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On a Theorem of Gaschütz

By Masatoshi IKEDA

In his paper "Über den Fundamentalsatz von Maschke zur Darstellungstheorie der endlichen Gruppen",[1] W. Gaschütz studied two types of G-Ω-modules, named M_u- and M_0-modules, where G and Ω are a finite group and an arbitrary domain of G-endomorphisms of the modules respectively. There he obtained a criterion for a G-Ω-module to be an M_u- or M_0-module, which is a generalization of the well-known theorem of I. Schur that every representation of a finite group of order g in a field with characteristic $p(\not| g)$ is completely reducible.

In the present note we take, instead of G and Ω, a Frobenius algebra A over a commutative ring R and a ring P which contains R in its centre respectively, and derive a criterion for an A-P-module to be an M_u- or M_0-module, which is essentially a generalization of Gaschütz's result.

Let R be a commutative ring with the unit element 1.

DEFINITION. A is called an *algebra* over R if A is an associative ring as well as a two-sided R-module with a right linearly independent R-basis $\{u_i\}$ which satisfies $u_i\omega=\omega u_i$ and $u_i 1=1u_i=u_i$ for every $\omega\in R$ and i.

Now let $\{u_i\}$ $(i=1,\cdots,n)$ be an R-basis of A and $u_iu_j=\sum_k \alpha_{i,j}^k u_k$ $(\alpha_{i,j}^k\in R)$; then we obtain the right and left regular representations with respect to $\{u_i\}$ in the usual manner.

DEFINITION. An algebra A over R is called a *Frobenius algebra* if A has a unit element and its right and left regular representations with respect to an R-basis are equivalent.

DEFINITION. Let $\{u_i\}$ $(i=1,\cdots,n)$ be an R-basis of an algebra A over R and $u_iu_j=\sum_k \alpha_{i,j}^k u_k$. Then the matrix $(\sum_k \alpha_{i,j}^k\lambda_k)_{i,j}$ is called a *parastrophic matrix* belonging to the basis $\{u_i\}$ and the parameters $\lambda_i\in R$ $(i=1,\cdots,n)$.

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1) W. Gaschütz, Math. Zeitschr. **56**, 1952.

Then we have

Lemma.[2] *An algebra A over R is a Frobenius algebra if and only if A has a non-singular parastrophic matrix. Moreover if A is a Frobenius algebra over R, then every matrix intertwining right and left regular representations is expressed as a parastrophic matrix belonging to suitable parameters.*

If A is a Frobenius algebra over R then, for every R-basis $\{u_i\}$, there exists an R-basis $\{v_i\}$ such that the right regular representation with respect to $\{v_i\}$ coincides with the left regular representation with respect to $\{u_i\}$. *We say that $\{v_i\}$ is dual to $\{u_i\}$.*

DEFINITION. Let A be an algebra over R and P a ring whose centre contains R.

i) A module $\mathfrak{m}$ is called an A-P-module if $\mathfrak{m}$ is a left A-module as well as a right P-module and satisfies

$$(a\omega)m = (am)\omega, \qquad (am)\rho = a(m\rho)$$

for every $a \in A$, $m \in \mathfrak{m}$, $\omega \in R$ and $\rho \in P$.

ii) An A-P-module $\mathfrak{m}$ on which the unit element of A acts as the identity operator is called an M_u-module if, for every A-P-module $\mathfrak{n}$ containing $\mathfrak{m}$, a direct decomposition $\mathfrak{n} = \mathfrak{m} + \mathfrak{m}'$ as a P-module implies a direct decomposition $\mathfrak{n} = \mathfrak{m} + \mathfrak{m}''$ as an A-P-module.

iii) An A-P-module $\mathfrak{m}$ on which the unit element of A acts as the identity operator is called an M_0-module if, for every A-P-module $\mathfrak{n}$ which contains an A-P-submodule $\mathfrak{n}'$ such that $\mathfrak{n}/\mathfrak{n}' \cong \mathfrak{m}$, a direct decomposition $\mathfrak{n} = \mathfrak{n}' + \mathfrak{m}'$ as a P-module implies a direct decomposition $\mathfrak{n} = \mathfrak{n}' + \mathfrak{m}''$ as an A-P-module.

