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A CRITERION FOR ALGEBRAIC INDEPENDENCE WITH SOME APPLICATIONS

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1. Introduction

It is a well-known matter of fact that complex numbers, which are too closely approximated by algebraic numbers, must be transcendental. Using this principle one can prove the transcendence of numbers defined by sufficiently well converging limit processes, say by certain infinite series or continued fractions.

A special question of this kind is the following one. Given a gap power series with non-zero algebraic coefficients and with positive radius of convergence. Under what conditions on the gaps and the coefficients is it true, that the power series takes on transcendental values at all non-zero algebraic points of its circle of convergence? Over some decades several authors treated this problem, but only in 1973 Cijsouw and Tijdeman [8] proved the following very general and satisfactory result in this direction.

Let $(e_k)_{k\geq 0}$ *denote a strictly increasing sequence of non-negative integers, and let* $(a_k)_{k\geq 0}$ *denote a sequence of non-zero algebraic numbers. Suppose that the power series* $\sum_{k\geqq 0} a_k z^{e_k}$ *has radius of convergence* $R>0$ *and defines the function* $f(z)$ *in* $|z|<R$. *Put* $S_k:=[\boldsymbol{Q}(a_0, \cdots, a_k): \boldsymbol{Q}]$, $A_k:=\max(1, \overline{|a_0|}, \cdots, \overline{|a_k|})$, $M_k:=\mathrm{lcm}(\mathrm{den}\ a_0, \cdots, \mathrm{den}\ a_k)$ *and suppose that*

$$S_k(e_k+\log A_k M_k) = o(e_{k+1}) \tag{1}$$

as $k\rightarrow\infty$. *Then for every algebraic* α *with* $0<|\alpha|<R$ *the number* $f(\alpha)$ *is transcendental.*

Here $\overline{|\beta|}$ denotes the maximum of the absolute values of all conjugates of an algebraic number β, whereas den β denotes the *denominator* of β, i.e. the least rational integer $b>0$ such that $b\beta$ is an (algebraic) integer.

On the one hand, as noticed above, the theorem of Cijsouw and Tijdeman completed the development of the transcendence results concerning gap power series with algebraic coefficients. On the other hand this theorem suggested the question, if one could even show the algebraic independence of $f(\alpha_1), \cdots, f(\alpha_t)$ for distinct non-zero algebraic $\alpha_1, \cdots, \alpha_t$ from the circle of convergence.

First partial results in this direction were found by Adams [1] and by his pu-

pil Pass [17]. Whereas both applied rather weak criteria for algebraic independence, Wylegala and the present author [6] succeeded shortly later to prove almost the full t-dimensional generalization of the Cijsouw-Tijdeman-Theorem:
Suppose that the hypotheses of the Cijsouw-Tijdeman-Theorem are satisfied. If $\alpha_1, \cdots, \alpha_t$ *are non-zero algebraic numbers of distinct absolute values less than R, then* $f(\alpha_1), \cdots, f(\alpha_t)$ *are algebraically independent.*

One application of this result concerns the Liouville function Λ defined in $|z|<1$ by

$$\Lambda(z) := \sum_{k\geq 1} z^{k!}. \tag{2}$$

If $\alpha_1, \cdots, \alpha_t$ are non-zero algebraic numbers of distinct absolute values less than 1, then $\Lambda(\alpha_1), \cdots, \Lambda(\alpha_t)$ are algebraically independent. (It should be remarked historically that $\Lambda\left(\frac{1}{10}\right)$ was one of the first real numbers, whose transcendence has been shown by Liouville in 1844 on the basis of his well-known approximation theorem.)

In [6] the proof of our result was based on a criterion for algebraic independence by Durand [9], who used the theorem on implicit functions in the complex domain, and his proof was technically not simple at all. Later other authors used our procedure to apply Durand's criterion to get results on algebraic independence, e.g. Cijsouw [7] and Zhu [20]. It is the main aim of this paper to present a new criterion for algebraic independence of complex numbers (in §2) and to give some applications (in §3). The proof of this criterion is much simpler than Durand's and generalizes an idea of Flicker [11]. In our context this idea was first applied by Shiokawa [18], later by Adams [2], Amou [3], Laohakosol and Ubolsri [13], Zhu [21], [22].

