

Title	On weakly transitive translation planes
Author(s)	Hiramine, Yutaka
Citation	Osaka Journal of Mathematics. 1985, 22(1), p. 55-60
Version Type	VoR
URL	https://doi.org/10.18910/12608
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

ON WEAKLY TRANSITIVE TRANSLATION PLANES

YUTAKA HIRAMINE

(Received November 28, 1983)

1. Introduction

Let π^{l_∞} be a translation plane of order p^r with p a prime. Let G be a subgroup of the translation complement and Δ a subset of l_∞ with $|\Delta|=p+1$. π is said to be Δ -transitive if the following conditions are satisfied (V. Jha [4]):

- (i) G leaves Δ invariant and acts transitively on $l_\infty-\Delta$.
- (ii) G fixes at least two points of Δ .
- (iii) G has a normal Sylow p -subgroup.

On Δ -transitive planes, V. Jha has proved the following theorem.

Theorem (V. Jha [4]). *If π^{l_∞} is Δ -transitive with $|\Delta|=p+1$, then π has order p^2 and $\Delta=\pi_0 \cap l_\infty$ where π_0 is a subplane of order p .*

If $(\pi^{l_\infty}, \Delta, G)$ satisfies the conditions (i) and (ii) above, π is said to be weakly transitive.

In his paper [4], V. Jha has conjectured that weakly transitive planes are the Hall planes of order p^2 , the Lorimer-Rahilly plane of order 16 and the Johnson-Walker plane of order 16.

In this paper we prove the following theorems on weakly transitive planes.

Theorem 1. *Let π^{l_∞} be a translation plane of order p^r with p a prime and Δ a subset of l_∞ with $|\Delta|=p+1$. If a subgroup G of the translation complement of π leaves Δ invariant and acts transitively on $l_\infty-\Delta$, then one of the following holds.*

- (i) $O_p(G)$ is semiregular on $\Delta-\{A\}$ for some point $A \in \Delta$.
- (ii) π has order p^2 .
- (iii) π has order p^3 and G is transitive on Δ .

The Lorimer-Rahilly plane of order 16 and the Johnson-Walker plane of order 16 are examples of the case (i). The Hall planes of order p^2 and the plane of order 25 constructed by M.L. Narayana Rao and K. Satyanarayana in [6] are examples of the case (ii). The desarguesian plane of order 27 is an example of the case (iii).

As an immediate corollary we have the following.

Theorem 2. *Suppose $(\pi^{l_\infty}, \Delta, G)$ with $|\Delta|=p+1$ is weakly transitive. If $O_p(G) \neq 1$, then π has order p^2 and $\Delta = F(O_p(G)) \cap l_\infty$.*

We note that if π^{l_∞} is Δ -transitive, then it satisfies the assumption of Theorem 2.

2. Proof of Theorem 1

We prove Theorem 1 by way of contradiction. Assume that $(\pi^{l_\infty}, \Delta, G)$ is a counterexample such that $p^r + |G|$ is minimal. Therefore $r \geq 3$ and $O_p(G) \neq 1$.

Throughout the paper we use the following notations.

T : the group of translations of π

$M(=O_p(G))$: the maximal normal p -subgroup of G

$F(H)$: the fixed structure consisting of points and lines of π fixed by a nonempty subset H of G .

n_p : the highest power of a prime p dividing a positive integer n

Γ : $l_\infty - \Delta$.

Other notations are taken from [1] and [2].

Lemma 1. *$F(M)$ is a subplane of π of order p and $\Delta = F(M) \cap l_\infty$.*

Proof. Let K be the pointwise stabilizer of Δ in G and assume that $M \not\leq K$. We denote by \bar{G} the restriction of G on Δ . Clearly $\bar{G} \triangleright \bar{M} \neq 1$ and as $|\Delta|=p+1$, \bar{M} is a Sylow p -subgroup of \bar{G} . By the Schur-Zassenhaus' theorem (Theorem 6.2.1 of [1]), there is a subgroup \bar{L} of \bar{G} such that $K < L$ and $|\bar{G}:\bar{L}|=p$, $\bar{G} = \bar{M}\bar{L}$.

Set $N = M \cap K$. We have $N \neq 1$, for otherwise π satisfies (i) of Theorem 1, contrary to the minimality of π . As $G \triangleright K$, $G \triangleright N$. It follows from the transitivity of G on Γ that N is $\frac{1}{2}$ -transitive on Γ .

