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# CONNECTEDNESS OF LEVELS FOR MOMENT MAPS
# ON VARIOUS CLASSES OF LOOP GROUPS

Dedicated to Jost-Hinrich Eschenburg on his sixtieth birthday

Augustin-Liviu Mare

### Abstract

The space $\Omega(G)$ of all based loops in a compact simply connected Lie group $G$ has an action of the maximal torus $T \subset G$ (by pointwise conjugation) and of the circle $S^1$ (by rotation of loops). Let $\mu: \Omega(G) \to (\mathfrak{t} \times i\mathbb{R})^*$ be a moment map of the resulting $T \times S^1$ action. We show that all levels (that is, pre-images of points) of $\mu$ are connected subspaces of $\Omega(G)$ (or empty). The result holds if in the definition of $\Omega(G)$ loops are of class $C^\infty$ or of any Sobolev class $H^s$, with $s \geq 1$ (for loops of class $H^1$ connectedness of regular levels has been proved by Harada, Holm, Jeffrey, and the author in [3]).

## 1. Introduction

Let $G$ be a compact simply connected Lie group and $T \subset G$ a maximal torus. The based loop group of $G$ is the space $\Omega(G)$ consisting of all smooth maps $\gamma: S^1 \to G$ with $\gamma(1) = e$. The assignments

$$T \times \Omega(G) \to \Omega(G), \quad (t, \gamma) \mapsto [S^1 \ni z \mapsto t\gamma(z)t^{-1}]$$

and

$$S^1 \times \Omega(G) \to \Omega(G), \quad (e^{i\theta}, \gamma) \mapsto [S^1 \ni z \mapsto \gamma(ze^{i\theta})\gamma(e^{i\theta})^{-1}]$$

define an action of $T \times S^1$ on $\Omega(G)$. In fact, the latter space is an infinite dimensional smooth symplectic manifold and the action of $T \times S^1$ is Hamiltonian. Let

$$\mu: \Omega(G) \to (\mathfrak{t} \oplus i\mathbb{R})^*$$

denote a moment map, where $\mathfrak{t} := \mathrm{Lie}(T)$ and $i\mathbb{R} = \mathrm{Lie}(S^1)$. Atiyah and Pressley [1] extended the celebrated convexity theorem of Atiyah and Guillemin–Sternberg and showed that the image of $\mu$ is the convex hull of its singular values. Their proof's idea is to determine first the image under $\mu$ of the subspace $\Omega_{\mathrm{alg}}(G) \subset \Omega(G)$ whose

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elements are restrictions of algebraic maps from $\mathbb{C}^*$ to the complexification $G^{\mathbb{C}}$ of $G$: they notice that $\mu(\Omega_{\text{alg}}(G))$ is a closed subspace of $(\mathfrak{t} \oplus i\mathbb{R})^*$; since $\Omega_{\text{alg}}(G)$ is dense in $\Omega(G)$ (by a theorem of Segal), they deduce from the continuity of $\mu$ that

$$\mu(\Omega(G)) = \mu(\Omega_{\text{alg}}(G)).$$

The goal of this paper is to extend to $\Omega(G)$ the well-known result which says that all levels of the moment map arising from a Hamiltonian torus action on a compact symplectic manifold are connected. That is, we will prove the following theorem.

**Theorem 1.1.** *For any $a \in \mu(\Omega(G))$, the pre-image $\mu^{-1}(a)$ is a connected topological subspace of $\Omega(G)$.*

REMARKS.   1.   A version of Theorem 1.1 has been proved in [3]. More specifically, instead of $\Omega(G)$ the authors consider there the space $\Omega_1(G)$ of all loops $S^1 \to G$ of Sobolev class $H^1$. They prove that all *regular* levels of $\mu \colon \Omega_1(G) \to (\mathfrak{t} \oplus i\mathbb{R})^*$ are connected. It is not obvious how to adapt that proof for singular levels or/and for loops of class $C^\infty$.
2.   One can easily see that our proof of Theorem 1.1 works for $\mu \colon \Omega_1(G) \to (\mathfrak{t} \oplus i\mathbb{R})^*$ (and even loops of Sobolev class $H^s$, with $s \geq 1$) as well. In other words, we can prove that *all* levels of $\mu \colon \Omega_1(G) \to (\mathfrak{t} \oplus i\mathbb{R})^*$ are connected topological subspaces of $\Omega_1(G)$. We decided to deal here with $\Omega(G)$ (smooth loops) rather than $\Omega_1(G)$ because the former is discussed in detail in our main reference [11], and the reader can make the connections directly.