2. The criterion

Let $\partial(\beta)$ denote the *degree* of an algebraic number β, and $H(\beta)$ the *height* of β, i.e. the maximum of the absolute values of the coefficients in the minimal polynomial of β over $\boldsymbol{Z}$. Finally denote by $s(\beta):=\partial(\beta)+\log H(\beta)$ the *size* of an algebraic number β. With these notations we can formulate the following

Criterion for algebraic independence. *Let* $\beta_1, \cdots, \beta_t$ *be given complex numbers, and let* $g\colon \boldsymbol{N}\to\boldsymbol{R}_+$ *satisfy* $g(n)\to\infty$ *as* $n\to\infty$. *Suppose that for each* $\tau\in\{1, \cdots, t\}$ *there exist an infinite set* $N_\tau\subset\boldsymbol{N}$ *and* τ *sequences* $(\beta_{1n})_{n\in N_\tau}, \cdots, (\beta_{\tau n})_{n\in N_\tau}$ *of algebraic numbers such that for each* $n\in N_\tau$ *the inequalities*

$$g(n)\sum_{\sigma=1}^{\tau-1}|\beta_\sigma-\beta_{\sigma n}|<|\beta_\tau-\beta_{\tau n}|\leq\exp(-g(n)\,[\boldsymbol{Q}(\beta_{1n}, \cdots, \beta_{\tau n}):\boldsymbol{Q}]\sum_{\sigma=1}^{\tau}\frac{s(\beta_{\sigma n})}{\partial(\beta_{\sigma n})}) \tag{3}$$

hold. Then $\beta_1, \cdots, \beta_t$ *are algebraically independent.*

The proof of this criterion depends heavily on the subsequent lemma due to Fel'dman [10].

Liouville-type estimate. *If $\beta_1, \cdots, \beta_t$ are algebraic numbers with $D := [\boldsymbol{Q}(\beta_1, \cdots, \beta_t) : \boldsymbol{Q}]$, and if $P \in \boldsymbol{Z}[X_1, \cdots, X_t]$ satisfies $P(\beta_1, \cdots, \beta_t) \neq 0$, then the following inequality holds*

$$(4) \qquad |P(\beta_1, \cdots, \beta_t)| \geq L(P)^{1-D} \prod_{\tau=1}^{t} (H(\beta_\tau)(1+\partial(\beta_\tau)))^{-\partial_\tau(P)D/\partial(\beta_\tau)} .$$

Here $\partial_\tau(P)$ denotes the degree of P with respect to X_τ, and $L(P)$, the *length* of P, denotes the sum of the absolute values of the coefficients of P.
For distinct algebraic numbers α, β we deduce immediately from (4)

$$(5) \qquad \begin{aligned} |\alpha-\beta| &> 2^{-\partial(\alpha)\partial(\beta)} (H(\alpha)(1+\partial(\alpha)))^{-\partial(\beta)} (H(\beta)(1+\partial(\beta)))^{-\partial(\alpha)} \\ &\geq e^{-c(\alpha)(\partial(\beta)+\log H(\beta))} = e^{-c(\alpha)s(\beta)} \end{aligned}$$

with some $c(\alpha) > 0$. The notation in the second line suggests α to be considered fixed. An obvious consequence of (5) is the

Criterion for transcendence. *Let β be a given complex number, and let $g: \boldsymbol{N} \to \boldsymbol{R}_+$ satisfy $g(n) \to \infty$ as $n \to \infty$. Suppose that there exists a sequence $(\beta_n)_{n \in N}$ of algebraic numbers with*

$$(6) \qquad 0 < |\beta - \beta_n| \leq \exp(-g(n)s(\beta_n))$$

for each $n \in \boldsymbol{N}$. Then β is transcendental.
From this criterion the reader can easily deduce the above-mentioned result of Cijsouw and Tijdeman.