Let Ψ be the set of N -orbits on Γ . Since there is no nontrivial homology of order p , N acts faithfully on Γ . As $N \neq 1$ and $|\Gamma|_p = p$, $|\Psi| = |\Gamma|/p = p^{r-1} - 1$. Hence Ψ coincides with the set of M -orbits on Γ .

Since $G = ML$, L is transitive on Ψ by the last paragraph. Hence L is transitive on Γ as $N < L$. From this $(\pi^{l_\infty}, \Delta, L)$ satisfies (ii) or (iii) of Theorem 1 by the minimality of $(\pi^{l_\infty}, \Delta, G)$. Therefore $(\pi^{l_\infty}, \Delta, G)$ also satisfies (ii) or (iii) of Theorem 1. This is a contradiction. Thus $M \leq K$.

Since $F(M) \cap \Gamma = \phi$, $F(M) \cap l_\infty = \Delta$, so that $F(M)$ is a subplane of π of order p .

Lemma 2. *If $p=2$, then r is even.*

Proof. Assume $p=2$. Let x be an involution in M . Since $F(x)$ contains Δ by Lemma 1, $F(x)$ is a subplane of π . By a Baer's theorem (Theorem 4.3 of [2]), $F(x)$ is of order $\sqrt{2^r}$. Thus r is even.

Lemma 3. *Let t be a prime p -primitive divisor of $p^{r-1}-1$ and let x be a nontrivial t -element of G . If x centralizes M , then $F(x) \cap \Delta = \phi$.*

Proof. Let $A \in F(x) \cap \Delta$ and set $U = T(A)$, the set of translations of T with center A . Clearly $|U| = p^r$. By Lemma 1, $|C_U(M)| = p$ as U is regular on the set of affine points on the line OA . Set $R = \langle x \rangle$. Since R normalizes $C_U(M)$ and $t \nmid p-1$, $C_U(R)$ contains $C_U(M)$.

If $C_U(R) \neq C_U(M)$, R acts trivially on $U/C_U(R)$ as $|U/C_U(R)| < p^{r-1}$ and t is a p -primitive divisor of $p^{r-1}-1$. Hence $[R, U] = 1$ by Theorem 5.3.2 of [1]. Therefore x is a homology with axis OA and so $t \mid (p^{r-1}-1, p^r-1) = p-1$, a contradiction. Thus $C_U(R) = C_U(M)$.

By Theorem 5.2.3 of [1], $U = C_U(R) \times [U, R]$. Since M centralizes R and normalizes U , it also normalizes $[U, R]$. Hence $1 \neq C_{[U, R]}(M) \leq C_U(M) = C_U(R)$, a contradiction. Thus $F(x) \cap \Delta = \phi$.

Lemma 4. *If $r=3$, then $p \equiv -1 \pmod{4}$.*

Proof. By a Baer's theorem and Lemma 1, $p \neq 2$ and $|M| = p$ as $r=3$. Assume $p \equiv 1 \pmod{4}$ and let t be an odd prime dividing $p+1$. Clearly t is a prime p -primitive divisor of $p^{r-1}-1 = p^2-1$. Since $|M| = p$ and $t \nmid p-1$, a Sylow t -subgroup R of G centralizes M . Applying Lemma 3, R is semi-regular on Δ . As $p+1 \mid |G|$ and t is arbitrary, the length of each G -orbit on Δ is divisible by $(p+1)/2$. Since π is a counterexample of Theorem 1, G has two orbits of length $(p+1)/2$ on Δ .

Let S be a Sylow 2-subgroup of G and let $X \in F(S) \cap \Delta$. Set $\pi_0 = F(M)$, $S_0 = S_{(0, l_\infty)}$ and $K = G_\Delta$, the pointwise stabilizer of Δ in G . Since M is a non-trivial normal subgroup of G , π_0 is G -invariant and isomorphic to $PG(2, p)$. The restriction of $\text{Aut}(PG(2, p))$ on the line at infinity is isomorphic to $PGL(2, p)$ in its usual 2-transitive permutation representation. Hence G/K is isomorphic to a subgroup of $PGL(2, p)$. As $|G/K|$ is divisible by $(p+1)/2$, G/K is isomorphic to a subgroup of the dihedral group of order $2(p+1)$ by a Dickson's theorem (Theorem 14.1 of [5]). Since G/K is not transitive on Δ , $|G/K| = (p+1)/2$ or $p+1$. Therefore $|S : S \cap K| = 1$ or 2 . Hence $S \cap K$ is semiregular on $F(M) \cap (OX - \{O, X\})$ and so $|S \cap K| \mid (p-1)_2$. From this, $|S| \leq 2(p-1)_2$. But, as $S \cap K \neq 1$, $S_0 \neq 1$ and so $|S/S_0| \geq |\Gamma|_2 = 2(p-1)_2$. This implies $|S| \geq 4(p-1)_2$, a contradiction.