We will give here an outline of the paper. In Section 2 we present basic notions and results concerning loop groups. In Section 3 we define the key ingredient of the proof of Theorem 1.1. This is a certain geometric invariant theory (shortly G.I.T.) quotient of $\Omega(G)$ with respect to the complexification $T^{\mathbb{C}} \times \mathbb{C}^*$ of $T \times S^1$. To define this quotient, we face the difficulty that the $S^1$ action on $\Omega(G)$ mentioned above does not extend to a $\mathbb{C}^*$ action (only the $T$ action extends canonically to a $T^{\mathbb{C}}$ action). However, for any $\gamma \in \Omega(G)$ there is a natural way to define the loop $u\gamma$ for any $u \in \mathbb{C}$ which is contained in the exterior of a disk with center at 0 and radius smaller than 1 (which depends on $\gamma$); if $|u| = 1$ then $u\gamma$ is given by the $S^1$ action on $\Omega(G)$ defined above. By putting $\gamma \sim gu\gamma$, where $u$ is as before and $g \in T^{\mathbb{C}}$ arbitrary, we obtain an equivalence relation $\sim$ on $\Omega(G)$. The G.I.T. quotient mentioned before is $A/\sim$, where $A$ consists of all elements of $\Omega(G)$ which are equivalent to elements of $\mu^{-1}(a)$. The main result of Section 3 is Proposition 3.5, which says that the natural map $\mu^{-1}(a)/(T \times S^1) \to A/\sim$ is bijective (the idea of the proof belongs to Kirwan, see [5, Chapter 7]). In Section 4 we note that the image of $(\mu^{-1}(a) \cap \Omega_{\text{alg}}(G))/(T \times S^1)$ under the map above is $(A \cap \Omega_{\text{alg}}(G))/\sim$. The former space is connected (by a result of [3]) and we prove that the latter is dense in $A/\sim$ (see Proposition 4.2). Consequently,

$A/\sim$ is a connected topological subspace of $\Omega(G)/\sim$. We deduce that $\mu^{-1}(a)/T \times S^1$ is connected. Hence $\mu^{-1}(a)$ is connected as well.

## 2. Notions of loop groups

In this section we collect results about loop groups which will be needed later. The details can be found in Pressley and Segal [11] and/or Atiyah and Pressley [1].

Like in the introduction, $G$ is a compact simply connected Lie group. We denote by $L(G)$ the space of all smooth maps $S^1 \to G$ (call them loops). The obvious multiplication makes it into a Lie group. By $\Omega(G)$ we denote the space of all loops which map $1 \in S^1$ to the unit $e$ of $G$. It can be naturally identified with the homogeneous space $L(G)/G$. In fact, the presentation of $\Omega(G)$ which is most appropriate for our goals is

(1) $$\Omega(G) = L(G^{\mathbb{C}})/L^+(G^{\mathbb{C}}).$$

Here $G^{\mathbb{C}}$ is the complexification of $G$ and $L(G^{\mathbb{C}})$ the set of all (smooth) loops $\alpha: S^1 \to G^{\mathbb{C}}$; by $L^+(G^{\mathbb{C}})$ we denote the subgroup of $L(G^{\mathbb{C}})$ consisting of all $\alpha$ as above which extend holomorphically for $|\zeta| \leq 1$ (this notion is explained in detail at the beginning of the next section). Since $L(G^{\mathbb{C}})$ is a complex Lie group and $L^+(G^{\mathbb{C}})$ a complex Lie subgroup, equation (1) shows that the manifold $\Omega(G)$ has a complex structure. More precisely, the complex structure $J_x$ at a point $x \in \Omega(G)$ is induced by the multiplication by $i$ in the tangent space $T_\alpha L(G^{\mathbb{C}})$, where $\alpha \in L(G^{\mathbb{C}})$ is such that $x = \alpha L^+(G^{\mathbb{C}})$.

Let us embed $G$ into some special unitary group $SU(N)$. We consider the Hilbert space $H := L^2(S^1, \mathbb{C}^N)$ and the corresponding "Grassmannian" $Gr(H)$. The latter consists of all closed vector subspaces of $H$ which satisfy certain supplementary properties; it turns out that $Gr(H)$ can be equipped with a Kähler (Hilbert) manifold structure (the details can be found in [11, Chapter 7]). An important subspace of $Gr(H)$ is $Gr_0(H)$. For the goals of our paper it is sufficient to mention that $Gr_0(H)$ contains $H_+$, which is the closed vector subspace of $H$ spanned by $S^1 \ni z \mapsto z^k v$, with $k \geq 0$ and $v \in \mathbb{C}^N$. Also, the connected component of $H_+$ in $Gr_0(H)$ consists of all vector subspaces $W$ of $H$ for which there exists $n \geq 0$ such that

$$z^n H_+ \subset W \subset z^{-n} H_+$$

and

$$\dim[(z^{-n} H_+)/W] = \dim[W/(z^n H_+)].$$

In other words, if $\mathcal{G}_n$ denotes the subspace of all $W$ which satisfy the last two equations, then the connected component of $H_+$ in $Gr_0(H)$ is $\bigcup_{n \geq 0} \mathcal{G}_n$. It is important to note that via the map

$$\mathcal{G}_n \ni W \mapsto W/z^n H_+,$$

the space $\mathcal{G}_n$ can be identified with the Grassmannian $Gr_{nN}(\mathbb{C}^{2nN})$ of all vector sub-spaces of dimension $nN$ in

$$(2) \qquad \mathbb{C}^{2nN} = z^{-n}H_+/z^n H_+.$$

Also note that we have the chain of inclusions

$$(3) \qquad \mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots.$$

Less obvious is the fact that for any $n \geq 0$, the canonical symplectic structure on the Grassmannian $\mathcal{G}_n$ makes this space into a symplectic submanifold of $Gr(H)$. The role of the above construction is revealed by the following result.