Proof of the algebraic independence criterion (by induction on τ). For $\tau = 1$ inequalities (3) reduce just to inequalities (6) such that the transcendence criterion yields the assertion in this case.

Let now $\tau \in \{2, \cdots, t\}$ and suppose $\beta_1, \cdots, \beta_{\tau-1}$ to be algebraically independent. Assume that $\beta_1, \cdots, \beta_\tau$ are algebraically dependent, and let further $Q \in \boldsymbol{Z}[X_1, \cdots, X_\tau] \setminus \{0\}$ be of minimal total degree with $Q(\beta_1, \cdots, \beta_\tau) = 0$. By Taylor expansion we get

$$(7) \qquad Q(X_1, \cdots, X_\tau) = \sum_{\nu_1+\cdots+\nu_\tau \geq 1} c(\nu_1, \cdots, \nu_\tau)(X_1-\beta_1)^{\nu_1} \cdots (X_\tau-\beta_\tau)^{\nu_\tau} .$$

If $c(0, \cdots, 0, 1) = \dfrac{\partial Q}{\partial X_\tau}(\beta_1, \cdots, \beta_\tau) = 0$, then $\dfrac{\partial Q}{\partial X_\tau}$ must be the zero polynomial, since otherwise this polynomial would have a total degree less than that of Q. But then Q is independent on X_τ and thus $\beta_1, \cdots, \beta_{\tau-1}$ would be algebraically dependent. Therefore we have $c(0, \cdots, 0, 1) \neq 0$. Using (3) and (7) we find for $n \to \infty$, $n \in N_\tau$

$$Q(\beta_{1n}, \cdots, \beta_{\tau n}) = c(0, \cdots, 0, 1)(\beta_{\tau n}-\beta_\tau)(1+o(1)), \tag{8}$$

and thus

$$Q(\beta_{1n}, \cdots, \beta_{\tau n}) \neq 0 \tag{9}$$

for all large $n \in N_\tau$. Because of (9) the Liouville-type estimate can be applied to $|Q(\beta_{1n}, \cdots, \beta_{\tau n})|$ which yields

$$|Q(\beta_{1n}, \cdots, \beta_{\tau n})| \geq \exp\Big(-c(Q)\, D_n \sum_{\sigma=1}^{\tau} \frac{s(\beta_{\sigma n})}{\partial(\beta_{\sigma n})}\Big) \tag{10}$$

for the same n as in (9). Here D_n denotes $[\boldsymbol{Q}(\beta_{1n}, \cdots, \beta_{\tau n}) : \boldsymbol{Q}]$, and $c(Q)>0$ is a constant depending only on Q. On the other hand, by (8) and the right-hand side of (3), we get

$$|Q(\beta_{1n}, \cdots, \beta_{\tau n})| \leq 2|c(0, \cdots, 0, 1)| \exp\Big(-g(n)\, D_n \sum_{\sigma=1}^{\tau} \frac{s(\beta_{\sigma n})}{\partial(\beta_{\sigma n})}\Big) \tag{11}$$

for all large $n \in N_\tau$. But (10) and (11) are contradictory for $n \in N_\tau$ large enough, and thus the proof is complete.

3. Some applications to gap power series

As a first application of our algebraic independence criterion we present a generalization of Zhu's results from [20], which reduces, for $m=1$, to our above-mentioned theorem with Wylegala from [6].

