$.

By [10], the following properties are shown:

When the reaction arises among some reactions, for each reactant the equation for
reaction-diffusion takes the form

\[ \frac{\partial C}{\partial t} = \text{div } D \text{ grad } C + q', \]

where $C$ is the concentration, $D$ is the diffusion coefficient and $q'$ is the
amount of material formed through chemical reactions per unit volume per unit time.
When a reaction arises among $n$ molecules of a substance $A$ and $n$ molecules of a
substance $B$ and does not reverse, that is to say, when the reaction is written as

\[ nA + nB \rightleftharpoons \text{product}, \]

then $q'$ of both equations for $A$ and $B$ are proportional to $-C_A^n C_B^n$, where $C_A$ and
$C_B$ are the concentrations of the substances $A$ and $B$, respectively. That is to
say, the concentrations $C_A$ and $C_B$ satisfy the equation

\begin{align*}
    \frac{\partial C_A}{\partial t} &= \text{div } D_A \text{ grad } C_A - k C_A^n C_B^n \\
    \frac{\partial C_B}{\partial t} &= \text{div } D_B \text{ grad } C_B - k C_A^n C_B^n,
\end{align*}

(1.3)
where $k$ is a positive constant. Here we omit the equation of the concentration of the product, since the concentration does not need to study $C_A$ and $C_B$ in our situation.

In this paper we consider (1.3) in case of $D_A = C_A^{-1}$ and $D_B = C_B^{m-1}$. Then equations (1.1) are equivalent to the equations (1.3). We make the following assumptions (A.I.) on the initial data $u_0$ and $v_0$:

(A.I.) (1) The functions $u_0$ and $v_0$ are nonnegative and continuous on $\mathbb{R}$, (2) they have compact support and are not identically zero on $\mathbb{R}$. Moreover, in this paper, we assume that every function is bounded and nonnegative.

If $u_0 = v_0$ on $\mathbb{R}$, the solutions $u$ and $v$ of (1.1) and (1.2) would coincide in $\mathbb{R} \times [0, \infty)$ and satisfy the following Cauchy problem with $p = 2n$:

\begin{align}
(1.4) & \quad u_t = (u^m)_{xx} - u^p \quad \text{in } \mathbb{R} \times (0, \infty) \\
(1.5) & \quad u(\cdot, 0) = u_0 \quad \text{on } \mathbb{R}.
\end{align}

As for the study of the large time behavior of solution for (1.4) and (1.5), it is important to investigate the large time behavior of supports and $L^\infty$-norms of the solutions. Therefore many authors have studied on supports and $L^\infty$-norms of the solutions (See [1], [5], [7]-[9], [11]-[13] etc.).

The support and $L^\infty$-norm of the solution $u$ of (1.4) and (1.5) have the following properties:

If $1 \leq p < m$, then $\bigcup_{t \geq 0} \text{supp } u(\cdot, t)$ is bounded in $\mathbb{R}$.

- $\inf \{ \text{supp } u(\cdot, t) \}, \sup \{ \text{supp } u(\cdot, t) \} \sim \log t$ if $1 < p = m$.
- $\inf \{ \text{supp } u(\cdot, t) \}, \sup \{ \text{supp } u(\cdot, t) \} \sim t^{(p-m)/(2p-2)}$ if $\max(1, p-2) < m < p$.
- $\inf \{ \text{supp } u(\cdot, t) \}, \sup \{ \text{supp } u(\cdot, t) \} \sim t^{\frac{p}{m+1}}$ if $1 < m < p-2$.

\begin{align}
\log(|u(\cdot, t)|_{\infty, R}) & \sim -t \quad \text{if } 1 = p < m, \\
|u(\cdot, t)|_{\infty, R} & \sim t^{-\frac{1}{p-1}} \quad \text{if } \max(1, p-2) < m \text{ and } 1 < p, \\
|u(\cdot, t)|_{\infty, R} & \sim t^{-\frac{1}{m+1}} \quad \text{if } 1 < m < p-2.
\end{align}

Here $a(t) \sim b(t)$ means that there exist two positive constants $c_1$ and $c_2$ satisfying

\[ c_1 a(t) \leq b(t) \leq c_2 a(t) \quad \text{for any sufficiently large } t. \]

In this paper, for the initial data $u_0$ and $v_0$ satisfying the following assumption, we consider the solutions of (1.1) and (1.2),

(A.II.) \quad $u_0 \equiv v_0$ on $\mathbb{R}$ and $0 \leq u_0 \leq v_0$ on $\mathbb{R}$.

The purposes of this paper is to investigate whether the large time behavior of the solutions for the system (1.1) differs from the behavior of the solutions for the equation (1.4).
The following is our main theorem.

**Theorem 1.1.** Let \( m > 1 \) and \( n \geq 1 \) and suppose that \( u_0 \) and \( v_0 \) satisfy (A.1) and (A.11).

The initial value problem (1.1)–(1.2) has a unique pair of solutions \( u \) and \( v \) which are nonnegative. Then, the support and \( L^n \)-norm of \( u \) and \( v \) have the following properties:

\[
\begin{align*}
\text{Let } & m > 1 \text{ and } n \geq 1 \text{ and suppose that } u_0 \text{ and } v_0 \text{ satisfy (A.1) and (A.11).} \\
\text{The initial value problem } & (1.1)–(1.2) \text{ has a unique pair of solutions } u \text{ and } v \text{ which are nonnegative. Then, the support and } L^n \text{-norm of } u \text{ and } v \text{ have the following properties:} \\
\text{1. If } & 2n - 1 < m, \text{ then } \text{supp } u(\cdot, t) \text{ is bounded in } \mathbb{R}, \\
\text{2. } & -\inf \{\text{supp } u(\cdot, t)\}, \sup \{\text{supp } u(\cdot, t)\} \sim \log t \text{ if } 2n - 1 = m, \\
\text{3. } & -\inf \{\text{supp } u(\cdot, t)\}, \sup \{\text{supp } u(\cdot, t)\} \sim t^{(1/2)(1 - (m - 1)/(n - 1))} \text{ if } 2n - 2 < m < 2n - 1, \\
\text{4. } & -\inf \{\text{supp } u(\cdot, t)\}, \sup \{\text{supp } u(\cdot, t)\} \sim \frac{1}{(m + 1)} \text{ if } m < 2n - 2.
\end{align*}
\]

In all of the above cases, the solution \( v \) satisfies

\[
-\inf \{\text{supp } v(\cdot, t)\}, \sup \{\text{supp } v(\cdot, t)\} \sim t^{1/(m + 1)}
\]

and

\[
|v(\cdot, t)|_{L^n} \sim t^{-1/(m + 1)}. 
\]

Therefore, the behavior of \( v \) is independent of the behavior of \( u \).

By Theorem 1.1 we see that the large time behaviors of \( u \) and \( v \) in our case are different from the behaviors in case of \( u_0 \equiv v_0 \). That is to say, the behavior of solutions for the system (1.1) is essentially different from one for the equation (1.3).

And we remark that the solutions \( v \) and \( u \) is similar to the solutions of (2.2) in Section 2 and (5.1) in Section 5, respectively (See Lemma 5.2).

In particular, we have:

**Corollary 1.2.** Under the assumptions of Theorem 1.1, the supports of \( u \) and \( v \) of the system has the following properties:

\[
\begin{align*}
\text{If } & 2n - 1 < m, \text{ then } \cup_{0 \leq t} \text{supp } u(\cdot, t) \text{ is bounded in } \mathbb{R}. \\
\text{If } & 1 < m \leq 2n - 1, \text{ then } \cup_{0 \leq t} \text{supp } u(\cdot, t) = \mathbb{R}. \\
\text{And, for all of the above cases, } & \cup_{0 \leq t} \text{supp } v(\cdot, t) = \mathbb{R}.
\end{align*}
\]

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2. **Notations and definitions**

Throughout this paper, we use the following notations and definitions.
For any measurable subsets $E$ of $\mathbb{R}$ or $\mathbb{R} \times [0, \infty)$, the usual norms of the spaces $L^q(E)$ for $1 \leq q \leq \infty$ are denoted by $| \cdot |_{E,q}$ and $C_0(E)$ is the space whose elements are compactly supported continuous functions in $E$. As for the other function spaces, we use the notations and the definitions in [14].

Next we shall define the solutions of (1.4) and (1.5). For $p \geq 1$ and $\mu \geq 0$, the operator $B^{(\mu)}$ is defined with the domain

$$D(B^{(\mu)}) = \{w \in L^1(\mathbb{R}) : (|u|^{m-1} u)_{xx} \in L^p(\mathbb{R})\}$$

by

$$B^{(\mu)} w = (|w|^{m-1} w)_{xx} - \mu |w|^p w \quad \text{for} \quad w \in D(B^{(\mu)}).$$

By [2], it is shown that $B^{(\mu)}$ is $m$-dissipative in $L^1(\mathbb{R})$. Therefore, by [6], it is shown that a contraction semigroup $T^{(\mu)}(t)$ on $L^1(\mathbb{R})$ is defined by

$$T^{(\mu)}(t) w = \lim_{\lambda \to 0^+} (1 - \lambda B^{(\mu)})^{-t/\lambda} w$$

for $t \geq 0$ and $w \in D(B^{(\mu)})$,

where $[\cdot]$ is the Gauss function. Then, for $w_0 \in L^1(\mathbb{R})$ we define the solutions $w$ of (1.4) with $w(\cdot, 0) = w_0$ by

$$(2.1) \quad w(\cdot, t) = T^{(\mu)}(t) w_0.$$

We also consider the equation:

$$(2.2) \quad \tilde{z}_t = (\tilde{z}^m)_{xx} \quad \text{in} \quad \mathbb{R} \times (0, \infty),$$

with initial condition

$$(2.3) \quad \tilde{z}(\cdot, 0) = \tilde{z}_0 \quad \text{on} \quad \mathbb{R}.$$

For $\tilde{z}_0 \in L^1(\mathbb{R})$, we define the solution of (2.3) and (2.4) by

$$(2.4) \quad \tilde{z}(\cdot, t) = T^{(\mu)}(t) \tilde{z}_0.$$

For a positive constant $M$ we see

$$(2.5) \quad \tilde{z}(x, t; M) \equiv t^{-1/(m+1)}(a^2 - b_m \cdot t^{-2/(m+1)} \cdot x^2)^{1/(m-1)}$$

for $(x, t) \in \mathbb{R} \times (0, \infty)$

where $a = a_m M^{(m-1)/(m+1)}$ with a certain constant $a_m$ and $b_m = (m-1)/(2m(m+1))$.

We call the function (2.5) the self-similar solution or the explicit solution (of (2.2)). We can observe that the self-similar solution satisfies (2.2) in $\mathbb{R} \times (0, \infty)$ and that the corresponding initial condition is $M \delta_0$, where $\delta_0$ is the delta function.

The Banach space $\mathcal{X}$ denotes $(L^1(\mathbb{R}))^2$ with the norm

$$| (u, v) |_{\mathcal{X}} = |u|_{1,R} + |v|_{1,R} \quad \text{for} \quad (u, v) \in \mathcal{X}.$$
We shall define an operator \( A \) with the domain
\[
D(A) = \{(u, v) \in \mathcal{X}: (|u|^{m-1}u)_{xx}, (|v|^{m-1}v)_{xx} \in L^1(\mathbb{R})\}
\]
by
\[
A(u, v) = (|(u|^{m-1}u)_{xx} - |v|^{n}u|^{n-1}u, (|v|^{m-1}v)_{xx} - |u|^{n}v|^{n-1}v).
\]

By Lemma 3.3 in [16] it is shown that \( A \) is \( m \)-dissipative in \( \mathcal{X} \) and that a contraction semigroup \( S(t), t \geq 0, \) on \( \mathcal{X} \) is defined by
\[
S(t) V = \lim_{\lambda \to 0} (1 - \lambda A)^{-t/\lambda} V \quad \text{for} \quad t \geq 0 \quad \text{and} \quad V \in \mathcal{X}.
\]

Then, we define the solutions \( u \) and \( v \) of (1.1)-(1.2) by
\[
(u(\cdot,t), v(\cdot,t)) = S(t)(u_0, v_0).
\]

3. The existence and the uniqueness of generalized solution

In this section we shall consider the following Cauchy problem:
\[
\begin{align*}
\frac{d}{dt}w &= (w^m)_{xx} - Pw^q \quad \text{in} \quad \mathbb{R} \times [0, \infty) \\
\w(\cdot,t) &= w_0 \quad \text{on} \quad \mathbb{R}
\end{align*}
\]

where \( q \geq 1, P \) is a function on \( \mathbb{R} \times [0, \infty) \) and \( w_0 \) is a continuous function on \( \mathbb{R} \). Throughout this section, we assume \( q \geq 1 \).

**Definition 3.1.** We say that \( w \) is a generalized solution of (3.1) if \( w \) belongs to \( C([0, \infty); L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times [0, \infty)) \), and for \( t_0, t_1, \) a.e. \( x_0 \) and a.e. \( x_1 \), such that \( 0 \leq t_0 < t_1, x_0 < x_1 \), the following integral identity holds:
\[
I(u, f, E) = \int_{t_0}^{t_1} \int_{x_0}^{x_1} \{w^m f_{xx} + w f_t - Pw^q f\} \, dx \, dt
\]
\[
- \left[ \int_{x_0}^{x_1} w f \, dx \right]_{t_0}^{t_1} - \left[ \int_{t_0}^{t_1} w^m f \, dt \right]_{x_0}^{x_1} = 0,
\]
for \( f \in C^{2,1}(E) \) satisfying
\[
f(x_0, t) = f(x_1, t) = 0 \quad \text{for} \quad t \in [t_0, t_1],
\]
where we set \( E = [x_0, x_1] \times [t_0, t_1] \).

Definition 3.1 is slightly different from ones in [9] and [11]-[13]. That is to say, they have assumed that the solutions are continuous in \( \mathbb{R} \times [0, \infty) \), while we do not assume such a continuity.

**Remark 3.2.** Under (A.I), the solutions of (1.1) and (1.2) defined in Section 2 are generalized solutions. Since this is shown by the following Lemma
3.3–3.5 and the standard argument, we omit the proof.

**Lemma 3.3.** Let \( P \in C^{1,1}(\mathbb{R} \times [0, \infty)) \cap L^\infty(\mathbb{R} \times [0, \infty)) \) and \( w_0 \in C_c(\mathbb{R}) \). Then there exists a unique generalized solution \( w \) of (3.1)–(3.2). Moreover, \( w \) is continuous in \( \mathbb{R} \times [0, \infty) \) satisfying

\[
0 \leq w(x, t) \leq \bar{z}(x, t+1; M), \quad |w_0|_{w, R} \text{ in } \mathbb{R} \times [0, \infty),
\]

where \( \bar{z}(x, t; M) \) is the self-similar solution such that

\[
|w_0(x)|_\infty \leq \bar{z}(\cdot, 1; M) \quad \text{on } \mathbb{R}.
\]

Proof. Let \( K = |w|_{w, R} + 1 \). Then we can see that there exists a sequence of smooth functions \( w_0 \) satisfying the following properties:

(i) \( \frac{1}{n} \leq w_0(x) \leq K \) for \( x \in (-n, n) \),

(ii) \( w_0(\pm n) = K \),

(iii) \( w_0 \) is strictly monotonically decreasing with respect to \( n \) and uniformly converges to \( w_0 \) in any finite intervals as \( n \to \infty \).

We shall consider the following boundary value problem of the form

\[
(3.4) \quad w_t = (w^m)_x -PWM \quad \text{in } Q_n \equiv (-n, n) \times (0, n),
\]

\[
(3.5) \quad w(\pm n, t) = K \quad \text{on } [0, n),
\]

\[
(3.6) \quad w(\cdot, 0) = w_0, \quad \text{on } [-n, n].
\]

Due to Theorem 4.4 in [14], we see that the problem (3.4)–(3.6) has a unique classical solution \( w_n \in C(\bar{Q}_n) \cap H^{s+a+1,1+a/2}_2(Q_n) \) \((0 < a < 1)\) satisfying

\[
(3.7) \quad 0 < w_n(x, t) \leq K \quad \text{for } (x, t) \in \bar{Q}_n.
\]

By the comparison theorem and (3.7), it follows that the sequence of the solutions \( w_n \) is monotonically decreasing with respect to \( n \). Therefore, for \( (x, t) \in \mathbb{R} \times [0, \infty) \), there exists \( \lim_{n \to \infty} w_n(x, t) \). Denote \( w \) the limit. For \( t_0, t_1, x_0, x_1 \) with \( 0 \leq t_0 < t_1 \), \( x_0 < x_1 \) and for \( f \in C^{2,1}([x_0, x_1] \times [t_0, t_1]) \) with \( f(x_0, t) = f(x_1, t) = 0 \), \( w_n \) satisfy the integral identity

\[
I(w_n, f, [x_0, x_1] \times [t_0, t_1]) = 0,
\]

and hence, \( w \) satisfies the integral identity

\[
(3.8) \quad I(w, f, [x_0, x_1] \times [t_0, t_1]) = 0.
\]

By a similar argument to the proofs of Theorem 6 and Theorem 8 in [12], we can prove that \( w \) belongs to \( C(\mathbb{R} \times [0, \infty)) \). By a similar argument to the proof of Theorem 3 in [12], \( w \) satisfies moreover,

\[
0 \leq w(x, t) \leq \bar{z}(x, t+1; M) \quad \text{and } 0 \leq w(x, t) \leq |w_0|_{w, R} \quad \text{for } (x, t) \in \mathbb{R} \times [0, \infty),
\]

where \( \bar{z}(x, t; M) \) is the self-similar solution such that

\[
|w_0(x)|_\infty \leq \bar{z}(\cdot, 1; M) \quad \text{on } \mathbb{R}.
where $\tilde{z}$ is the self-similar solution such that
\[ w_0(x) \leq \tilde{z}(x, 1; M) \quad \text{for} \quad x \in \mathbb{R}. \]

Therefore, $w$ is a required generalized solution.

To prove the uniqueness we let $\bar{w}$ be another generalized solution of (3.1) and (3.2). Set (cf. Theorem 2 in [13])
\[ A_n = A_n(x, t) = \int_0^1 m \{ \theta w_n + (1 - \theta) \bar{w} \}^{m-1} d\theta \]
and
\[ C_n = C_n(x, t) = \int_0^1 qP \{ \theta w_n + (1 - \theta) \bar{w} \}^{q-1} d\theta. \]

Let $T \in (0, n)$ and let $r \in (0, n)$ be a point where
\[ I(\bar{w}, f, [-r, r] \times [0, T]) = 0 \]
holds for $f \in C^2([0, T])$ with
\[ f(\pm r, t) = 0 \quad \text{for} \quad t \in [0, T]. \]

Then $w_n$ and $\bar{w}$ satisfy
\[
\begin{aligned}
\left[ \int_{-r}^r \{ w_n(x, t) - \bar{w}(x, t) \} f(x, t) \, dx \right]_0^T & \\
= -\left[ \int_{-r}^r \{ (w_n(x, t))^m - (\bar{w}(x, t))^m \} f_n(x, t) \, dx \right]_0^T & \\
+ \int_0^T \int_{-r}^r \{ A_n(x, t) f_{xx} + f_x - C_n(x, t) f \} \{ w_n - \bar{w} \} \, dx \, dt.
\end{aligned}
\]

By (3.7), there exist two sequences of smooth positive functions $A_{nk}(x, t)$ and $C_{nk}(x, t)$ with the following properties:

\[
\lim_{k \to \infty} A_{nk}(x, t) = A_n(x, t) \quad \text{a.e. in} \quad [-r, r] \times [0, T],
\]
\[
\frac{1}{2} \delta_n \leq A_{nk}(x, t) \leq K_{1n}
\]
for $k \geq 1$ and a.e. in $[-r, r] \times [0, T]$,
\[
\lim_{k \to \infty} C_{nk}(x, t) = C_n(x, t) \quad \text{a.e. in} \quad [-r, r] \times [0, T],
\]
and
\[
0 \leq C_{nk}(x, t) \leq K_{2n}
\]
for $k \geq 1$ and a.e. in $[-r, r] \times [0, T]$,

where
\[
\delta_n = \{ \min_{x, t \in \mathbb{R}} \{ |w_n(x, t)| \} \}^{m-1},
\]
\[
K_{1n} = 2m \left[ \max_{x, t \in \mathbb{R}} \{ \min \{ |w_n|, |\bar{w}| \} \} \right]^{m-1}.
\]
Then the first boundary value problem
\begin{align*}
\frac{\partial f}{\partial t} + A_{nkr} f_{xx} - C_{nkr} f &= 0 \quad \text{in } [-r, r] \times [0, T], \\
f(r^+, T) &= f_0(r^+) \quad \text{on } [-r, r], \\
f(\pm r, t) &= 0 \quad \text{on } [0, T]
\end{align*}

has a unique classical solution \(f=f_{nkr}\), here \(f_0\) is an arbitrary smooth function such that
\[
\text{supp } f_0 \subset (-r, r) \quad \text{and} \quad |f_0|_{L^\infty} \leq 1.
\]
Substituting the function \(f=f_{nkr}\) into (3.9), we observe that
\[
\left\{ \frac{u}{\Lambda} (x, t) - w(x, t) \right\} f_0(x, t) dx + \int_{-r}^{r} A_{n} - A_{nkr} \right) (w_n - \bar{w}) f_{xx} dx dt
\]
Taking the limit as \(h \to \infty, n \to \infty\) and \(r \to \infty\) in this order we get by Lemma 3.6 in [12],
\[
\int_{-r}^{r} \{w(x, T) - \bar{w}(x, T)\} f_0(x) dx = 0,
\]
which implies \(w(\cdot, T) = \bar{w}(\cdot, T)\) a.e. on \(R\).
Since \(T\) is arbitrary, we conclude \(w = \bar{w}\) a.e. in \(R \times [0, \infty)\). Q.E.D.

