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THE DORFMEISTER-NEHER THEOREM ON ISOPARAMETRIC HYPERSURFACES

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Abstract

A new proof of the homogeneity of isoparametric hypersurfaces with six simple principal curvatures [4] is given in a method applicable to the multiplicity two case.

1. Introduction

The classification problem of isoparametric hypersurfaces is remaining in some cases of four and six principal curvatures ([3], [5], [9]). The homogeneity in the case $(g, m) = (6, 1)$ was proved by Dorfmeister-Neher [4]. A shorter proof was given in [7], but some argument was insufficient (pointed out by Xia Qiaoling). Moreover, we found it difficult to extend the method to the case $(g, m) = (6, 2)$.

In the present paper, we show that a delicate change of signs of some vectors at anti-podal points on a leaf, which is related to the back ground symmetry caused by a spin action, is essential. This investigation is also indispensable to attack on the case $m = 2$. Before treating this overwhelmingly difficult case, a complete short proof for $m = 1$ will give us an overview how to settle the problem in the case $m = 2$ [9].

§2–§5 consist of reviews of [6] and [7]. We do not repeat the proofs in [6], but give those of [7] in a refined manner. The shape operators of each focal submanifold $M_{\pm}$ consist of an S^1-family of isospectral transformations with simple eigenvalues $\pm\sqrt{3}$, $\pm 1/\sqrt{3}$, 0. There are many such S^1-families (see §2), but in §6–§9, we narrow down them by using both local and global properties of isoparametric hypersurfaces, and conclude that non-homogeneous cases cannot occur.

2. Preliminaries

We refer readers to [11] for a nice survey of isoparametric hypersurfaces. Here we review fundamental facts and the notation given in [6]. Let M be an isoparametric hypersurface in the unit sphere S^{n+1}, with a unit normal vector field ξ. We denote the Riemannian connection on S^{n+1} by $\tilde{\nabla}$, and that on M by ∇. The principal curvatures of M are given by constants $\lambda_1 \geq \cdots \geq \lambda_n$, and the curvature distribution for $\lambda \in \{\lambda_\alpha\}$

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is denoted by $D_\lambda(p)$, $m_\lambda = \dim D_\lambda(p)$. In our situation, D_λ is completely integrable and a leaf L_λ of D_λ is an m_λ-dimensional sphere of S^{n+1}. Choose a local orthonormal frame $e_1, \dots, e_n$ consisting of unit principal vectors corresponding to $\lambda_1, \dots, \lambda_n$. We express

$$\tilde{\nabla}_{e_\alpha} e_\beta = \Lambda^\sigma_{\alpha\beta} e_\sigma + \lambda_\alpha \delta_{\alpha\beta}\xi, \quad \Lambda^\gamma_{\alpha\beta} = -\Lambda^\beta_{\alpha\gamma}, \tag{1}$$

where $1 \le \alpha, \beta, \sigma \le n$, using the Einstein convention. The curvature tensor $R_{\alpha\beta\gamma\delta}$ of M is given by

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= (1 + \lambda_\alpha\lambda_\beta)(\delta_{\beta\gamma}\delta_{\alpha\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta}) \\ &= e_\alpha(\Lambda^\delta_{\beta\gamma}) - e_\beta(\Lambda^\delta_{\alpha\gamma}) + \Lambda^\sigma_{\beta\gamma}\Lambda^\delta_{\alpha\sigma} - \Lambda^\sigma_{\alpha\gamma}\Lambda^\delta_{\beta\sigma} - \Lambda^\sigma_{\alpha\beta}\Lambda^\delta_{\sigma\gamma} + \Lambda^\sigma_{\beta\alpha}\Lambda^\delta_{\sigma\gamma}. \end{aligned} \tag{2}$$

From the equation of Codazzi, we obtain

$$e_\beta(\lambda_\alpha) = \Lambda^\beta_{\alpha\alpha}(\lambda_\alpha - \lambda_\beta), \quad \text{for} \quad \alpha \ne \beta, \tag{3}$$

and if $\lambda_\alpha, \lambda_\beta, \lambda_\gamma$ are distinct, we have

$$\Lambda^\gamma_{\alpha\beta}(\lambda_\beta - \lambda_\gamma) = \Lambda^\beta_{\gamma\alpha}(\lambda_\alpha - \lambda_\beta) = \Lambda^\alpha_{\beta\gamma}(\lambda_\gamma - \lambda_\alpha). \tag{4}$$

Moreover,

$$\Lambda^\gamma_{ab} = 0, \quad \Lambda^\gamma_{aa} = \Lambda^\gamma_{bb}, \quad \text{if} \quad \lambda_a = \lambda_b \ne \lambda_\gamma \quad \text{and} \quad a \ne b, \tag{5}$$

hold, and since λ_α is constant on M, it follows from (3),

$$\Lambda^\gamma_{\alpha\alpha} = 0 \quad \text{if} \quad \lambda_\gamma \ne \lambda_\alpha. \tag{6}$$

When the number g of principal curvatures is six, the multiplicity m of λ_i is independent of i and takes values 1 or 2 [1]. In the following, let $(g, m) = (6, 1)$. As is well known, $\lambda_i = \cot(\theta_1 + (i-1)\pi/6)$, $1 \le i \le 6$, $0 < \theta_1 < \pi/6$, modulo π. Since the homogeneity is independent of the choice of θ_1, we take

$$\theta_1 = \frac{\pi}{12} = -\theta_6, \quad \theta_2 = \frac{\pi}{4} = -\theta_5, \quad \theta_3 = \frac{5\pi}{12} = -\theta_4$$

so that

$$\lambda_1 = -\lambda_6 = 2 + \sqrt{3}, \quad \lambda_2 = -\lambda_5 = 1, \quad \lambda_3 = -\lambda_4 = 2 - \sqrt{3}. \tag{7}$$

Note that we choose $\theta_i \in (-\pi/2, \pi/2)$. By (5) and (6), a leaf $L_i = L_i(p)$ of $D_i(p) = D_{\lambda_i}(p)$ is a geodesic of the corresponding curvature sphere.

For $a = 6$ or 1, define the focal map $f_a : M \to S^7$ by

$$f_a(p) = \cos\theta_a p + \sin\theta_a \xi_p,$$

which collapses $L_a(p)$ into a point $\bar{p} = f_a(p)$. Then we have

$$df_a(e_j) = \sin\theta_a(\lambda_a - \lambda_j)e_j, \tag{8}$$

where the right hand side is considered as a vector in $T_{\bar{p}}S^7$ by a parallel translation in S^7. We always use such identification. The rank of f_a is constant and we obtain the focal submanifold M_a of M:

$$M_a = \{\cos\theta_a p + \sin\theta_a \xi_p \mid p \in M\}.$$

By (8), the tangent space of M_a is given by $T_{\bar{p}}M_a = \bigoplus_{j \neq a} D_j(q)$ for any $q \in f_a^{-1}(\bar{p})$. An orthonormal basis of the normal space of M_a at $\bar{p}$ is given by

$$\eta_q = -\sin\theta_a q + \cos\theta_a \xi_q, \quad \zeta_q = e_a(q) \tag{9}$$

for any $q \in L_a(p) = f_a^{-1}(\bar{p})$.