Theorem 1. *For $\mu=1, \cdots, m$ let $(e_{\mu k})_{k\geq 0}$ denote a strictly increasing sequence of non-negative integers, and $(a_{\mu k})_{k\geq 0}$ a sequence of non-zero algebraic numbers. Suppose that the power series $\sum_{k\geqq 0} a_{\mu k} z^{e_{\mu k}}$ has radius of convergence $R_\mu>0$ and defines $f_\mu(z)$ in $|z|<R_\mu$, for $\mu=1, \cdots, m$. Define $S_k := [\boldsymbol{Q}(a_{10}, \cdots, a_{1k}, \cdots, a_{m0}, \cdots, a_{mk}) : \boldsymbol{Q}]$, $A_k := \max(1, \overline{|a_{10}|}, \cdots, \overline{|a_{mk}|})$, $M_k := \mathrm{lcm}(\mathrm{den}\ a_{10}, \cdots, \mathrm{den}\ a_{mk})$ and suppose that for $k\to\infty$*

$$e_{\mu+1,k} = o(e_{\mu k}) \quad (\mu = 1, \cdots, m-1) \tag{12}$$

$$S_k(e_{1k}+\log A_k M_k) = o(e_{m,k+1}). \tag{13}$$

If, for $\mu=1, \cdots, m$, the non-zero algebraic numbers $\alpha_{\mu 1}, \cdots, \alpha_{\mu t_\mu}$ have distinct absolute values less than R_μ, then the numbers $f_1(\alpha_{11}), \cdots, f_1(\alpha_{1t_1}), \cdots, f_m(\alpha_{m1}), \cdots, f_m(\alpha_{mt_m})$ are algebraically independent.

For the proof we need the following lemma on algebraic numbers.

Lemma. *Let β be an algebraic number. Then*

$$H(\beta) \leq (2(\mathrm{den}\ \beta) \max(1, \overline{|\beta|}))^{\partial(\beta)} \tag{14}$$

and, if $\beta \neq 0$,

$$(15) \qquad |\beta| \geq ((\mathrm{den}\,\beta) \max(1, \overline{|\beta|}))^{-\delta(\beta)} .$$

Inequality (14) is Lemma 1 from [8], and (15) can be found in [19], p. 6.

Proof of Theorem 1. First of all we assert for each complex number z with $0<|z|<R_\mu$

$$(16) \qquad f_\mu(z) - \sum_{k<n} a_{\mu k} z^{e_{\mu k}} \sim a_{\mu n} z^{e_{\mu n}}$$

as $n \to \infty$. Namely we have by (15) and the definition of the S_k, A_k, M_k in Theorem 1

$$(17) \qquad |a_{\mu n}| \geq (A_n M_n)^{-S_n} .$$

Further, supposing $\varepsilon \in \boldsymbol{R}_+$ such that $|z|(\frac{1}{R_\mu}+\varepsilon)<1$ holds, we have from Cauchy-Hadamard's formula

$$(18) \qquad |\sum_{k>n} a_{\mu k} z^{e_{\mu k}}| \leq \sum_{k>n} \left(|z|\left(\frac{1}{R_\mu}+\varepsilon\right)\right)^{e_{\mu k}} \leq c_1\left(|z|\left(\frac{1}{R_\mu}+\varepsilon\right)\right)^{e_{\mu, n+1}} ,$$

where $c_1 \in \boldsymbol{R}_+$ depends only on z, R_μ and ε. In virtue of the conditions (12), (13) of Theorem 1 the asymptotic equality (16) follows from (17) and (18). Now, for $\mu=1, \cdots, m$; $\tau=1, \cdots, t_\mu$, let us define

$$\Delta_{\mu\tau}(n) := |f_\mu(\alpha_{\mu\tau}) - \sum_{k<n} a_{\mu k} \alpha_{\mu\tau}^{e_{\mu k}}| .$$

Our next aim is to compare the numbers $\Lambda_{\mu\tau}(n)$. To do this we may suppose w.l.o.g.