**Proposition 2.1.** (a) *The map*

$$(4) \qquad \Omega(G) \to Gr(H), \quad \gamma \mapsto \gamma H_+$$

*is an embedding, which induces on $\Omega(G)$ a structure of symplectic manifold. Together with the complex structure $J$ defined above, this makes $\Omega(G)$ into a Kähler manifold.*
(b) *The image of $\Omega_{\mathrm{alg}}(G)$ (see the introduction) under the embedding* (4) *is contained in $\bigcup_{n\geq 0} \mathcal{G}_n$.*

Based on point (b), we identify $\Omega_{\mathrm{alg}}(G)$ with a subspace of $\bigcup_{n\geq 0} \mathcal{G}_n$. The inclu-sions (3) induce the filtration

$$\Omega_{\mathrm{alg}}(G) = \bigcup_{n\geq 0} \Omega_n, \quad \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \cdots,$$

where

$$\Omega_n := \Omega_{\mathrm{alg}}(G) \cap \mathcal{G}_n.$$

The space $\Omega_n$ is a closed subvariety of the Grassmannian $\mathcal{G}_n$. We refer to the topology on $\Omega_{\mathrm{alg}}(G)$ induced by the filtration above as the *direct limit topology*. There is another natural topology on $\Omega_{\mathrm{alg}}(G)$, namely the subspace topology, induced by the inclusion $\Omega_{\mathrm{alg}}(G) \subset \Omega(G)$.

The following proposition can be proved with the same arguments as Proposition 2.1 of [3] (the result is also mentioned in [2, Section 2]).

**Proposition 2.2.** *The direct limit topology on $\Omega_{\mathrm{alg}}(G)$ is finer than the subspace topology.*

Let us consider again the $T \times S^1$ action on $\Omega(G)$ described at the beginning of the paper, and the corresponding moment map $\mu: \Omega(G) \to (\mathfrak{t} \oplus i\mathbb{R})^*$. This is uniquely

determined up to an additive constant, which will be made more precise momentarily (a standard moment map is described explicitly in [1, Section 3], but we will not need that expression here). For the moment, we would like to deduce from Proposition 2.2 a result which will be useful later. Namely, let us take $a \in \mu(\Omega(G))$; by [3, Proposition 3.4], $\mu^{-1}(a) \cap \Omega_{\mathrm{alg}}(G)$ is a connected subspace of $\Omega_{\mathrm{alg}}(G)$ with respect to the direct limit topology. We deduce:

**Proposition 2.3.** *For any $a \in \mu(\Omega_{\mathrm{alg}}(G))$, the space $\mu^{-1}(a) \cap \Omega_{\mathrm{alg}}(G)$ is a connected topological subspace of $\Omega(G)$.*

There is also an action of $T \times S^1$ on each $\mathcal{G}_n$, $n \geq 0$, which can be described as follows. We fix a basis, say $b_1, \ldots, b_N$, of $\mathbb{C}^N$, and consider the induced basis $z^k b_j$, $-n \leq k \leq n-1$, $1 \leq j \leq N$, of $\mathbb{C}^{2nN}$ (see equation (2)). The action of $T$ on $\mathcal{G}_n$ is induced by

$$(5) \qquad\qquad t \cdot (z^k b_j) := z^k(t b_j),$$

for any $t \in T$ and $k, j$ as above; the action of $S^1$ is induced by

$$(6) \qquad\qquad e^{i\theta} \cdot (z^k b_j) := (e^{i\theta}z)^k b_j = z^k e^{ik\theta} b_j$$

for all $e^{i\theta} \in S^1$. This $T \times S^1$ action is the restriction of an obvious $T^{\mathbb{C}} \times \mathbb{C}^*$ action: namely, in equation (5) we take $t \in T^{\mathbb{C}}$ and in equation (6) we replace $e^{i\theta}$ by an arbitrary element of $\mathbb{C}^*$. The $T^{\mathbb{C}} \times \mathbb{C}^*$ action turns out to be linear with respect to the Plücker embedding of $\mathcal{G}_n$ (see [1, Section 4]). Thus, the $T \times S^1$ action is Hamiltonian. We pick

$$\mu_n \colon \mathcal{G}_n \to (\mathfrak{t} \oplus i\mathbb{R})^*$$

a moment map, which is again uniquely determined up to an additive constant. We can arrange the constants in such a way that if $m < n$ then

$$\mu_n|_{\mathcal{G}_m} = \mu_m.$$

The reason is that $\mathcal{G}_m$ is a $T \times S^1$-invariant symplectic submanifold of $\mathcal{G}_n$. We obtain the map $\tilde{\mu} \colon \bigcup_{n \geq 0} \mathcal{G}_n \to (\mathfrak{t} \oplus i\mathbb{R})^*$ such that $\tilde{\mu}|_{\mathcal{G}_n} = \mu_n$, for all $n \geq 0$. The map $\tilde{\mu}$ is uniquely determined up to an additive constant. The following proposition relates the moment maps $\mu$ and $\tilde{\mu}$.

**Proposition 2.4.** *We can choose $\mu$ and $\tilde{\mu}$ such that*

$$\mu|_{\Omega_{\mathrm{alg}}(G)} = \tilde{\mu}|_{\Omega_{\mathrm{alg}}(G)}.$$

Proof. The idea of the proof is that there exists a submanifold $Gr_\infty(H)$ of $Gr(H)$ acted on smoothly by $T \times S^1$ and such that

- $Gr_0(H) \subset Gr_\infty(H)$ and the inclusion is $T \times S^1$ equivariant
- there exists $\hat{\mu} \colon Gr_\infty(H) \to \mathfrak{t} \oplus i\mathbb{R}$ which is a moment map for the $T \times S^1$ action
- the image of $\Omega(G)$ under the inclusion (4) is contained in $Gr_\infty(H)$.