Now, the connection $\bar{\nabla}$ on M_a is induced from the connection $\tilde{\nabla}$, that is

$$\frac{1}{\sin\theta_a(\lambda_a - \lambda_j)}\tilde{\nabla}_{e_j}X = \bar{\nabla}_{e_j}\tilde{X} + \bar{\nabla}^{\perp}_{e_j}\tilde{X}, \quad \lambda_j \neq \lambda_a,$$

where X is a tangent field on S^7 in a neighborhood of p, and $\tilde{X}$ is the one near $\bar{p}$ translated from X. Note that $\bar{\nabla}^{\perp}_{e_j}\tilde{X}$ denotes the normal component in S^7. In particular, we have for $j \neq a$,

$$\bar{\nabla}_{e_j}\tilde{e}_k = \frac{1}{\sin\theta_a(\lambda_a - \lambda_j)} \sum_{l \neq a} \Lambda^l_{jk} e_l, \tag{10}$$

$$\bar{\nabla}^{\perp}_{e_j}\tilde{e}_k = \frac{1}{\sin\theta_a(\lambda_a - \lambda_j)}\{\Lambda^a_{jk} e_a + \sin\theta_a(1 + \lambda_j\lambda_a)\delta_{jk}\eta_p\}, \tag{11}$$

using $\langle \lambda_j \xi_p - p, \eta_p \rangle = \sin\theta_a(1 + \lambda_j\lambda_a)$. In the following, we identify $\tilde{e}_k$ with e_k. Denote by B_N the shape operator of M_a with respect to the normal vector N. Then from (10) and (11), we obtain:

Lemma 2.1 ([6] (Lemma 3.1)). *When we identify* $T_{\bar{p}}M_a$ *with* $\bigoplus_{j=1}^{5} D_{a+j}(p)$ *where the indices are modulo* 6, *the second fundamental tensors* B_{η_p} *and* B_{ζ_p} *at* $\bar{p}$ *are given respectively by*

$$B_{\eta_p} = \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3} \end{pmatrix},$$

$$B_{\zeta_p} = \begin{pmatrix} 0 & b_{a+1\,a+2} & b_{a+1\,a+3} & b_{a+1\,a+4} & b_{a+1\,a+5} \\ b_{a+1\,a+2} & 0 & b_{a+2\,a+3} & b_{a+2\,a+4} & b_{a+2\,a+5} \\ b_{a+1\,a+3} & b_{a+2\,a+3} & 0 & b_{a+3\,a+4} & b_{a+3\,a+5} \\ b_{a+1\,a+4} & b_{a+2\,a+4} & b_{a+3\,a+4} & 0 & b_{a+4\,a+5} \\ b_{a+1\,a+5} & b_{a+2\,a+5} & b_{a+3\,a+5} & b_{a+4\,a+5} & 0 \end{pmatrix},$$

where

$$b_{jk} = \frac{1}{\sin\theta_a(\lambda_a - \lambda_j)}\Lambda^a_{jk} = \frac{1}{\sin\theta_a(\lambda_j - \lambda_a)}\Lambda^k_{ja}, \quad a = 6,\ 1. \tag{12}$$

In fact, from (11) it follows $B_{\eta_p}(e_j) = \mu_j e_j$, where for a is, say 6,

$$\mu_j = \frac{1 + \lambda_j\lambda_6}{\lambda_6 - \lambda_j}, \quad \mu_1 = \sqrt{3} = -\mu_5,\ \mu_2 = \frac{1}{\sqrt{3}} = -\mu_4,\ \mu_3 = 0, \tag{13}$$

and $b_{jk} = b_{kj}$ follows from (4). In the following, we denote $M_+ = M_6$ and $M_- = M_1$. Note that both are minimal. It is easy to see that any unit normal vector is written as η_q in (9) for some $q \in L_6(p)$, and we have immediately:

Lemma 2.2 ([10], [6]). *The shape operators are isospectral*, *i.e.*, *the eigenvalues of* B_N *are* $\pm\sqrt{3}$, $\pm 1/\sqrt{3}$, 0, *for any unit normal* N.

For a fixed $p \in f_a^{-1}(\bar{p})$, all the shape operators for unit normals at $\bar{p}$ are expressed as

$$L(t) = \cos t\, B_{\eta_p} + \sin t\, B_{\zeta_p}, \quad t \in [0, 2\pi). \tag{14}$$

The homogeneous hypersurfaces M^h with $(g, m) = (6, 1)$ are given as the principal orbits of the isotropy action of the rank two symmetric space $G_2/SO(4)$, where two singular orbits correspond to the focal submanifolds $M^h_\pm$. In [6], we show that the shape operators of M_+^h and M_-^h are given respectively by:

$$
(15)\quad
\begin{aligned}
&\cos t\begin{pmatrix} \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3} \end{pmatrix} + \sin t\begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 & 0 \end{pmatrix}, \\
&\cos t\begin{pmatrix} \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3} \end{pmatrix} + \sin t\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned}
$$

These imply that $M_\pm$ are *not* congruent to each other.

Note that there *exist* many other one parameter families of isospectral operators $\cos t B_\eta + \sin t A$, where, for instance, A is given by

$$
(16)\quad
\begin{aligned}
&\begin{pmatrix} 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ -\sqrt{\frac{3}{2}} & 0 & 0 & 0 & \sqrt{\frac{3}{2}} \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 & 0 \end{pmatrix}, \\
&\begin{pmatrix} 0 & \frac{5}{3\sqrt{3}} & 0 & \frac{2}{3\sqrt{3}} & 0 \\ \frac{5}{3\sqrt{3}} & 0 & \frac{4}{3\sqrt{3}} & 0 & -\frac{2}{3\sqrt{3}} \\ 0 & \frac{4}{3\sqrt{3}} & 0 & \frac{4}{3\sqrt{3}} & 0 \\ \frac{2}{3\sqrt{3}} & 0 & \frac{4}{3\sqrt{3}} & 0 & -\frac{5}{3\sqrt{3}} \\ 0 & -\frac{2}{3\sqrt{3}} & 0 & -\frac{5}{3\sqrt{3}} & 0 \end{pmatrix},
\end{aligned}
$$

and so forth. We see in the homogeneous case, the kernel does not depend on t, while it depends in other cases. In the following, we show that all the latter cases are not admissible to the shape operators of the focal submanifolds of isoparametric hypersurfaces with $(g, m) = (6, 1)$.

