



Title	On fixed point free involutions of $S^2 \times S^1$
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Citation	Osaka Mathematical Journal. 1962, 14(1), p. 145-152
Version Type	VoR
URL	https://doi.org/10.18910/12621
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ON FIXED POINT FREE INVOLUTIONS OF $S^1 \times S^2$

Dedicated to Professor K. Shoda on his sixtieth birthday

BY

YOKO TAO

Introduction

In 1958, J. H. C. Whitehead [10] generalized the sphere theorem by C. D. Papakyriakopoulos in the following way:

WHITEHEAD'S SPHERE THEOREM. *Let M be an orientable 3-manifold, compact or not, with boundary which may be empty, such that $\pi_2(M) \neq 0$. Then there exists a 2-sphere S semi-linearly embedded in M , such that $S \neq 0^{(1)}$ in M .*

As the example $S^1 \times P^{2(2)}$ (S^k means k -sphere) shows, the above sphere theorem does not hold generally for non-orientable 3-manifolds. Therefore it remains as a question that for what 3-manifolds the sphere theorem does not hold? This problem naturally leads to the fixed point free involution (homeomorphism on itself of order 2) of $S^1 \times S^2$ as Theorem 2 of §3 in this paper shows.

The main purpose of this paper is to prove the following

Theorem 1. *If T is a fixed point free involution of $S^1 \times S^2$, and if M is the 3-manifold obtained by identifying x and Tx in $S^1 \times S^2$, then M is either homeomorphic to (1) $S^1 \times S^2$, or (2) 3-dimensional Klein Bottle⁽³⁾ (we denote it by K^3), or (3) $S^1 \times P^2$, or (4) $P^3 \#^{(4)} P^3$.*

This theorem may be regarded as an analogy of the following

Theorem (G. R. LIVESAY [4]). *If T is a fixed point free involution*

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- 1) $\neq 0$ means not homotopic to a constant.
 - 2) P^2 is the real projective plane.
 - 3) 3-dimensional Klein Bottle is defined as follows: let S_0, S_1 be the boundaries of $S^2 \times [0, 1]$. Then S_0, S_1 have the orientations induced from the orientation of $S^2 \times [0, 1]$. Let f be an orientation preserving homeomorphism from S_0 to S_1 . Identifying S_0 with S_1 by f in $S^2 \times [0, 1]$, we obtain a non-orientable closed 3-manifold which we call 3-dim. Klein Bottle.
 - 4) P^3 is the projective space. $P^3 \# P^3$ is defined as follows: Let E', E'' be two open 3-cells in P^3, P^3 respectively. Matching the boundaries of $P^3 - E'$ and $P^3 - E''$, we obtain a new closed 3-manifold which we denote $P^3 \# P^3$.

of S^3 , then the space obtained by identifying x and Tx in S^3 is the projective space.

Theorem 2 follows almost immediately from Theorem 1.

§ 1

According to E. E. Moise [7], we may suppose that $S^1 \times S^2$ and M have fixed triangulations and that T is simplicial on some subdivision of the triangulation of $S^1 \times S^2$ (See Chap. 1. of [4]). Therefore we stand throughout this paper on the semi-linear point of view: i.e., a 2-sphere will be considered as a 2-sphere semi-linearly embedded in M and any curve will be considered as polygonal, any homeomorphism as a semi-linear homeomorphism and so on.

Lemma 1. *Let E, E_1, E_2 be disks in a connected closed 3-manifold M , such that they have a common boundary c and $E_1 \cap E_2 = E \cap E_1 = E \cap E_2 = c$. If any two of 2-spheres $S = E_1 \cup E_2$, $S_1 = E \cup E_1$ and $S_2 = E \cup E_2$ separate M , then the other one also separates M .*

Proof. Suppose S and S_1 separate M . Let A, B be two components of $M - S$, and let A_1, B_1 be two components of $M - S_1$. Since $E \cap S = \partial E = c$, $\text{Int } E \subset A$ or $\text{Int } E \subset B$. Here we suppose $\text{Int } E \subset A$. In the same way, we may suppose $\text{Int } E_2 \subset A_1$. Take a point P on $\text{Int } E_2$. Then, there exist two points P_1, P_2 sufficiently close to P , such that $P_1 \in A_1 \cap A$ and $P_2 \in A_1 \cap B$.