$$(19) \qquad 0<|\alpha_{\mu 1}|<\cdots<|\alpha_{\mu t_\mu}|<R_\mu$$

for each $\mu \in \{1, \cdots, m\}$. For each fixed μ with $t_\mu \geq 2$ we see from (16) and (19)

$$(20) \qquad \Delta_{\mu\tau}(n)/\Delta_{\mu\sigma}(n) \to \infty \quad \text{as} \quad n \to \infty$$

for $1 \leq \sigma < \tau \leq t_\mu$. If $\mu \geq 2$ we compare $\Delta_{\mu_\tau}(n)$ with all numbers $\Delta_{\nu\sigma}(n)$ having $\nu < \mu$. Replacing n by $n-1$ in (18) we find

$$(21) \qquad \Delta_{\mu\sigma}(n) \leq c_2 \left(|\alpha_{\nu\sigma}|\left(\frac{1}{R_\nu}+\varepsilon\right)\right)^{e_{\nu n}}$$

for each n large enough. On the other hand using (16) and the Cauchy-Hadamard formula, we get

$$(22) \qquad \Delta_{\mu\tau}(n) \geq \frac{1}{2}|a_{\mu n}|\,|\alpha_{\mu\tau}|^{e_{\mu n}} \geq \frac{1}{2}\left(|\alpha_{\mu\tau}|\left(\frac{1}{R_\mu}-\varepsilon\right)\right)^{e_{\mu n}}$$

for infinitely many n. In (21) and (22) $\varepsilon\in\boldsymbol{R}_+$ is supposed to satisfy the conditions $|\alpha_{\nu\sigma}|(\frac{1}{R_\nu}+\varepsilon)<1, \varepsilon<\frac{1}{R_\mu}$. Because of $e_{\mu n}=o(e_{\nu n})$, see (12), we find from (21) and (22)

$$\Delta_{\mu\tau}(n)/\Delta_{\nu\sigma}(n)\to\infty \tag{23}$$

as $n\to\infty$ over a certain infinite set depending only on μ, τ. Now we identify the $\beta_1, \cdots, \beta_t$ in our criterion with the $f_1(\alpha_{11}), \cdots, f_1(\alpha_{1t_1}), \cdots, f_m(\alpha_{m1}), \cdots, f_m(\alpha_{mt_m})$ and the approximating algebraic numbers $\beta_{1n}, \cdots, \beta_{tn}$ with the corresponding partial sums $\sum_{k<n} a_{1k}\,\alpha_{11}^{e_{1k}}, \cdots, \sum_{k<n} a_{mk}\,\alpha_{mt_m}^{e_{mk}}$. Given μ, τ with $\mu\in\{1, \cdots, m\}, \tau\in\{1, \cdots, t_\mu\}$ we know from (20) and (23) that there exists an infinite subset, say $N_{\mu\tau}$, of $\boldsymbol{N}$, such that the left-hand inequality in (3) is satisfied for each $n\in N_{\mu\tau}$, if g is chosen appropriately.

To ensure the validity of the right-hand inequality in (3) we deduce from (18) firstly

$$\Delta_{\mu\tau}(n)\leq c_3\left(|\alpha_{\mu\tau}|\left(\frac{1}{R_\mu}+\varepsilon\right)\right)^{e_{\mu n}}\leq\exp\left(-c_4\,e_{\mu n}\right) \tag{24}$$

for each $n\in\boldsymbol{N}$ large enough, with some positive constants c_3, c_4 which are independent on n. Secondly we get from (14) for each n

$$[\boldsymbol{Q}(\beta_{1n}, \cdots, \beta_{\tau n})\colon\boldsymbol{Q}]\sum_{\sigma=1}^{\tau}\left(1+\frac{\log H(\beta_{\sigma n})}{\partial(\beta_{\sigma n})}\right)\leq c_5\,S_{n-1}(e_{1,n-1}+\log A_{n-1}\,M_{n-1}) \tag{25}$$

where $c_5>0$ is again independent on n. Using both hypotheses (12), (13) of Theorem 1 we see from (24) and (25) that the right-hand inequality in (3) is also satisfied, maybe with a "smaller" g, even for all $n\in\boldsymbol{N}$ large enough. Of course, g could be given explicitly. Thus Theorem 1 is proved.