3. Isospectral operators and Gauss equation

By Lemma 2.2, $L(t) = \cos t B_\eta + \sin t B_\zeta$ is isospectral and so can be written as

$$L(t) = U(t)L(0)U^{-1}(t) \tag{17}$$

for some $U(t) \in O(5)$. Moreover, this implies the Lax equation

$$L_t(t) = \frac{d}{dt}L(t) = [H(t), L(t)], \tag{18}$$

where

$$H(t) = U_t(t)U(t)^{-1} \in \mathfrak{o}(5).$$

In particular, we have $L(0) = B_\eta$, and

$$L_t(t) = -\sin t B_\eta + \cos t B_\zeta = L\left(t + \frac{\pi}{2}\right), \tag{19}$$

and hence for $L_t(0) = B_\zeta = (b_{ij})$, $b_{ij} = b_{ji}$, and $H(0) = (h_{ij})$, $h_{ji} = -h_{ij}$, we can express

$$\begin{aligned} B_\zeta &= L_t(0) = [H(0), B_\eta] \\ &= \begin{pmatrix} 0 & -\frac{2}{\sqrt{3}}h_{12} & -\sqrt{3}h_{13} & -\frac{4}{\sqrt{3}}h_{14} & -2\sqrt{3}h_{15} \\ \frac{2}{\sqrt{3}}h_{21} & 0 & -\frac{1}{\sqrt{3}}h_{23} & -\frac{2}{\sqrt{3}}h_{24} & -\frac{4}{\sqrt{3}}h_{25} \\ \sqrt{3}h_{31} & \frac{1}{\sqrt{3}}h_{32} & 0 & -\frac{1}{\sqrt{3}}h_{34} & -\sqrt{3}h_{35} \\ \frac{4}{\sqrt{3}}h_{41} & \frac{2}{\sqrt{3}}h_{42} & \frac{1}{\sqrt{3}}h_{43} & 0 & -\frac{2}{\sqrt{3}}h_{45} \\ 2\sqrt{3}h_{51} & \frac{4}{\sqrt{3}}h_{52} & \sqrt{3}h_{53} & \frac{2}{\sqrt{3}}h_{54} & 0 \end{pmatrix}. \end{aligned} \tag{20}$$

Note that the eigenvectors of $L(t)$ are given by

$$e_j(t) = U(t)e_j(0), \tag{21}$$

which implies

$$\nabla_{d/dt}e_j(t) = H(t)e_j(t). \tag{22}$$

Here we have

$$\nabla_{d/dt} = c_0 \nabla_{e_6}, \quad c_0 = \frac{\sqrt{2}(\sqrt{3}-1)}{4}, \tag{23}$$

because L_6 has radius $|\sin\theta_6| = c_0$. Hence we obtain

$$H(0) = (c_0 \Lambda^i_{6j}(0)), \tag{24}$$

where i denotes the row and j denotes the column indices. Moreover, denoting the (i, j) component of $L(t+\pi/2)$ by $b_{ij}(t)$ where $b_{ji}(t) = b_{ij}(t)$, we have

$$\begin{aligned}\left(\nabla_{e_6} L\left(t+\frac{\pi}{2}\right)\right)_{ij} &= e_6(b_{ij}(t)) - b_{kj}(t)\Lambda^k_{6i}(t) - b_{ik}(t)\Lambda^k_{6j}(t)\\ &= e_6(b_{ij}(t)) + \Lambda^i_{6k}(t) b_{kj}(t) - b_{ik}(t)\Lambda^k_{6j}(t).\end{aligned}$$

Because $L_t(t+\pi/2) = c_0\nabla_{e_6} L(t+\pi/2)$, $L_t(\pi/2) = -B_\eta$ and $L(\pi/2) = B_\zeta$, multiplying $-c_0$ to the both sides and putting $t = 0$, we obtain

$$B_\eta = -c_0 e_6(B_\zeta) - [H(0), B_\zeta]. \tag{25}$$

Now, rewrite (20) as

$$H(0) = \begin{pmatrix} 0 & -\frac{\sqrt{3}}{2}b_{12} & -\frac{1}{\sqrt{3}}b_{13} & -\frac{\sqrt{3}}{4}b_{14} & -\frac{1}{2\sqrt{3}}b_{15} \\ \frac{\sqrt{3}}{2}b_{21} & 0 & -\sqrt{3}b_{23} & -\frac{\sqrt{3}}{2}b_{24} & -\frac{\sqrt{3}}{4}b_{25} \\ \frac{1}{\sqrt{3}}b_{31} & \sqrt{3}b_{32} & 0 & -\sqrt{3}b_{34} & -\frac{1}{\sqrt{3}}b_{35} \\ \frac{\sqrt{3}}{4}b_{41} & \frac{\sqrt{3}}{2}b_{42} & \sqrt{3}b_{43} & 0 & -\frac{\sqrt{3}}{2}b_{45} \\ \frac{1}{2\sqrt{3}}b_{51} & \frac{\sqrt{3}}{4}b_{52} & \frac{1}{\sqrt{3}}b_{53} & \frac{\sqrt{3}}{2}b_{54} & 0 \end{pmatrix},$$

and substitute this into (25). Then we have the following formulas which we use later:

$$[1.1] \quad \sqrt{3} = 2\left(\frac{\sqrt{3}}{2}b_{12}^2 + \frac{1}{\sqrt{3}}b_{13}^2 + \frac{\sqrt{3}}{4}b_{14}^2 + \frac{1}{2\sqrt{3}}b_{15}^2\right),$$

$$[2.2] \quad \frac{1}{\sqrt{3}} = 2\left(-\frac{\sqrt{3}}{2}b_{21}^2 + \sqrt{3}b_{23}^2 + \frac{\sqrt{3}}{2}b_{24}^2 + \frac{\sqrt{3}}{4}b_{25}^2\right),$$

$$[3.3] \quad 0 = 2\left(-\frac{1}{\sqrt{3}}b_{31}^2 - \sqrt{3}b_{32}^2 + \sqrt{3}b_{34}^2 + \frac{1}{\sqrt{3}}b_{35}^2\right),$$

$$[4.4]\quad -\frac{1}{\sqrt{3}} = 2\left(-\frac{\sqrt{3}}{4}b_{41}^2 - \frac{\sqrt{3}}{2}b_{42}^2 - \sqrt{3}b_{43}^2 + \frac{\sqrt{3}}{2}b_{45}^2\right),$$

$$[5.5]\quad -\sqrt{3} = -2\left(\frac{1}{2\sqrt{3}}b_{51}^2 + \frac{\sqrt{3}}{4}b_{52}^2 + \frac{1}{\sqrt{3}}b_{53}^2 + \frac{\sqrt{3}}{2}b_{54}^2\right),$$

$$[1.2]\quad 0 = -c_0e_6(b_{12}) + \frac{4}{\sqrt{3}}b_{13}b_{32} + \frac{3\sqrt{3}}{4}b_{14}b_{42} + \frac{5}{4\sqrt{3}}b_{15}b_{52},$$

$$[1.3]\quad 0 = -c_0e_6(b_{13}) - \frac{\sqrt{3}}{2}b_{12}b_{23} + \frac{5\sqrt{3}}{4}b_{14}b_{43} + \frac{\sqrt{3}}{2}b_{15}b_{53},$$

$$[1.4]\quad 0 = -c_0e_6(b_{14}) - \frac{2}{\sqrt{3}}b_{13}b_{34} + \frac{2}{\sqrt{3}}b_{15}b_{54},$$

$$[1.5]\quad 0 = -c_0e_6(b_{15}) + \frac{\sqrt{3}}{4}b_{12}b_{25} - \frac{\sqrt{3}}{4}b_{14}b_{45},$$