Suppose S_2 does not separate M . Then we can take a simple arc w in M which starts from P_1 and ends in P_2 , such that $w \cap S_2 = \phi$, $w \cap \text{Int } E_1$ consists of an even number of points, $Q_1, Q_2, \dots, Q_{2n-1}, Q_{2n}$. Let w_i ($i=1, 2, \dots, n$) be the subarcs of w from Q_{2i-1} to Q_{2i} . Then we replace w_i by w'_i , such that w'_i is an arc from Q_{2i-1} to Q_{2i} on $\text{Int } E_1$, and $w'_i \cap w'_j = \phi$, if $i \neq j$. For convenience, we denote w by the same letter w after the deformation. Then shifting each w'_i slightly into A_1 , we can delete the intersection $w \cap E_1$, keeping $w \cap S_2 = \phi$ and getting any new intersections of w and E_1 . Hence P_1 is joined with P_2 in $M - S$ by an arc, which contradicts that S separates M . Therefore S_2 must separate M .

Thus Lemma 1 is proved.

Lemma 2. *Let $S^1 \times S^2$ be obtained from $I \times S^2$, where I is the closed interval $[0, 1]$, by identifying its boundaries $0 \times S^2$ and $1 \times S^2$. Let S be a 2-sphere semi-linearly embedded in $S^1 \times S^2$, such that $S \cap (0 \times S^2) = \phi$ and S does not separate $S^1 \times S^2$. Then S is isotopic to $0 \times S^2$ in $S^1 \times S^2$.*

Proof. Let S^3 be a 3-sphere obtained from $I \times S^2$ by filling in the

boundaries $0 \times S^2$ and $I \times S^2$ with two 3-cells e_1^3, e_2^3 . Since S is semi-linearly embedded in S^3 , by Alexander's theorem⁽⁵⁾ ([1], [2], [6]), S divides S^3 into two 3-cells E_1^3, E_2^3 such that $S^3 = E_1^3 \cup E_2^3$ and $E_1^3 \cap E_2^3 = \partial E_1^3 = \partial E_2^3 = S$. Since S does not separate $S^1 \times S^2$, we may suppose that $\text{Int } E_1^3 \supset 0 \times S^2$ and $\text{Int } E_2^3 \supset 1 \times S^2$. Therefore there exists a homeomorphism $h : I \times S^2 \rightarrow E_1^3 - \text{Int } e_1^3$ by Alexander's theorem.

Thus Lemma 2 is proved.

Hereafter we suppose throughout this paper that $S^1 \times S^2$ is obtained from $I \times S^2$ by identifying its boundaries $0 \times S^2$ and $1 \times S^2$.

Lemma 3. *There exists a 2-sphere S^* in $S^1 \times S^2$ which is isotopic to $0 \times S^2$, such that $S^* \cap TS^* = \phi$ or $TS^* = S^*$.*

Proof. Let $S = 0 \times S^2$. If $TS \neq S$, nor $S \cap TS = \phi$, then we may suppose $S \cap TS$ consists of a finite number of simple closed curves c_1, c_2, \dots, c_n . If otherwise, by a small isotopic simplicial deformation of S , we obtain $S \cap TS$ in such a form. Let c be one of the innermost intersection curves on TS : i.e., there exists a disk E on TS , such that $c = \partial E$ and $\text{Int } E \cap S = \phi$. c divides S into two disks E_1, E_2 such that $E_1 \cup E_2 = S$ and $E_1 \cap E_2 = \partial E_1 = \partial E_2 = c$. Since there is no intersection curves on $\text{Int } TE$, we may suppose, without loss of generality, $TE \subseteq E_1$ (equality holds, if and only if $c = Tc$). Let $S_1 = E \cup E_1, S_2 = E \cup E_2$. Then one of S_1 or S_2 does not separate $S^1 \times S^2$. For, if both S_1 and S_2 separate $S^1 \times S^2$, then S, S_1 and S_2 satisfy the conditions of Lemma 1. Therefore S separates $S^1 \times S^2$ by the conclusion of Lemma 1, which contradicts the first assumption.

(1) S_1 does not separate $S^1 \times S^2$.

If $c = Tc$, then $TS_1 = T(E \cup E_1) = TE \cup TE_1 = E_1 \cup E = S_1$. Hence S_1 is an invariant 2-sphere under T .