Theorem. *Let A be a Frodenius algebra over a commutative ring R with an R-basis containing the unit element of A and P a ring whose centre contains R. Then an A-P-module $\mathfrak{m}$ is an M_u- or M_0-module if and only if there exists a P-endomorphism β of $\mathfrak{m}$ such that $\sum_i u_i \beta v_i$ is the identity endomorphism of $\mathfrak{m}$ for every R-basis $\{u_i\}$ of A and its dual basis $\{v_i\}$.*

Proof. 1) Proof of sufficiency. Let $\mathfrak{n}$ be an A-P-module which contains $\mathfrak{m}$ and $\mathfrak{n} = \mathfrak{m} + \mathfrak{m}'$ as a P-module. By our assumption, there exists a P-endomorphism β of $\mathfrak{m}$. Let β^* be a P-endomorphism which

2) The proof of this lemma is quite similar to that of footnotes 6) and 7) in Nakayama & Nesbitt: Note on symmetric algebras, Annals of Math. **39**, 1938.

coincides with β on $\mathfrak{m}$ and $\beta^*\mathfrak{m}'=0$. Then $\sum_i u_i\beta^*v_i=\varepsilon$ is a P-endomorphism and $\varepsilon m=(\sum_i u_i\beta^*v_i)m=\sum_i u_i\beta^*(v_i m)=\sum_i u_i\beta(v_i m)$ $=(\sum_i u_i\beta v_i)m=m$ for every $m\in\mathfrak{m}$, by our assumption. Moreover it can easily be seen that $\varepsilon\mathfrak{n}=\mathfrak{m}$. Therefore $\varepsilon^2=\varepsilon$. Now we show that ε is an A-P-endomorphism. Let n be an arbitrary element of $\mathfrak{n}$ and a an arbitrary element of A. Since $\{v_i\}$ is dual to $\{u_i\}$, if $a(u_1,\cdots,u_n)$ $=(u_1,\cdots,u_n)(\alpha_{i,j})$, then $\begin{pmatrix} v_1\\ \vdots \\ v_n\end{pmatrix}a=(\alpha_{i,j})\begin{pmatrix} v_1\\ \vdots \\ v_n\end{pmatrix}$. Then

$$(a\varepsilon)n=(a\sum_i u_i\beta^*v_i)n=\sum_i au_i(\beta^*v_in)=\sum_{i,k}(u_k\alpha_{k,i})(\beta^*v_in)\,.$$

By the definition of A-P-modules and the fact that β^* is a P-endomorphism,

$$(u_k\alpha_{k,i})(\beta^*v_in)=u_k((\beta^*v_in)\alpha_{k,i})=u_k(\beta^*((v_in)\alpha_{k,i}))=(u_k\beta^*(v_i\alpha_{k,i}))n\,.$$

Therefore

$$(a\varepsilon)n=\sum_{i,k}(u_k\beta^*(v_i\alpha_{k,i}))n=(\sum_k u_k\beta^*(\sum_i v_i\alpha_{k,i}))n\,.$$

On the other hand

$$(\varepsilon a)n=(\sum_k u_k\beta^*(v_ka))n=(\sum_k u_k\beta^*(\sum_i v_i\alpha_{k,i}))n\,.$$

Thus $a\varepsilon=\varepsilon a$ and consequently ε is an A-P-endomorphism. Therefore we have the direct decomposition of $\mathfrak{n}$: $\mathfrak{n}=\mathfrak{m}+(1-\varepsilon)\mathfrak{n}$, where 1 is the identity endomorphism of $\mathfrak{n}$. This shows that $\mathfrak{m}$ is an M_u-module.