$$[2.3]\quad 0 = -c_0e_6(b_{23}) - \frac{5}{2\sqrt{3}}b_{21}b_{13} + \frac{3\sqrt{3}}{2}b_{24}b_{43} + \frac{7}{4\sqrt{3}}b_{25}b_{53},$$

$$[2.4]\quad 0 = -c_0e_6(b_{24}) - \frac{3\sqrt{3}}{4}b_{21}b_{14} + \frac{3\sqrt{3}}{4}b_{25}b_{54},$$

$$[2.5]\quad 0 = -c_0e_6(b_{25}) - \frac{2}{\sqrt{3}}b_{21}b_{15} + \frac{2}{\sqrt{3}}b_{23}b_{35},$$

$$[3.4]\quad 0 = -c_0e_6(b_{34}) - \frac{7}{4\sqrt{3}}b_{31}b_{14} - \frac{3\sqrt{3}}{2}b_{32}b_{24} + \frac{5}{2\sqrt{3}}b_{35}b_{54},$$

$$[3.5]\quad 0 = -c_0e_6(b_{35}) - \frac{\sqrt{3}}{2}b_{31}b_{15} - \frac{5\sqrt{3}}{4}b_{32}b_{25} + \frac{\sqrt{3}}{2}b_{34}b_{45},$$

$$[4.5]\quad 0 = -c_0e_6(b_{45}) - \frac{5}{4\sqrt{3}}b_{41}b_{15} - \frac{3\sqrt{3}}{4}b_{42}b_{25} - \frac{4}{\sqrt{3}}b_{43}b_{35}.$$

These are nothing but another description of a part of the Gauss equations (2) [8].

4. Global properties

An isoparametric hypersurface M can be uniquely extended to a closed one [2]. We recall now the global properties of M.

Let $p \in M$ and let γ be the normal geodesic at p. We know that $\gamma \cap M$ consists of twelve points $p_1, \dots, p_{12}$ which are vertices of certain dodecagon: see Fig. 1, where indices are changed from [6, pp. 197–198] and [7, Lemma 3.2].

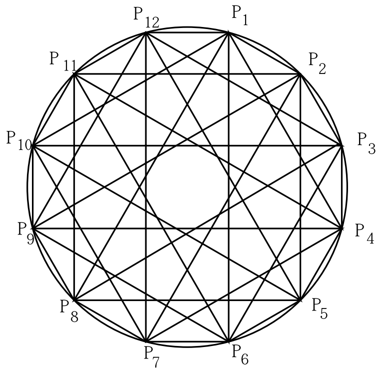

Fig. 1.

Lemma 4.1 ([6]). *We have the relations*

$$D_i(p_1) = D_{2-i}(p_2) = D_{i+4}(p_3) = D_{4-i}(p_4) = D_{i+2}(p_5) = D_{6-i}(p_6),$$
$$D_i(p_j) = D_i(p_{j+6}), \quad j = 1, \dots, 6$$

where the equality means "be parallel to with respect to the connection of S^7*", and the indices are modulo* 6.

From these, some relations among $\Lambda^{\gamma}_{\alpha\beta}$'s are obtained as follows. Denote by $p(t)$ the point on $L_6(p)$ such that $p_1 = p(0)$, parametrized by the center angle where the center means that of a circle on a plane. Similarly, we denote by $q(t)$ the point on $L_2(p_2)$ parametrized from $p_2 = q(0)$. Note that $e_6(p_1)$ is parallel with $e_2(p_2)$. Extend e_6 and e_2 as the unit tangent vectors of $p(t)$ and $q(t)$, respectively. Consider the normal geodesic γ_t at $p(t)$, then $q(t) = L_2(p_2) \cap \gamma_t$. Here $e_3(p(t))$ is parallel with $e_5(q(t))$. Then we have

$$\frac{1}{\sin\theta_6} \nabla_{d/dt} e_3(p(t)) = \frac{\sin\theta_2}{\sin\theta_6} \frac{1}{\sin\theta_2} \nabla_{d/dt} e_5(q(t)).$$

Therefore the D_j component of $(\nabla_{e_6} e_3)(p_1)$ is the D_{2-j} component of $(\nabla_{e_2} e_5)(p_2)$ multiplied by $\sin\theta_2/\sin\theta_6$. We denote such relation by

$$\Lambda^{j}_{63}(p_1) \sim \Lambda^{2-j}_{25}(p_2),$$

up to sign. A similar argument at every p_m implies the global correspondence among $\Lambda^{\gamma}_{\alpha\beta}$'s:

p_1	p_2	p_3	p_4	p_5	p_6
1	1	5	3	3	5
2	6	6	2	4	4
3	5	1	1	5	3
4	4	2	6	6	2
5	3	3	5	1	1
6	2	4	4	2	6

Table 1.

Lemma 4.2 ([6]). *For a frame consisting of principal vectors around each* p_m, *we have the correspondence* $\Lambda^i_{jk}(p_m) \sim \Lambda^{i'}_{j'k'}(p_n)$ *where* i, j, k *at* p_m *correspond to* i', j', k' *at* p_n *in* Table 1.

5. The kernel of the shape operators

For $p \in M$ and $\bar{p} \in M_+$, let

$$E_{\bar{p}} = \mathrm{span}\{\mathrm{Ker} L(t) \mid t \in [0, 2\pi)\} = \mathrm{span}_{t\in[0,2\pi)}\{e_3(t)\}.$$

The following proposition proved in [6] is crucial.

Proposition 5.1 ([6] (Proposition 4.2)). *M is homogeneous if and only if* $\dim E_{\bar{p}} = 1$ *for any* p.

Next, recall

$$\mu_i = \frac{1 + \lambda_i \lambda_6}{\lambda_6 - \lambda_i} = c_1 \frac{\lambda_3 - \lambda_i}{\lambda_6 - \lambda_i}, \quad c_1 = 2 + \sqrt{3}. \tag{26}$$

The second equality follows from $\lambda_6 = -1/\lambda_3 = -(2 + \sqrt{3})$. Put

$$c_2 = \frac{1}{\sin\theta_6(\lambda_3 - \lambda_6)} = -\frac{\sqrt{2}(\sqrt{3}+1)}{4}, \quad (\sin\theta_6 = -\sqrt{2}(\sqrt{3}-1)/4).$$

Lemma 5.2. *Take* $p \in f_6^{-1}(\bar{p})$ *and identify* $T_{\bar{p}}M_+$ *with* $\bigoplus_{j=1}^5 D_j(p)$. *Then we have*

$$B_\zeta(e_3) = c_2 \nabla_{e_3} e_6, \tag{27}$$

$$B_\eta(\nabla_{e_6} e_3) = c_1 \nabla_{e_3} e_6, \tag{28}$$

$$B_\zeta(\nabla_{e_6} e_3) = c_2 \nabla_{e_6} \nabla_{e_3} e_6. \tag{29}$$

Similar formulas hold for the shape operators C_N *of* M_-, *if we replace* 6 *by* 1, *and* 3 *by* 4.