If $c \neq Tc$, then $TE \subseteq E_1$. Take a simple closed curve c' on E_1 so close to c that the ring domain R bounded by c and c' on E_1 has no intersection with TS except c (Fig. 1). Then span a disk E' on c' so close to E that $S'_1 = (E_1 - R) \cup E'$ does not separate $S^1 \times S^2, E' \cap TE' = \phi, E' \cap TS = \phi$ and $E' \cap S = \partial E' = c'$. From the way of construction of $S'_1, S'_1 \cap TS'_1$ consists of a subset of $\{c_1, c_2, \dots, c_n\}$. we shall denote by $n(S \cap TS)$ the number of intersection curves of $S \cap TS$. Then it follows that $n(S'_1 \cap TS'_1) < n(S \cap TS)$, because the former is diminished at least by 2 (c and Tc) from the latter.

5) Alexander's theorem: Let S be a polygonal 2-sphere in the 3-sphere S^3 . Then $S^3 = e_1 \cup e_2$ and $e_1 \cap e_2 = \partial e_1 = \partial e_2 = S$ where e_1, e_2 are topological 3-cells.

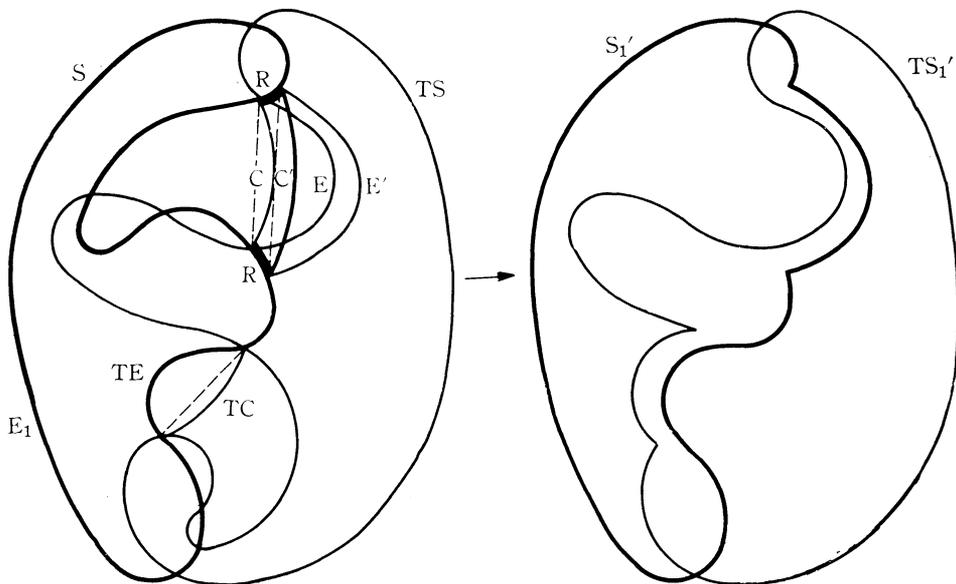


Fig. 1

(2) S_2 does not separate $S^1 \times S^2$.

If $c = Tc$, then we can take a simple closed curve c' and a disk E' so close to c and E that they satisfy the following conditions: (i) $c' \subset E_2$ and the domain R bounded by c and c' on E_2 has no intersection with TS , (ii) $E' \cap TS = \phi$ and $E' \cap S = \partial E' = c'$, (iii) $S'_2 = (E_2 - R) \cup E'$ does not separate $S^1 \times S^2$. Furthermore we can take E' such that $E' \cap TE' = \phi$, because $TE' \cap S = \phi$, $TE' \cap TS = Tc'$ (Fig. 2). Then $n(S'_2 \cap TS'_2) <$

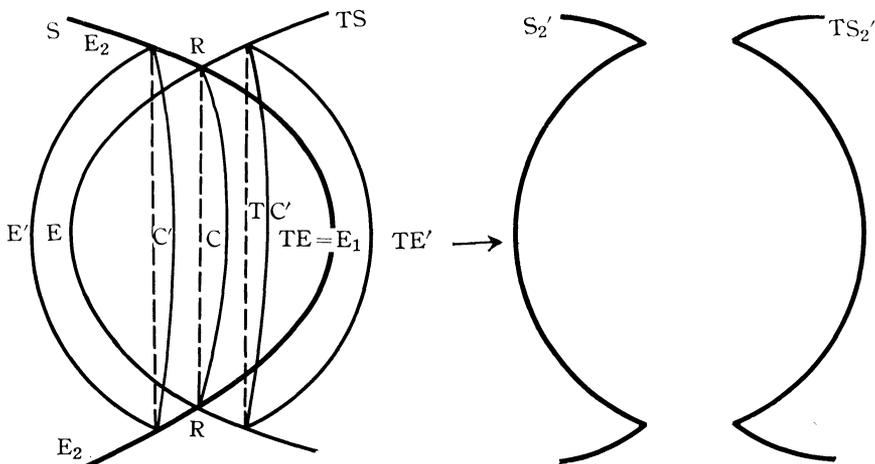


Fig. 2

$n(S \cap TS)$, because c is deleted and there arise no new intersection curves.