It is worth noticing that $Gr(H)$ does not admit a smooth action of $T \times S^1$; only its subspace $Gr_\infty(H)$ does (see [11, Section 7.6]). This is why we need to use the latter space in our proof.

We deduce that $\hat{\mu}|_{\Omega(G)}$ differs from $\mu$ by a constant; the same can be said about $\hat{\mu}|_{\mathcal{G}_n}$ and $\mu_n$, for any $n \geq 0$. The result follows. $\qquad\square$

Let us consider again the $T^{\mathbb{C}} \times \mathbb{C}^*$ action on $\mathcal{G}_n$ defined above. Any of the inclusions $\mathcal{G}_n \subset \mathcal{G}_{n+1}$ is equivariant. Thus, we have an action of $T^{\mathbb{C}} \times \mathbb{C}^*$ on $\bigcup_{n \geq 0} \mathcal{G}_n$. The same group acts on $\Omega_{\mathrm{alg}}(G)$, as follows. We take into account that

$$(7) \qquad \Omega_{\mathrm{alg}}(G) = L_{\mathrm{alg}}(G^{\mathbb{C}})/L_{\mathrm{alg}}^+(G^{\mathbb{C}})$$

where $L_{\mathrm{alg}}(G^{\mathbb{C}})$ is the space of all algebraic maps $\alpha \colon \mathbb{C}^* \to G^{\mathbb{C}}$ and $L_{\mathrm{alg}}^+(G^{\mathbb{C}})$ the subgroup consisting of those $\alpha$ which can be extended holomorphically to $\mathbb{C}$. Then the action we are referring to is

$$(8) \qquad T^{\mathbb{C}} \times \mathbb{C}^* \times \Omega_{\mathrm{alg}}(G) \ni (g, u, \alpha L_{\mathrm{alg}}^+(G^{\mathbb{C}})) \mapsto [\mathbb{C}^* \ni \zeta \mapsto g\alpha(u\zeta)]L_{\mathrm{alg}}^+(G^{\mathbb{C}}).$$

The following result will be needed later.

**Proposition 2.5.** *The inclusion $\Omega_{\mathrm{alg}}(G) \subset \bigcup_{n \geq 0} \mathcal{G}_n$ defined in Proposition 2.1 (b) is $T^{\mathbb{C}} \times \mathbb{C}^*$ equivariant.*

Proof. Take $\gamma \in \Omega_{\mathrm{alg}}(G)$, which is of the form

$$S^1 \ni z \mapsto \gamma(z) = \sum_{-k_0 \leq k \leq k_0} A_k z^k,$$

where $k_0 \geq 0$. Here $A_k$ are $N \times N$ matrices with entries in $\mathbb{C}$. The subspace $\gamma H_+$ of $H$ has the property that

$$z^n H_+ \subset \gamma H_+ \subset z^{-n} H_+,$$

for some $n \geq 0$. Any element $v$ of $H_+$ has a Fourier expansion of the form $v = \sum_{m \geq 0} v_m z^m$, where $v_m \in \mathbb{C}^N$, for all $m \geq 0$. Then

$$\gamma v = \sum_{m \geq 0, k \in \mathbb{Z}} (A_k v_m) z^{k+m}.$$

The corresponding element of $(\gamma H_+)/z^n H_+$ is

$$[\gamma v] = \gamma v \bmod z^n H_+ = \sum_{m \geq 0, -k_0 \leq k \leq k_0, k+m \leq n-1} (A_k v_m) z^{k+m}.$$

In this sum we have $m \leq n - k - 1 \leq n + k_0 - 1$. Thus, to describe all elements of $\gamma H_+ / z^m H_+$, it is sufficient to take $v$ of the form

$$v = \sum_{0 \leq m \leq n + k_0 - 1} v_m z^m.$$

If $t \in T^{\mathbb{C}}$, then

$$t \cdot [\gamma v] = \sum_{m \geq 0, -k_0 \leq k \leq k_0, k+m \leq n-1} t(A_k v_m) z^{k+m} = \sum_{m \geq 0, -k_0 \leq k \leq k_0, k+m \leq n-1} (t A_k) v_m z^{k+m}$$

$$= [(t \cdot \gamma) v].$$

Consequently, $t \cdot (\gamma H_+) = (t \cdot \gamma) H_+$. If $u \in \mathbb{C}^*$, then

$$u \cdot [\gamma v] = \sum_{m \geq 0, -k_0 \leq k \leq k_0, k+m \leq n-1} (A_k v_m)(uz)^{k+m} = \sum_{m \geq 0, -k_0 \leq k \leq k_0, k+m \leq n-1} (u^k A_k u^m v_m) z^{k+m}.$$