Proof. From (12) follows (27). Using (4), we have (28):

$$B_\eta(\nabla_{e_6} e_3) = \Lambda^i_{63}\mu_i e_i = c_1\Lambda^i_{63}\frac{\lambda_3 - \lambda_i}{\lambda_6 - \lambda_i}e_i = c_1\Lambda^i_{36}e_i = c_1\nabla_{e_3}e_6. \tag{30}$$

Taking the covariant derivative of (27) where $\nabla_{d/dt} = c_0\nabla_{e_6}$ by (23), we obtain

$$c_2\nabla_{e_6}\nabla_{e_3}e_6 = \nabla_{e_6}(B_\zeta(e_3)) = -\frac{1}{c_0}B_\eta(e_3) + B_\zeta(\nabla_{e_6}e_3) = B_\zeta(\nabla_{e_6}e_3). \qquad \square$$

REMARK 5.3. (27) implies that $\dim E_{\bar{p}} = 1$ holds if and only if $\nabla_{e_6}e_3$ vanishes at a point of $f^{-1}(\bar{p})$. Moreover, (28) implies that $\nabla_{e_6}e_3$ vanishes if and only if $\nabla_{e_3}e_6$ vanishes.

When $\nabla_{e_6}e_3(p) \not\equiv 0$, we have $\dim E_{\bar{p}} \geq 2$, since $e_3(p)$ and $\nabla_{e_6}e_3(p)$ ($\in E_{\bar{p}}$) are mutually orthogonal. We denote E instead of $E_{\bar{p}}$, when it causes no confusion. Let $E^\perp$ be the orthogonal complement of E in $T_{\bar{p}}M_+$. Moreover, put

$$W = W_{\bar{p}} = \mathrm{span}_{t\in[0,2\pi)}\{\nabla_{e_3}e_6(t)\},$$

where we regard W as a subspace of $T_{\bar{p}}M_+$ by a parallel displacement. The following lemmas are significant.

Lemma 5.4 ([7] (Lemma 4.2)). $W \subset E^\perp$.

Proof. We can express $L(t)$ with respect to the basis $e_i(p)$, $i = 1, \dots 5$, as in Lemma 2.1,

$$L(t) = \begin{pmatrix} \sqrt{3}c & sb_{12} & sb_{13} & sb_{14} & sb_{15} \\ sb_{12} & \frac{1}{\sqrt{3}}c & sb_{23} & sb_{24} & sb_{25} \\ sb_{13} & sb_{23} & 0 & sb_{34} & sb_{35} \\ sb_{14} & sb_{24} & sb_{34} & -\frac{1}{\sqrt{3}}c & sb_{45} \\ sb_{15} & sb_{25} & sb_{35} & sb_{45} & -\sqrt{3}c \end{pmatrix}, \quad \begin{cases} c = \cos t, \\ s = \sin t. \end{cases}$$

Let $e_3(t) = {}^t(u_1(t), \dots, u_5(t))$ belong to the kernel of $L(t)$. Then the third component of $L(t)(e_3(t))$ must satisfy

$$\frac{\sin t}{\sin\theta_6}\frac{1}{\lambda_3 - \lambda_6}\sum_{i=1}^{5}\Lambda^i_{36}(p)u_i(t) = 0.$$

Thus we obtain

$$\langle \nabla_{e_3} e_6(p), e_3(t) \rangle = 0 \tag{31}$$

for *all* t, which means $\nabla_{e_3} e_6(p) \in E^{\perp}$. □

By the analyticity and the definition of E and W, we can express

$$\begin{aligned} E &= \mathrm{span}\{e_3(q), \nabla_{e_6}^k e_3(q), \ k = 1, 2, \dots\}, \\ W &= \mathrm{span}\{\nabla_{e_3} e_6(q), \nabla_{e_6}^k \nabla_{e_3} e_6(q), \ k = 1, 2, \dots\}, \end{aligned} \tag{32}$$

at any fixed point $q \in L_6$, where $\nabla_{e_6}^k$ means k-th covariant differential in the direction e_6. Thus we have by Lemma 5.4,

$$\langle \nabla_{e_6}^k e_3, \nabla_{e_6}^l \nabla_{e_3} e_6 \rangle = 0, \quad k, l = 0, 1, 2, \dots. \tag{33}$$

Lemma 5.5 ([7] (Lemma 4.3)). *For any t, $L(t)$ maps E onto $W \subset E^{\perp}$.*

Proof. We can express $L(t) = \cos t L(t_0) + \sin t L_t(t_0)$ for any t_0. Then $L(t_0)(e_3(t_0)) = 0$ and $L_t(t_0)(e_3(t_0)) = c_2 \nabla_{e_3} e_6(t_0)$ (see (27)) imply

$$L(t)(e_3(t_0)) = (\cos t L(t_0) + \sin t L_t(t_0))(e_3(t_0)) = c_2 \sin t \nabla_{e_3} e_6(t_0) \in W.$$

Moreover, (27) implies that this is an onto map. □

Lemma 5.6 ([7] (Lemma 4.4)). $\dim E \leq 3$.

Proof. Take any $p \in f_6^{-1}(\bar{p})$. Since $\mathrm{Ker} B_{\eta_p} = D_3(p) \subset E$, we have $\dim B_{\eta}(E) = \dim E - 1$. Because $B_{\eta_p}(E)$ is a subspace of $E^{\perp}$, the lemma follows from $\mathbb{R}^5 \cong T_{\bar{p}} M_+ = E \oplus E^{\perp}$. □

The following is obvious:

Lemma 5.7. *As a function of $\bar{p} \in M_+$,* $\dim E$ *is lower-semi-continuous*.

Let $d = \max_{\bar{p} \in M_+} \dim E_{\bar{p}}$. We know that $1 \leq d \leq 3$ and M is homogeneous when $d = 1$. At a point $\bar{q}$ on the focal submanifolds $M_- = M_1$, denote $F_{\bar{q}} = \mathrm{span}_{q(t) \in L^1(q)}\{e_4(q(t))\}$. The argument on M_+ holds for M_- if we replace E by F and pay attention to the change of indices. Especially, $\dim E = 1$ holds on M_+ if and only if $\dim F = 1$ holds on M_-, because $\Lambda_{36}^j = 0$ holds for all j if and only if $\Lambda_{14}^j = 0$ holds for all j, by the global correspondence in §4. Note that, however, not everything is symmetric on $M_{\pm}$. Indeed, for homogeneous hypersurfaces with six principal curvatures, M_+ and M_- are *not* congruent (§2, [6]).

6. Description of E

In this section, we discuss what happens if we suppose $\dim E \neq 1$. Lemma 5.5 suggests that the matrix expression of $L(t)$ can be simplified if we use the decomposition $T_{\bar{p}}M_+ = E \oplus E^\perp$.