If $c \neq Tc$, then $n(S_2 \cap TS_2) = n(E_2 \cap TE_2) < n(S \cap TS)$, because c and Tc are not contained in $S_2 \cap TS_2$.

As has been shown there exists in both cases (1), (2), a 2-sphere S' which does not separate $S^1 \times S^2$ such that $n(S' \cap TS') < n(S \cap TS)$ or $TS' = S'$. Furthermore, from the way of our construction of S' , we have by a small deformation of S' $S' \cap S = \phi$ without changing any other situations. Therefore it follows from Lemma 2 that S' is isotopic to S . Since $n(S \cap TS)$ is a non negative integer, we can find by proceeding with the above procedure a 2-sphere S^* which is isotopic to S and $S^* \cap TS^* = \phi$ or $TS^* = S^*$, in a finite step.

Thus Lemma 3 is proved.

§ 2

Proof of Theorem 1. By Lemma 3, there exists a 2-sphere S which is semi-linearly embedded in $S^1 \times S^2$ and is isotopic to $0 \times S^2$, such that $S \cap TS = \phi$ or $S = TS$. we divide our proof into the following two cases:

(1) $S \cap TS = \phi$,

(2) $S = TS$ and there is no 2-sphere S' which is isotopic to $0 \times S^2$ and $S' \cap TS' = \phi$.

(1) $S \cap TS = \phi$. Since S is isotopic to $0 \times S^2$, we may suppose $S = 0 \times S^2$. Then $S^1 \times S^2 - (S \cup TS)$ consists of two components A, B . Here A and B are homeomorphic to $I \times S^2$ by Lemma 2. Then the following two cases are possible:

(a) $TA = A$, (b) $TA = B$.

Case (a). Let p be a map from $S^1 \times S^2$ onto M defined by $px = pTx$. Then $p: S^1 \times S^2 \rightarrow M$ is a double covering. Let M_A, M_B be closed 3-manifolds obtained from pA, pB by filling in the boundary 2-sphere $S' = p(S \cup TS)$ with 3-cells respectively. Filling in $\partial A = S \cup TS$ with two 3-cells, we obtain from A a 3-sphere S^3 . Since T is a fixed point free involution of A , T is extended naturally to a fixed point free involution T' of S^3 . Then, by Theorem 3 of [4], T' is equivalent to the antipodal map: i.e. there exists a homeomorphism $h: S^3 \rightarrow S^3$ such that $hT'h^{-1}$ is an antipodal map. Hence $M_A = P^3$. In the same way, it follows that $M_B = P^3$. Therefore $M = P^3 \# P^3$ (For example Fig. 3).

Case (b). In this case, M is homeomorphic to the manifold obtained from A by matching S and TS by the homeomorphism T . Therefore M is either homeomorphic to $S^1 \times S^2$ or K^3 .

(2) By Lemma 2, we may suppose $S = \frac{1}{2} \times S^2$. There are the following two cases to be considered:

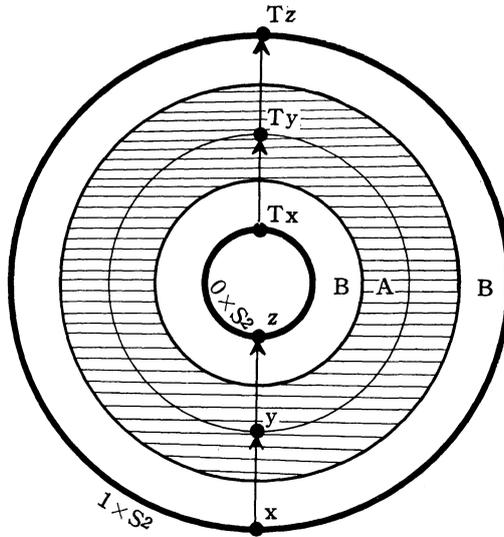


Fig. 3

(c). For all number ε , where $0 < \varepsilon < \frac{1}{2}$, there exists a point Q such that $Q \in [\frac{1}{2}, \frac{1}{2} + \varepsilon] \times S^2$, $TQ \notin [\frac{1}{2}, 1) \times S^2$.