This is the same as $[(u \cdot \gamma) \tilde{v}]$, where

$$\tilde{v} = \sum_{0 \leq m \leq n + k_0 - 1} u^m v_m z^m.$$

Consequently, $u \cdot (\gamma H_+) = (u \cdot \gamma) H_+$. $\qquad\square$

Finally, let us pick $B \subset G^{\mathbb{C}}$ a Borel subgroup with $T \subset B$. The presentation (7) of $\Omega_{\mathrm{alg}}(G)$ allows us to define on the latter space a natural action of the group

$$\mathcal{B}_+ := \{\alpha \in L_{\mathrm{alg}}^+(G^{\mathbb{C}}) : \alpha(0) \in B\}$$

on $\Omega_{\mathrm{alg}}(G)$. The orbit decomposition is

$$\Omega_{\mathrm{alg}}(G) = \bigcup_{\lambda \in \check{T}} C_\lambda,$$

where the union is disjoint and

$$C_\lambda := \mathcal{B}_+ \lambda$$

is called a *Bruhat cell*. Here $\check{T}$ denotes the lattice of group homomorphisms $S^1 \to T$. The space $C_\lambda$ is really a (finite dimensional) cell, being homeomorphic to $\mathbb{C}^r$ for some $r$. In this paper, by $\overline{C_\lambda}$ we will always mean the closure of $C_\lambda$ in the direct limit topology (see above). The following property of the Bruhat cells will be needed later.

**Proposition 2.6.** *For any $\lambda \in \check{T}$, there exists $n \geq 1$ such that $\overline{C_\lambda}$ is contained in $\mathcal{G}_n$ as a $T^{\mathbb{C}} \times \mathbb{C}^*$-invariant closed subvariety.*

This can be proved as follows. There exists $n \geq 1$ such that $C_\lambda \subset \mathcal{G}_n$, because $\mathcal{B}_+$ leaves each $\Omega_k$, $k \geq 0$, invariant (see [6, Lemma 3.3.2]). The space $C_\lambda$ is a locally Zariski closed subspace of $\Omega_{\mathrm{alg}}(G)$ (see [8, Proposition 2.13 and Theorem 3.1]), thus also of $\Omega_n$ and of $\mathcal{G}_n$. Consequently, the closures of $C_\lambda$ in the Zariski, respectively differential topology of $\mathcal{G}_n$ are equal.

Another result concerning the Bruhat cells is the following proposition (cf. [1, Section 1], see also [3, Proof of Proposition 3.4]).

**Proposition 2.7.** *For any* $\lambda_1, \lambda_2 \in \check{T}$ *there exists* $\lambda \in \check{T}$ *such that*

$$C_{\lambda_1} \subset \overline{C_\lambda}$$

*and*

$$C_{\lambda_2} \subset \overline{C_\lambda}.$$

*Consequently, for any* $x, y \in \Omega_{\mathrm{alg}}(G)$ *there exists* $\lambda \in \check{T}$ *such that both* $x$ *and* $y$ *are in* $\overline{C_\lambda}$.

## 3. The equivalence relation $\sim$

We begin with the following definition. Take $0 < r \leq 1$. We say that a free loop $S^1 \to G^{\mathbb{C}}$ *extends holomorphically for* $|\zeta| \geq r$ if it is the restriction of a map

$$\alpha \colon \{\zeta \in \mathbb{C} \cup \infty \colon |\zeta| \geq r\} \to G^{\mathbb{C}}$$

which is continuous, holomorphic on $\{\zeta \in \mathbb{C} \cup \infty \colon |\zeta| > r\}$ and smooth on $\{\zeta \in \mathbb{C} \colon |\zeta| = r\}$; the same terminology is adopted if we take $r \geq 1$ and replace "$\geq$" and "$>$" by "$\leq$", respectively "$<$" (and also $\mathbb{C} \cup \infty$ by $\mathbb{C}$).

Let $L^-(G^{\mathbb{C}})$ denote the subspace of $L(G^{\mathbb{C}})$ consisting of those $\alpha$ which extend holomorphically for $|\zeta| \geq 1$ in the sense of the definition above. One knows that any $\alpha \in L(G^{\mathbb{C}})$ can be written as

$$\alpha = \alpha_- \lambda \alpha_+,$$

where $\alpha_- \in L^-(G^{\mathbb{C}})$, $\alpha_+ \in L^+(G^{\mathbb{C}})$, and $\lambda$ is a group homomorphism $S^1 \to T$ (see [11, Theorem 8.1.2]). By using the presentation (1), the elements of $\Omega(G)$ are cosets of the form $\alpha_- \lambda L^+(G^{\mathbb{C}})$, where $\alpha_-$ and $\lambda$ are as above. The following lemma will be used later.

**Lemma 3.1.** *Take* $\alpha_-$, $\beta_-$ *in* $L^-(G^{\mathbb{C}})$ *and* $\lambda, \mu \colon S^1 \to T$ *group homomorphisms such that*

$$\alpha_- \lambda L^+(G^{\mathbb{C}}) = \beta_- \mu L^+(G^{\mathbb{C}}).$$

*Let* $r$ *be a strictly positive real number.*

(a) *Assume that $r < 1$. If $\alpha_-$ extends holomorphically for $|\zeta| \geq r$ then $\beta_-$ extends holomorphically for $|\zeta| \geq r$ as well.*

(b) *Assume that $r \geq 1$ or $r < 1$ and $\alpha_-$ extends holomorphically for $|\zeta| \geq r$. For any $u \in \mathbb{C}^*$ with $|u| \geq r$ we have*

$$[S^1 \ni z \mapsto \alpha_-(uz)]\lambda L^+(G^{\mathbb{C}}) = [S^1 \ni z \mapsto \beta_-(uz)]\mu L^+(G^{\mathbb{C}}).$$

Proof. We have

$$(9) \qquad\qquad\qquad \alpha_- \lambda = \beta_- \mu \alpha_+,$$

where $\alpha_+ \in L^+(G^{\mathbb{C}})$.