Lemma 6.1. *When* $\dim E = d$, *we can express* $L = L(t)$ *as*

$$L = \begin{pmatrix} 0_d & R \\ {}^tR & S \end{pmatrix},$$

with respect to the decomposition $T_{\bar{p}}M_+ = E \oplus E^\perp$, *where* 0_d *is* d *by* d, R *is* d *by* $5-d$ *and* S *is* $5-d$ *by* $5-d$ *matrices. The kernel of* L *is given by*

$$\begin{pmatrix} X \\ 0 \end{pmatrix} \in E, \quad {}^tRX = 0.$$

The eigenvectors for μ_i $(\neq 0)$ *in* (13) *are given by*

$$\begin{pmatrix} \dfrac{1}{\mu_i} RY \\ Y \end{pmatrix}$$

where $Y \in E^\perp$ *is a solution of*

$$({}^tRR + \mu_i S - \mu_i^2 I)Y = 0. \tag{34}$$

Proof. The first part follows from Lemma 5.5. Let $\begin{pmatrix} X \\ Y \end{pmatrix}$ be an eigenvector of L with respect to μ_i, where $X \in E$ and $Y \in E^\perp$, abusing the notation $X = \begin{pmatrix} X \\ 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 \\ Y \end{pmatrix}$. Then we have

$$\begin{pmatrix} 0_d & R \\ {}^tR & S \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} RY \\ {}^tRX + SY \end{pmatrix} = \mu_i \begin{pmatrix} X \\ Y \end{pmatrix},$$

and hence

$$\begin{cases} RY = \mu_i X, \\ {}^tRX + SY = \mu_i Y. \end{cases}$$

For $\mu_3 = 0$, $Y = 0$ and ${}^tRX = 0$ hold since the kernel belongs to E. When $\mu_i \neq 0$, multiplying μ_i to the second equation and substitute the first one into it, we obtain (34).

Then the eigenvector of L for an eigenvalue μ_i is given by

$$\begin{pmatrix} \frac{1}{\mu_i} RY \\ Y \end{pmatrix}. \qquad \square$$

7. dim E = 2

In this section, we suppose $\dim E = 2$ occurs at some point $\bar{p} \in M_+$. Then we have the decomposition $T_{\bar{p}} M_+ = E^2 \oplus V^2 \oplus W^1$ (the upper indices mean dimensions), where $W = B_\eta(E) = B_\zeta(E)$ by Lemma 5.5.

For a continuous frame $e_3(t) \in D_3(t)$ along L_6, $D_3(t+\pi) = D_3(t)$ implies $e_3(t+\pi) = \varepsilon e_3(t)$, $\varepsilon = \pm 1$. Then we have $\nabla_{e_6} e_3(t+\pi) = \varepsilon \nabla_{e_6} e_3(t)$, and it follows

$$\begin{aligned} \nabla_{e_3} e_6(t+\pi) &= \frac{1}{c_1} L(t+\pi)(\nabla_{e_6} e_3(t+\pi)) \\ &= -\frac{1}{c_1} L(t)(\varepsilon \nabla_{e_6} e_3(t)) = -\varepsilon \nabla_{e_3} e_6(t). \end{aligned}$$

Since $\nabla_{e_3} e_6(t) \in W$ never vanishes (Remark 5.3), and so has a constant direction, we have $\varepsilon = -1$.

In the following, we mean by a continuous frame $e_i(t)$ along L_6, a frame on L_6 minus a point. This is because we may have $e_i(t + 2\pi) = -e_i(t)$, which occurs as $O(5)$ acts on the shape operator via spin action. Fortunately, this does not affect the argument.

Consider a continuous frame $e_i(t)$ along L_6, and express $\nabla_{e_6} e_3(t) = \Lambda^i_{63}(t) e_i(t)$. Then putting $f(t) = (\Lambda^1_{63}(t))^2 - (\Lambda^5_{63}(t))^2$, we have $f(t+\pi) = -f(t)$ since $\nabla_{e_6} e_3(t+\pi) = -\nabla_{e_6} e_3(t)$ and $D_i(t+\pi) = D_{6-i}(t)$ holds. Thus at some point $p = p(t_0)$ of L_6, $f(t_0) = 0$ occurs. Here by the Gauss equation [3.3], or from

$$\begin{aligned} 0 &= \langle \nabla_{e_6} e_3(t), L(t)(\nabla_{e_6} e_3)(t) \rangle \\ &= \sqrt{3}\{(\Lambda^1_{63}(t))^2 - (\Lambda^5_{63}(t))^2\} + \frac{1}{\sqrt{3}}\{(\Lambda^2_{63}(t))^2 - (\Lambda^3_{63}(t))^2\}, \end{aligned}$$

we have also $(\Lambda^2_{63}(t_0))^2 - (\Lambda^4_{63}(t_0))^2 = 0$. Thus we may put at p,

$$\begin{aligned} \nabla_{e_6} e_3 &= x(e_1 + e_5) + y(e_2 + e_4), \\ \nabla_{e_3} e_6 &= \sqrt{3} x(e_1 - e_5) + \frac{y}{\sqrt{3}}(e_2 - e_4) \end{aligned} \tag{35}$$

by rechoosing the directions of $e_i = e_i(p)$, $i = 1, 2, 4, 5$, if necessary. Normalizing the right hand side, we define

$$X_1 = \alpha(e_1 + e_5) + \beta(e_2 + e_4) \in E,$$
$$Z_1 = \frac{1}{\sigma}\{\sqrt{3}\alpha(e_1 - e_5) + \frac{\beta}{\sqrt{3}}(e_2 - e_4)\} \in W$$

where $\alpha^2 + \beta^2 = 1/2$ and $\sigma = 2(3\alpha^2 + \beta^2/3)$, and $\nabla_{e_6} e_3 = aX_1$ and $\nabla_{e_3} e_6 = bZ_1$ hold for some a and b. Note that $B_\eta(X_1) = \sqrt{\sigma} Z_1$. Since V is orthogonal to e_3, X_1, Z_1, we have an orthonormal basis of V given by

$$X_2 = \frac{1}{\sigma}\left\{\frac{\beta}{\sqrt{3}}(e_1 - e_5) - \sqrt{3}\alpha(e_2 - e_4)\right\},$$
$$Z_2 = \beta(e_1 + e_5) - \alpha(e_2 + e_4),$$

where $B_\eta(X_2) = 1/\sqrt{\sigma} Z_2$ holds. Since V is parallel,

$$X_2(t) := X_2(0) = X_2, \quad Z_2(t) := Z_2(0) = Z_2$$

is an orthonormal frame of V at any $p(t)$. Now express $X_2(\pi) = X_2(0)$ and $Z_2(\pi) = Z_2(0)$ via basis at $p(\pi)$. Namely, choosing $e_i(\pi) = e'_i \in D_i(\pi) = D_{6-i}(0)$ suitably, we can express

$$\begin{aligned} X_2(\pi) &= \frac{1}{\sigma'}\left\{\frac{\beta'}{\sqrt{3}}(e'_1 - e'_5) - \sqrt{3}\alpha'(e'_2 - e'_4)\right\} \\ &= \frac{1}{\sigma}\left\{\frac{\beta}{\sqrt{3}}(e_1 - e_5) - \sqrt{3}\alpha(e_2 - e_4)\right\}, \end{aligned} \tag{36}$$

$$\begin{aligned} Z_2(\pi) &= \beta'(e'_1 + e'_5) - \alpha'(e'_2 + e'_4) \\ &= \beta(e_1 + e_5) - \alpha(e_2 + e_4), \end{aligned} \tag{37}$$