**Lemma 3.4.** Let \(w_i(\cdot, t)\) \((i=1, 2)\) be functions on \(R\) with compact support and let \(w_i(\cdot, t)\) \((i=1, 2)\) be generalized solutions for
\[
\frac{\partial w_i}{\partial t} = (w_i^n)_{xx} - P_i \mathbf{w}^d \quad \text{in } R \times [0, \infty)
\]
with \(w_i(\cdot, 0) = w_{i,0}\) where \(P_i(i=1, 2)\) are functions on \(R \times [0, \infty)\). Then, we have
\[
|w_i(\cdot, t) - w_2(\cdot, t)|_{1, R} \leq |w_{i,0} - w_{2,0}|_{1, R} + \int_{0}^{t} |P_i(\cdot, s) w_i(\cdot, s) - P_2(\cdot, s) w_2(\cdot, s)|_{1, R} ds 
\]
for \(t \in [0, \infty)\).

The proof is given in a quite similar way as in the proof of uniqueness part in Lemma 3.3 and omitted.
Combining Lemma 3.3 with Lemma 3.4 and using approximation procedure we can prove the following.

**Lemma 3.5.** Let \( P \) be a function on \( \mathbb{R} \times [0, \infty) \) and let \( w_0 \) be a compactly supported function on \( \mathbb{R} \). Then, there exists a unique generalized solution \( w \) of (3.1) and (3.2) and \( w \) satisfies
\[
0 \leq w(x, t) \leq \tilde{z}(x, t+1; M), \quad |w_0|_{\infty, \mathbb{R}},
\]
for \( t \geq 0 \) and a.e. \( x \in \mathbb{R} \),

where \( \tilde{z} \) is the self-similar solution such that
\[
0 \leq \tilde{w}_0(\cdot) \leq \tilde{z}(\cdot, 1; M) \text{ a.e. on } \mathbb{R}.
\]

4. Comparison theorems

In this section, we shall define the generalized supersolutions and the generalized subsolutions and give some comparison results.

By the same argument as in the proof of Lemma 4.1, 4.2 in [16], we can show the next lemma, the proof being omitted.

**Lemma 4.1.** Let \( u_0, v_0, \bar{u}_0 \) and \( \bar{v}_0 \) belong to \( L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and satisfy
\[
0 < u_0 < \bar{u}_0 < \bar{v}_0 < v_0 \text{ a.e. on } \mathbb{R}.
\]
Let \((u, v)\) and \((\bar{u}, \bar{v})\) be two pairs of solutions of (1.1) with initial data \((v_0, v_0)\) and \((\bar{u}_0, \bar{v}_0)\), respectively. Then, the functions satisfy, \( t \geq 0 \),
\[
0 \leq u(\cdot, t) \leq \bar{u}(\cdot, t) \leq \bar{v}(\cdot, t) \leq v(\cdot, t) \text{ a.e. on } \mathbb{R}.
\]

**Definition 4.2.** Let \( G \) be a connected open subset of \( \mathbb{R} \times (0, \infty) \). A function \( w \) belonging to \( C([0, \infty); L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times [0, \infty)) \) is called a generalized super (sub) solution of (3.1) in \( G \), if for \( t_0, t_1 \), a.e. \( x_0 \) and a.e. \( x_1 \) such that \( 0 \leq t_0 < t_1 \), \( x_0 < x_1 \) and \([x_0, x_1] \times [t_0, t_1] \subset \bar{G} \), the following integral inequality holds (see (3.3)):
\[
I(w, f, E) \leq 0
\]
for \( f \in C^2(G) \) with \( f(x_0, t) = f(x_1, t) = 0, t_0 \leq t \leq t_1 \), where we recall \( E = [x_0, x_1] \times [t_0, t_1] \).

**Lemma 4.3.** Let \( P \) and \( w_0 \) be functions in \( L^\infty(\mathbb{R} \times [0, \infty)) \) and \( C_0(\mathbb{R}) \), respectively. Let \( w \) be a generalized solution of (3.1) with \( w(\cdot, 0) = w_0 \) and let \( \bar{w} \) be a generalized super (sub) solution of (3.1). Then, if
\[
w_0 \leq \bar{w}(\cdot, 0) \text{ a.e. on } \mathbb{R},
\]
\[
(\geq \bar{w}(\cdot, 0))
\]
we have, for $t \geq 0$,

$$w(\cdot, t) \leq \varnothing(\cdot, t) \quad \text{a.e. on } \mathbb{R}.$$  

($\geq \varnothing(\cdot, t)$)

Proof. By a quite similar argument obtaining uniqueness part in Lemma 3.3 we can prove

$$\int_{\mathbb{R}} (w(x, t) - \varnothing(x, t)) f_0(x) \, dx \leq 0$$

for any function $f_0(x)$, which yields the desired result. The details are omitted. Q.E.D.

For $T > 0$, let $l$ be a smooth function in $[T, \infty)$ such that

$$l(T) > 0 \quad \text{and} \quad l'(\cdot) > 0 \quad \text{in } [T, \infty).$$

and let

$$G = \{(x, t): t > T \quad \text{and} \quad x \in (-l(t), l(t))\}.$$  

$S_i$ and $S_{-i}$ denote the subsets $\{(l(t), t); t \in [T, \infty)\}$ and $\{(-l(t), t); t \in [T, \infty)\}$ of $\mathbb{R} \times [0, \infty)$ respectively.

**Lemma 4.4.** Let $w$ be a generalized solution of (3.1) with $w(\cdot, 0) = w_0 \in C_0(\mathbb{R})$ and let $\varnothing$ be a generalized super (sub) solution of (3.1) in $G$ that belonging to $C(\mathcal{G})$.

$P$ in (3.1) satisfies that

$$(S_i \cup S_{-i}) \cap \text{supp } P = \emptyset,$$

Then $w$ is Hölder continuous in some neighborhood of $S_i \cup S_{-i}$. Moreover, if

$$w(\cdot, T) \leq \varnothing(\cdot, T) \quad \text{a.e. on } [-l(T), l(T)]$$  

($\geq \varnothing(\cdot, T)$)

and if

$$w(l(t), t) < \varnothing(l(t), t) \quad \text{for } t \geq T,$$

($> \varnothing(l(t), t)$)

$$w(-l(t), t) < \varnothing(-l(t), t) \quad \text{for } t \geq T,$$

($> \varnothing(-l(t), t)$)

then we have

$$w(x, t) \leq \varnothing(x, t) \quad \text{for } t \geq T \quad \text{and a.e. } x \in [-l(t), l(t)].$$  

($\geq \varnothing(x, t)$)
Proof. Let an arbitrary constant $T_1 > T$ be fixed. There exists a constant $\delta = \delta(T, T_1, P) \in (0, T)$ such that

$$(E_{\delta}^+(t) \cup E_{\delta}^-(t)) \cap \text{supp } P = \phi \quad \text{for} \quad t \in [T, T_1],$$

where, for $\delta > 0$ and $t \geq T$, $E_{\delta}^+(t)$ and $E_{\delta}^-(t)$ denote $[l(t) - \delta, l(t) + \delta] \times [t - \delta, t + \delta]$ and $[-l(t) - \delta, -l(t) + \delta] \times [t - \delta, t + \delta]$ respectively.

Then, there exists a sequence of smooth functions $P_j \in L^\infty(\mathbb{R} \times [0, \infty))$ such that

$$|P_j|_{L^\infty(\mathbb{R} \times [0, \infty))} \leq |P|_{L^\infty(\mathbb{R} \times [0, \infty))} \quad \text{for} \quad j \geq 1,$$

$$\lim_{j \to \infty} P_j(x, t) = P(x, t) \quad \text{for a.e. } (x, t) \in \mathbb{R} \times [0, \infty)$$

and

$$(E_{5/6}^+(t) \cup E_{5/6}^-(t)) \cap \text{supp } P = \phi$$

for $j \geq 1$ and $t \in [T, T_1]$.

Let $w_{j\alpha}$ satisfy the properties (i)–(iii) in the proof of Lemma 3.3. For $j \geq 1$, let $w_{j\alpha}$ be the classical solutions of

$$w_t = (w^m)_{xx} - P_j w^\xi \quad \text{in } Q_n$$

with (3.5) and (3.6), where $Q_n$ denotes $(-n, n) \times (0, n)$.

Then, since $w_{j\alpha}$ are positive in $\overline{Q}_n$, $w_{j\alpha}$ are smooth in $Q_n$.

Now, we shall show the uniformly Holder continuity of the solutions $w_{j\alpha}$ in $E^i(t)$ and $E^{\xi}(t)$.

We shall omit $(\pm)$ and $(-)$ from $E^i((\pm)(\cdot))$ and $E^{\xi}((\cdot))$, respectively.

We fix $t_0 \in [T, T_1]$ and $(x_1, t_1) \in E_{25/2}(t_0)$ arbitrarily.

Let $\psi_0$ be a smooth function such that

$$\psi_0(x) \begin{cases} 
1 & \text{on } [-1, 1] \\
0 & \text{on } (-2, -1) \cup (1, 2) \\
0 & \text{on } (-\infty, -2) \cup (2, \infty), 
\end{cases}$$

and we set

$$\psi_{\delta}(x, t) = \frac{t - (t_0 - (5/6) \delta)}{t_1 - (t_0 - (5/6) \delta)} \psi_0 \left( \frac{x - x_1}{\delta/24} \right)$$

in $\mathbb{R} \times \mathbb{R}$.

We set also

$$\phi(y) = Ny(2 - y) \quad \text{for } y \geq 0,$$

with $N = (4m/(m - 1))(|w_0|_{L^\infty} + 1)^{m-1}$. 
Let an arbitrary \( j \geq 1 \) and an arbitrary \( n \) satisfying \( E_{38/6}(t_0) \subset Q_* \) be fixed. We shall omit \( j \) and \( n \) from \( w_{j, n} \) and \( P_j \). Setting \( (m/(m-1)) w^{m-1} = \phi(q) \), we see \( 0 \leq q \leq 1/4 \) and

\[
q_t = (m-1) \phi q_{xx} + \left[ (m-1) \phi \frac{\phi''}{\phi'} + \phi' \right] (q_x)^2 - \lambda \frac{P \phi^2}{\phi'},
\]

in \((-n, n) \times [t_0 - \frac{5}{6} \delta, \infty)\)

with \( \beta = (m+n-2)/(m-1) \) and \( \lambda = m((m-1)/m)^\theta \).

We differentiate (4.2) with respect to \( x \), multiply by \( q_x \psi_T^2 \) and consider a point \((x_2, t_2)\) of \( E_{18} \) where the function \( z = (q_x \psi_T^2)^2 \) attains a maximum in \([-n, n] \times [t_0 - (5\delta/6), \infty)\). Since we may assume \( t_2 > t_0 - (5\delta/6) \) without loss of generality, we observe that

\[
z_t(x_2, t_2) \geq 0, \quad z_x(x_2, t_2) = 0 \quad \text{and} \quad z_{xx}(x_2, t_2) \leq 0.
\]

Then, at such a point we have the following inequality:

\[
\begin{align*}
&\psi_T^2 (q_x)^4 \\
\leq &\left[ \psi_T^2 \psi_{tt} + 3(m-1) \phi (\psi_{tx})^2 - (m-1) \phi \psi_T \psi_{txx} \right] (q_x)^2 \\
&- \left[ (m+1) \phi' + 2(m-1) \phi \frac{\phi''}{\phi'} \right] \psi_T \psi_{tx}(q_x)^3.
\end{align*}
\]

Set

\[
a_1 = |\psi_T|_{E_{18}/6}(t_0), \quad a_2 = |\psi_T|_{E_{18}/6}(t_0) \quad \text{and} \quad a_3 = |\psi_T|_{E_{18}/6}(t_0).
\]

Note that

\[
0 \leq q \leq 1/4 \quad \text{in} \quad [-n, n] \times [t_0 - \frac{5}{6} \delta, \infty)
\]

and

\[
0 \leq \frac{3}{2} N \leq \phi'(q) \leq 2N, \quad \phi''(q) = -2N
\]

and \( |\phi''/\phi'| \leq \frac{4}{3} \) in \([-n, n] \times [t_0 - \frac{5}{6} \delta, \infty)\).

By (4.3) and (4.4) we obtain

\[
\psi_T^2 (q_x)^4 \leq C_1 (q_x)^2 + C_2 \psi_T |q_x|^3,
\]

with

\[
C_1 = \frac{1}{2N m} (a_1 + N(m-1) a_3 + 3N(m-1) a_2),
\]

\[
C_2 = \frac{a_2}{3N m} (7m-1).
\]
Therefore we have
\[ x(x, t) \leq |x|_{m, E(Q; \Omega)}(t_0) \leq 2 \left( C_1 + \frac{C^2_2}{2} \right) \] for \( (x, t) \in E(Q; \Omega)(t_0) \),
and hence,
\[ \left| \frac{m}{m-1} (w_n^{m-1})(x, t) \right| \leq 8N^2 \left( C_1 + \frac{C^2_2}{2} \right) \] for \( (x, t) \in E(Q; \Omega)(t_0) \).

By (4.5) and Theorem 8 in [13], for \( j \geq 1 \) and \( n \), such that \( \cup_{t_0 \in \tau, \tau_1} E_{8b^0}(t_0) \subset Q_n \), the solutions \( w_{j_n} \) satisfy
\[ \langle w_{j_n} \rangle_{k_n, E_{8b^0}(t_0)} + \langle w_{j_n} \rangle_{k_n, E_{8b^0}(t_0)} \leq C_8, \quad \text{for } t_0 \in [T, T_1], \]
where \( \alpha = \min(1, 1/(m-1)) \) and \( C_8 \) is a positive constant depending only on \( |w_{0}|_{m, \Omega}, |P|_{m, \Omega \times (0, \infty)} \) and \( \delta \). Set \( E^{(+)} = \cup_{t_0 \in \tau, \tau_1} E_{b^0}(t_0) \) and \( E^{(-)} = \cup_{t_0 \in \tau, \tau_1} E_{b^0}(t_0) \). By (4.6) and Ascoli-Arzelà theorem, for each \( j \geq 1 \), a subsequence of the solutions \( w_{j_n} \) uniformly converges to \( w_j \) on \( \Omega^+ \cup \Omega^- \) as \( n \to \infty \). Moreover, we obtain
\[ \langle w_j \rangle_{k, \Omega^+} + \langle w_j \rangle_{k, \Omega^-} \leq C_8 \] for \( i = +, - \).

By Lemma 3.3, Lemma 3.5, Lebegue’s convergence theorem and Gronwall’s inequality, we have
\[ \lim_{j \to \infty} \sup_{t \in [0, \tau], \tau_1} |w_j(\cdot, t) - w(\cdot, t)|_{1, R} = 0. \]

By (4.7), (4.8) and Ascoli-Arzelà theorem, there exists a subsequence of the solutions \( w_j \) which uniformly converges to \( w \) on \( E^{(+)} \cup E^{(-)} \).

Therefore, the generalized solution \( w \) is Hőlder continuous in \( \Omega^+ \cup \Omega^- \).

Let \( \bar{w} \in C(G) \) be a generalized supersolution in \( G \).

There exists a positive constant \( \eta \) which has the following property; For any sufficiently large \( n \) and \( j \), \( w_{j_n} \) satisfy that
\[ w_{j_n}(x, t) < \bar{w}(x, t) \]
for \( t \in [T, T_1] \) and \( |x| \in [l(t) - \eta, l(t)] \).

For any integer \( H \geq 1 \) and \( h = 0, 1, 2, \ldots, H - 1 \), we set \( t^{(H)}_h = T + (T_1 - T) h/H \) and \( G^{(H)}_h = \left[ -l(t^{(H)}_h), l(t^{(H)}_h) \right] \times \left[ t^{(H)}_h, t^{(H)}_{h+1} \right] \).

We fix a large \( H \) such that
\[ 0 \leq l(t^{(H)}_h) - l(t^{(H)}_{h+1}) \leq \eta \quad \text{for } h = 0, 1, \ldots, H - 1. \]

Repeating the same argument obtaining uniqueness part in Lemma 3.3 we can prove
where $f \in C^{2,1}(G_0)$ be an arbitrary function such that $f(\pm l(t_0), t) = 0$ for $t \in [t_0, t_1]$ and consequently

$$\int_{-l(t_0)}^{l(t_0)} (w(x, t_1) - \bar{w}(x, t_1))_+ \, dx \leq 0. \tag{4.11}$$

By (4.9) and (4.11) we have

$$w(\cdot, t_1) \leq \bar{w}(\cdot, t_1) \quad \text{a.e. on } [-l(t_1), l(t_1)]. \tag{4.12}$$

From (4.12), the same argument yields

$$w(\cdot, t_2) \leq \bar{w}(\cdot, t_2) \quad \text{a.e. on } [-l(t_2), l(t_2)].$$

Repeating this procedure we arrive at

$$w(\cdot, T_1) \leq \bar{w}(\cdot, T_1) \quad \text{a.e. on } [-l(T_1), l(T_1)].$$

Since $T_1 > T$ is arbitrary, we conclude

$$w(\cdot, t) \leq \bar{w}(\cdot, t) \quad \text{for } t \geq T \text{ and a.e. on } [-l(t), l(t)].$$

Q.E.D.

**Lemma 4.5.** Let $w_0$ in (3.2) belong to $C_0(R)$.

Let $w$ be a generalized solution of (3.1) and (3.2). And let $\bar{w}$ be a generalized supersolution of (3.1) in $G$ and be continuous and positive in $\bar{G}$. Suppose that:

$$w(\cdot, T) \leq \bar{w}(\cdot, T) \quad \text{a.e. on } [-l(T), l(T)],$$

$$\text{supp } w \cap (S_1 \cup S_{-1}) = \phi.$$

Then, $w$ and satisfy

$$w(x, t) \leq \bar{w}(x, t) \text{ for } t \geq T \text{ and a.e. } x \in [-l(t), l(t)].$$

The proof is given in a quite similar way as in the one of Lemma 4.4 and omitted.

Finally, we state for following.

**Lemma 4.6.** Let $l(t) \equiv l_0$ on $[T, \infty)$.

Let $P$ in (3.1) and $w_0$ in (3.2) belong to $L^\infty(R \times [0, \infty)) \cap C^{2,1}(R \times [0, \infty))$ and $C_0(R)$ respectively.