Another problem, often treated in the literature, is also touched by Theorem 1, namely to construct systems of continuum-many algebraically independent real or complex numbers. The first such system was found by von Neumann [14]. Corollary 1 below gives another such system; its special case $\alpha=\frac{1}{2}$ is due to Kneser [12].

Corollary 1. *For each algebraic number α with $0<|\alpha|<1$ the set*

$$\{\sum_{k\geq1}\alpha^{[k^{k+\lambda}]}\colon \lambda \text{ real}, 0\leq\lambda<1\}$$

is algebraically independent.

Proof. Let $\lambda_1, \cdots, \lambda_m$ be m distinct real numbers from the interval $[0,1[$ and suppose w.l.o.g.

$$0\leq\lambda_m<\cdots<\lambda_1<1\,. \tag{26}$$

Defining $e_{\mu k}:=[k^{k+\lambda_\mu}]$ we see immediately from (26) that condition (12) in Theorem 1 is satisfied. Since we have to take all $a_{\mu k}$ equal to 1, condition (13) of Theorem 1 reduces to $e_{1k}=o(e_{m,k+1})$, which holds by $1+\lambda_m>\lambda_1$, compare (26).

Our criterion in §2 contains also the first two theorems from [3] as well as their corollaries. Here we derive only his first result.

Theorem 2. *Suppose that the hypotheses of Theorem* 1, *apart from* (12), *are satisfied and that condition* (13) *is replaced by*

$$(13') \qquad S_k(\max_\mu e_{\mu k}+\log A_k M_k) = o(\min_\mu e_{\mu,k+1}) .$$

Let $\alpha_1, \cdots, \alpha_m$ *be algebraic numbers satisfying* $0<|\alpha_\mu|<R_\mu$ *for* $\mu=1, \cdots, m$, *and suppose additionally*

$$(27) \qquad a_{\mu k}\, \alpha_\mu^{e_{\mu k}} = o(|a_{\mu+1,k}\, \alpha_{\mu+1}^{e_{\mu+1,k}}|) \quad (\mu = 1, \cdots, m-1)$$

as $k\to\infty$. *Then* $f_1(\alpha_1), \cdots, f_m(\alpha_m)$ *are algebraically independent.*

Proof. For $\tau=1, \cdots, m$ put $\beta_\tau:=f_\tau(\alpha_\tau)$, $\beta_{\tau n}:=\sum_{k<n} a_{\tau k}\, \alpha_\tau^{e_{\tau k}}$. From (16) we get

$$(28) \qquad |\beta_\tau-\beta_{\tau n}|\sim|a_{\tau n}|\,|\alpha_\tau|^{e_{\tau n}}$$

as $n\to\infty$, for $\tau=1, \cdots, m$. Therefore, if $\tau\geq 2$,

$$\sum_{\sigma=1}^{\tau-1} |\beta_\sigma-\beta_{\sigma n}|\sim\sum_{\sigma=1}^{\tau-1} |a_{\sigma n}|\,|\alpha_\sigma|^{e_{\sigma n}}$$

and here the right-hand side is $o(|a_{\tau n}|\,|\alpha_\tau|^{e_{\tau n}})$ as $n\to\infty$, by (27). This estimate, combined with (28), shows that the left-hand inequality in (3) is satisfied for some appropriate function g. Using (14) again we see

$$[\boldsymbol{Q}(\beta_{1n}, \cdots, \beta_{\tau n}): \boldsymbol{Q}] \sum_{\sigma=1}^{\tau} \frac{s(\beta_{\sigma n})}{\partial(\beta_{\sigma n})}\leq c_6\, S_{n-1}(\max_{\sigma\leq\tau} e_{\sigma,n-1}+\log A_{n-1} M_{n-1})$$

which is $o(\min_\mu e_{\mu n})$ as $n\to\infty$, by (13′). On the other hand we have from (28)

$$|\beta_\tau-\beta_{\tau n}|\leq\exp(-c_7\, e_{\tau n})\leq\exp(-c_7 \min_\mu e_{\mu n})$$

as $n\to\infty$, with some constant $c_7>0$. Thus the right-hand inequality in (3) is also true, maybe with some "smaller" g, and Theorem 2 is proved.
From Theorem 2 we deduce Cijsouw's main result in [7], which again generalizes our result from [6].