Next we show that $\mathfrak{m}$ is also an M_0-module. Let $\mathfrak{n}$ be an A-P-module which contains an A-P-submodule $\mathfrak{n}'$ such that $\mathfrak{n}/\mathfrak{n}'\cong\mathfrak{m}$ and $\mathfrak{n}=\mathfrak{n}'+\mathfrak{m}'$ as a P-module. Since $\mathfrak{m}'\cong\mathfrak{m}$ as a P-module, we can see β as a P-endomorphism of $\mathfrak{m}'$. Let β^* be a P-endomorphism of $\mathfrak{n}$ which coincides with β on $\mathfrak{m}'$ and $\beta^*\mathfrak{n}'=0$. From our assumption, $(\sum_i u_i\beta^*v_i)n\equiv n \pmod{\mathfrak{n}'}$ for $n\in\mathfrak{n}$. In the same way as above, we see that the P-endomorphism $\sum_i u_i\beta^*v_i=\varepsilon$ is an A-P-endomorphism and $\varepsilon^2=\varepsilon$. Therefore $\varepsilon'=1-\varepsilon$ is also an A-P-endomorphism and $\varepsilon'^2=\varepsilon'$. Moreover it is easy to see that $\varepsilon'\mathfrak{n}=\mathfrak{n}'$. Consequently we have that $\mathfrak{n}=\mathfrak{n}'+\varepsilon\mathfrak{n}$ and $\mathfrak{m}$ is an M_0-module.

2. Proof of necessity. Let M_A be a module satisfying the following conditions:

(i) M_A is a module of linear forms $\sum_i x_{u_i}a_{u_i}(a_{u_i}\in\mathfrak{m})$.

(ii) $\sum_i x_{u_i}a_{u_i}+\sum_i x_{u_i}b_{u_i}=\sum_i x_{u_i}(a_{u_i}+b_{u_i})$.

(iii) $(\sum_i x_{u_i}a_{u_i})\rho = \sum_i x_{u_i}(a_{u_i}\rho)$ for $\rho \in P$.

(iv) $u_j(\sum_i x_{u_i}a_{u_i}) = \sum_i x_{u_i}(\sum_k a_{u_k}\alpha^k_{i,j})$, if $u_i u_j = \sum_k u_k \alpha^k_{i,j}$.

(v) $a(\sum_i x_{u_i}a_{u_i}) = \sum_j (u_j(\sum_i x_{u_i}a_{u_i}))\alpha_j$, if $a = \sum_j u_j\alpha_j$.

Then it is not hard to verify that M_A is an A-P-module.

Now, since $\{v_i\}$ is dual to $\{u_i\}$, $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = P^{-1}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, where $P = (\sum_k \alpha^k_{i,j}\lambda_k)_{i,j}$ is a non-singular parastrophic matrix belonging to $\{u_i\}$ and $\{\lambda_i\}$. We write $P = (p_{i,j})_{i,j}$ and $P^{-1} = (p^*_{i,j})_{i,j}$. We assume that $\sum_i u_i\eta_i = 1$, the unit element of A. Then the mapping β: $\sum_i x_{u_i}a_{u_i} \to \sum_i x_{u_i}(\sum_j a_{u_j}\eta_j)\lambda_i$ satisfies our condition, that is, β is a P-endomorphism and $\sum_i u_i\beta v_i$ is the identity endomorphism of M_A. Since β is obviously a P-endomorphism, we are only to prove that $\sum_i u_i\beta v_i$ is the identity endomorphism.

$$\begin{aligned}(\sum_j u_j\beta v_j)(\sum_i x_{u_i}a_{u_i}) &= \sum_j u_j\beta(v_j\sum_i x_{u_i}a_{u_i}) = \sum_j u_j\beta((\sum_k u_k p^*_{j,k})(\sum_i x_{u_i}a_{u_i})) \\ &= \sum_j u_j\beta(\sum_i x_{u_i}(\sum_{h,k} a_{u_h}\alpha^h_{i,k}p^*_{j,k})) = \sum_j u_j(\sum_i x_{u_i}(\sum_{h,k,l} a_{u_h}p^*_{j,k}\alpha^h_{l,k}\eta_l)\lambda_i).\end{aligned}$$