Let $w$ be the generalized solution of (3.1) and (3.2). And let $\bar{w}$ be a generalized subsolution of (3.1) in $G$ and be continuous in $\bar{G}$.

Suppose that:

$$w(\cdot, T) \geq \bar{w}(\cdot, T) \quad \text{on } [-l_0, l_0]$$

$$w(l_0, t) \geq \bar{w}(l_0, t) \text{ and } w(-l_0, t) \geq \bar{w}(-l_0, t) \text{ for } t \geq T.$$
Then, \( w \) and \( \bar{w} \) satisfy

\[
w \geq \bar{w} \quad \text{in} \quad \mathcal{G}.
\]

The proof is standard and omitted.

5. **The large time behavior of the solutions for**

\[
w_t = (w^m)_{xx} - \lambda (t+1)^{-n/(m+1)} w^n
\]

In this section, we consider the large time behavior of solutions for the following equation:

\[
(5.1) \quad w_t = (w^m)_{xx} - \lambda (1+t)^{-n/(m+1)} w^n \quad \text{in} \quad \mathbb{R} \times (0, \infty)
\]

with initial condition

\[
(5.2) \quad w(\cdot, 0) = w_0 \quad \text{on} \quad \mathbb{R}.
\]

where \( \lambda > 0 \) and \( w_0 \) satisfies (A.I.) in Introduction.

In order to investigate the large time behavior of the generalized solution for (5.1), we shall derive an estimate of \( |(w^m)_{x}(\cdot, t)|_{\infty, \mathbb{R}} \). The following is proved similarly as in the proof of Lemma 3.1 in [8].

**Lemma 5.1.** Let \( w \) be the generalized solution of (5.1)-(5.2). Then we have

\[
|(w^m)_{x}(\cdot, t)|_{\infty, \mathbb{R}} \leq C(t^{-1} |w(\cdot, t/2)|_{\infty, \mathbb{R}})^{1/2} \quad \text{for} \quad t > 0,
\]

where \( C \) is a positive constant independent of \( t \) and \( w_0 \).

Our main result in this section is as follows.

**Lemma 5.2.** Let \( p^* = nm/(m+1-n) \). Let \( w_0 \) satisfy the assumptions (A.I.) and \( w \) be the generalized solution of (5.1) and (5.2). Then, the support and \( L^\infty \)-norm of \( w \) have the following properties:

\[
\begin{align*}
\cup_{t \geq 0} \text{supp } w(\cdot, t) \text{ is bounded in } \mathbb{R}, & \quad \text{if } 2n-1 < m. \\
- \inf \{ \text{supp } w(\cdot, t) \}, \sup \{ \text{supp } w(\cdot, t) \} \sim \log t & \quad \text{if } m = 2n-1. \\
- \inf \{ \text{supp } w(\cdot, t) \}, \sup \{ \text{supp } w(\cdot, t) \} \sim t^{(p^*-m)/(2p^*-2)} & \quad \text{if } 2n-2 < m < 2n-1. \\
- \inf \{ \text{supp } w(\cdot, t) \}, \sup \{ \text{supp } w(\cdot, t) \} \sim t^{1/(m+1)} & \quad \text{if } m < 2n-2. \\
\log(|w(\cdot, t)|_{\infty, \mathbb{R}}) \sim -t^{m/(m+1)} & \quad \text{if } n = 1. \\
|w(\cdot, t)|_{\infty, \mathbb{R}} \sim t^{-1/(p^*-1)} & \quad \text{if } 2n-2 < m \text{ and } n > 1. \\
|w(\cdot, t)|_{\infty, \mathbb{R}} \sim t^{-1/(m+1)} & \quad \text{if } 1 < m < 2n-2.
\end{align*}
\]
Proof. For simplicity we assume that \( \omega_0(0) > 0 \).

(I) Case: \( n = 1 \).

Let \( \mathcal{W} \) be a generalized solution of (2.2) with initial condition \( \mathcal{W}(\cdot, 0) = \omega_0 \).

We set for \( t \geq 0 \)

\[
\rho(t) = \exp \left( -\lambda \int_0^t (1+s)^{-\nu/(m+1)} \, ds \right)
\]

and

\[
\nu(t) = \int_0^t \rho(s)^{m-1} \, ds.
\]

Then, we can observe

\[
w(x, t) = \rho(t) \mathcal{W}(x, \nu(t)) \quad \text{for} \quad (x, t) \in \mathbb{R} \times (0, \infty),
\]

and the result follows from [17].

(II) Case: \( 2n-1 < m \) and \( n > 1 \).

Let \( \mathcal{W}^* \) be a generalized solution of (1.4) with \( \rho = \rho^* \) and with \( \mathcal{W}^*(\cdot, 0) = \omega_0 \).

In Introduction we describe the estimate of \( \text{supp } \mathcal{W}^*(\cdot, t) \) and \( |\mathcal{W}^*(\cdot, t)|_{\infty, \mathbb{R}} \), which is the required one also for \( \mathcal{W}(t) \). Suppose that \( \lambda \) is so large to satisfy:

\[
(5.3) \quad \mathcal{W}^*(x, t)^{(m-1)/(m+1-n)} \leq \lambda^{1/(n+1)} (1+t)^{1/(m+1-n)} \quad \text{for} \quad (x, t) \in \mathbb{R} \times (0, \infty).
\]

For such a constant \( \lambda \), since \( \mathcal{W} \) is a generalized subsolution of (3.1) with \( P = \mathcal{W}^*(m-1)/(m+1-n) \) and \( q = n \) we obtain by Lemma 3.3, Lemma 4.3 and (5.3) that \( \mathcal{W}^* \geq \mathcal{W} \) in \( \mathbb{R} \times [0, \infty) \). Let \( a \) and \( b \) be positive constants satisfying \( a^{n-1} b^2 = 1 \) and set \( \mathcal{W}(x, t) = a \mathcal{W}(bx, t) \) for \( (x, t) \in \mathbb{R} \times (0, \infty) \). Then \( \mathcal{W} \) is a generalized solution of the following equation:

\[
w_t = (\mathcal{W}^m)_{xx} - \mu (\mathcal{W}^n) \quad \text{in} \quad \mathbb{R} \times (0, \infty).
\]

Therefore, we obtain the upper estimates of \( \text{supp } \mathcal{W}(\cdot, t) \) and \( |\mathcal{W}(\cdot, t)|_{\infty, \mathbb{R}} \).

Let \( \tilde{h}_a \in (0, 1) \) and \( \tilde{h} \) be the solution of the following Cauchy problem:

\[
\begin{align*}
(\tilde{h})'' &= \mu (\tilde{h}^n - \tilde{h}) \quad \text{on} \quad (0, \infty) \\
\tilde{h}(0) &= \tilde{h}_a, \tilde{h}'(0) &= 0, \text{ where } \mu = \lambda^{(n-2)/(n-1)} \left( \frac{m+1}{(n-1)(m+1-n)} \right)^{1/(n-1)}
\end{align*}
\]

It is easy to observe that \( \tilde{h} \) has a zero point, and let \( \delta \) be the first nontrivial and nonnegative solution \( h \) for

\[
\begin{align*}
\mu (h^n - h) &= (h^m)_{xx} \quad \text{on} \quad (-\delta, \delta), \\
h(\delta) &= h(-\delta) = 0
\end{align*}
\]

such that

\[
(5.4) \quad \omega_0 \geq h \quad \text{on} \quad (-\delta, \delta).
\]
Let $\tau$ be the solution of the following Cauchy problem:

\begin{equation}
\begin{cases}
\tau'(t) = -\lambda(1+t)^{-n/(m+1)} \tau^m & \text{on } (0, \infty), \\
\tau(0) = \tau_0,
\end{cases}
\end{equation}

where $\tau_0 = \min\{1, ((m+1)/\lambda(n-1)(m-n+1))^{-1/(n-1)}\}$.

Then, we observe that

\begin{equation}
(\tau h)_{s}\leq ((\tau h)^{m})_{s}\lambda(1+t)^{-n/(m+1)} (\tau h)^{m} \text{ in } (-\delta, \delta) \times (0, \infty).
\end{equation}

By (5.4), (5.6) and Lemma 4.6, we get that $h_\tau \leq w$ in $(-\delta, \delta) \times (0, \infty)$. Moreover, the decay rate of $\tau$ in $t$ is equal to the one which we want to show. Therefore, we have a lower estimate of $|w(\cdot, t)|_{w, R}$.

(III) Case: $m = 2n - 1$.

Let $c > 0$ and set $\bar{\lambda} = \lambda(m-1)^{(m+2)/(m-1)}$. We consider the Cauchy Problem:

\begin{equation}
\begin{cases}
(q^{\bar{\lambda}})' + cq' + q = 5q^\bar{\lambda} & \text{on } [0, \eta) \\
q(0) = \bar{\lambda}^{-1/(n-1)}, 0 < q < \bar{\lambda}^{-1/(n-1)}, q' < 0 \text{ on } [0, \eta),
\end{cases}
\end{equation}

where $\eta$ is a positive constant. This problem has a solution for some $\eta > 0$ and the behavior of it is known (See [1]). Using this we can construct desired generalized supersolutions and subsolutions (See [5]).

(IV) Case: $2n - 2 < m < 2n - 1$.

Set

\begin{equation}
w_*(x, t) = A(1+t)^{-a}(D-x^2(1+t)^{-t})^{1/(m-1)}
\end{equation}

for $(x, t) \in R \times (0, \infty)$

where $a = 1/(p_\star - 1)$, $b = (p_\star - m)/(p_\star - 1)$ and $A$ and $D$ are positive constants. Then we obtain that

\begin{equation}
L(w_*) = w_*(x, t)_{ss} + \lambda(1+t)^{-n/(m+1)} w_*(x, t)
\end{equation}

for $(x, t) \in R \times (0, \infty)$.

Since $a(m-1) + b - 1 = 0$, we have

\begin{equation}
L(w_*) \leq -A(m-1)^{-1} x^2(1+t)^{-a(m-1)} \psi_a^{-1}(m-1)
\end{equation}

for $(x, t) \in R \times (0, \infty)$.
where $\psi=\psi(x,t)=(D-x^2(1+t)^{-\delta})_+$. Since $2a>b$, there exists a positive constant $A$ such that $2a>4m(m-1)^{-1}A^{m-1}>b$. Let such an $A$ be fixed. Moreover, there exists a sufficiently small constant $D>0$ such that $a-2(m-1)^{-1}mA^{m-1}-\lambda A^{m-1}D^{(m-1)/(m-1)}>0$ and $w_0(x)\geq A(D-x^{2})^{1/(m-1)}$ on $R$ and we shall fix such a constant $D$. Then we have $L(w_\#)\leq 0$ in $R\times(0,\infty)$. Since $w_\#$, $(w_\#)_x \in L^1(R \times [0,T])$ for $T>0$, $w_\#$ is a generalized subsolution of (5.1). Therefore, we have by Lemma 3.3 and Lemma 4.3, that $w_\# \leq w$ in $R\times(0,\infty)$. Thus, we obtain the lower estimates of $w(\cdot,t)$ and $|w(\cdot,t)|_{\infty,R}$.

Let $w_\#$ be a solution of (5.5) with the initial condition $w(\cdot,0)=|w_0|_{\infty,R}$. By Lemma 3.2, there exists a smooth function $l$ in $[0,\infty)$ such that

$l(0)>0$, $l' \geq 0$ in $[0,\infty)$,

$\sup w(\cdot,t) \subset (-l(t), l(t))$ for $t \geq 0$.

Set $G=\{(x,t): t>0$ and $x \in (-l(t), l(t))\}$. Since $w_\#$ is a generalized supersolution of (5.1) in $G$, we find by Lemma 4.5 an upper estimate of $|w(\cdot,t)|_{\infty,R}$. On the other hand, we see by Lemma 5.1.

\[(5.10) |(w^{m-1})_x(\cdot,t)|_{\infty,R} \leq Ct^{-(1/2)(1+(m-1)/(p+\delta))}, \]

By a similar argument to one which is used for the porous medium equation, we can show that $\sup w(\cdot,t)$ is an interval $(\zeta_1(t), \zeta_2(t))$ for large $t$ and that

\[(5.11) \zeta_i'(t) = \frac{m}{m-1} (w^{m-1})_x(\zeta_i(t), t) \text{ for large } t, i = 1, 2. \]

By (5.10) and (5.11), we see

\[(5.12) |\zeta_i(t)| \leq Ct^{-(1/2)(1+(m-1)/(p+\delta))}, \]

where $C$ is a positive constant. Integrating (5.12) from 0 to $t$, we have the desired estimates of $|\zeta_1(t)|$ and $|\zeta_2(t)|$.

(V) Case: $m<2n-2$.

We set again

$w_\#(x,t) = A(1+t)^{-\delta}(D-x^2(1+t)^{-\delta})_+^{1/(m-1)}$ for $(x,t) \in R \times (0,\infty)$.

We put for $\varepsilon>0$

\[(5.13) a = 1/(m+1)+\varepsilon, b = 2/(m+1)-\varepsilon(m-1), A = \left(\frac{m-1}{2m(m+1)}\right)^{1/(m-1)}, \]

\[D = (\lambda^{-1} A^{1+n} \varepsilon)^{1/(m-1)}, \quad E = AD^{1/(m-1)}. \]

Then, for any sufficiently small $\varepsilon>0$, we get

\[(5.14) w_0(x) \geq EX(\varepsilon D_{-1/2}^{1/2})_2(x) \text{ for } x \in R. \]
where $\chi$ is a characteristic function. We fix such an $\epsilon > 0$. By (5.9), (5.13) and (5.14), we see that $w_\ast$ is a generalized subsolution of (5.1), and $w \geq w_\ast$ in $\mathbb{R} \times [0, \infty)$. Set $T_1 = (\epsilon \lambda^{-1} A^{1-n} 2^{-i})^{-1/4} - 1$, which $q = (2n - 2 - m)/(m + 1) > 0$. Then we see

$$w(x, T_1) \geq A(1 + T_1)^{-\sigma(1)} (D(1) - x^2(1 + T_1)^{-k(1)})^{1/(m-1)}$$
onumber

on $\mathbb{R}$, with $a(1) = 1/(m + 1) + \epsilon/2$, $b(1) = 2/(m + 1) - \epsilon(m - 1)/2$ and $D(1) = D(1 + T_1)^{-\epsilon(m-1)/2}$. Setting

$$w(x, t) = A(1 + t)^{-\sigma(j)} (D(j) - x^2(1 + t)^{-k(j)})^{1/(m-1)}$$

in $\mathbb{R} \times (0, \infty)$. We see

$$L(w_{(\ast);1}) \leq 0 \text{ in } \mathbb{R} \times [T_1, \infty)$$

and

$$w_{(\ast);1}, (w_{(\ast);1})_{x^2} \in L^1(\mathbb{R} \times [0, T]) \text{ for } T > 0.$$}

Thus, $w_{(\ast);1}(x, t)$ is a generalized subsolution of (5.1) in $\mathbb{R} \times [T_1, \infty)$, and we have by Lemma 4.3 $w(\ast;1) \leq w$ in $\mathbb{R} \times [T_1, \infty)$.

For any positive integer $j$, we put $a(j) = 1/(m + 1) + \epsilon 2^{-j}$, $b(j) = 2/(m + 1) - \epsilon(m - 1) 2^{-j}$, $T_j = (\epsilon \lambda^{-1} A^{1-n} 2^{-j})^{-1/4} - 1$ and $D(j + 1) = D(j) (1 + T_j)^{-\epsilon(m-1)2^{-j-1}}$. Setting

$$w_{(\ast);1}(x, t) = A(1 + t)^{-\sigma(j)} (D(j) - x^2(1 + t)^{-k(j)})^{1/(m-1)}$$

in $\mathbb{R} \times (0, \infty)$. We get similarly $w \geq w_{(\ast);j}$ in $\mathbb{R} \times [T_j, \infty)$. Therefore, we get for $t \in [T_j, T_{j+1}]$, $j \geq 1$,

$$(1 + t)^{-1/(m+1)} \inf \{ \supp w(\cdot, t) \} \geq A D(j)^{1/2} (1 + T_{j+1})^{-\epsilon(m-1)2^{-j}}$$

and

$$(1 + t)^{-1/(m+1)} \sup \{ \supp w(\cdot, t) \} \geq A D(j)^{1/2} (1 + T_{j+1})^{-\epsilon(m-1)2^{-j}}$$

and

$$(1 + t)^{1/(m+1)} |w(\cdot, t)|_{w, R} \geq A D(j)^{1/(m-1)} (1 + T_{j+1})^{-2^{-j}}.$$}

Since $\lim_{j \to \infty} D(j) = D(\infty) > 0$ and $\lim_{j \to \infty} (1 + T_{j+1})^{-2^{-j}} = 1$, we obtain the lower estimates of $\supp w(\cdot, t)$ and $|w(\cdot, t)|_{w, R}$.

Let $w_\ast$ be a generalized solution of (2.1) with initial condition $w_\ast(\cdot, 0) = w_0$. Then we can show that $w_\ast$ is a generalized supersolution of (5.1). Therefore, we obtain the upper estimates of $\supp w(\cdot, t)$ and $|w(\cdot, t)|_{w, R}$. Q.E.D.

6. Regularity and semiconvexity of the solution for (1.4) and (1.5) in case of $\inf_{x \in \mathbb{R}} w_0(x) > 0$

In this section, we let the function $w_0$ belong to $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and satisfy
(6.1) \[ \inf_{x \in \mathbb{R}} w_0(x) > 0. \]

We shall consider the regularity and the semiconvexity of the solution for (1.4) and (1.5) under this assumption.

By Theorem 0 in [8], we know the following proposition.

**Proposition 6.1.** Let \( w_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) satisfy (6.1). Then, there exists a unique classical solution of (1.4) and (1.5), and the solution is smooth in \( \mathbb{R} \times (0, \infty) \).

Now we shall show main results of this section.

**Lemma 6.2.** Let \( w_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) satisfy (6.1) and let \( w \) be the classical solution of (1.4) and (1.5). Then, \( w \) satisfies

\[
\chi(t) \leq D_{n,m} (t+D_2)^{-(m-1)/(2n-1)-1} \times \{D_2^{(m-1)/(2n-1)}-(t+D_2)^{-(m-1)/(2n-1)}\}^{-1/2}
\]

for \((x, t) \in \mathbb{R} \times (0, \infty)\), where \( D_{n,m} \) is a positive constant depending only on \( m \) and \( n \).

**Proof.** Set \( W=(m/(m-1)) w^{m-1} \). Then, \( W \) satisfies the equation:

\[
W_t = (m-1) WW_{xx} + |W_x|^2 - \lambda W^\beta \quad \text{in} \quad \mathbb{R} \times (0, \infty),
\]

with \( \beta=(2n+m-2)/(m-1) \) and \( \lambda=m^{\beta-1}(m-1)^{\beta} \).

By the comparison theorem, we have

\[
\mu(t+C^*)^{-(m-1)/(2n-1)} \geq W(x, t) \geq \mu(t+C^*)^{-(m-1)/(2n-1)} \quad \text{in} \quad \mathbb{R} \times [0, \infty),
\]

with \( \mu=\{(2n-1)/(m-1)\}^{-(m-1)/(2n-1)} \), \( C^*=\{(2n-1)^{-1} \text{inf}_{x \in \mathbb{R}} w_0(x) \}^{-2n+1} \) and \( C^*=\{(2n-1)^{-1} \text{inf}_{x \in \mathbb{R}} w_0(x) \}^{-2n+1} \).

Set

\[
\bar{Q}(x, t) = (t+2C^*)^\beta W(x, t) - \mu(t+2C^*)^{-(m-1)/(2n-1)} \quad \text{in} \quad \mathbb{R} \times [0, \infty),
\]

where \( \alpha=(2n+m-2)/(2n-1) \).