Corollary 2. *Suppose that the hypotheses of the Cijsouw-Tijdeman-Theorem are satisfied. If* $\alpha_1, \cdots, \alpha_t$ *are non-zero algebraic numbers of distinct absolute values less than R, and if l is any non-negative integer, then the numbers* $f^{(\lambda)}(\alpha_\tau)$ $(\tau=1, \cdots,$

t; $\lambda=0, \cdots, l$) *are algebraically independent.*

Proof. We identify the $(l+1)t$ functions $f_1, \cdots, f_{(l+1)t}$ of Theorem 2 with t successive blocks $f, f', \cdots f^{(l)}$ of $l+1$ functions. From the gap condition (1) of the Cijsouw-Tijdeman-Theorem and from

$$f^{(\lambda)}(z)=\sum_{\substack{k\geq 0\\ e_k\geq\lambda}} a_k e_k\cdot\cdots\cdot(e_k-\lambda+1)\, z^{e_k-\lambda}$$

condition (13′) of Theorem 2 is easily deduced. The $\alpha_1, \cdots, \alpha_t$ from Corollary 2 are supposed to satisfy w.l.o.g.

$$0<|\alpha_1|<\cdots<|\alpha_t|<R, \tag{29}$$

and the α's from Theorem 2 are chosen to be $\alpha_1, \cdots, \alpha_1, \cdots, \alpha_t, \cdots, \alpha_t$ where each of the t blocks consists of $l+1$ equal numbers α_τ. Condition (27) of Theorem 2 is also satisfied because of

$$\frac{a_k e_k\cdot\cdots\cdot(e_k-\lambda+1)\,\alpha_\tau^{e_k-\lambda}}{a_k e_k\cdot\cdots\cdot(e_k-\lambda)\,\alpha_\tau^{e_k-\lambda-1}}=\frac{\alpha_\tau}{e_k-\lambda}\to 0 \quad\text{as}\quad k\to\infty,$$

if $0\leq\lambda<l$, and because of

$$\frac{a_k e_k\cdot\cdots\cdot(e_k-l+1)\,\alpha_{\tau-1}^{e_k-l}}{a_k\,\alpha_\tau^{e_k}}=\frac{e_k\cdot\cdots\cdot(e_k-l+1)}{\alpha_{\tau-1}^{l}}\left(\frac{\alpha_{\tau-1}}{\alpha_\tau}\right)^{e_k}\to 0 \quad\text{as}\quad k\to\infty$$

for $\tau=2, \cdots, t$, compare (29). Thus Corollary 2 is proved.

REMARK 1. The Liouville function Λ from (2) shows that, at least in general, the hypothesis on the distinctness of the absolute values of the α's in the above theorems and corollaries cannot be weakened to the distinctness of the α's themselves. Namely, if $t\geq 2$ and some quotient $\frac{\alpha_\sigma}{\alpha_\tau}$ with $\sigma\neq\tau$ is a root of unity, then the numbers

$$\Lambda^{(\lambda)}(\alpha_\tau)\quad(\tau=1, \cdots, t;\ \lambda=0, \cdots, l) \tag{30}$$

are easily seen to be algebraically dependent. On the other hand Nishioka [15] showed that the numbers (30) are indeed algebraically independent for non-zero algebraic $\alpha_1, \cdots, \alpha_t$, less than 1 in absolute value, if no quotient $\frac{\alpha_\sigma}{\alpha_\tau}$ with $\sigma\neq\tau$ is a root of unity.

Using the same method, Nishioka [16] could describe precisely the conditions on the distinct non-zero algebraic numbers $\alpha_1, \cdots, \alpha_t$ of absolute value less than R, which guarantee the algebraic independence of the numbers $f(\alpha_1), \cdots, f(\alpha_t)$ in our theorem with Wylegala [6] mentioned in §1. In her paper she had only to suppose additionally that all power series coefficients $a_0, a_1, \cdots$ of f belong to a fixed algebraic number field.