because $D_1(\pi) \oplus D_5(\pi) = D_1(0) \oplus D_5(0)$, and $D_2(\pi) \oplus D_4(\pi) = D_2(0) \oplus D_4(0)$ hold. Hence from (37) $|\alpha'| = |\alpha|$, $|\beta'| = |\beta|$, and $\sigma' = \sigma(\pi) = \sigma(0)$ follow. Thus we may consider

$$\begin{cases} \beta'(e'_1 - e'_5) = \beta(e_1 - e_5), \\ \beta'(e'_1 + e'_5) = \beta(e_1 + e_5), \end{cases} \quad \begin{cases} \alpha'(e'_2 - e'_4) = \alpha(e_2 - e_4), \\ \alpha'(e'_2 + e'_4) = \alpha(e_2 + e_4), \end{cases}$$

and from $D_i(\pi) = D_{6-i}(0)$, it follows

$$\begin{cases} \beta' e'_1 = -\beta e_5, \\ -\beta' e'_5 = \beta e_1, \\ \beta' e'_1 = \beta e_5, \\ \beta' e'_5 = \beta e_1, \end{cases} \quad \begin{cases} \alpha' e'_2 = -\alpha e_4, \\ -\alpha' e'_4 = \alpha e_2, \\ \alpha' e'_2 = \alpha e_4, \\ \alpha' e'_4 = \alpha e_2. \end{cases}$$

However then, we have $\alpha = \beta = 0$, a contradiction.

Thus we conclude:

Proposition 7.1. $\dim E = 2$ *does not occur at any point of* M_+.

8. dim E = 3

By the previous proposition, $\dim E = 3$ occurs on M_+ if $\dim E > 1$.

Proposition 8.1. *When* $\dim E = 3$, *at any point* p *of* L_6, E *and* $E^\perp$ *are expressed via* $e_i = e_i(p)$ *as*

$$E = \operatorname{span}\left\{e_3,\ \alpha(e_1+e_5)+\beta(e_2+e_4),\ \frac{\beta}{\sqrt{3}}(e_1-e_5)-\sqrt{3}\alpha(e_2-e_4)\right\},$$
$$E^\perp = \operatorname{span}\left\{\sqrt{3}\alpha(e_1-e_5)+\frac{\beta}{\sqrt{3}}(e_2-e_4),\ \beta(e_1+e_5)-\alpha(e_2+e_4)\right\},$$

for suitable α, β *satisfying* $\alpha^2+\beta^2 \neq 0$.

Proof. Since $e_3, e_1+e_5, e_2+e_4, e_1-e_5, e_2-e_4$ generate a frame of TM_+, we can choose $X_1, X_2 \in E$ as

$$X_1 = \alpha(e_1+e_5)+\beta(e_2+e_4)+\gamma(e_1-e_5),$$
$$X_2 = x(e_1+e_5)+y(e_2+e_4)+z(e_1-e_5)+w(e_2-e_4).$$

Then $Z_i = B_\eta(X_i) \in E^\perp$ are given by

$$Z_1 = \sqrt{3}\alpha(e_1-e_5)+\frac{1}{\sqrt{3}}\beta(e_2-e_4)+\sqrt{3}\gamma(e_1+e_5),$$
$$Z_2 = \sqrt{3}x(e_1-e_5)+\frac{1}{\sqrt{3}}y(e_2-e_4)+\sqrt{3}z(e_1+e_5)+\frac{1}{\sqrt{3}}w(e_2+e_4).$$

Because $0 = \langle X_1, Z_1\rangle = 2\sqrt{3}\alpha\gamma$, changing the sign of e_5, if necessary, we may assume $\gamma = 0$, i.e.,

$$X_1 = \alpha(e_1+e_5)+\beta(e_2+e_4) \in E,$$
$$Z_1 = \sqrt{3}\alpha(e_1-e_5)+\frac{\beta}{\sqrt{3}}(e_2-e_4) \in E^\perp. \tag{38}$$

Next from $0 = \langle X_1, Z_2\rangle = \sqrt{3}\alpha z + \beta w/\sqrt{3}$, and $0 = \langle X_2, Z_2\rangle = 2(\sqrt{3}xz + (1/\sqrt{3})yw)$, $\alpha y - \beta x = 0$ holds unless $z = w = 0$, and then $x(e_1+e_5)+y(e_2+e_4)$ is proportional to

X_1. Thus we may rechoose

$$X_2 = z(e_1 - e_5) + w(e_2 - e_4) = \frac{\beta}{\sqrt{3}}(e_1 - e_5) - \sqrt{3}\alpha(e_2 - e_4) \in E, \tag{39}$$

and

$$Z_2 = \beta(e_1 + e_5) - \alpha(e_2 + e_4) \in E^{\perp}. \tag{40}$$

When $z = w = 0$, we have $\mathrm{span}\{X_1, X_2\} = \mathrm{span}\{e_1 + e_5, e_2 + e_4\}$ and $\mathrm{span}\{Z_1, Z_2\} = \mathrm{span}\{e_1 - e_5, e_2 - e_4\}$. Here, in order to fit in with the expression (39) and (40), we change the sign of e_4, and may consider

$$X_2 = e_2 - e_4, \quad Z_2 = e_2 + e_4, \tag{41}$$

corresponding to $\beta = 0$. □

Note that X_1, X_2, Z_1, Z_2 are mutually orthogonal. Then the orthonormal frames of E and $E^{\perp}$ are given respectively, by

$$\begin{aligned} &e_3, \quad X_1 = \alpha(e_1 + e_5) + \beta(e_2 + e_4), \\ &X_2 = \frac{1}{\sqrt{\sigma}}\left(\frac{\beta}{\sqrt{3}}(e_1 - e_5) - \sqrt{3}\alpha(e_2 - e_4)\right) \end{aligned} \tag{42}$$

and

$$\begin{aligned} Z_1 &= \frac{1}{\sqrt{\sigma}}\left(\sqrt{3}\alpha(e_1 - e_5) + \frac{\beta}{\sqrt{3}}(e_2 - e_4)\right), \\ Z_2 &= \beta(e_1 + e_5) - \alpha(e_2 + e_4), \end{aligned} \tag{43}$$

where we put

$$\alpha^2 + \beta^2 = \frac{1}{2}, \quad \sigma = 2\left(3\alpha^2 + \frac{\beta^2}{3}\right). \tag{44}$$