Then, \( \bar{Q} \) is positive in \( \mathbb{R} \times [0, \infty) \) and we obtain the following estimates:

\[
\bar{Q}(x, t) \leq \mu \frac{m-1}{2n-1} \left( \frac{2C^*}{C^*} \right)^\alpha (2C^* - C^*) \quad \text{in} \quad \mathbb{R} \times [0, \infty)
\]

and
(6.4) \[ \tilde{Q}(x, t) \geq \mu \frac{m-1}{2n-1} C_* \text{ in } \mathbb{R} \times [0, \infty). \]

Setting
\[ r(s) = ((2C_*)^{1-\sigma} - (\alpha - 1) s)^{-1/(\sigma-1)} \text{ on } [0, S_*] \]
with \( S_* = (2C_*)^{1-\sigma}(\alpha - 1)^{-1} \), we find
\[
\begin{cases}
  r' = r^* & \text{on } (0, S_*) \\
  r(0) = 2C_*.
\end{cases}
\]

Using \( r(s) \) we set
\[ Q(x, s) = \tilde{Q}(x, r(s) - 2C_*) \text{ in } \mathbb{R} \times (0, S_*). \]

Then
\[
(6.5) \quad Q_s = (m-1) (Q + \mu r) Q_{ss} + |Q_s|^2 + \lambda \mu^\beta r^\sigma \\
- \lambda \mu^\beta r^\sigma (1 + \mu r^{-1} Q)^\beta + \alpha r^\sigma Q \\
\text{in } \mathbb{R} \times (0, S_*).
\]

Set \( N = 8 \mu (m-1) (2n-1)^{-1} (2C_*/C^*)^{\sigma} (2C_*/C^*) \) and \( \phi(y) = Ny(1-y) \). By a quite similar argument obtaining (4.5), since we can prove
\[
(6.6) \quad |Q_s(x, s)| \leq \frac{N}{m^\beta} \text{ for } (x, s) \in \mathbb{R} \times (0, S_*).
\]

Therefore we omit the proof of (6.6).

Since
\[ s = \frac{1}{(\alpha - 1)} \{(2C_*)^{1-\sigma} - (t + 2C_*)^{1-\sigma}\}, \]

it follows from (6.6) that
\[
|W_s(x, t)| \leq D_{s, m} \left[ \left( \frac{C_*}{C^*} \right)^{\sigma} (2C_*/C^*) \right]^{1/2} \left\{ (2C_*/C^*)^{1-\sigma} - (2C_*/t)^{1-\sigma} \right\}^{1/2} \\
\times (t + 2C_*)^{-\sigma} \text{ in } \mathbb{R} \times (0, \infty),
\]
for a certain \( D_{s, m} > 0 \).

**Lemma 6.3.** Let \( m \geq 2n \). Let \( w_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) satisfy (6.1) and let \( w \) be the classical solution of (1.4) and (1.5). Then, the solution \( w \) satisfies the following inequality:
\[
(w^{m-1})_{ss}(x, t) \geq -K(t + D_2)^{-\sigma - 1/2} \\
\times \left\{ D_2^{-\sigma - 1/2} - (t + D_2)^{-\sigma - 1/2} \right\}^{-1} \text{ in } \mathbb{R} \times (0, \infty),
\]

Q.E.D.
with \( K = \max((m-1)/(m(m+1)), D_{n,m} D_1) \), where \( D_1 \) and \( D_2 \) are the constants in Lemma 6.2 and \( D_{n,m} \) is a positive constant depending only on \( n \) and \( m \).

Proof. We differentiate the equation in (6.5) twice with respect to \( x \) and set \( P = Q_{xx} \) to get

\[
(6.7) \quad P_x = (m-1) (Q + \mu r) P_{xx} + 2m Q_x P_x + (m+1) P^2 \\
- \alpha r^{a-1} ((1 + \mu^{-1} r^{-1} Q)^{a-1} - 1) P \\
- (\beta-1) \alpha r^{a-2} (1 + \mu^{-1} r^{-1} Q)^{a-2} (Q_x)^2
\]

in \( \mathcal{R} \times (0, S_*) \).

We shall consider the differential operator:

\[
(6.8) \quad \mathcal{L}(\theta) = \theta_{xx} - (m-1) (Q + \mu r) \theta_{xx} - 2m Q_x \theta_x - (m+1) \theta^2 \\
+ \alpha r^{a-1} ((1 + \mu^{-1} r^{-1} Q)^{a-1} - 1) \theta
\]

in \( \mathcal{R} \times (0, S_*) \).

Then from (6.4), (6.6) and (6.7) it follows that

\[
\mathcal{L}(P) \geq -m C_* N / m (\beta-1) \alpha r^{a-1} (1 + \mu^{-1} r^{-1} Q)^{a-2}
\]

in \( \mathcal{R} \times (0, S_*) \).

Let \( k > 0 \) and \( 0 < \eta < S_* \). We substitute \( \hat{P} = -k/(s-\eta) \) into (6.7) to get (note that \( 1 < \beta \leq 2 \))

\[
\mathcal{L}(\hat{P}) \leq k \left( \frac{m+1}{(s-\eta)^2} \right)^2 (\beta-1) \alpha r^{a-1} (1 + \mu^{-1} r^{-1} Q)^{a-2} \frac{k}{s-\eta}
\]

in \( \mathcal{R} \times (\eta, S_*) \).

If the positive constant \( k \) satisfies

\[
\left( \frac{m-1}{2m-1} C_* \right)^{-1} \frac{N}{m} \leq k \leq (m+1) k^2,
\]

we have

\[
(6.9) \quad \mathcal{L}(P) \geq \mathcal{L}(\hat{P}) \text{ in } \mathcal{R} \times (\eta, S_*).
\]

By Theorem 5.1 in Chapter 7 of [14] \( P \) is bounded in \( \mathcal{R} \times [\eta, (S_*+\eta)/2] \) and there exists a positive constant \( \varepsilon_0 \) such that

\[
(6.10) \quad P \geq \hat{P} \text{ in } \mathcal{R} \times (\eta, \eta+\varepsilon_0).
\]

Thus, choosing

\[
k = \max \left\{ \frac{1}{m+1}, \left( \frac{m-1}{2m-1} C_* \right)^{-1} \frac{N}{m} \right\},
\]

we have
\[ P \geq P = -\frac{k}{s-\eta} \text{ in } \mathcal{R} \times (\eta, S_\eta). \]

Since \( \eta \) is an arbitrary constant such that \( 0 < \eta < S_\eta \), we get

\[ Q_{ss} \geq -\frac{k}{s} \text{ in } \mathcal{R} \times (0, S_\eta). \]

Therefore it follows that

\[ \frac{m}{m-1}(w^{m-1})_{ss} \geq -(t+2C_\eta)^{-s} k[(2C_\eta)^{1-s} - (t+2C_\eta)^{1-s}]^{-1} \]

in \( \mathcal{R} \times (0, \infty) \).

Q.E.D.

### 7. The estimates of \( v \) in \((\text{supp } u)^c\)

In this section we shall consider the estimates of \( v \) in \((\text{supp } u)^c\), where \( u \) and \( v \) are the solutions for (1.1) and (1.2). Throughout this section, we assume \( 2n-2 < m \) and that \( u_0 \) and \( v_0 \) satisfy (A.I) and (A.II). Set \( d_0 = (b_m)^{-1/2} a_m (|v_0|_{1, \mathcal{R}} - |u_0|_{1, \mathcal{R}})^{1/(m+1)} \), where \( a_m \) and \( b_m \) are the constants in (2.5).

For \( d \in (0, d_0) \), we set \( h = h(d) = b_m^{1/(m-1)} (d_0^2 - d^2)^{1/(m-1)} \).

**Lemma 7.1.** Let \( u \) and \( v \) be the solutions of (1.1) and (1.2).

For \( d \in (0, d_0) \) and \( \varepsilon \in (0, d_0 - d) \) such that \( h(d + \varepsilon) \geq (3/4) h(d) \), there exists a positive constant \( T_1 = T_1(d, \varepsilon) \) satisfying the property:

\[ \text{supp } u(x, t) \subseteq \left[ -\frac{d}{3} t^{1/(m+1)}, \frac{d}{3} t^{1/(m+1)} \right] \text{ for } t \geq T_1. \]

Moreover, for \( t \geq T_1 \), there exist \( x_1 = x_1(d, \varepsilon, t) \in [dt^{1/(m+1)}, (d + \varepsilon) t^{1/(m+1)}] \) and \( x_2 = x_2(d, \varepsilon, t) \in [-(d + \varepsilon) t^{1/(m+1)}, -dt^{1/(m+1)}] \) such that

\[ v(x_i, t) \geq \frac{2}{3} h(d) t^{-1/(m+1)} \text{ for } i = 1, 2. \]

**Remark 7.2.** For \( \eta \in (0, \varepsilon) \), we put

\[ G = \{ (x, t) : t \geq T_1, x \in [-(d + \eta) t^{1/(m+1)}, (d + \eta) t^{1/(m+1)}] \}. \]

Applying Lemma 4.4 to the solution \( v \), we see that \( v \) is continuous in

\[ \{ (x, t) : t \geq T_1, x \in [-(d + \varepsilon) t^{1/(m+1)}, -dt^{1/(m+1)}] \cup [dt^{1/(m+1)}, (d + \varepsilon) t^{1/(m+1)}] \}. \]

Proof of Lemma 7.1. For \( \lambda > 0 \) and \( (f_0, g_0) \in \mathcal{X} \) such that \( f_0(x) \) and \( g_0(x) \geq 0 \) a.e. \( x \in \mathcal{R} \), we put \( (f_\lambda, g_\lambda) = (I - \lambda \mathcal{A})^{-1} (f_0, g_0) \), where \( \mathcal{A} \) and \( \mathcal{X} \) are the operator and the space in Section 2 respectively. Since \( (|f_\lambda|^{m-1} f_\lambda)_{xx} \) and \( (|g_\lambda|^{m-1} g_\lambda)_{xx} \) belong to \( L^1(\mathcal{R}) \), we have that
\[ \int_{R} g(x) \, dx - \int_{R} f(x) \, dx = \int_{R} g_0(x) \, dx - \int_{R} f_0(x) \, dx. \]

Therefore, by the definition of the solutions \( u \) and \( v \) in Section 2, we get that
\[ |v(\cdot, t)|_{1,R} - |u(\cdot, t)|_{1,R} = M \text{ for } t \geq 0, \]
where \( M = |v_0|_{1,R} - |u_0|_{1,R}. \)

Let \( w \) be the generalized solution of (1.4) \((p=2n)\) with \( w(\cdot, 0) = u_0. \) Then, since \( 2n-2 < m, \) the solution \( w \) satisfies the following properties:
\[ \lim_{t \to \infty} |w(\cdot, t)|_{1,R} = 0 \]
and there exists a positive constant \( T_{1*} \) such that
\[ \text{supp } w(\cdot, t) \subseteq \left[ -\frac{d}{3} t^{\frac{1}{(m+1)}}, \frac{d}{3} t^{\frac{1}{(m+1)}} \right] \text{ for } t \geq T_{1*}. \]

Therefore, by Lemma 4.1, the solution \( u \) satisfies that
\[ \text{supp } u(\cdot, t) \subseteq \left[ -\frac{d}{3} t^{\frac{1}{(m+1)}}, \frac{d}{3} t^{\frac{1}{(m+1)}} \right] \text{ for } t \geq T_{1*} \]
and
\[ \lim_{t \to \infty} |u(\cdot, t)|_{1,R} = 0. \]

By (7.4), there exists \( T_{2*} = T_{2*}(\varepsilon, \theta) \geq T_{1*} \) such that
\[ \text{supp } u(\cdot, t) \subseteq \left[ -\frac{1}{24} h(d) \varepsilon \right] \text{ for } t \geq T_{2*}. \]

Let \( v^* \) be the generalized solution of (2.2) with \( v^*(\cdot, T_{2*}) = v(\cdot, T_{2*}) \) in \( R \times [T_{2*}, \infty). \) From Lemma 4.1 it follows that
\[ v(x, t) \leq v^*(x, t) \text{ for } t \geq T_{2*} \text{ and a.e. } x \in R. \]

By [7] and [15], the solution \( v^* \) satisfies that
\[ \sup \{\text{supp } v^*(\cdot, t)\}, -\inf \{\text{supp } v^*(\cdot, t)\} \sim t^{\frac{1}{(m+1)}} \]
and
\[ \lim_{t \to \infty} \left\{ t^{\frac{1}{(m+1)}} |v^*(\cdot, t) - v(\cdot, t)|_{1,R} \right\} = 0, \]
where \( v(x, t) = \bar{v}(x, t; |v(\cdot, T_{2*})|_{1,R}) \) and \( \bar{v} \) is the self-similar solution ((2.5)).

From (7.7) and (7.8), it follows that
\[ \lim_{t \to \infty} |v^*(\cdot, t) - v(\cdot, t)|_{1,R} = 0. \]
Now we shall show the existence of the point \( x_v \).

By Remark 7.2 and (7.3), the negation of our conclusion is as follows:

There exist a constant \( \varepsilon \in (0, d_0 - d) \) and a sequence of the points \( t_n \in [T_{2\delta}, \infty) \) such that

\[
\lim_{n \to \infty} h(d + \varepsilon) \geq \frac{3}{4} h(d), \quad \text{for} \quad t_n \to \infty
\]

and

\[
(7.10) \quad \psi(x, t_n) < \frac{2}{3} h(d) t_n^{-1/(m+1)}
\]

for \( n \geq 1 \) and \( x \in [dt_n^{1/(m+1)}, (d + \varepsilon) t_n^{1/(m+1)}] \).

From (7.6), (7.8) and (7.10), it follows that

\[
(7.11) \quad \psi(t, t_n) < \int_{R^1(I(t_n))} \psi(x, t_n) \, dx + |\psi^*(\cdot, t_n) - \psi(\cdot, t_n)|_{1,R} + \frac{2}{3} h\varepsilon,
\]

where

\[
I(t) = [dt_n^{1/(m+1)}, (d + \varepsilon) t_n^{1/(m+1)}] \quad \text{for} \quad t > 0.
\]

For any positive constant \( \varepsilon \) such that \( h(d + \varepsilon) > (3/4) h(d) \), the function \( \psi \) satisfies that

\[
(7.12) \quad \psi(x, t) \geq \frac{3}{4} h(d) t^{-1/(m+1)}
\]

for \( t > 0 \) and \( x \in I(t) \).

Since any generalized solutions \( z \) of (2.2) conserve the total mass:

\[
|z(\cdot, t)|_{1,R} = |z(\cdot, 0)|_{1,R},
\]

it follows from (7.1) and (7.5) that

\[
(7.13) \quad |\psi(\cdot, t)|_{1,R} + \frac{1}{24} h\varepsilon
\]

\[
> |\psi(\cdot, T_{2\delta})|_{1,R} = |\psi^*(\cdot, t)|_{1,R} \quad \text{for any} \quad t \geq T_{2\delta}.
\]

By (7.12) and (7.13) we have

\[
(7.14) \quad |\psi(t, t_n)|_{1,R} \geq \int_{R^1(I(t_n))} \psi(x, t_n) \, dx + \frac{3}{4} h\varepsilon - |\psi^*(\cdot, t_n) - \psi(\cdot, t_n)|_{1,R} - \frac{1}{24} h\varepsilon \quad \text{for} \quad n \geq 1.
\]

From (7.11) and (7.14) it follows that

\[
(7.15) \quad |\psi^*(\cdot, t_n) - \psi(\cdot, t_n)|_{1,R} \geq \frac{1}{48} h\varepsilon \quad \text{for} \quad n \geq 1.
\]
The property (7.15) contradicts with (7.9). Thus, for any sufficiently large \( t \), there exists a point \( x_1 \) having the required property.

By the similar argument, we can show the existence of the point \( x_2 \).

Q.E.D.

In the proof of the following lemma, since we shall use the scheme employed in the proof of Lemma 3.1 in [8], we omit the proof.

**Lemma 7.3.** For \( d \in (0, d_0) \), there exist two positive constants \( T_2 = T_2(d) \) and \( C_2 = C_2(v_0, d) \) such that

\[
|v^{-1}(x, t)| \leq C_2 t^{-m/(m+1)} \quad \text{for} \quad t \geq T_2
\]

and a.e. \( x \in (-\infty, -dt^{1/(m+1)}] \cup [dt^{1/(m+1)}, \infty) \).

By Lemma 7.1 and Lemma 7.3, we can show the following lemma.

**Lemma 7.4.** For \( d \in (0, d_0) \), there exists a positive constant \( T_3 = T_3(d, v_0) \) such that

\[
v(dt^{1/(m+1)}, t), \quad v(-dt^{1/(m+1)}, t) > \frac{1}{2} h(d) t^{-1/(m+1)}
\]

for \( t \geq T_3 \).

**Proof.** Fix \( d \in (0, d_0) \) and choose

(7.16) \( \varepsilon = \frac{1}{2} \left\{ \left( \frac{2}{3} h(d) \right)^{m-1} - \left( \frac{1}{2} h(d) \right)^{m-1} \right\} C_2^{-1} \),

where \( C_2 \) is the constant in Lemma 7.3. Let \( T_3 = \max(T_1, T_2) \). Then, it follows from Lemma 7.1 and (7.16) that

\[
v^{-1}(dt^{1/(m+1)}, t) \geq v^{-1}(x_1, t) - C_2 t^{-m/(m+1)} (x_1 - dt^{1/(m+1)})
\]

\[
\geq \left( \frac{2}{3} h(d) t^{-1/(m+1)} \right)^{m-1} - C_2 \varepsilon t^{-1/(m+1)}
\]

\[
\geq \left( \frac{1}{2} h(d) t^{1/(m+1)} \right)^{m-1} \quad \text{for} \quad t \geq T_3,
\]

where \( x_1 \) is the point in Lemma 7.1.

By a similar argument, we obtain

\[
v^{-1}(-dt^{1/(m+1)}, t) > \left( \frac{1}{2} h(d) t^{1/(m+1)} \right)^{m-1} \quad \text{for} \quad t \geq T_3.
\]

Q.E.D.

### 8. Proof of Theorem 1.1.

**Part I (Case \( m < 2n - 2 \)).**

Let \( w^* \) be the generalized solution of (2.2) with \( w^*(\cdot, 0) = v_0 \). By [17], it is
shown that the solution $w^*$ satisfies

\begin{equation}
|w^*(\cdot, t)|_{\infty, R} \sim t^{-1/(m+1)}
\end{equation}

and

\begin{equation}
\sup(\text{supp } w^*(\cdot, t)), -\inf(\text{supp } w^*(\cdot, t)) \sim t^{1/(m+1)}.
\end{equation}

Let $w$ be the generalized solution of (1.4) ($p=2n$) with $w(\cdot, 0)=v_0$. By the result stated in Introduction, the solution $w$ satisfies

\begin{equation}
|w(\cdot, t)|_{\infty, R} \sim t^{-1/(m+1)},
\end{equation}

and

\begin{equation}
\sup(\text{supp } w(\cdot, t)), -\inf(\text{supp } w(\cdot, t)) \sim t^{1/(m+1)}.
\end{equation}

On the other hand, by Lemma 4.1 we have

\begin{equation}
v(x, t) \leq w(x, t) \leq w^*(x, t)
\end{equation}

for $t \geq 0$ and a.e. $x \in \mathbb{R}$.

It follows from (8.1)–(8.5) that

\begin{equation}
|v(\cdot, t)|_{\infty, R} \sim t^{-1/(m+1)}
\end{equation}

and

\begin{equation}
\sup(\text{supp } v(\cdot, t)), -\inf(\text{supp } v(\cdot, t)) \sim t^{1/(m+1)}.
\end{equation}

Therefore we have the estimates of $|v(\cdot, t)|_{\infty, R}$ and $\text{supp } v(\cdot, t)$.

By (8.6), there exists a positive constant $\lambda$ such that

\begin{equation}
(v(x, t))^\ast \leq \lambda (1+t)^{-n/(m+1)}
\end{equation}

for $t \geq 0$ and a.e. $x \in \mathbb{R}$.