(a) The loops $\lambda$ and $\mu$ are one-parameter subgroups in $T$, thus they have obvious (holomorphic) extensions to group homomorphisms $\mathbb{C}^* \to T^{\mathbb{C}}$. From (9) we deduce that $\beta_-$ is the restriction of a function holomorphic on the annulus

$$\{\zeta \in \mathbb{C} : r < |\zeta| < 1\}$$

and continuous on the closure of this space. Consequently, the map $\xi \mapsto \beta_-(1/\xi)$ extends holomorphically for $|\xi| \leq 1/r$, that is, $\beta_-$ extends holomorphically for $|\zeta| \geq r$. Indeed, let us consider again the embedding $G \subset SU(N)$, as in Section 2. The resulting embedding $G^{\mathbb{C}} \subset \mathrm{Mat}^{N \times N}(\mathbb{C})$ is holomorphic. We use the following claim:

**Claim.** *If $f : \{\xi \in \mathbb{C} : |\xi| \leq 1/r\} \to \mathbb{C}$ is a continuous function which is holomorphic on $\{\xi \in \mathbb{C} : |\xi| < 1/r, |\xi| \neq 1\}$, then $f$ is holomorphic on $\{\xi \in \mathbb{C} : |\xi| < 1/r\}$.*

This can be proved by comparing the Laurent series of $f$ on $\{\xi \in \mathbb{C} : |\xi| < 1\}$, respectively $\{\xi \in \mathbb{C} : 1 < |\xi| < 1/r\}$. The series are equal, since the coefficients of both of them are equal to $(1/(2\pi i)) \int_{|\xi|=1} f(\xi)/\xi^k d\xi$, $k \in \mathbb{Z}$ (by a uniform continuity argument). Thus, the radius of convergence of the first of the two series (which is actually the Taylor series of $f$ around 0) is at least equal to $1/r$. The claim is proved.

(b) From equation (9) we deduce that $\alpha_+$ extends holomorphically to $\mathbb{C}$. The reason is that the entries of the $N \times N$ matrix $\alpha_+ = \mu^{-1}\beta_-^{-1}\alpha_- \lambda$ are $\mathbb{C}$-valued functions which are continuous on $\mathbb{C}$ and holomorphic on $\mathbb{C} \setminus \{\xi \in \mathbb{C} : |\xi| = 1\}$; by the same argument as in the claim above, they are holomorphic on the whole $\mathbb{C}$. Again from equation (9), we deduce that

$$\alpha_-(uz)\lambda(uz) = \beta_-(uz)\mu(uz)\alpha_+(uz),$$

for all $z \in S^1$. The map $S^1 \ni z \mapsto \alpha_+(uz)$ is in $L^+(G^{\mathbb{C}})$. We only need to notice that

$$\lambda(uz) = \lambda(z)\lambda(u), \quad \mu(uz) = \mu(z)\mu(u). \qquad\qquad \square$$

DEFINITION 3.2. (a) Take $x \in \Omega(G)$ and $u \in \mathbb{C}^*$. We say that the pair $(u, x)$ is *admissible* if

- $|u| \geq 1$

or

- $|u| < 1$ and $x = \alpha_- \lambda L^+(G^{\mathbb{C}})$, where $\alpha_- \in L^-(G^{\mathbb{C}})$ extends holomorphically for $|\zeta| \geq |u|$ and $\lambda \colon S^1 \to T$ is a group homomorphism.

If $(u, x)$ is as above and $g \in T^{\mathbb{C}}$, then

$$gux := g[S^1 \ni z \mapsto \alpha_-(uz)]\lambda L^+(G^{\mathbb{C}})$$

is an element of $\Omega(G)$.

(b) Take $x, y \in \Omega(G)$. We say that

$$x \sim y$$

if there exist $u \in \mathbb{C}^*$ and $g \in T^{\mathbb{C}}$ such that $(u, x)$ is an admissible pair and $y = gux$.

REMARK. We can also express $gux$ as

$$gux := g[S^1 \ni z \mapsto (\alpha_-\lambda)(uz)]L^+(G^{\mathbb{C}}),$$

because $\lambda$ is a group homomorphism $\mathbb{C}^* \to T^{\mathbb{C}}$.

Note that by Lemma 3.1, the definition of $gux$ in part (a) is independent of the choice of the representative $\alpha_-\lambda$ of $x \in L(G^{\mathbb{C}})/L^+(G^{\mathbb{C}})$. The following lemma shows that $\sim$ is an equivalence relation.

**Lemma 3.3.** (a) *If $x \in \Omega(G)$, $u \in \mathbb{C}^*$ and $g \in T^{\mathbb{C}}$ such that $(u, x)$ is admissible, then $(u^{-1}, gux)$ is admissible and we have*

$$g^{-1}u^{-1}(gux) = x.$$

(b) *If $x \in \Omega(G)$, $u_1, u_2 \in \mathbb{C}^*$, and $g_1, g_2 \in T^{\mathbb{C}}$ such that $(u_1, x)$ and $(u_2, g_1u_1x)$ are admissible, then $(u_1u_2, x)$ is admissible and*

$$(g_1g_2)(u_1u_2)x = g_2u_2(g_1u_1x).$$

Proof. (a) We can assume that $g = 1$. We write $x = \alpha_-\lambda L^+(G^{\mathbb{C}})$. Assume first that $|u| \geq 1$. The loop $S^1 \ni z \mapsto \alpha_-(uz)$ extends holomorphically for $|\zeta| \geq 1/|u|$ by $\zeta \mapsto \alpha_-(u\zeta)$. The case $|u| < 1$ is even easier to analyze. Verifying that $u^{-1}(ux) = x$ is equally easy.