Since $\sum_l u_l\eta_l = 1$, $\sum_l \alpha^h_{l,k}\eta_l = \delta_{k,h}$ and consequently

$$\begin{aligned}(\sum_j u_j\beta v_j)(\sum_i x_{u_i}a_{u_i}) &= \sum_j u_j(\sum_i x_{u_i}(\sum_k a_{u_k}p^*_{j,k}\lambda_i)) = \sum_{i,j} x_{u_i}(\sum_m(\sum_k a_{u_k}p^*_{j,k}\lambda_m)\alpha^m_{i,j}) \\ &= \sum_{i,j} x_{u_i}(\sum_k a_{u_k}p^*_{j,k}(\sum_m \lambda_m\alpha^m_{i,j})) = \sum_{i,j} x_{u_i}(\sum_k a_{u_k}p^*_{j,k}p_{i,j}) \\ &= \sum_i x_{u_i}(\sum_k a_{u_k}(\sum_j p_{i,j}p^*_{j,k})) = \sum_i x_{u_i}(\sum_k a_{u_k}\delta_{i,k}) = \sum_i x_{u_i}a_{u_i}.\end{aligned}$$

Thus β satisfies our condition.

Next we show that M_A contains A-P-modules M and N such that $M \cong \mathfrak{m}$ and $M_A/N \cong \mathfrak{m}$. The module $M = \{\sum_i x_{u_i}(u_i a) \mid a \in \mathfrak{m}\}$ is P-isomorphic to $\mathfrak{m}$ by the correspondence $\mathfrak{m} \ni a \leftrightarrow \sum_i x_{u_i}(u_i a) \in M$. For, if $\sum_i x_{u_i}(u_i a) = 0$ then $u_i a = 0$ for all i and consequently $a = 1$ $a = (\sum_i u_i\eta_i)a = \sum_i (u_i a)\eta_i = 0$. Therefore this correspondence is one-to-one and obviously P-isomorphism. Moreover this correspondence is A-isomorphism. For $u_j(\sum_i x_{u_i}(u_i a)) = \sum_i x_{u_i}(\sum_k (u_k a)\alpha^k_{i,j}) = \sum_i x_{u_i}((\sum_k u_k\alpha^k_{i,j})a) = \sum_i x_{u_i}(u_i(u_j a))$, that is, $u_j a$ corresponds to $u_j(\sum_i x_{u_i}(u_i a))$. Therefore M is A-P-isomorphic to $\mathfrak{m}$. Since A has an R-basis containing 1, say $w_1 = 1, w_2, \cdots, w_n$, we can construct the module M'_A satisfying (i), $\cdots$, (v) with respect to $\{w_i\}$. Let Q be a non-singular matrix such that $(u_i) = (w_i)Q'$. Then it is not hard to see that M'_A

and M_A are A-P-isomorphic by the correspondence φ: $M'_A \ni \sum_i x_{w_i} a_{w_i} \rightarrow \sum_i x_{u_i} b_{u_i}$, where $(b_{u_1}, \cdots, b_{u_n}) = (a_{w_1}, \cdots, a_{w_n})Q'$. By φ, M corresponds to $M' = \{\sum_i x_{w_i}(w_i a) | a \in \mathfrak{m}\}$. It is obvious that $M'_A = M' + M''$ as a P-module, where $M'' = \{\sum_i x_{w_i} a_{w_i} | a_{w_1} = 0\}$. Therefore we have that $M_A = M + \varphi M''$ as a P-module and consequently $M_A = M + M'''$ as an A-P-module if $\mathfrak{m}$ is an M_u-module. Next we consider the mapping ψ: $M_A \ni \sum_i x_{u_i} a_{u_i} \rightarrow \sum_i u_i(\sum_j a_{u_j} p^*_{i,j}) \in \mathfrak{m}$. Since $(p^*_{i,j}) = P^{-1}$ is non-singular, the linear equation $\sum_j x_j p^*_{i,j} = a\eta_i (a \in \mathfrak{m}, i = 1, \cdots, n)$ have a unique solution $\{a_j\}$ in $\mathfrak{m}$. Then $\sum_i x_{u_i} a_i$ corresponds to $\sum_i u_i(a\eta_i) = (\sum_i u_i \eta_i)a = 1\ a = a$. This shows that ψ is an "onto" mapping. Furthermore it is easy to see that ψ is a P-homomorphism. We show that ψ is an A-P-homomorphism.