For such a $\lambda$, we let $u_\ast$ be the generalized solution of (5.1) with $u_\ast(\cdot, 0)=u_0$. By (8.7), the solution $u_\ast$ is a subsolution of (3.1) with $P=v^\ast$ and $q=n$. Hence, by Lemma 4.1 and Lemma 4.3, we have

\begin{equation}
u_\ast(x, t) \leq u(x, t) \leq w(x, t)
\end{equation}

in $\mathbb{R} \times [0, \infty)$. By Lemma 5.2, the solution $u_\ast$ satisfies

\begin{equation}
|u_\ast(\cdot, t)|_{\infty, R} \sim t^{-1/(m+1)}
\end{equation}

and

\begin{equation}
\sup(\text{supp } u_\ast(\cdot, t)), -\inf(\text{supp } u_\ast(\cdot, t)) \sim t^{1/(m+1)}.
\end{equation}

From (8.3), (8.4) and (8.8)–(8.10), we obtain the required estimates of $|u(\cdot, t)|_{\infty, R}$ and $\text{supp } u(\cdot, t)$. 
Part II (Case of $2n - 2 < m$).

The following lemma gives some lower estimates of $v$ in a subset of $(\text{supp } u)^c$.

**Lemma 8.1.** Let

$$
\zeta(t) = \begin{cases} 
\zeta_0 t^{(2n-m)/2(2n-1)} & \text{if } 2n \neq m, \\
\zeta_0 (\log(t))_+ & \text{if } 2n = m,
\end{cases}
$$

$$
k(t) = A(t+K)^{1/(m+1)} + \zeta(t) \quad \text{for } t > 0
$$

and let

$$
v^{(1)}(x, t) = \left( \frac{m-1}{2m(m+1)} \right)^{1/(m-1)} t^{-1/(m+1)} (A^2 - (x-k(t))^2 t^{-2/(m+1)})^{1/(m-1)} \text{ in } \mathbb{R} \times [0, \infty).
$$

Then, $u$ and $v$ satisfy

$$
\text{supp } u(\cdot, t) \subset [-\zeta(t), \zeta(t)] \quad \text{for } t \geq T_1
$$

and

$$
v(x, t), v(-x, t) \geq v^{(1)}(x, t)
$$

for $t \geq T_1$ and a.e. $x \in (-\infty, At^{1/(m+1)}]$

for certain positive constants $k$, $\zeta_0$ and $T_1$.

**Proof.** Let $w$ be the generalized solution of (1.4) with $w(\cdot, 0) = u_0$. Then, as is noted in Introduction, it holds for sufficiently large $\zeta_0$ and $T_{1*}$ that

$$
\text{supp } w(\cdot, t) \subset [-\zeta(t), \zeta(t)] \quad \text{for } t \geq T_{1*}.
$$

By Lemma 4.1, we have also

$$
\text{(8.11)} \quad \text{supp } u(\cdot, t) \subset [-\zeta(t), \zeta(t)] \quad \text{for } t \geq T_{1*}.
$$

We set

$$
\text{(8.12)} \quad A = (2^{m-1}+1)^{-1/2} d_0,
$$

where $d_0$ is the constant in Section 7. By Lemma 7.4, (8.12) and $2n - 2 < m$, there exists a positive constant $T_{2*} = T_{2*}(A) \geq T_{1*}$ such that

$$
\text{(8.13)} \quad v(At^{1/(m+1)}, t), v(-At^{1/(m+1)}, t) > \frac{1}{2} h(A) t^{-1/(m+1)} \quad \text{for } t \geq T_{2*}
$$

and

$$
\text{(8.14)} \quad \text{supp } u(\cdot, t) \subset [-\frac{1}{2} At^{1/(m+1)}, \frac{1}{2} At^{1/(m+1)}] \quad \text{for } t \geq T_{2*},
$$

where $h$ is the function in Section 7.
Here, we take
\[ K = 2^{n+1} T^*_2 \]
and set
\[ G_{T^*_2} = \{(x, t); t \geq T^*_2, x \in [-A t^{1/(m+1)}, A t^{1/(m+1)}]\}. \]

We shall compare the function \( v^{(1)}_* \) in this lemma and the solution \( v \) in \( G_{T^*_2} \).

By (8.15), the function \( v^{(1)}_* \) satisfies
\[ v^{(1)}_*(x, T^*_2) = 0 \quad \text{for} \quad x \in [-A T^{1/(m+1)}_{T^*_2}, A T^{1/(m+1)}_{T^*_2}]. \]

Since
\[ k(t) > A t^{1/(m+1)} \quad \text{for} \quad t > 0 \]
we have
\[ v^{(1)}_*(A t^{1/(m+1)}, t) < \frac{1}{2} h(A) t^{-1/(m+1)} \quad \text{for} \quad t \geq T^*_2 \]
\[ v^{(1)}_*(-A t^{1/(m+1)}, t) = 0 \quad \text{for} \quad t \geq T^*_2 \]
and
\[ v^{(1)}_*(x, t) \geq 0 \quad \text{for} \quad t \geq T^*_2 \quad \text{and} \quad a.e. \quad x \in [-A t^{1/(m+1)}_{T^*_2}, A t^{1/(m+1)}_{T^*_2}]. \]

By (8.11), the solution \( u \) satisfies
\[ (\text{supp } u) \cap (\text{supp } v^{(1)}_*) = \emptyset \quad \text{in} \quad G_{T^*_2}. \]

Since the function \( ((m-1)/(2m(m+1))) t^{1/(m-1)} \cdot t^{-1/(m+1)}(A^2 - x^2) t^{-2/(m+1)} \) satisfies (2.2) a.e. in \( \mathbb{R} \times (0, \infty) \), we obtain by (8.19) and (8.20)
\[ (8.21) \]
That is, \( v^{(1)}_* \) is a subsolution of (3.1) with \( P = v^* \) and \( q = n \) in \( G_{T^*_2} \). Thus, by (8.13), (8.14), (8.16)–(8.18) and Lemma 4.4, we conclude \( v^{(1)}_* \leq v \) in \( G_{T^*_2} \).

Q.E.D.

**Lemma 8.2.** There exist a positive constant \( T_2 \) and a positive function \( \rho \) defined on \([T_2, \infty)\) such that
\[ v(x, t) \geq \rho(t) > 0 \quad \text{for} \quad t \geq T_2 \quad \text{and} \quad a.e. \quad x \in [-A t^{1/(m+1)}_{T^*_2}, A t^{1/(m+1)}_{T^*_2}], \]
where \( A \) is the constant in Lemma 8.1.
Proof. Case: $2n - 2 < m \leq 2n$.

Let $w$ be the generalized solution of (1.4) with $w(\cdot, 0) = u_0$. Then, by [8], [9], [12] and [13], there exists a nonnegative constant $T_{1*}$ such that for each $t \geq T_{1*}$, \{x \in \mathbb{R}: w(x, t) > 0\} is an open interval in $\mathbb{R}$ containing $x = 0$. By Lemma 8.1, there exists a constant $T_{2*} \geq \max(T_{1*}, T_1)$ such that

$$\tag{8.22} v(x, t) \geq \phi_0(x; t) \quad \text{for} \quad t \geq T_{2*} \quad \text{and a.e.} \quad x \in [-At^{1/(m+1)}, At^{1/(m+1)}],$$

where

$$\phi_0(x; t) = \left(\frac{m-1}{2m(m+1)}\right)^{1/(m+1)} t^{-1/(m+1)} \left(\frac{A^2}{9} - \frac{2}{3} At^{1/(m+1)} \right)^{-2/(m+1)} +$$

and $A$ and $T_1$ are the constants in Lemma 8.1.

For each $s \geq T_{2*}$, let $w(\cdot, \cdot; s)$ be the generalized solution of (1.4) with $w(\cdot, s; s) = \max\{w(\cdot, s), \phi_0(\cdot; s)\}$ in $\mathbb{R} \times [s, \infty)$ and let $\phi(\cdot, \cdot; s)$ be the generalized solution of (1.4) with $\phi(\cdot, s; s) = \phi_0(\cdot; s)$ in $\mathbb{R} \times [s, \infty)$. Then, by (8.21), Lemma 4.1 and Lemma 4.3, we have

$$\tag{8.23} \phi(x, t; s) \leq w(x, t; s) \leq v(x, t) \quad \text{for} \quad s \geq T_{2*}, \text{any} \ t \geq s \quad \text{and a.e.} \quad x \in \mathbb{R}.$$ 

We observe by [9], [12] and [13] that for each $t \geq s$, \{x \in \mathbb{R}: \phi(x, t; s) > 0\} is an open interval in $\mathbb{R}$.

Since $m \leq 2n$, the result in Introduction implies that there exists a constant $T_{3*}(\geq T_{2*})$ such that

$$\{x \in \mathbb{R}: w(x, t) > 0\} \cap \{x \in \mathbb{R}: \phi(x, t; s) > 0\} \neq \emptyset \quad \text{for} \quad t \geq T_{3*}.$$ 

Let $E_0(t) = \{x \in \mathbb{R}: w(x, t) > 0\}$, $E_d(t) = \{x \in \mathbb{R}: \phi(x, t; s) > 0\}$ and $E_g(t) = \{x \in \mathbb{R}: \psi^*(x, t) > 0\}$, where $\psi^*$ is the function in Lemma 8.1. Since $E_0(t)$, $E_g(t)$ and $E_d(t)$ extend as $t$ increases we obtain by [9], [12] and [13] that

$$\tag{8.24} E_0(t) \cup E_g(t) \cup \bigcup_{\phi \in [0, At^{1/(m+1)}]} E_\phi(t) \supseteq [0, At^{1/(m+1)}] \quad \text{for} \quad t \geq T_{3*}.$$ 

Since $[0, At^{1/(m+1)}]$ is compact there exists a finite sequence $\{s_j\}_{j=1}^j \subset [T_{2*}, t]$ such that

$$\tag{8.25} E_0(t) \supseteq E_{s_j}(t) \cup \bigcup_{j=1}^j E_{s_j}(t) \supseteq [0, At^{1/(m+1)}].$$

By Lemma 3.3, $w$ and $\phi(\cdot, \cdot; s)$ are continuous in $\mathbb{R} \times [0, \infty)$ and $\mathbb{R} \times [s, \infty)$ respectively. When we put

$$\rho_+(t) = \min\{\max\{w(x, t), \psi^*(x, t), \phi(x, t; s_1), \ldots, \phi(x, t; s_j)\}: x \in [0, At^{1/(m+1)}]\} \quad \text{for} \quad t \geq T_{3*},$$

we have
Lemma 4.1, (8.23) and (8.25) imply that the function $\rho_+$ is positive in $[T_{3*}, \infty)$ and satisfies

\begin{equation}
\begin{aligned}
v(x, t) &\geq \rho_+(t) \\
& \text{for } t \geq T_{3*} \text{ and a.e. } x \in [0, A^{1/(m+1)}].
\end{aligned}
\end{equation}

By a similar argument, we find a positive constant $T_{4*}$ and a positive function $\rho_-$ defined on $[T_{4*}, \infty)$ such that

\begin{equation}
\begin{aligned}
v(x, t) &\geq \rho_-(t) \\
& \text{for } t \geq T_{4*} \text{ and a.e. } x \in [-A^{1/(m+1)}, 0].
\end{aligned}
\end{equation}

Case: $m > 2n$. By a result stated in Introduction and Lemma 4.1, there exists a positive constant $b$ such that

\begin{equation}
\supp u(\cdot, t) \subset \left[-\frac{b}{2},\frac{b}{2}\right] \text{ for } t \geq 0.
\end{equation}

By (8.28) and Lemma 4.4, $v$ is continuous in $(R \setminus [-2b/3, 2b/3]) \times [0, \infty)$. Hence, by Lemma 8.1, there exist two positive constants $a$ and $T_{5*}$ such that

\begin{equation}
v(x, t) > at^{-m/((m+1)\cdot(m-1))}
\end{equation}

\begin{equation}
\text{for } t \geq T_{5*} \text{ and } |x| \geq b.
\end{equation}

For $T \geq 0$, we let $w(x, t; T)$ be the solution of

\begin{equation}
\begin{aligned}
w_t &= (w^m)_{xx} - \lambda w \text{ in } R \times (T, \infty) \\
\end{aligned}
\end{equation}

with

\begin{equation}
w(\cdot, T; T) = v(\cdot, T) \text{ on } R,
\end{equation}

where $\lambda = |v_0|_{R^m}$. By Remark 3.2, Lemma 3.5 and 4.1, we have

\begin{equation}
|u^n(\cdot, t) v^{n-1}(\cdot, t)|_{R^m} \leq \lambda \text{ for } t \geq 0,
\end{equation}

and hence, by the comparison theorem,

\begin{equation}
w(\cdot, t; T) \leq v(\cdot, t) \text{ for } t \text{ and } T \text{ with } t \geq T \geq 0.
\end{equation}

If we can show

\begin{equation}
\bigcup_{x \geq 0} \bigcup_{t \geq T} (\operatorname{supp} w(\cdot, t; T))^0 \supset [-b, b],
\end{equation}

this together with (8.32) will give the desired result. In fact, for $x_0 \in [-b, b]$ there exist $t(x_0) > T(x_0)$ such that

\begin{equation}
w(x_0, t(x_0); T(x_0)) > 0.
\end{equation}
Then, there exists an open interval $I(x_0)$ such that
\[ x_0 \in I(x_0) \subset (\text{supp } w(\cdot, t(x_0); T(x_0)))^\circ. \]
Since $\text{supp } w(\cdot, t; T(x_0))$ is monotonically non-decreasing with respect to $t$, we get
\[ I(x_0) \subset (\text{supp } w(\cdot, t; T(x_0)))^\circ \]
for $t \geq t(x_0)$.

Since there exist finite points $\{x_j\}_{j=1}^f \subset [-b, b]$ such that $\bigcup_{j=1}^f I(x_j) \supset [-b, b]$, we obtain
\[ v(x, t) \geq h(t) > 0 \]
for any $x \in [-b, b]$ and any $t \geq t^*$, where $t^* = \max_j t(x_j)$ and $h(t) = \min \max \limits_x w(x, t; T(x_j))$.

In order to show (8.32), we assume
\[ \bigcup \bigcup_{t \geq 0} (\text{supp } w(\cdot, t; T))^\circ \supset [-b, b]. \]
Then there exists a point $x_* \in [-b, b]$ such that
\[ w(x_*, t; T) = 0 \text{ for } x \text{ and } T \text{ with } t > T \geq 0. \]
Therefore, $v_0$ and $u_0$ satisfy $v_0(x_*) = u_0(x_*) = 0$. Note that
\[ (8.33) \quad \int_{x_*}^{x_*} v_0(y) \, dy > \int_{x_*}^{x_*} u_0(y) \, dy \]
or
\[ \int_{-\infty}^{x_*} v_0(y) \, dy > \int_{-\infty}^{x_*} u_0(y) \, dy. \]
We assume (8.33) without loss of generality.
Let
\[ v_{\#0} = u_0 \chi_{(-\infty, x_*)} + v_0 \chi_{(x_*, \infty)}, \]
\[ u_{\#0} = u_0 \]
and let $v_*$ and $u_*$ be the solutions of (1.1) with $u_*(\cdot, 0) = v_{\#0}$ and $u_*(\cdot, 0) = u_{\#0}$.
Then, by Lemma 4.1 we have for $t \geq 0$
\[ (8.34) \quad v(x, t) \geq v_*(x, t) \geq u_*(x, t) \geq u(x, t) \geq 0 \]
for a.e. $x \in \mathbb{R}$.

For $T \geq 0$, we let $w_*(\cdot, \cdot; T)$ be the solution of (8.30) with $w_*(\cdot, T; T) = v_*(\cdot, T)$.
Then, since $T \geq 0$ $w(\cdot, \cdot; T) \geq w_*(\cdot, \cdot; T)$ in $\mathbb{R} \times (T, \infty)$, we get
\[ w_*(x_*, t; T) = 0 \text{ for } t \text{ and } T \text{ with } t > T \geq 0. \]

For \( T \geq 0 \), let \( z_*(\cdot, \cdot; T) \) be the solution of (2.2) in \( \mathbb{R} \times (T, \infty) \) with \( z_*(\cdot, T; T) = v_*(\cdot, T) \).

Then, we have

\[ (8.35) \quad w_*(x, t; T) = e^{\lambda(T-t)} z_*(x, \int_0^t e^{-(m-1)\lambda s} ds; T). \]

By the comparison theorem and (8.35), \( z_* \) satisfies

\[ (8.36) \quad z_*(x, s+T; T) = 0 \quad \text{for} \quad s \leq \left(0, \frac{1}{(m-1)\lambda}\right) \]

and for \( t \geq T \),

\[ (8.37) \quad z_*(\cdot, t; T) \geq v_*(\cdot, t) \text{ a.e. on } \mathbb{R}. \]

For each \( T \geq 0 \), let

\[ v_1(\cdot, \cdot; T) = z_*(\cdot, \cdot; T) \chi_{[z_*, \infty)}. \]

Then, we shall show that \( v_1(\cdot, \cdot; T) \) is the solution of (2.2) in \( \mathbb{R} \times (T, T+1/(\lambda(m-1))) \) with \( v_1(\cdot, T; T) = v_*(\cdot, T) \chi_{[z_*, \infty)}. \) For this we take \( t_0, t_1 \in (T, T+1/(\lambda(m-1))] \) with \( t_0 < t_1 \) and \( x_0, x_1 \in \mathbb{R} \) with \( x_0 < x_1 \).

For \( \delta > 0 \), we set

\[ \rho_\delta(x) = \begin{cases} 
\exp(-((\delta^2 - (x-(x_*+\delta))^2)^{-1}) & \text{if } |x-(x_*+\delta)| < \delta, \\
0 & \text{if } |x-(x_*+\delta)| \geq \delta
\end{cases} \]

and

\[ h_\delta(x) = \int_{-\infty}^x \rho_\delta(y) dy. \]

For \( f \in C^{2,1}([t_0, t_1] \times [x_0, x_1]) \) with \( f(x_0, t) = f(x_1, t) = 0, z_* \) satisfies

\[ I(z_*, fh_\delta, [t_0, t_1] \times [x_0, x_1]) = 0. \]

Since there exists a constant \( C = C(t_0, t_1) \) such that

\[ |(z_*^{-1})_*(\cdot, t; T)|_{\infty,\mathbb{R}} \leq C \quad \text{for} \quad t \in [t_0, t_1], \]

we have

\[ 0 = I(z_*, fh_\delta, [t_0, t_1] \times [x_0, x_1]) = \int_{t_0}^{t_1} \int_{x_0}^{x_1} \{-(z_*^{-1})_*(f_x h_\delta + f \rho_\delta) + z_* f, h_\delta \} dx dt - \left[ \int_{x_0}^{x_1} z_* h_\delta f \right]_0^{t_1}, \]

and, letting \( \delta \to 0, \)
\[(8.38) \quad I(x_*, \chi_{\mathcal{F}(x, \omega)}, f, [t_0, t_1] \times [x_0, x_1]) = 0.\]

Therefore \(v_1(\cdot, \cdot; T)\) is the solution of (2.2) in \(\mathbb{R} \times (T, T+1/(\lambda(m-1)))\).

Let \(w\) be the solution of (1.4) with \(w(\cdot, 0) = u_0\) and let \(v_2 = w \chi_{(-\infty, x_*)}\). By Lemma 4.1, we see
\[
v_2(x, t) \geq w(x, t) \quad \text{for} \quad t \geq 0 \quad \text{and a.e.} x \in \mathbb{R},
\]
and hence, by (8.36) and (8.37),
\[
w(x_*, t) = 0 \quad \text{for} \quad t \geq 0.
\]

By a similar argument we can show that \(v_2\) is the solution of (1.4) \((p=2n)\) with \(v_2(\cdot, 0) = u_0 \chi_{(-\infty, x_*)}\).

For \(T \geq 0\), let \(\vartheta(\cdot, \cdot; T) = v_1(\cdot, \cdot; T) + v_2\) and let \(\bar{u} = v_2\). Then \(\vartheta(\cdot, \cdot; T)\) and \(\bar{u}\) are the solutions of (1.1) in \(\mathbb{R} \times (T, T+1/(\lambda(m-1)))\). First we take \(T=0\).