(b) We can assume that $g_1 = g_2 = 1$. Again we write $x = \alpha_-\lambda L^+(G^{\mathbb{C}})$. It is sufficient to analyze the case when $|u_1u_2| < 1$. Thus, at least one of the numbers $|u_1|$ and $|u_2|$ is strictly less than 1. We distinguish the following two cases.

CASE 1. $|u_2| < 1$. The loop $S^1 \ni z \mapsto \alpha_-(u_1 z)$ is well defined and extends holomorphically for $|\xi| \geq |u_2|$. Let $\tilde{\alpha} \colon \{\xi \in \mathbb{C} \cup \infty \colon |\xi| \geq |u_2|\} \to G^{\mathbb{C}}$ be an extension of this loop. The map $\hat{\alpha} \colon \{\zeta \in \mathbb{C} \cup \infty \colon |\zeta| \geq |u_1 u_2|\} \to G^{\mathbb{C}}$ given by

$$\hat{\alpha}(\zeta) = \begin{cases} \tilde{\alpha}(u_1^{-1}\zeta), & \text{if} \quad |u_1 u_2| \leq |\zeta| \leq |u_1|, \\ \alpha_-(\zeta), & \text{if} \quad |u_1| \leq |\zeta| \end{cases}$$

is the desired extension of $\alpha_-$ for $|\zeta| \geq |u_1 u_2|$ (note that $\hat{\alpha}$ is holomorphic on $|\zeta| > |u_1 u_2|$, since it is continuous and is holomorphic on the complement of the circle $\{\zeta \in \mathbb{C} \colon |\zeta| = |u_1|\}$).

CASE 2. $|u_2| \geq 1$. This implies $|u_1| < 1$. We notice that the pair $(u_1, u_2 x)$ is admissible: indeed, by hypothesis, the loop $S^1 \ni z \mapsto \alpha_-(u_2 z)$ extends holomorphically for $|u_2 \zeta| \geq |u_1|$, hence also for $|\zeta| \geq |u_1|$. The pair $(u_2, x)$ is admissible too. From the result proved in Case 1 we deduce that $(u_1 u_2, x)$ is admissible.

The equation $u_2(u_1 x) = (u_1 u_2)x$ is straightforward. $\qquad\square$

The following result relates the equivalence relation $\sim$ to the $T \times S^1$ action on $\Omega(G)$ (see Section 1).

**Lemma 3.4.** *Take* $\gamma \in \Omega(G)$. *If* $\theta \in \mathbb{R}$, *then the pair* $(e^{i\theta}, \gamma)$ *is admissible. If* $t \in T$, *then the loop* $te^{i\theta}\gamma$ *given by* Definition 3.2 (b) *can be expressed as*

$$te^{i\theta}\gamma = t\gamma^\theta t^{-1}.$$

*Here the right-hand side is given by*

$$(t\gamma^\theta t^{-1})(z) = t\gamma(ze^{i\theta})\gamma(e^{i\theta})^{-1}t^{-1},$$

*for all* $z \in S^1$.

Proof. There exist $\alpha_- \in L^-(G^{\mathbb{C}})$ and $\lambda \colon S^1 \to T$ a group homorphism such that the image of $\gamma$ under the isomorphism (1) is $\alpha_- \lambda L^+(G^{\mathbb{C}})$. This means that

$$\alpha_- \lambda = \gamma \alpha_+,$$

for some $\alpha_+ \in L^+(G^{\mathbb{C}})$. We deduce that for any $z \in S^1$ we have

$$[t\alpha_-(ze^{i\theta})\lambda(z)]\lambda(e^{i\theta}) = [t\gamma^\theta(z)t^{-1}]t\gamma(e^{i\theta})\alpha_+(ze^{i\theta}).$$

In other words, via the isomorphism (1), to $t\gamma^\theta t^{-1}$ corresponds the coset of

$$t[S^1 \ni z \mapsto \alpha_-(ze^{i\theta})]\lambda,$$

which is the same as $te^{i\theta}(\alpha_- \lambda L^+(G^{\mathbb{C}}))$. $\qquad\square$

We now denote by $A$ the set of all $x \in \Omega(G)$ with $x \sim y$, for some $y \in \mu^{-1}(a)$. We are interested in the quotient space $A/\sim$ and the (natural) map $\mu^{-1}(a)/(T \times S^1) \to A/\sim$ which assigns to the coset of $x \in \mu^{-1}(a)$ the equivalence class of $x$. By Lemma 3.4, this map is well defined.