Lemma 5. *Let S be a 2-group acting faithfully on an elementary abelian p -group W of order p^r with $p^r \equiv -1 \pmod{4}$. If an element $x \in S$ inverts W , then $S = \langle x \rangle \times S_1$ for a subgroup S_1 of S .*

Proof. We may assume that $S \leq GL(r, p)$ and $x = -I$, where I is the unit matrix of degree r . Since r is odd, $\det(x) = (-1)^r = -1$ and so $x \in SL(r, p)$.

Since $2|p-1$ and $4 \nmid p-1$, $\langle x \rangle \times SL(r, p)$ is a normal subgroup of $GL(r, p)$ of odd index. Thus $S = \langle x \rangle \times S_1$, where $S_1 = S \cap SL(r, p)$.

Lemma 6. *Let S be a Sylow 2-subgroup of G . If $r=3$, then the length of every S -orbit on Δ is divisible by $|\Delta|_2$.*

Proof. By Lemma 4, $p \equiv -1 \pmod{4}$. Since G is transitive on Γ , $|\Gamma| = p(p^2-1)|G|$ and so $2(p+1)_2 || S/S_0|$, where S is a Sylow 2-subgroup of G and $S_0 = S_{(0, l_\infty)}$. Hence $|S_x| \geq 2 \times |S_0|$ for some point $X \in \Delta$. Here S_x denotes the stabilizer of X in S . Let $Y \in F(S_x) \cap (\Delta - \{X\})$.

First we show that $S_0 \neq 1$. Assume that $S_0 = 1$ and let u be an involution in $Z(S_x)$. By a Baer's theorem, any involution in S is a homology. Hence either u is a (X, OY) -homology or u is a (Y, OX) -homology. In either case $C_S(u) \leq S_x$. As $u \in Z(S_x)$, $C_S(u) = S_x$. In particular $|S_x| \geq 4$.

We note that either $S_{(x, oY)} = 1$ or $S_{(y, oY)} = 1$, for otherwise $S_0 \neq 1$ by Lemma 4.22 of [2]. Let $A \in \{X, Y\}$ such that $S_{(B, oA)} = 1$, where $\{B\} = \{X, Y\} - \{A\}$. Then S_x acts faithfully on $T(A)$. In particular every involution in S_x fixes no affine point on $OA - \{O\}$. Therefore every involution in S_x inverts $T(A)$. From this S_x has exactly one involution. But, by Lemma 5, S_x contains a subgroup isomorphic to $Z_2 \times Z_2$, a contradiction. Thus $S_0 \neq 1$.

Let z be an involution in S_0 . Since O is the only affine fixed point of z , z inverts T . As $(p-1)_2 = 2$, $\langle z \rangle$ is a unique Sylow 2-subgroup of $G_{(0, l_\infty)}$.

Set $V = S_x$. If $V_{(x, oY)} = 1$, then V acts faithfully on $T(Y)$ and moreover z inverts $T(Y)$. By Lemma 5, V contains a subgroup U such that $z \in U$ and U is isomorphic to $Z_2 \times Z_2$. By Lemma 4.22 of [2], we obtain a contradiction. Hence $V_{(x, oY)} \neq 1$.

Let u be an involution in $V_{(x, oY)}$. Then, as $u \in Z(V)$, we have $C_S(u) = V$. Assume $|V| > 4$. $\bar{V} = V/\langle u \rangle$ normalizes $T(Y)$ and z inverts $T(Y)$. Hence $\bar{V} = \langle \bar{z} \rangle \times \bar{L}$ for a subgroup L of V with $u \in L$ by Lemma 5. Since $L_{(0, l_\infty)} = 1$ and $u \in L$, L acts faithfully on $T(X)$ and u inverts $T(X)$. Hence $L = \langle u \rangle \times Z$ for a subgroup Z of L by Lemma 5. As $|L| \geq 4$, Z contains an involution. Therefore $Z_{(0, l_\infty)} \neq 1$ or $Z_{(y, oX)} \neq 1$, a contradiction. Thus $|V| = 4$.