Since
\[
\vartheta(\cdot, 0; 0) = v_{\vartheta_0} \geq u_{\vartheta_0} \geq \bar{u}(\cdot, 0) \quad \text{a.e. on} \quad \mathbb{R},
\]
we have by Lemma 4.1 that for \(t \in [0, 1/(\lambda(m-1))]\)
\[(8.39) \quad \vartheta(\cdot, t; 0) \geq v_{\vartheta}(\cdot, t) \geq u_{\vartheta}(\cdot, t) \geq \bar{u}(\cdot, t) \quad \text{a.e. on} \quad \mathbb{R}.
\]

Since \(\bar{u} = \vartheta\) if \(x < x_\vartheta, v_\vartheta = u_\vartheta\) in \((-\infty, x_\vartheta] \times [0, 1/(\lambda(m-1))]\).

By induction, we conclude
\[v_\vartheta = u_\vartheta = w \chi_{(-\infty, x_\vartheta]} \text{ in } (-\infty, x_\vartheta] \times [0, \infty).
\]

By a result stated in Introduction, \(\bigcup_{t \geq 0} \text{supp } \vartheta(\cdot, t)\) is bounded in \(\mathbb{R}\), while \(\bigcup_{t \geq 0} \text{supp } v_\vartheta(\cdot, t) \chi_{(-\infty, x_\vartheta]}\) is not bounded in \(\mathbb{R}\) by Lemma 7.1. This contradicts to the above equality and (8.32) is now proved.

Q.E.D.

For \(T, N > 0\), we shall consider the following boundary value problem:

\[(8.40) \quad w_t = (w^m)_{xx} - w^{2n} \quad \text{in} \quad Q_T^N,
\]
\[(8.41) \quad w(\pm N\zeta(t), t) = 4t^{-1/(2n-1)} \quad \text{on} \quad [T, \infty),
\]
where \(Q_T^N = \{(x, t) \in \mathbb{R} \times [0, \infty): t \geq T, x \in [-N\zeta(t), N\zeta(t)]\}\) and \(\zeta\) is the function in Lemma 8.1.

**Lemma 8.3.** If \(2n-2 < m \leq 2n\), there exist two positive constants \(T_3, N\) and a positive classical solution \(w_3\) of (8.40) and (8.41) in \(Q_T^N\) such that
\[
\begin{align*}
\text{for } t \geq T_3 \quad \text{and a.e. } x \in [-N\zeta(t), N\zeta(t)],
\quad v(\pm N\zeta(t), t) > 5t^{-1/(2n-1)} \quad \text{in} \quad [T_3, \infty),
\end{align*}
\]
\[
\begin{align*}
u(x, t) \leq w_3(x, t) \leq v(x, t) \quad \text{for } t \geq T_3 \quad \text{and a.e. } x \in [-N\zeta(t), N\zeta(t)].
\end{align*}
\]
where $w$ is the generalized solution of (1.4) with $w(\cdot, 0) = u_0$.

**Proof.** By a result in Introduction, there exist two positive constants $N \geq 2$ and $T_{1\ast}$ such that

(8.42) \[ \text{supp } w(\cdot, t) \subseteq \left[ -\frac{N}{2} \xi(t), \frac{N}{2} \xi(t) \right] \text{ for } t \geq T_{1\ast}. \]

Hence, there exists a positive constant $T_{2\ast} \geq \max(T_{1\ast}, T_1)$ such that

(8.43) \[ v^{(1)}(N_\xi(t), t) > 5t^{-1/(2a-1)} \text{ for } t \geq T_{2\ast}, \]

where $T_1$ and $v^{(1)}$ are the constant and the function in Lemma 8.1, respectively.

By (8.42), Lemma 4.1 and Lemma 4.4 $v$ is continuous in $\mathbb{R} \times [T_{2\ast}, \infty) \setminus Q^{2N/3}_{T_{2\ast}}$ and we have by (8.43) and Lemma 8.1 that

(8.44) \[ v(N_\xi(t), t) > 5t^{-1/(2a-1)} \text{ for } t \geq T_{2\ast}. \]

Let $T_{3\ast} = \max(T_{2\ast}, T_2)$, where $T_2$ is the constant in Lemma 8.2. Then, by Lemma 4.1, Lemma 8.1 and Lemma 8.2, $v$ satisfies

(8.45) \[ v(x, T_{3\ast}) \geq \max \{ v^{(1)}(x, T_{2\ast}), v^{(1)}(-x, T_{3\ast}), \rho(T_{3\ast}) \}, \]

where $A$ is the constant in Lemma 8.1 and $\rho$ is the function in Lemma 8.2. By Theorem 0 in [8] $w$ is smooth in $\{(x, t) \in \mathbb{R} \times (0, \infty) : w(x, t) > 0\}$ and by (8.42), (8.44), (8.45) there exists a positive function $w_{0b} \in H^{2+\beta}([-N_\xi(T_{3\ast}), N_\xi(T_{3\ast})])$, $0 < \beta < 1$, such that

(8.46) \[ w(x, T_{3\ast}) \leq w_{0b}(x) \leq v(x, T_{3\ast}) \]

for a.e. $x \in [-N_\xi(T_{3\ast}), N_\xi(T_{3\ast})]$ and that $w_{0b}$ satisfies the compatibility condition of first order for (8.40) and (8.41) in $Q^N_{T_{3\ast}}$.

Since $w_{0b}$ is positive on $[-N_\xi(T_{3\ast}), N_\xi(T_{3\ast})]$, we can show by Theorem 6.1 of Section 5 in [14] and the change of variables the existence of the positive solution $w_3 \in H^{2+\beta, 1+\beta/2}_{loc}(Q^N_{T_{3\ast}})$ of (8.40) and (8.41) with $w_3(\cdot, T_{3\ast}) = w_{0b}$ in $Q^N_{T_{3\ast}}$. Since $w_3$ is a generalized solution of (1.4) in $Q^N_{T_{3\ast}}$, we obtain by (8.42), (8.46), Lemma 4.1 and Lemma 4.5 that

(8.47) \[ u(x, t) \leq w(x, t) \leq w_3(x, t) \]

for $t \geq T_{3\ast}$ and a.e. $x \in [-N_\xi(t), N_\xi(t)]$.

Thus, $w_3$ is a subsolution of (3.1) with $P = u^n$ and $q = n$ in $Q^N_{T_{3\ast}}$ and by (8.43),
(8.46), (8.47) and Lemma 4.4 we have

\begin{equation}
(8.48) \quad u(x, t) \leq w_b(x, t) \leq v(x, t)
\end{equation}

for \( t \geq T_{3b} \) and a.e. \( x \in [-N\xi(t), N\xi(t)] \).

Q.E.D.

**Lemma 8.4.** If \( 2n < m \), there exists a positive constant \( b \) such that

\[\sup_{[-b, b]} u(\cdot, t) \leq b \]

for \( t \geq 0 \).

Moreover, suppose that for such a constant \( b \), there exist three functions \( u, v \) and \( w^* \in H^{1,2,1+b/2}([-b, b] \times [T_4, \infty)) \), \( 0 < \beta < 1 \), \( T_4 > 0 \), satisfying the following properties:

- \( u \) is a generalized supersolution for \( w_t = (w^m)_{xx} - w^n \) in \([-b, b] \times [T_4, \infty) \), \( \psi \) is a generalized subsolution for \( w_t = (w^m)_{xx} - w^n \) in \([-b, b] \times [T_4, \infty) \) and \( w^* \) is a generalized solution for \( w_t = (w^m)_{xx} - w^n \) in \([-b, b] \times [T_4, \infty) \).

\[
\begin{align*}
\frac{1}{2} t^{-1/(2m-1)} &< u(\pm b, t) \leq w^*(\pm b, t) \leq v(\pm b, t) \\
&< 2t^{-1/(2m-1)} \leq v(\pm b, t) \quad \text{for } t \geq T_4,
\end{align*}
\]

\[
\begin{align*}
u(x, T_4) &\leq u(x, T_4) \leq w^*(x, T_4) \leq v(x, T_4) \\
&\leq v(x, T_4) \quad \text{for a.e. } x \in [-b, b],
\end{align*}
\]

\[
\begin{align*}
u(x, t) &\leq w^*(x, t) \leq v(x, t) \leq v(x, t) \\
&< v(x, t) \quad \text{for } t \geq T_4 \text{ and a.e. } x \in [-b, b],
\end{align*}
\]

\[
\begin{align*}
0 &< u \leq w^* \leq v \text{ in } [-b, b] \times [T_4, \infty).
\end{align*}
\]

Then, those functions satisfy that

\[
\begin{align*}
u(x, t) &\leq u(x, t) \leq w^*(x, t) \leq v(x, t) \leq v(x, t) \\
&< v(x, t) \quad \text{for } t \geq T_4 \text{ and a.e. } x \in [-b, b].
\end{align*}
\]

**Proof.** We know already that there exists a positive constant \( b \) such that

\[\sup_{[-b, b]} u(\cdot, t) \leq b \]

for \( t \geq 0 \). Let us fix such a constant \( b \). We shall consider the following initial boundary value problems:

\begin{align}
(8.49) & \quad \mathcal{L}(w; q) \equiv w_t - (w^m)_{xx} + q^w w^n = 0 \text{ in } [-b, b] \times [T_4, \infty), \\
(8.50) & \quad w(b, t) = w(-b, t) = \frac{1}{2} t^{-1/(2m-1)} \text{ on } [T_4, \infty), \\
(8.51) & \quad w(x, T_4) = u_{0b} \text{ on } [-b, b]
\end{align}

and

\begin{align}
(8.52) & \quad \mathcal{L}(w; q) = 0 \text{ in } [-b, b] \times [T_4, \infty), \\
(8.53) & \quad w(b, t) = w(-b, t) = 2t^{-1/(2m-1)} \text{ on } [T_4, \infty),
\end{align}
(8.54) \[ w(x, T_4) = v_{ob} \text{ on } [-b, b] \]

where \( q \) is an arbitrary function belonging to \( L^\infty([-b, b] \times [T_4, \infty)) \), \( v_{ob} = 2T_4^{-1/(2m-1)} \) and \( u_{ob} \) is the convolution of \( \max \{ u(\cdot, T_4), (1/2) T_4^{-1/(2m-1)} \} \) and an appropriate mollifier. Therefore, \( u_{ob} \) and \( v_{ob} \) satisfy the compatibility condition of first order for (8.49)-(8.51) and (8.52)-(8.54), respectively.

Let \( u_0 = v_0 = w^* \). Then there exist two sequences of the positive functions \( u_j \) and \( v_j \in L^{2+b, \infty}([-b, b] \times [T_4, \infty)) \) satisfying the following properties:

For \( j \geq 1 \), \( v_j \) is the classical solution of \( \mathcal{L}(w; u_{j-1}) = 0 \) with (8.53) and (8.54) in \([-b, b] \times [T_4, \infty) \) and \( u_j \) is the classical solution of \( \mathcal{L}(w; v_{j-1}) = 0 \) with (8.50) and (8.51).

We can prove that there exist \( \lim_{j \to \infty} v_j(x, t) \) and \( \lim_{j \to \infty} u_j(x, t) \) in \([-b, b] \times [T_4, \infty) \).

Setting \( v_j(x, t) = \lim_{j \to \infty} v_j(x, t) \) and \( u_j(x, t) = \lim_{j \to \infty} u_j(x, t) \), we can obtain

(8.55) \[ u(x, t) \leq u_j(x, t) \leq w^*(x, t) \leq v_j(x, t) \leq v(x, t) \]

for \( t \leq T_4 \) and a.e. \( x \in [-b, b] \).

Let \( \tilde{v}_0 = v \) and let \( \tilde{u}_0 = u \). Then, for any \( j \geq 1 \), \( \tilde{u}_j \) is the classical solution for \( \mathcal{L}(w; \tilde{u}_{j-1}) = 0 \) with (8.50) and (8.51), \( \tilde{v}_j \) is the classical solution for \( \mathcal{L}(w; \tilde{v}_{j-1}) = 0 \) with (8.53) and (8.54). We can prove that there exist \( \lim_{j \to \infty} \tilde{v}_j(x, t) \) and \( \lim_{j \to \infty} \tilde{u}_j(x, t) \) in \([-b, b] \times [T_4, \infty) \). Setting \( \tilde{v}_j(x, t) = \lim_{j \to \infty} \tilde{v}_j(x, t) \) and \( \tilde{u}_j(x, t) = \lim_{j \to \infty} \tilde{u}_j(x, t) \), we obtain

(8.56) \[ \tilde{u}_j \leq u \leq \tilde{v}_j \leq |v_{ob}| \infty \times [-b, b] \times [T_4, \infty) \]

Therefore we can observe that

(8.57) \[ \tilde{u}_j(x, t) = u_j(x, t) \text{ for } t \geq T_4 \text{ and a.e. } x \in [-b, b] \],

(8.58) \[ \tilde{v}_j(x, t) = v_j(x, t) \text{ for } t \geq T_4 \text{ and a.e. } x \in [-b, b] \].

From (8.55)-(8.58) we conclude

\[ u(x, t) \leq u_j(x, t) \leq w^*(x, t) \leq v_j(x, t) \leq v(x, t) \]

for \( t \geq T_4 \) and a.e. \( x \in [-b, b] \).

Q.E.D.

Let \( T > 0 \) and let

\[ G_T = \{(x, t) \in \mathbb{R} \times [T, \infty): x \in [-At^{1/(m+1)}, At^{1/(m+1)}]\} \]

where \( A \) is the constants in Lemma 8.1. Let us consider the following initial boundary value problem (I.B.):
First, we shall construct some convenient initial functions.

**Lemma 8.5.** There exist a positive constant $T_5$ and two positive functions $v$ and $u$ such that

\[
\begin{align*}
&v_0 = \frac{(m-1)}{9} A^\nu(x, t) \quad \text{in } G_T, \\
&u_0 = \frac{(m-1)}{9} A^\nu(x, t) \quad \text{in } G_T, \\
&v(\pm At^\nu(m+1), t) = \frac{8}{9} \left( \frac{m-1}{2m(m+1)} \right)^{\nu(m-1)} A^{2\nu(m+1)} t^{-\nu(m+1)} \quad \text{on } [T, \infty), \\
&u(\pm At^\nu(m+1), t) = t^{-\nu(m+1)-2} \quad \text{on } [T, \infty), \\
&v(\cdot, T) = v_0 \quad \text{on } [-AT^\nu(m+1), AT^\nu(m+1)], \\
&u(\cdot, T) = u_0 \quad \text{on } [-AT^\nu(m+1), AT^\nu(m+1)].
\end{align*}
\]

and for $T \geq T_5$ $v(\cdot, T)$ and $u(\cdot, T)$ satisfy the compatibility condition of first order for (I.B.), where $A$ is the constant in Lemma 8.1 and $w$ is the generalized solution of (1.4) with $w(\cdot, 0) = u_0$.

**Proof.** Case: $2n-2 < m \leq 2n$.

Let $\alpha$ be the solution of the Cauchy problem

\[
\begin{align*}
\alpha' &= \alpha^{m+2n-1} - \alpha^2 \quad \text{in } (0, \infty) \\
\alpha(0) &= (1/2)^{\nu(m-1)}.
\end{align*}
\]

Then, the solution $\alpha$ is monotone decreasing with respect to $t$ and satisfies

\[
\alpha(t) \sim t^{-\nu(2n-1)}.
\]

We set

\[
\vartheta_\alpha(x, t) = \alpha(t) \exp(x^2 \alpha^{2n-1}(t)).
\]
Then it follows that

\[(8.60) \quad \mathcal{L}(w_\ast) = \mathcal{L}_{\ast}(w_\ast) - (\mathcal{L}_{\ast}^p)_{xx} + \mathcal{L}_{\ast}^{2p} = -\exp(x^2 \alpha^{2p-1}) \left\{(2m-1) - (2n-1) \alpha^{2p-1} x^2 \right\} \alpha^{2p-1} \]
\[+ \alpha^{2p} \exp(x^2 \alpha^{2p-1}) \left\{(2n-1) x^2 \alpha^{2p-1} - (2n-1) x^2 \right\} \alpha^{2p-1}\]
\[\text{in } \mathbb{R} \times (0, \infty).\]

By (8.59) and (8.60), there exists a positive constant \(T_{1\ast}\) such that \(\mathcal{L}(w_\ast) \leq 0\) for \(t \geq T_{1\ast} + T_3\) and \(x \in \left[-N \xi(t), N \xi(t)\right]\) and that

\[(8.61) \quad w_\ast(x, t) \geq w_\ast(x, t + T_{1\ast})\]
\[\text{for } t \geq T_3 \text{ and } x \in \left[-N \xi(t), N \xi(t)\right].\]

where \(T_3, N\) and \(w_\ast\) are two constants and the function in Lemma 8.3 respectively and \(\xi\) is the function in Lemma 8.1.

Let us fix the positive constant \(T_{1\ast}\) satisfying (8.61) and let \(w_\ast(x, t) = \mathcal{L}_{\ast}(x, t + T_{1\ast}).\)

By (8.61), Lemma 8.3 and the comparison theorem, we obtain

\[(8.62) \quad w_\ast(x, t) \leq w(t, x, t) \leq v(x, t)\]
\[\text{for } t \geq T_3 \text{ and a.e. } x \in \left[-N \xi(t), N \xi(t)\right].\]

On the other hand, we see by Lemma 4.5 and Lemma 8.3 that

\[(8.63) \quad w(x, t) \leq (2n-1)^{-1/2} \left(t + d_1\right)^{-1/2}\]
\[\text{for } (x, t) \in Q^r_{T_3},\]

with \(d_1 = (2n-1)^{-1}(|w(\cdot, T_3)|_{\infty, \mathbb{R}})^{-2n+1} - T_3.\)

By (8.63), Lemma 4.1 and Lemma 8.3, we obtain

\[(8.64) \quad u(x, t) \leq (2n-1)^{-1/2} \left(t + d_1\right)^{-1/2}\]
\[\text{for } t \geq T_3 \text{ and a.e. } x \in \mathbb{R}.\]

By (8.62) and (8.63), there exists a positive constant \(T_{2\ast} \geq T_3\) such that

\[(8.65) \quad u(x, t) \leq (2n-1)^{-1/2} \left(t + d_1\right)^{-1/2}(2n-1) < \alpha(t + T_{1\ast})\]
\[\leq w_\ast(x, t) \leq w_\ast(x, t) \leq v(x, t)\]
\[\text{for } t \geq T_{2\ast} \text{ and a.e. } x \in \left[-N \xi(t), N \xi(t)\right].\]

Here, let

\[v_\ast(x, t) = \begin{cases} 
\frac{8}{9} (\frac{m-1}{2m(m+1)})^{1/(m-1)} A t^{1/(m-1)} t^{-1/(m+1)} & \text{if } x \in [0, T_{2\ast}] \\
\frac{1}{2} (x_\ast^{(2)}(x, t) \text{ if } \left| x \right| \in (N \xi(t), A t^{1/(m+1)}-1) \\
\end{cases}\]
and let
\[ u^*(x; t) = \max \{w(x, t), t^{-2-1/(m+1)}\}. \]

Then, by Lemma 8.1, there exists a positive constant \( T_{3*} \geq T_{2*} \) such that for \( t \geq T_{3*} \), \( \psi^* \) and \( u^* \) satisfy the properties of this lemma except these regularities.

Let \( \psi^*(\cdot; t) \) be the convolution of \( \psi^*(\cdot; t) \) and an appropriate mollifier and let \( u^*(\cdot; t) \) be the convolution of \( u^*(\cdot; t) \) and an appropriate mollifier. Then, for \( t \geq T_{3*} \), \( \psi(\cdot; t) \) and \( u(\cdot; t) \) satisfy the properties of this lemma.

In case of \( 2n < m \).

Let \( \rho \) be the function in Lemma 8.2.

Let \( T_{4*} \) be the positive constant such that
\[ T_{4*} \geq T_1 \quad \text{and} \quad T_2, \]
\[ 8 \frac{m-1}{9 \left(2m(m+1)\right)^{1/(m-1)}} \quad A^{2/(m-1)} t^{-1/(m+1)} \quad (2n-1)^{-1/(2m-1)} \quad t^{-1/(2n-1)} \]
for any \( t \geq T_{4*} \),

where \( T_1 \) and \( A \) are the constants in Lemma 8.1 and \( T_2 \) is the constant in Lemma 8.2.