$$\psi(u_j(\sum_i x_{u_i} a_{u_i})) = \psi(\sum_i x_{u_i}(\sum_k a_{u_k} \alpha^k_{i,j})) = \sum_i u_i(\sum_m (\sum_k a_{u_k} \alpha^k_{m,j}) p^*_{i,m})$$
$$= \sum_i u_i(\sum_k a_{u_k}(\sum_m \alpha^k_{m,j} p^*_{i,m})).$$

Since $P = (p_{i,j})$ interwines right and left regular representations, we have $\sum p^*_{i,m} \alpha^k_{m,j} = \sum \alpha^i_{j,m} p^*_{m,k}$ and consequently

$$\psi(u_j(\sum_i x_{u_i} a_{u_i})) = \sum_i u_i(\sum_k a_{u_k}(\sum_m \alpha^i_{j,m} p^*_{m,k})) = \sum_{k,m} (\sum_i u_i \alpha^i_{j,m}) a_{u_k} p^*_{m,k}$$
$$= u_j(\sum_m u_m(\sum_k a_{u_k} p^*_{m,k})) = u_j \psi(\sum_i x_{u_i} a_{u_i}).$$

This shows that ψ is an A-P-homomorphism and consequently M_A contains an A-P-submodule N auch that $M_A/N \cong \mathfrak{m}$. Moreover, as was shown above, the P-submodule $N' = \{\sum_i x_{u_i} a_{u_i} | \sum_i a_{u_i} p^*_{i,j} = a\eta_i, a \in \mathfrak{m}\}$ is mapped onto $\mathfrak{m}$ by ψ. Therefore $M_A = N + N'$ as a P-module and consequently $M_A = N + N''$ as an A-P-module if $\mathfrak{m}$ is an M_0-module. Thus we have that M_A is directly decomposable into $\mathfrak{m}$ and an A-P-module. Since M_A has a P-endomorphism β satisfying our condition, we can easily construct a P-endomorphism satisfying our condition for $\mathfrak{m}$.

Next we show that our result is essentially a generalization of Gaschütz's result. Let $\mathfrak{m}$ be a G-module, where $G = \{g_i | i = 1, \cdots, n\}$ is a finite group and Ω an arbitrary domain of G-endomorphisms of $\mathfrak{m}$. Let P be the ring of endomorphisms generated by Ω and the identity endomorphism of $\mathfrak{m}$, and C the centre of P. Then the group ring $G(C)$ of G over C is a Frobenius algebra with a C-basis containing the unit element of G. Furthermore $\{g_i^{-1}\}$ is a dual basis to $\{g_i\}$. Considering $\mathfrak{m}$ as $G(C)$-P-module in the natural way, we have

Theorem. (Gaschütz). *Let* $G = \{g_i \mid i = 1, \cdots, n\}$, $\mathfrak{m}$ *and* Ω *be a finite group, a G-module and an arbitrary domain of G-endomorphisms of* $\mathfrak{m}$ *respectively. Then G-*Ω*-module* $\mathfrak{m}$ *is an* M_u*- or* M_0*-module if and only if* $\mathfrak{m}$ *has an* Ω*-endomorphism* β *such that* $\sum g_i \beta g_i^{-1}$ *is the identity endomorphism of* $\mathfrak{m}$.

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