As $V \leq S_Y$ and $F(V) \cap l_\infty = \{X, Y\}$, we have $V = S_Y$. Since V is isomorphic to $Z_2 \times Z_2$ and $C_S(u) = V$, S is dihedral or semidihedral by a lemma of [7]. Therefore any involution in S is S -conjugate to an involution in V . Hence, if $S_Q \neq 1$ for some $Q \in \Delta$, then $Q = X^s$ or Y^s for some $s \in S$. Thus $|S_Q| = |V| = 4$. Therefore $|Q^s| \geq 2|\Gamma|_2/4 = (p+1)_2$ for all $Q \in \Delta$.

Lemma 7. $r \neq 3$.

Proof. Assume that $r=3$. Let t be an odd prime dividing $p+1$. Then

t is a prime p -primitive divisor of p^2-1 . Let R be a Sylow t -subgroup of G . Since G is transitive on Γ , $p(p^2-1)=|\Gamma||G|$ and so $R \neq 1$. By Lemma 1, $|M|=p$ as $r=3$. Hence R centralizes M . Applying Lemma 3, R acts semi-regularly on Δ . Since t is arbitrary, using Lemma 6 we have that G acts transitively on Δ . As π is a counterexample, this is a contradiction. Thus we have the lemma.

Lemma 8. *There exists a prime p -primitive divisor t of $p^{r-1}-1$ such that $t||G|$ and $t \nmid |C_G(M)|$.*

Proof. $|G|$ is divisible by $p^{r-1}-1$ as $|\Gamma||G|$. By Lemmas 1 and 7, $r-1 \geq 3$ and by Lemma 2, $(p, r-1) \neq (2, 6)$. It follows from a Zsigmondy's theorem (Theorem 6.2 of [5]) that there exists a prime p -primitive divisor t of $p^{r-1}-1$.

Assume $t||C_G(M)|$ and let R be a Sylow t -subgroup of $C_G(M)$. By Lemma 3, R is semiregular on Δ . Hence $t|p+1$ and so $t|p^2-1$. Since t is a p -primitive divisor of $p^{r-1}-1$, we have $r-1=2$, contrary to Lemma 7.

Lemma 9. *Each M -orbit on Γ is of length p .*

Proof. Since $p||\Gamma|$, $p^2 \nmid |\Gamma|$ and M is $\frac{1}{2}$ -transitive on Γ , using Lemma 1 each M -orbit on Γ has length p .

Proof of Theorem 1.

Let t be a prime as in Lemma 8 and let R be a Sylow t -subgroup of G . By Lemma 8, $R \neq 1$ and acts faithfully on M . Since t is a p -primitive divisor of $p^{r-1}-1$, we have $|M| \geq p^{r-1}$. Hence, by Proposition 6.12 of [3], $p^r=16$. From this, $p=2$, $t=7$ and $|M| \geq 8$.

Let $A \in \Gamma$ and set $N=M_A$. By Lemma 1, $F(N) \supset \Delta \cup \{A\}$. Therefore $F(N)$ is a subplane of order 4. Let $B \in l_\infty - F(N) \cap l_\infty$. Clearly $F(N_B)=\pi$ and so $N_B=1$. By Lemma 9, $|M:N|=2$ and $|N:N_B|=2$. Hence $|M|=4$, a contradiction. Thus we have Theorem 1.

References

- [1] D. Gorenstein: *Finite groups*, Harper and Row, New York, 1968.
- [2] D.R. Hughes and F.C. Piper: *Projective planes*, Springer-Verlag, Berlin-Heidelberg-New York, 1973.
- [3] V. Jha: *On tangentially transitive translation planes and related systems*, *Geom. Dedicata* **4** (1975), 457-483.
- [4] V. Jha: *On Δ -transitive translation planes*, *Arch. Math.* **37** (1981), 377-384.
- [5] H. Lüneburg: *Translation planes*, Springer-Verlag, Berlin-Heidelberg-New York, 1980.
- [6] M.L. Narayana Rao and K. Satyanarayana: *A new class of square order planes*,

- J. Combin. Theory Ser. A **35** (1983), 33–42.
- [7] M. Suzuki: *A characterization of simple groups $LF(2, F)$* , J. Fac. Sci. Univ. Tokyo Sect. I **6** (1951), 259–293.

Department of Mathematics
College of General Education
Osaka University
Toyonaka, Osaka 560
Japan