(d). There exists a number ε such that $0 < \varepsilon < \frac{1}{2}$ and $T([\frac{1}{2}, \frac{1}{2} + \varepsilon] \times S^2) \subset [\frac{1}{2}, 1) \times S^2$.

Case (c). Since $\frac{1}{2} \times S^2$ is invariant under T , there exists two numbers α, β such that $0 < \beta < \alpha < \frac{1}{2}$, and $T([\frac{1}{2} - \beta, \frac{1}{2} + \beta] \times S^2) \subset [\frac{1}{2} - \alpha, \frac{1}{2} + \alpha]$. Then by the assumption, there exists $\gamma \times S^2$ with $0 < \gamma < \beta$, such that $T(\gamma \times S^2) \cap (\gamma \times S^2) = \emptyset$. Hence this case does not actually occur.

Case (d). Cut $S^1 \times S^2$ by $\frac{1}{2} \times S^2$. Then it is homeomorphic to $I \times S^2$ and we may suppose that T is a fixed point free involution of $I \times S^2$. Then T restricted to the boundary $i \times S^2$ ($i=0,1$) is an antipodal map A on 2-sphere S^2 . Hence, by Lemma 3.1 of [3] T is equivalent to $e \times A: I \times S^2 \rightarrow I \times S^2$, where $e: I \rightarrow I$ is the identity. Matching again the boundary of $I \times S^2$, we obtain that M is homeomorphic to $S^1 \times P^2$.

Thus Theorem 1 is proved.

§ 3

A connected closed 3-manifold M is said to be *irreducible* if every 2-sphere which is semi-linearly embedded in M and separates M bounds a 3-cell in M .

Using Theorem 1, we obtain the following

Theorem 2. *Under Poincaré Hypothesis⁽⁶⁾, the following two propositions are equivalent:*

(I) *Let \tilde{M} be the orientable double covering of a connected closed 3-manifold M . If M is irreducible, then \tilde{M} is irreducible.*

(II) *$S^1 \times P^2$ is the only one connected closed 3-manifold for which the sphere theorem does not hold.*

Proof. First suppose that (I) is true. Let M be a connected closed 3-manifold for which the sphere theorem does not hold: i.e., $\pi_2(M) \neq 0$ but every 2-sphere semi-linearly embedded in M is homotopic to a constant in M . Then M is irreducible. For, if M is reducible, then $\pi_1(M)$ is a free product of two non trivial groups. Then it follows from Whitehead's theorem (Theorem 1.1 of [9]) that there exists a 2-sphere S semi-linearly embedded in M , such that $S \neq 0$ in M , which contradicts the assumption. Therefore by (I), \tilde{M} is irreducible. From Milnor's result ([5], [8]) and Poincaré Hypothesis, \tilde{M} is homeomorphic to (1) $S^1 \times S^2$, or (2) is aspherical, or (3) has a non trivial finite fundamental group. Case (2) or (3) does not occur. For, if \tilde{M} is aspherical, then $\pi_2(M) \approx \pi_2(\tilde{M}) = 0$, which contradicts the assumption. If $\pi_1(\tilde{M})$ is finite, then the universal covering $\tilde{\tilde{M}}$ of \tilde{M} is S^3 . Hence $\pi_2(M) \approx \pi_2(\tilde{M}) \approx \pi_2(\tilde{\tilde{M}}) = 0$, which contradicts the assumption. Therefore \tilde{M} is homeomorphic to $S^1 \times S^2$. Since M is irreducible and non orientable, it follows from Theorem 1 that M is homeomorphic to $S^1 \times P^2$.

Next, suppose that (II) is true. Suppose \tilde{M} is reducible. Then by Whitehead's theorem, $\pi_2(\tilde{M}) \neq 0$. Therefore $\pi_2(M) \neq 0$. If there exists a 2-sphere semi-linearly embedded in M , such that $S^2 \neq 0$, then by Whitehead's theorem, M is reducible or $M = S^1 \times S^2$ or $M = K^3$. If there is no 2-sphere semi-linearly embedded in M , such that $S^2 \neq 0$, then it follows from (II) that $M = S^1 \times P^2$. On the other hand, by Theorem 1, if $M = S^1 \times S^2$, or $= K^3$, or $= S^1 \times P^2$, then $\tilde{M} = S^1 \times S^2$, which contradicts the first assumption. Hence M is reducible. Therefore (I) is true under the assumption of (II).

Thus Theorem 2 is proved.

(Received March 17, 1962)

6) Poincaré Hypothesis: Every simply connected closed 3-manifold is the 3-sphere.

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