Consider an arc c of L_6 containing $p = p(0)$ and $p(\pi)$. Since X_1, X_2 are given at each point of L_6 by (38), (39) and (40), using a continuous frame $e_i(t)$ and a continuous function $\alpha(t)$, $\beta(t)$ along c, we have a continuous frame $e_3(t)$, $X_1(t)$ and $X_2(t)$ of E, and $Z_1(t)$ and $Z_2(t)$ of $E^\perp$ along c. With respect to this moving frame, we can express

$$L(t) = B_{\eta_t} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\sigma(t)} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{\sqrt{\sigma(t)}} \\ 0 & \sqrt{\sigma(t)} & 0 & 0 & u(t) \\ 0 & 0 & \dfrac{1}{\sqrt{\sigma(t)}} & u(t) & 0 \end{pmatrix} \tag{45}$$

for $\eta_t = \eta_{p(t)}$. In fact, from $L(t)(e_i(t)) = \mu_i e_i(t)$, we know $L(t)(X_1(t)) = \sqrt{\sigma(t)} Z_1(t)$ and $L(t)(X_2(t)) = 1/\sqrt{\sigma(t)} Z_2(t)$. Moreover, it is easy to see $\langle L(t)(Z_i(t)), Z_i(t)\rangle = 0$. Then putting $u(t) = \langle L(t)(Z_1(t)), Z_2(t)\rangle$, we have (45). Note that $\sigma(t) + 1/\sigma(t) + u(t)^2 = 10/3$ follows from $\|L(t)\|^2 = 20/3$. Moreover, by using the notation in §6, (45) implies that $T(t) = {}^tR(t)R(t)$ has eigenvalues $\sigma(t)$, $1/\sigma(t)$ with eigenvectors $Z_1(t), Z_2(t) \in E^\perp$, respectively. Note that even if $\sigma(t) = 1/\sigma(t)$ holds, $Z_1(t)$ and $Z_2(t)$ (thus, $X_1(t)$ and $X_2(t)$) are continuously chosen so that the $S(t)$ part in (45) be described as above where $u(t)^2 = 4/3 \neq 0$.

Next, we show:

Proposition 8.2. *$\sigma(t)$ is constant and takes values* 1, 1/3 *or* 3.

Proof. We have $L(\pi) = -L(0)$ from $L(t) = \cos t B_\eta + \sin t B_\zeta$, and $T(\pi) = T(0)$ from $T(t) = {}^tR(t)R(t)$. This implies $\sigma = \sigma(\pi) = \sigma(0)$. When $\sigma(t)$ is not identically 1, we may consider $\sigma \neq 1$, and as an eigenvector of $T(0)$ for σ, $Z_1(\pi)$ is parallel to $Z_1(0)$. Then from

$$\begin{cases} L(\pi)(X_1(\pi)) = \sqrt{\sigma} Z_1(\pi), \\ L(0)(X_1(0)) = \sqrt{\sigma} Z_1(0), \end{cases}$$

we have

$$X_1(\pi) = \varepsilon X_1(0), \quad Z_1(\pi) = -\varepsilon Z_1(0), \quad \varepsilon = \pm 1.$$

Similarly from

$$\begin{cases} L(\pi)(X_2(\pi)) = \dfrac{1}{\sqrt{\sigma}} Z_2(\pi), \\[2ex] L(0)(X_2(0)) = \dfrac{1}{\sqrt{\sigma}} Z_2(0), \end{cases}$$

we have, unless $\alpha\beta \not\equiv 0$,

$$X_2(\pi) = -\varepsilon X_2(0), \quad Z_2(\pi) = \varepsilon Z_2(0),$$

where we use $e_i(\pi) \in D_{6-i}(0)$ by the global correspondence in (42) and (43). However, since $E^\perp$ is parallel along L_6, and the pair $Z_1(t)$, $Z_2(t)$ is a continuous orthonormal frame of $E^\perp$ by the remark before the proposition, this contradicts the fact that a continuous frame preserves the orientation. Therefore, only the cases $\sigma \equiv 1, 1/3, 3$ remain. □

9. Final result

Proposition 9.1. *When* $\dim E = 3$, $\sigma \equiv 1$ *does not occur.*

Proof. In this case, $3\alpha^2 = \beta^2$ follows from (44), and hence by a suitable choice of directions of e_i's, we have

$$E = \text{span}\{e_3, e_1 + \sqrt{3}e_4, \sqrt{3}e_2 + e_5\},$$
$$E^\perp = \text{span}\{\sqrt{3}e_1 - e_4, e_2 - \sqrt{3}e_5\}.$$

Since B_ζ maps E onto $E^\perp$, $b_{14} = b_{25} = 0$ follows, i.e., $\Lambda^4_{16} = \Lambda^5_{26} = 0$ holds. These imply $\Lambda^2_{63} = \Lambda^4_{63} = 0$ by the global correspondence. However, since $\nabla_{e_6} e_3$ is a combination of $e_1 + \sqrt{3}e_4$ and $\sqrt{3}e_2 + e_5$, this implies $\nabla_{e_6} e_3 = 0$, a contradiction. □

In the last possible case, we have by Proposition 8.1,

$$E = \text{span}\{e_3, e_1 + e_5, e_2 - e_4\}, \quad E^\perp = \text{span}\{e_1 - e_5, e_2 + e_4\},$$

and this holds everywhere by a continuous choice of e_i's. Since E is mapped onto $E^\perp$ by $B_\zeta = (b_{ij})$, we have

$$b_{15} = b_{24} = 0, \quad b_{12} + b_{25} = b_{14} + b_{45}. \tag{46}$$

On the other hand, for another focal submanifold M_-, the remaining possible case is also this case when $\dim F = 3$. (For the definition of F, see the end of §5.) Because $\nabla_{e_3} e_6(p) \sim \nabla_{e_1} e_4(q) \in E^\perp \cap F$, where $p = p_1$ and $q = p_3$ in Fig. 1, identifying the vectors at q with those at p as in Table 1, we may consider

$$\begin{aligned} F &= \{e_4(q), e_5(q) - e_3(q), e_6(q) + e_2(q)\} \\ &= \{e_6(p), e_1(p) - e_5(p), e_2(p) + e_4(p)\}, \\ F^\perp &= \{e_5(q) + e_3(q), e_6(q) - e_2(q)\} \\ &= \{e_1(p) + e_5(p), e_2(p) - e_4(p)\}. \end{aligned}$$

Here, some signature might be opposite, which does not matter. The importance is

$$c_{35} = c_{26} = 0$$

holds since C_ζ maps F onto $F^\perp$, where $c_{ij} = (1/(\sin\theta_1(\lambda_i - \lambda_1)))\Lambda_{i1}^j$ is the components of the shape operator C_ζ of M_- for $\zeta = e_1$ (see Lemma 2.1). Then the latter implies $b_{12} = 0$, and by the global correspondence, we have $b_{45} = 0$, and hence it follows from (46),

$$b_{14} = b_{25}.$$

Next from the Gauss equation [1.2] in §3, $b_{13}b_{32} = 0$ follows. When $b_{13} = 0$, [1.1] implies $b_{14}^2 = 2$, and hence $b_{25}^2 = 2$, but this contradicts [2.2]. Thus we have $b_{23} = 0$. Since this holds identically by the analyticity, $b_{14} = b_{25} = 0$ follows from the global correspondence, and the second row of B_ζ vanishes, contradicts [2.2]. Therefore we obtain:

Proposition 9.2. $\dim E = 3$ *does not occur.*

Finally, the kernel of the shape operators of the focal submanifolds of isoparametric hypersurfaces with $(g, m) = (6, 1)$ is independent of the normal directions, and by Proposition 4.2 of [6], we obtain:

Theorem 9.3 ([4]). *Isoparametric hypersurfaces with* $(g, m) = (6, 1)$ *are homogeneous*.

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