We set
\[ w^*(x) = \max \{w(x, T_{4*}), \min(\rho(T_{4*}), (2n-1)^{-1/(2m-1)} T_{4*}^{-1/(2n-1)})\} \]
for \( x \in \mathbb{R} \).

By Theorem 0 in [8], there exists a unique positive classical solution \( w^* \) of (1.4) with \( w^*(\cdot, T_{4*}) = w^* \) in \( \mathbb{R} \times [T_{4*}, \infty) \) satisfying the following properties:
\[ (2n-1)^{-1/(2m-1)} (t+d_4)^{-1/(2n-1)} \leq w^*(x, t) \leq (2n-1)^{-1/(2m-1)} t^{-1/(2n-1)} \]
for \( (x, t) \in \mathbb{R} \times [T_{4*}, \infty) \),

where
\[ d_4 = (2n-1)^{-1} \min(\rho(T_{4*}), (2n-1)^{-1/(2m-1)} T_{4*}^{-1/(2n-1)})^{-2n+1} \quad T_{4*}. \]

By Lemma 4.1 and the comparison theorem, the solution \( w^* \) satisfies
\[ u(x, t) \leq w(x, t) \leq w^*(x, t) \quad \text{for} \quad t \geq T_{4*} \]
and a.e. \( x \in \mathbb{R} \),

and \( w^* \) is a generalized subsolution of (3.1) with \( P = u^* \) and \( q = n \) in \( G_{T_{4*}} \). Then, by (8.66), (8.67), Lemma 4.4 and the definitions of \( \rho \) and \( T_{4*} \), we have
\[ u(x, t) \leq w^*(x, t) \leq v(x, t) \]
\[ u(x, t) \leq w^*(x, t) \leq w(x, t) \]
for \( t \geq T^* \) and a.e. \( x \in [-At^{(m+1)}, At^{(m+1)}] \).

Now, by a result in Introduction, Lemma 4.1 and Lemma 8.1 there exist two positive constants \( b \) and \( T_{s^*} \geq T_{s^*} \) such that

\[
\text{supp } u(\cdot, t) \subseteq \left[ -\frac{b}{2}, \frac{b}{2} \right] \text{ for } t \geq T_{s^*},
\]

\[ A T_{s^*}^{(m+1)} > b \geq 1, \]

\[ v(x, t) > 5 t^{-1/(2a-1)} \]

for \( t \geq T_{s^*} \) and \( x \in [-At^{(m+1)}, -b] \cup [b, At^{(m+1)}] \).

We set

\[ \xi(t) = 2b \left( \frac{T+1}{t+1} \right)^{\frac{(m-2a)/(2a-1)}{2}} - b, \]

\[ l(t) = \frac{1}{S} (2b - \xi(t))^{-p}, \]

with \( P = 2Kb^{2}(T+1)^{(m-2a)/(2a-1)} + 2 \), where \( T > T_{s^*} \) \( K \) and \( S \) are constants chosen later.

We set further

\[ W(x, t) = \frac{m}{m-1} (w^*(x, t))^{m-1}, \]

\[ U(x, t) = \frac{m}{m-1} (w^*(x, t))^{m-1} \left( 1 - l(x - \xi)^{l} \right) \]

and

\[ V(x, t) = \frac{m}{m-1} (w^*(x, t))^{m-1} \left( 1 + l(x - \xi)^{p} \right). \]

Now, let us consider the differential operator:

\[
\mathcal{M}(F; H) = F_t - (m-1) FF_{xx} - (F_x)^2 + \lambda H^{(m-1)} F^{(n+m-3)/(m-1)},
\]

with \( \lambda = m((m-1)/m)(2a+m-3)/(m-1) \).

Then we observe that

(8.69) \( \mathcal{M}(W; W) = 0 \) in \( \mathbb{R} \times (T_{s^*}, \infty) \).

We shall find \( T, K \) and \( S \) such that

(8.70) \( \mathcal{M}(U; V) \geq 0 \) in \( [-b, b] \times [T, \infty) \),

(8.71) \( \mathcal{M}(V; U) \leq 0 \) in \( [-b, b] \times [T, \infty) \).

First, we shall consider (8.70).

We see
(8.72) \[ \mathcal{M}(U; V) = [W_t(1-l(x-\xi)\xi)_t - (m-1) WW_{xx}(1-l(x-\xi)\xi)_t]^2 \]
\[-(W_x)^2(1-l(x-\xi)\xi)_t^2 \]
\[+\lambda W^{2(n+m-2)/(m-1)}(1+l(x-\xi)\xi)_t^2(1-l(x-\xi)\xi)^{(m-3)/(m-1)} \]
\[+ [W - l'(x-\xi)\xi + Pl\xi'(x-\xi)^{P-1} - W P^2 P'(x-\xi)\xi^{P-2} \]
\[+(m-1) P(P-1) W^2 l(x-\xi)\xi^{P-1}(1-l(x-\xi)\xi)] \]
\[+[2m Pl(x-\xi)_t W W_t(1-l(x-\xi)\xi)] \]
\[= I + II + III \text{ in } [-b, b] \times [T, \infty). \]

Let \( S>1. \) Then, we see for \( y \in [0, 1/S] \)

(8.73) \[ (1-y)^{(1-\frac{1}{2})/(\frac{m}{m-1})} + (1+y)^{(1-\frac{1}{2})/(\frac{m}{m-1})} \geq \left(1+\frac{1}{S}\right)^{(1-\frac{1}{2})/(\frac{m}{m-1})} \times \left\{ \frac{1}{\frac{1}{m-1}} \left(1-\frac{2n-2}{S} - \left(\frac{2n-2}{S} - \frac{1}{S^2}\right) \right) \right\} y. \]

Let

\[ C_{m0}(S) = \left(1+\frac{1}{S}\right)^{(1-\frac{1}{2})/(\frac{m}{m-1})} \times \left\{ \frac{1}{\frac{1}{m-1}} \left(1-\frac{2n-2}{S} - \left(\frac{2n-2}{S} - \frac{1}{S^2}\right) \right) \right\} \]

and let \( S \) be a constant such that

(8.74) \[ C_{m0}(S) \geq \frac{1}{2(m-1)}. \]

By (8.69), (8.73) and (8.74) we have

(8.75) \[ I \geq \frac{\lambda}{2(m-1)} W^{(2n+m-2)/(m-1)} - (m-1) WW_{xx} + (W_x)^2 \]
\[ \times l(x-\xi)\xi^{P-2}(1-l(x-\xi)\xi)_t^2 \text{ in } [-b, b] \times [T, \infty), \]

and hence, by (8.75) and Lemma 6.3, there exists a positive constant \( T>T_{s*} \)

such that

(8.76) \[ I \geq 0 \text{ in } [-b, b] \times [T, \infty). \]

To treat \( II \) we observe

(8.77) \[ l'(t) \leq 0. \]

Then, we have

(8.78) \[ II \geq (m-1) P(P-1) l(x-\xi)\xi^{P-2} W \left\{ 1 - \left(\frac{P}{(m-1)(P-1)}\right) l(x-\xi)\xi^P \right\} \]
\[-\frac{1}{(m-1)(P-1)} W(x-\xi)_t + 2b \left(\frac{m-2n}{2n-1}\right) (T+1)^{(m-2n)/(2n-1)} \]
Here, by choosing $T$ larger, if necessary, we have

\[(8.79)\]
\[W(x, t) \geq \left( \frac{2}{3} \right)^{m-1} \frac{m}{m-1} (2n-1)^{-\frac{m}{2}+\frac{1}{m}} t^{-(m-1)/(2m-1)} \text{ for } t \geq T.\]

Now, we choose $S, K$ such that

\[(8.80)\]
\[S \geq 3 \left( 1 + \frac{2}{m-1} \right)\]

and

\[(8.81)\]
\[K \geq \left( \frac{3}{2} \right)^{m-1} \frac{3}{m} (2n-1)^{(m+2n-2)/(2m-1)} (m-1).\]

Then, by (8.78)–(8.81) we obtain

\[(8.82)\]
\[II \geq \frac{1}{3} (m-1) P(P-1) l(x-\xi)^2 W^2\]
\[\text{ in } [-b, b] \times [T, \infty).\]

It follows from (8.80)–(8.82) that

\[(8.83)\]
\[II + III \geq (m-1) P(P-1) l(x-\xi)^2 W^2 \left\{ \frac{1}{3} \left( m \right) \frac{m}{m-1} K \frac{|W_x|}{W} \right\}\]
\[\text{ in } [-b, b] \times [T, \infty).\]

By choosing $T$ larger, if necessary, it follow from Lemma 6.2, (8.79) and (8.83) that

\[(8.84)\]
\[II + III \geq 0 \text{ in } [-b, b] \times [T, \infty).\]

Thus, from (8.76) and (8.84) we conclude (8.70).

If we choose $S$ satisfying (8.74), (8.80) and

\[(8.85)\]
\[S > \left( 1 - \left( \frac{3}{4} \right)^{m-1} \right)^{-1}, (2^{m-1}-1)^{-1},\]

then by (8.79) we have that

\[(8.86)\]
\[U(\pm b, t) > \frac{m}{m-1} \left( \frac{1}{2} \right)^{m-1} (2n-1)^{-(m-1)/(2n-1)} t^{-(m-1)/(2n-1)} \]
\[\text{ in } [-b, b] \times [T, \infty).\]

By the definitions of the functions $l$ and $\xi$, we see

\[(8.87)\]
\[V(x, T) = U(x, T) = W(x, T) \text{ on } [-b, b].\]
By a quite similar argument obtaining (8.70), since we can prove (8.71), we shall omit the proof of (8.71).

Further, (8.67) and (8.85) we have

\[(8.88)\]
\[V(\pm b, t) < 2^{m-1} \frac{m}{m-1} (2n-1)^{-(m-1)/2n-1} t^{-(m-1)/2n-1} \]

for \( t \geq T \).

Thus, (8.68), (8.70), (8.71), (8.86)-(8.88) and Lemma 8.4 we obtain

\[(8.89)\]
\[u(x, t) \leq \left( \frac{m-1}{m} U(x, t) \right)^{1/(m-1)} \leq w^*(x, t) \]
\[\leq \left( \frac{m-1}{m} V(x, t) \right)^{1/(m-1)} \leq v(x, t) \]

for \( t \geq T \) and a.e. \( x \in [-b, b] \).

Similarly, setting

\[\hat{U}(x, t) = W(x, t) \left( 1 - l(-x - \xi(t)) \right)^p\]

and

\[\hat{V}(x, t) = W(x, t) \left( 1 + l(-x - \xi(t)) \right)^p\]

these have the same estimates as \( U \) and \( V \), respectively, and we obtain

\[(8.90)\]
\[u(x, t) \leq \left( \frac{m-1}{m} \hat{U}(x, t) \right)^{1/(m-1)} \leq w^*(x, t) \]
\[\leq \left( \frac{m-1}{m} \hat{V}(x, t) \right)^{1/(m-1)} \leq v(x, t) \]

for \( t \geq T \) and a.e. \( x \in [-b, b] \).

By the definitions of the functions \( l \) and \( \xi \), there exists a positive constant \( \theta \) such that

\[(8.91)\]
\[l(t) (x - \xi(t)) \geq \frac{1}{S} \left( \frac{5}{2} b \right)^p \left( \frac{b}{4} \right)^p \]
\[= \frac{1}{S} \left( \frac{1}{10} \right)^p \quad \text{in } [0, b] \times [T_\theta, \infty) .\]

Here, let

\[v^*(x; t) = \begin{cases} 
8 \left( \frac{m-1}{2m(m+1)} \right)^{1/(m-1)} A^{2/(m-1)} t^{-1/(m+1)} \\
9 \frac{m-1}{2m(m+1)} \right)^{1/(m-1)} A^{2/(m-1)} t^{-1/(m+1)} \\
\text{if } |x| \in [A t^{1/(m+1)} - 1, \infty) \\
\text{if } |x| \in (b, A t^{1/(m+1)} - 1) \\
\text{if } |x| \in [0, b] \\
\end{cases} \]

and let
then, by (8.68) and (8.91), there exists a positive constant $T_{7*} \geq T_{6*}$ such that for $t \geq T_{7*}$, $v_*(\cdot; t)$ and $u_*(\cdot; t)$ satisfy the properties of this lemma except these regularities.

Let $v(\cdot; t)$ be the convolution of $v_*(\cdot; t)$ and an appropriate mollifier and let $u(\cdot; t)$ be the convolution of $u_*(\cdot; t)$ and an appropriate mollifier. Then, for $t \geq T_{7*}$, $v(\cdot; t)$ and $u(\cdot; t)$ satisfy the properties of this lemma. Q.E.D.

Lemma 8.6. Let $T_6$ be the constant such that

\[
T_6 \geq \max \{ T_9, 1, T_6^{(m+2-2\varepsilon)/(m+1)} \left( \frac{1}{2} H \right) \left( \frac{2n-1}{m+1} \right) > \frac{1}{m+1},
\]

\[
T_6^{(m+1)/2} < \frac{1}{9} H, \quad \frac{1}{m+1} > 2T_6^{2n+m/(m+1)} |v_0|_{\infty, b}^{-1},
\]

\[
AT_6^{1/(m+1)} > N(\xi(T_6) and b),
\]

where $A, N$ and $b$ are the constants in Lemma 8.1, Lemma 8.3 and Lemma 8.4 respectively, $\xi$ is the function in Lemma 8.1 and $H= ((m-1)/(2n(m+1)))^{1/(m+1)} A^{2n/(m+1)}$.

Let the constant $T$ in (I.B.) be $T_6$ and set $u_0 = u(\cdot; T_6)$ and $v_0 = v(\cdot; T_6)$ in (I.B.), where $u$ and $v$ are the functions in Lemma 8.5.

Then, there exists a unique pair of positive classical solutions $u_6$ and $v_6 \in H^{2+\varepsilon, 1+\varepsilon}(G_{T_6})$ of (I.B.), satisfying the following properties:

\[ u(x, t) \leq u_6(x, t) \leq v_6(x, t) \leq v(x, t) \]

for $t \geq T_6$ and a.e. $x \in [-At^{1/(m+1)}, At^{1/(m+1)}]$ and there exists a positive constant $h_6$ such that

\[ v_6(x, t) \geq h_6 \cdot t^{-1/(m+1)} \]

for $t \geq T_6$ and a.e. $x \in [-At^{1/(m+1)}, At^{1/(m+1)}]$,

where $u$ and $v$ are the solutions of (1.1) and (1.2).

Proof. Since $u(\cdot; T_6)$ and $v(\cdot; T_6)$ are positive functions satisfying the compatibility condition of first order for (I.B.), then by Theorem 7.1 of Section 7 in [14] and the change of variables there exists a unique pair of positive classical solutions $u_6$ and $v_6 \in H^{2+\varepsilon, 1+\varepsilon}(G_{T_6})$ of (I.B.).

Let us consider the following initial boundary value problem:

\[ w_{ij} = (w^m)_{xj} - w^m \text{ in } G_{T_6}, \]

\[ w(\pm At^{1/(m+1)}, t) = \frac{1}{2} Ht^{-1/(m+1)} \text{ on } [T_6, \infty), \]
There exists a positive function $w_0 \in H^{2+\beta}(\mathbb{R}^+ \times \Omega)$ satisfying the compatibility condition of first order for (8.92)–(8.94) and the following property:

\[(8.95) \quad u(\cdot, T_0) \leq w_0 \leq v(\cdot, T_0) \text{ on } [-AT_0^{1/(m+1)}, AT_0^{1/(m+1)}].\]

Then, by Theorem 4.1 of Section 4 in [14] and the change of variables, there exists a unique positive classical solution $w_0 \in H^{2+\beta,1+\beta/2}_{\text{loc}}(G_{T_0})$ for (8.92)–(8.94). By Lemma 8.5 and (8.95) we see

$w(\cdot, T_0) \leq w_0$ on $[-AT_0^{1/(m+1)}, AT_0^{1/(m+1)}]$,

and hence, by Lemma 4.5,

\[(8.96) \quad w \leq w_0 \text{ in } G_{T_0}.\]

Let $\mathcal{L}(p; q)$ be the differential operator defined in (8.49).

By Theorem 4.1 of Section 4 in [14] and the change of variables there exists a unique positive classical solution $v_1 \in H^{2+\beta,1+\beta/2}_{\text{loc}}(G_{T_0})$

\[(8.97) \quad \mathcal{L}(w; w_0) = 0 \text{ in } G_{T_0},\]

\[(8.98) \quad w(\pm AT_0^{1/(m+1)}, t) = \frac{8}{9} H_t^{-1/(m+1)} \text{ on } [T_0, \infty),\]

\[(8.99) \quad w(\cdot, T_0) = v(\cdot, T_0) \text{ on } [-AT_0^{1/(m+1)}, AT_0^{1/(m+1)}].\]

Since $v_1$ is a subsolution of $\mathcal{L}(v_1; u) = 0$ in $G_{T_0}$, Lemma 4.4, Lemma 8.1, Lemma 8.6, the comparison theorem and (8.95) give

\[(8.100) \quad w_0(x, t) \leq v_1(x, t) \leq v(x, t) \quad \text{for } t \geq T_0 \text{ and a.e. } x \in [-AT_0^{1/(m+1)}, AT_0^{1/(m+1)}].\]

By a similar argument, there exists a unique positive classical solution $u_1 \in H^{2+\beta,1+\beta/2}_{\text{loc}}(G_{T_0})$ of

\[(8.101) \quad \mathcal{L}(w; v_1) = 0 \text{ in } G_{T_0},\]

\[(8.102) \quad w(\pm AT_0^{1/(m+1)}, t) = t^{-1/(m+1)-2} \text{ on } [T_0, \infty),\]

\[(8.103) \quad w(\cdot, T_0) = u(\cdot, T_0) \text{ on } [-AT_0^{1/(m+1)}, AT_0^{1/(m+1)}].\]

By (8.100) $u_1$ is a supersolution for $\mathcal{L}(u_1; v) = 0$ in $G_{T_0}$, and Lemma 4.5, Lemma 8.6, the comparison theorem and (8.96) give

\[(8.104) \quad u(x, t) \leq u_1(x, t) \leq w_0(x, t) \quad \text{for } t \geq T_0 \text{ and a.e. } x \in [-AT_0^{1/(m+1)}, AT_0^{1/(m+1)}].\]
By a similar argument, there exist two sequences of positive functions \( u_j \) and \( v_j \in H^{2+\beta,1+1/\beta}(G_{T_0}) \) satisfying the following property:

For each \( j \geq 2, v_j \) is a unique classical solution for \( \mathcal{L}(v_j; u_{j-1}) = 0 \) with (8.98) and (8.99) in \( G_{T_0} \) and \( u_j \) is a unique classical solution for \( \mathcal{L}(u_j; v_j) = 0 \) with (8.102) and (8.103) in \( G_{T_0} \).