Remark 2. From our algebraic independence criterion in §2 one can easily deduce the main result of Adams [2], i.e. his Theorem 4. He used this result to generalize theorems of the present author [4], [5], resp. of Laohakosol and Ubolsri [13] concerning algebraic independence of certain continued fractions.

References

[1] W.W. Adams: *On the algebraic independence of certain Liouville numbers,* J. Pure Appl. Algebra **13** (1978), 41–47.

[2] W.W. Adams: *The algebraic independence of certain Liouville continued fractions,* Proc. Amer. Math. Soc. **95** (1985), 512–516.

[3] M. Amou: *On algebraic independence of special values of gap series,* Tohoku Math. J. **37** (1985), 385–393.

[4] P. Bundschuh: *Über eine Klasse reeller transzendenter Zahlen mit explizit angebbarer g-adischer und Kettenbruch-Entwicklung,* J. Reine Angew. Math. **318** (1980), 110–119.

[5] P. Bundschuh: *Transcendental continued fractions,* J. Number Theory **18** (1984), 91–98.

[6] P. Bundschuh und F.J. Wylegala: *Über algebraische Unabhängigkeit bei gewissen nichtfortsetzbaren Potenzreihen,* Arch. Math. **34** (1980), 32–36.

[7] P.L. Cijsouw: *Gap series and algebraic independence,* EUT-Report Eindhoven 84-WSK-**03** (1984), 111–119.

[8] P.L. Cijsouw and R. Tijdeman: *On the transcendence of certain power series of algebraic numbers,* Acta Arith. **23** (1973), 301–305.

[9] A. Durand: *Indepéndance algébrique de nombres complexes et critère de transcendance,* Compositio Math. **35** (1977), 259–267.

[10] N.I. Fel'dman: *Estimate for a linear form of logarithms of algebraic numbers,* Mat. Sb. (N.S.) **76** (1968), 304–319 (Russian); Engl. transl.: Math. USSR Sb. **5** (1968), 291–307.

[11] Y. Flicker: *Algebraic independence by a method of Mahler,* J. Austral. Math. Soc. Ser. A **27** (1979), 173–188.

[12] H. Kneser: *Eine kontinuumsmächtige, algebraisch unabhängige Menge reeller Zahlen,* Bull. Soc. Math. Belg. **12** (1960), 23–27.

[13] V. Laohakosol and P. Ubolsri: *Some algebraically independent continued fractions,* Proc. Amer. Math. Soc. **95** (1985), 169–173.

[14] J. von Neumann: *Ein System algebraisch unabhängiger Zahlen,* Math. Ann. **99** (1928), 134–141.

[15] K. Nishioka: *Proof of Masser's conjecture on the algebraic independence of values of Liouville series,* Proc. Japan Acad. Ser. A **62** (1986), 219–222.

[16] K. Nishioka: *Conditions for algebraic independence of certain power series of algebraic numbers,* Compositio Math. **62** (1987), 53–61.

[17] R. Pass: *Results concerning the algebraic independence of sets of Liouville numbers.* Thesis Univ. of Maryland, College Park, 1978.

[18] I. Shiokawa: *Algebraic independence of certain gap series,* Arch. Math. **38** (1982),

438–442.

[19] M. Waldschmidt: Nombres transcendants, Springer, Berlin-Heidelberg-New York, 1974.

[20] Y.C. Zhu: *On the algebraic independence of certain power series of algebraic numbers,* Chin. Ann. Math. Ser. B **5** (1984), 109–117.

[21] Y.C. Zhu: *Criteria of the algebraic independence of complex numbers,* Kexue Tongbao **29** (1984), 61–62.

[22] Y.C. Zhu: *Algebraic independence property of values of certain gap series,* Kexue Tongbao **30** (1985), 293–297.

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