By (8.100), (8.104), Lemma 4.4, Lemma 4.5 and the comparison theorem we see

\[
\begin{align*}
  u(x, t) &\leq \cdots \leq u_2(x, t) \leq u_1(x, t) \leq w_2(x, t) \\
  &\leq v_2(x, t) \leq v_1(x, t) \leq \cdots \leq v(x, t) \\
  \text{for } t \geq T_0 \text{ and a.e. } x \in [-AT^{(m+1)}, AT^{(m+1)}].
\end{align*}
\]

Similarly as in (8.57), (8.58) we can prove

\[
\begin{align*}
  v_j(x, t) &\to v(x, t) \text{ for } (x, t) \in G_{T_0}, \\
  u_j(x, t) &\to u(x, t) \text{ for } (x, t) \in G_{T_0},
\end{align*}
\]

and

\[
\begin{align*}
  u(x, t) &\leq u_2(x, t) \leq u_1(x, t) \leq v_2(x, t) \leq v_1(x, t) \leq v(x, t) \\
  \text{for } t \geq T_0 \text{ and a.e. } x \in [-AT^{(m+1)}, AT^{(m+1)}].
\end{align*}
\]

Setting

\[
\begin{align*}
  \dot{v}(y, s) &= e^{s/(m+1)} v_b(e^{s/(m+1)} y, e^s) \\
  \dot{u}(y, s) &= e^{s/(m+1)} u_b(e^{s/(m+1)} y, e^s),
\end{align*}
\]

we see that the functions \( \dot{v} \) and \( \dot{u} \in H^{2+\beta,1+1/\beta}([-A, A] \times [\log T_0, \infty)) \) satisfy

\[
\begin{align*}
  \dot{v}_b &= (\dot{v}_b)_{yy} + \frac{y}{m+1} \dot{v}_y + \frac{1}{m+1} \dot{v} - e^{s/(m+2-2A/(m+1))} \dot{u}^n \dot{v}^m \\
  &\quad \text{in } [-A, A] \times [\log T_0, \infty), \\
  \dot{u}_b &= (\dot{u}_b)_{yy} + \frac{y}{m+1} \dot{u}_y + \frac{1}{m+1} \dot{u} - e^{s/(m+2-2A/(m+1))} \dot{v}^n \dot{u}^m \\
  &\quad \text{in } [-A, A] \times [\log T_0, \infty), \\
  \dot{v}(\pm A, s) &= \frac{8}{9} H \text{ on } [\log T_0, \infty), \\
  \dot{u}(\pm A, s) &= e^{-2s} \text{ on } [\log T_0, \infty), \\
  \dot{v}(y, \log T_0) &= T_0^{1/(m+1)} v(y T_0^{1/(m+1)}; T_0) \text{ on } [-A, A], \\
  \dot{u}(y, \log T_0) &= T_0^{1/(m+1)} u(y T_0^{1/(m+1)}; T_0) \text{ on } [-A, A].
\end{align*}
\]

Let \( S^* \) be an arbitrary positive constant such that \( S^* > \log T_0 \) and let \( \chi \) be an
arbitrary positive constant such that

\[(8.107)\quad 1 - \frac{\lambda}{m+1} \geq e^{-2\lambda/(m+1)}, \quad 2 > e^{(m+2-2\lambda)\lambda/(m+1)}, \quad \lambda 2me^{(m-2\lambda+2)\lambda/(m+1)}(|v_0|_{L^\infty R} e^{2\lambda/(m+1)})^{m-1} \leq 1.\]

Let \( S_j = \lambda j + \log T_0. \) Then, there exists a finite sequence of positive classical solutions \( v_\lambda(\cdot, S_j) \in H^{2+\beta}([-A, A]), j = 1, \ldots, [\lambda^{-1}(S^* - \log T_0)], \) of the problem

\[(8.108)\quad \psi_\lambda(\cdot, S_j) - \lambda (\psi_\lambda)_y(\cdot, S_j) - \frac{\lambda}{m+1} y \psi_\lambda(\cdot, S_j) \]

\[- \frac{\lambda}{m+1} \psi_\lambda(\cdot, S_j) = \psi_\lambda(\cdot, S_{j-1}) - \lambda e^{(m+2-2\lambda)\lambda/(m+1)} \]

\[\times \psi_\lambda(\cdot, S_{j-1}) \psi_\lambda(\cdot, S_{j-1}) \text{ on } [-A, A]\]

with

\[(8.109)\quad \psi_\lambda(\pm A, S_j) = \frac{8}{9} H,\]

where \( \psi_\lambda(\cdot, S_0) = \psi(\cdot, S_0) \) on \([-A, A]\). Also, for \( j = 1, \ldots, [\lambda^{-1}(S^* - \log T_0)], \) there exists a positive classical solution \( \psi_\lambda(\cdot, S_j) \in H^{2+\beta}([-A, A]) \) of the problem

\[(8.110)\quad \psi_\lambda(\cdot, S_j) - \lambda (\psi_\lambda)_y(\cdot, S_j) - \frac{\lambda}{m+1} y \psi_\lambda(\cdot, S_j) \]

\[- \frac{\lambda}{m+1} \psi_\lambda(\cdot, S_j) = \psi_\lambda(\cdot, S_{j-1}) - \lambda e^{(m+2-2\lambda)\lambda/(m+1)} \]

\[\times \psi_\lambda(\cdot, S_{j-1}) \psi_\lambda(\cdot, S_{j-1}) \text{ on } [-A, A]\]

with

\[(8.111)\quad \psi_\lambda(\pm A, S_j) = e^{-2j/},\]

where \( \psi_\lambda(\cdot, S_0) = \psi(\cdot, S_0) \) on \([-A, A]\).

In fact, by (8.109), see see

\[ \psi_\lambda(y, S_0) - \lambda e^{(m+2-2\lambda)\lambda/(m+1)} \psi_\lambda(y, S_0) \psi_\lambda(y, S_0) > 0 \text{ for } y \in [-A, A], \]

\[ \psi_\lambda(y, S_0) - \lambda e^{(m+2-2\lambda)\lambda/(m+1)} \psi_\lambda(y, S_0) \psi_\lambda(y, S_0) > 0 \text{ for } y \in [-A, A] \]

and

\[ 0 \leq \psi_\lambda(y, S_0) \leq \psi_\lambda(y, S_0) \leq |v_0|_{L^\infty R} T_0^{1/(m+1)} \text{ for } y \in [-A, A]. \]

Hence by Theorem 5.1 of Section 8 in [18] and the comparison theorem we find a required unique positive classical solution \( \psi_\lambda(\cdot, S_0), \psi_\lambda(\cdot, S_0) \in H^{2+\beta}([-A, A]) \) satisfying

\[ 0 \leq \psi_\lambda(y, S_0) \leq \psi_\lambda(y, S_0) \leq \psi_\lambda(y, S_0) \leq |v_0|_{L^\infty R} T_0^{1/(m+1)} \text{ for } y \in [-A, A]. \]
By induction, there exist two finite sequences of the positive classical solutions \( \dot{u}_\lambda(\cdot, S_j) \) and \( \dot{\theta}_\lambda(\cdot, S_j) \in H^{2+\#}([-A, A]) \) with the property

\[
0 < \dot{u}_\lambda(y, S_j) \leq \dot{\theta}_\lambda(y, S_j) \leq e^{2j/(m+1)} |v_0|_\infty R T_\delta^{(m+1)}
\]

for \( j = 1, 2, \ldots, [\lambda^{-1}(S^*-\log T_\delta)] \)

and \( y \in [-A, A] \).

Further, we can show

\[
|\dot{u}_\lambda(\cdot, S_j)|_{-T-A, A} \leq \frac{1}{2} H
\]

for \( j = 0, 1, \ldots, [\lambda^{-1}(S^*-\log T_\delta)] \).

Indeed, setting \( E_j = |\dot{u}_\lambda(\cdot, S_j)|_{-T-A, A} \) for \( j = 0, 1, \ldots, [\lambda^{-1}(S^*-\log T_\delta)] \), we see by (8.109), (8.112), Lemma 8.5 and the definitions of \( T_\delta \) that

\[
E_1 \leq \frac{1}{2} H
\]

and by induction, we have (8.113).

For each \( j = 0, 1, \ldots, [\lambda^{-1}(S^*-\log T_\delta)] \), we set \( d_j = \min \{ \dot{\theta}_\lambda(y, S_j) : y \in [-A, A] \} \).

We shall show that

\[
d_j - E_j \geq \delta_0
\]

for \( j = 0, 1, \ldots, [\lambda^{-1}(S^*-\log T_\delta)] \),

where \( \delta_0 = \min(d_0 - E_0, 7H/18) \).

Let \( y_{j*} \) and \( y_{j*}^\# \) be the points satisfying \( d_j = \dot{\theta}_\lambda(y_{j*}, S_j) \) and \( E_j = \dot{u}_\lambda(y_{j*}^\#, S_j) \), respectively.

In case of \( y_{j*}^\# \) and \( y_{j*} \in (-A, A) \), we observe by (8.109) and (8.112) that

\[
(d_j - E_j) \left(1 - \frac{\lambda}{m+1} \right)
\]

\[
\geq (d_{j-1} - \lambda e^{(m+2-2\eta)/(m+1)} S_j \dot{E}_{j-1}^\# - (E_{j-1} - \lambda e^{(m+2-2\eta)/(m+1)} S_j \dot{E}_{j-1}^\#) - (E_{j-1} - \lambda e^{(m+2-2\eta)/(m+1)} S_j \dot{E}_{j-1}^\#)
\]

In case of \( y_{j*} = \pm A \), we get by (8.113)

\[
d_j - E_j \geq \frac{8}{9} H - \frac{1}{2} H = \frac{7}{18} H
\]

In case of \( y_{j*} = \pm A \) and \( y_{j*}^\# = \pm A \), we have by (8.112) and the definitions of \( \lambda \) and \( T_\delta \),

\[
\dot{u}_\lambda(x, S_{j-1})^\# \dot{\theta}_\lambda(x, S_{j-1})^{(n-1)} \leq 2e^{(m+2-2\eta)/(m+1)} S_{j-1} e^{-2S_{j-1}^{(n-1)}} |v_0|_{\infty R} T_{\delta}^{(n-1)/(m+1)} e^{(m+2-2\eta)/(m+1)} S_j^{-1} \log T_\delta
\]
\[
\frac{1}{m+1} \quad \text{for } y \in [-A, A]
\]
and that \( E_{j-1} \geq e^{-2s_j} > e^{-2s_j} = E_j \)

Then, since
\[
d_j \left(1 - \frac{\lambda}{m+1}\right) \geq d_{j-1} - \lambda 2e^{m/(m+1)s_{j-1} - 2s_j} \cdot |v_0|^{\frac{1}{m+1}} \cdot d_{j-1}
\]
we have

\[d_j - E_j > d_{j-1} - E_j > d_{j-1} - E_{j-1}\]  

(8.117)

By (8.115)–(8.117) we conclude (8.114).

For \( j = 1, 2, \ldots, [\lambda^{-1}(S^* - \log T_{\theta})]\), let
\[
f_j(y) = \int_{s_{j-1}}^{s_j} (\dot{\theta}_s(y, s) - \dot{\theta}_s(y, S_j)) \, ds
\]
\[+ \lambda e^{(m-2n+2)/(m+1)s_j} \cdot \{ \dot{u}^\nu(y, S_{j-1}) \dot{\theta}^\nu(y, S_{j-1})
\]
\[-\dot{u}^\nu(y, S_j) \dot{\theta}^\nu(y, S_j) \}

and let
\[
g_j(y) = \int_{s_{j-1}}^{s_j} (\dot{u}_s(y, s) - \dot{u}_s(y, S_j)) \, ds
\]
\[+ \lambda e^{(m-2n+2)/(m+1)s_j} \cdot \{ \dot{u}^\nu(y, S_{j-1}) \dot{u}^\nu(y, S_{j-1})
\]
\[-\dot{u}^\nu(y, S_j) \dot{u}^\nu(y, S_j) \}.

Then we observe that for \( y \in [-A, A] \)

\[\dot{\theta}(y, S_j) - \lambda (\dot{w}^\nu)(y, S_j) - \frac{\lambda}{m+1} y \dot{\theta}_s(y, S_j)
\]
\[\frac{\lambda}{m+1} \cdot \dot{\theta}(y, S_j) = \dot{\theta}(y, S_{j-1}) - \lambda e^{(m-2n+2)/(m+1)s_j}
\]
\[\times \dot{u}^\nu(y, S_{j-1}) \dot{\theta}^\nu(y, S_{j-1}) + f_j(y)
\]
for \( j = 1, 2, \ldots, [\lambda^{-1}(S^* - \log T_{\theta})]\)

and

\[\ddot{u}(y, S_j) - \lambda (\ddot{u}^\nu)(y, S_j) - \frac{\lambda}{m+1} y \dot{u}_s(y, S_j)
\]
\[\frac{\lambda}{m+1} \cdot \dot{u}(y, S_j) = \dot{u}(y, S_{j-1}) - \lambda e^{(m-2n+2)/(m+1)s_j} \dot{\theta}^\nu(y, S_{j-1})
\]
\[\times \dot{u}^\nu(y, S_{j-1}) + g_j(y).
\]

Let
We subtract (8.108) from (8.118) and multiply $\varphi_k(\dot{\psi}(y, S_j) - \dot{\psi}_k(y, S_j))$. We shall denote by (8.120) the resulted equation. Similarly we subtract (8.109) from (8.119) and multiply $\varphi_k(\dot{u}(y, S_j) - \dot{u}_k(y, S_j))$. We shall denote by (8.121) the resulted equation. Adding (8.120) and (8.121) and integrating over $[-A, A]$ we have

$$
\int_{-A}^{A} (\dot{\psi}(y, S_j) - \dot{\psi}_k(y, S_j)) \varphi_k(\dot{\psi}(y, S_j) - \dot{\psi}_k(y, S_j)) \, dy \\
+ \int_{-A}^{A} (\dot{u}(y, S_j) - \dot{u}_k(y, S_j)) \varphi_k(\dot{u}(y, S_j) - \dot{u}_k(y, S_j)) \, dy \\
\leq \left(1 - \frac{\lambda}{m+1}\right)^{-1} \left\{ |\dot{\psi}(\cdot, S_{j-1}) - \dot{\psi}_k(\cdot, S_{j-1})|_{1,[-A, A]} \\
+ |\dot{u}(\cdot, S_{j-1}) - \dot{u}_k(\cdot, S_{j-1})|_{1,[-A, A]} \right\} \\
+ \left(1 - \frac{\lambda}{m+1}\right)^{-1} \left\{ |f_j|_{1,[-A, A]} + |g_j|_{1,[-A, A]} \right\}.
$$

Letting $k \to \infty$, we have that

$$
(8.122) \quad |\dot{\psi}(\cdot, S_j) - \dot{\psi}_k(\cdot, S_j)|_{1,[-A, A]} + |\dot{u}(\cdot, S_j) - \dot{u}_k(\cdot, S_j)|_{1,[-A, A]} \\
\leq \left(1 - \frac{\lambda}{m+1}\right)^{-1} \left\{ |\dot{\psi}(\cdot, S_{j-1}) - \dot{\psi}_k(\cdot, S_{j-1})|_{1,[-A, A]} \\
+ |\dot{u}(\cdot, S_{j-1}) - \dot{u}_k(\cdot, S_{j-1})|_{1,[-A, A]} \right\} \\
+ \left(1 - \frac{\lambda}{m+1}\right)^{-1} \left\{ |f_j|_{1,[-A, A]} + |g_j|_{1,[-A, A]} \right\}.
$$

Setting

$$
L_0 = \max \left\{ |\dot{u}|_{2+\beta, [-A, A] \times \{\log \tau e, s^*\}}, \, |\dot{\psi}|_{2+\beta, [-A, A] \times \{\log \tau e, s^*\}} \right\}.
$$

We obtain by (8.107) that

$$
(8.123) \quad |f_j|_{1,[-A, A]} + |g_j|_{1,[-A, A]} \leq \frac{2L_0 A}{1 + \frac{\beta}{2}} \chi^{1+(\beta/2)} \\
+ 4n A e^{\frac{m}{2}/(m+1)} s^* L_0^\beta \chi^2 \leq L \chi^{1+(\beta/2)},
$$

where

$$
L = 2 \left(1 + \frac{\beta}{2}\right)^{-1} L_0 A + 4n e^{\frac{m}{2}/(m+1)} s^* L_0^\beta (m+1)^{1-(\beta/2)} A.
$$
By (8.107), (8.122) and (8.123) we have

\[(8.124) \quad \left| \hat{\theta}(\cdot, S_j) - \hat{\theta}_*(\cdot, S_j) \right|_{L^1([-A, A])} + \left| \hat{u}(\cdot, S_j) - \hat{u}_*(\cdot, S_j) \right|_{L^1([-A, A])} \leq 2S_* e^{2(m+1)S_*} L \lambda^{\frac{3}{2}} \text{ for } j = 1, 2, \ldots, [\lambda^{-1}(S_* - \log T_0)].\]

Letting \( \lambda \to 0 \) we have by (8.114), (8.124)

\[
\min \{ \hat{\theta}(y, s) : y \in [-A, A] \} - |\hat{u}(\cdot, s)|_{L^1([-A, A])} \geq \delta_0 > 0 \quad \text{for } s \geq \log T_0.
\]

Thus, by changing the variables \( \varepsilon^s = t \), \( ye^{s/(m+1)} = x \), we conclude

\[
\min \{ v_\delta(x, t) : x \in I(t) \} - |u_\delta(\cdot, t)|_{L^1(t)} \geq \delta_0 t^{-\frac{1}{2}(m+1)} \text{ for any } t \geq T_0,
\]

where \( I(t) = [-AT_0^{1/(m+1)}, AT_0^{1/(m+1)}] \),

which proves the lemma. Q.E.D.

By Lemma 8.1 and Lemma 8.6, there exists two positive constants \( h_* \) and \( h^* \) such that

\[(8.125) \quad h_* t^{-\frac{1}{2}(m+1)} \leq v(x, t) \leq h^* t^{-\frac{1}{2}(m+1)} \text{ for } t \geq T_0 \text{ and a.e. } x \in \text{supp } u(\cdot, t).
\]

Let \( u_\delta \) and \( u_{0\delta} \in C_0(\mathbb{R}) \) be the functions such that

\[
0 \leq u_{0\delta}(x) \leq u(x, T_0) \leq u_\delta(x) \quad \text{for a.e. } x \in \mathbb{R},
\]

\[
u_{0\delta} \equiv 0 \text{ in } \mathbb{R}.
\]

Let \( u^* \) be the generalized solution of

\[(8.126) \quad u_i^* = (u^* m)_{xx} - (h_* t^{-\frac{1}{2}(m+1)})^n u^* \text{ in } \mathbb{R} \times [T_0, \infty)
\]

with

\[
u^*(\cdot, T_0) = u_{0\delta}^* \text{ in } \mathbb{R}
\]

and let \( u_* \) be the generalized solution of

\[(8.127) \quad u_{i1}^* = (u^*_{m})_{xx} - (h_* t^{-\frac{1}{2}(m+1)})^n u^* \text{ in } \mathbb{R} \times [T_0, \infty)
\]

with

\[
u_*^*(\cdot, T_0) = u_{0\delta\delta} \text{ in } \mathbb{R}.
\]

By (8.125) \( u \) is a subsolution of (8.127) in \( \mathbb{R} \times [T_0, \infty) \) and a supersolution of (8.126) in \( \mathbb{R} \times [T_0, \infty) \), and we obtain by Lemma 4.3
Therefore, by Lemma 5.2, we arrive at the desired estimates of $|u(\cdot, t)|_{\infty, R}$ and supp $u(\cdot, t)$.

By Lemma 8.1, we obtain also the desired estimates of $|v(\cdot, t)|_{\infty, R}$ and supp $v(\cdot, t)$.

The proof of Theorem 1.1 is now complete. Q.E.D. of Theorem 1.1.

9. Proof of Corollary 1.2

It suffices to prove

$\cup_{t \geq 0} \text{supp } u(\cdot, t) = R$ in case of $m = 2n - 2$.

There exists a positive constant $h^*$ such that

$v(x, t) \leq h^*(t+1)^{-1/(m+1)}$ for $t \geq 0$ and a.e. $x \in R$.

Let $u_\#$ be the generalized solution of

$u_{\# t} = (u_\#)^{m/2} - h_\#(t+1)^{-n/(m+1)} u_\#$ in $R \times [0, \infty)$

with

$u_\#(\cdot, 0) = \min(u_0(x), 1)$ in $R$.

Let $u_{**}$ be the generalized solution of

$u_{** t} = (u_{**})^{m/2} - h_{**}(t+1)^{-n/(m+1)} u_{**}$ in $R \times [0, \infty)$

with

$u_{**}(\cdot, 0) = \min(u_0(x), 1)$ in $R$,

where $\hat{n} = n - 1/4$.

By Lemma 4.3, we obtain

(9.1) $u_{**}(x, t) \leq u(\cdot, t) \leq u(x, t)$

for $t \geq 0$ and a.e. $x \in R$.

Therefore, by Lemma 5.2 we have $\cup_{t \geq 0} \text{supp } u_{**}(\cdot, t) = R$, and by (9.1) we conclude that $\cup_{t \geq 0} \text{supp } u(\cdot, t) = R$. Q.E.D.

References