**Proposition 3.5.** *The natural map*

$$\mu^{-1}(a)/(T \times S^1) \to A/\sim$$

*is bijective.*

Proof. Only the injectivity has to be proved. We have to show that if $x, y \in \mu^{-1}(a)$ with $x \sim y$, then $y = te^{i\theta}x$, where $(t, e^{i\theta}) \in T \times S^1$. By Definition 3.2 we have

$$y = gux$$

for some $u \in \mathbb{C}^*$ and $g \in T^{\mathbb{C}}$. We write $g = \exp(w_1)\exp(iw_2)$ and $u = e^{i\alpha}e^{-\beta}$, where $w_1, w_2 \in \mathfrak{t}$ and $\alpha, \beta \in \mathbb{R}$ (here we see $-\beta$ as $i(i\beta)$). Since the pair $(u, x)$ is admissible and $|u| = |e^{-\beta}|$, the pair $(e^{-\beta}, x)$ is admissible too. By Lemma 3.3 (b) we have

$$y = \exp(w_1)e^{i\alpha}(\exp(iw_2)e^{-\beta}x).$$

Thus, it is sufficient to assume that

$$y = \exp(iw_2)e^{-\beta}x.$$

Moreover, without loss of generality we assume that

$$\beta \geq 0,$$

because if contrary we write $x = \exp(-iw_2)e^{\beta}y$ (by Lemma 3.3 (a)). Let us consider the function $h\colon [0, 1] \to \mathbb{R}$,

$$h(s) = [\mu(\exp(isw_2)e^{-s\beta}x) - a](w_2, i\beta),$$

where $0 \leq s \leq 1$. Notice that $h(s)$ is well defined for any $s$ with $0 \leq s \leq 1$: indeed, the pair $(e^{-\beta}, x)$ is admissible hence, because $e^{-s\beta} \geq e^{-\beta}$, the pair $(e^{-s\beta}, x)$ is admissible too. Since $\mu(x) = \mu(y)$, we have $h(0) = h(1) = 0$. Consequently, there exists $s_0$ in the interval $(0, 1)$ such that $h'(s_0) = 0$. We denote

(10)                          $$x_0 := \exp(is_0w_2)e^{-s_0\beta}x.$$

**Claim.** *We have*

$$\left.\frac{d}{ds}\right|_{s_0} \exp(isw_2)e^{-s\beta}x = J_{x_0}((w_2, i\beta) . x_0),$$

*where $J_{x_0}$ is the complex structure at $x_0$ (see Section 2) and*

$$(w_2, i\beta) \cdot x_0 := \frac{d}{ds}\bigg|_0 [\exp(sw_2)e^{is\beta}x_0]$$

*arising from the infinitesimal action of $T \times S^1$ on $\Omega(G)$.*

The claim can be proved as follows. Write $x_0 = \alpha_-\lambda L^+(G^{\mathbb{C}})$, where $\alpha_- \in L^-(G^{\mathbb{C}})$ and $\lambda \colon S^1 \to T$ is a group homomorphism. By using Lemma 3.4 and the remark following Definition 3.2, we have

$$\exp(isw_2)e^{-s\beta}x = \exp(i(s-s_0)w_2)e^{-(s-s_0)\beta}x_0 = \exp(i(s-s_0)w_2)(\alpha_-\lambda)_{-(s-s_0)}L^+(G^{\mathbb{C}}).$$

Here we have denoted

$$(\alpha_-\lambda)_{-(s-s_0)}(z) := (\alpha_-\lambda)(e^{-(s-s_0)\beta}z)$$

for all $s$ in the interval $(0, 1)$ and all $z \in S^1$. By the definition of the complex structure $J$ (see Section 2), it is sufficient to prove that

$$(11) \qquad \frac{d}{ds}\bigg|_{s_0} [\exp(i(s-s_0)w_2)(\alpha_-\lambda)_{-(s-s_0)}] = i\frac{d}{ds}\bigg|_0 [\exp(sw_2)(\alpha_-\lambda)_{is}],$$

where

$$(\alpha_-\lambda)_{is}(z) := (\alpha_-\lambda)(e^{is\beta}z)$$

for all $s \in \mathbb{R}$ and all $z \in S^1$. By using the Leibniz rule, the left-hand side of (11) is

$$\frac{d}{ds}\bigg|_0 [\exp(isw_2)(\alpha_-\lambda)_{-s}] = i\frac{d}{ds}\bigg|_0 [\exp(sw_2)](\alpha_-\lambda) + i\frac{d}{ds}\bigg|_0 [(\alpha_-\lambda)_{is}].$$

Here we have used that

$$\frac{d}{ds}\bigg|_0 [\exp(isw_2)] = iw_2 = i\frac{d}{ds}\bigg|_0 [\exp(sw_2)]$$

and also that

$$\frac{d}{ds}\bigg|_0 (\alpha_-\lambda)(e^{-s\beta}z) = i\frac{d}{ds}\bigg|_0 (\alpha_-\lambda)(e^{is\beta}z),$$

for all $z \in S^1$ (the last equation follows from the fact that $\alpha_-\lambda$ is holomorphic on the exterior of a closed disk with center at 0 and radius strictly smaller than 1). The claim is proved.

From the claim we deduce as follows:

$$h'(s_0) = (d\mu)_{x_0}\left(\frac{d}{ds}\bigg|_{s_0} (\exp(isw_2)e^{-s\beta}x)\right)(w_2, i\beta)$$

$$= \omega_{x_0}\left(\frac{d}{ds}\bigg|_{s_0} (\exp(isw_2)e^{-s\beta}x), (w_2, i\beta) . x_0\right)$$

$$= \omega_{x_0}(J_{x_0}((w_2, i\beta) . x_0), (w_2, i\beta) . x_0)$$

$$= \langle (w_2, i\beta) . x_0, (w_2, i\beta) . x_0\rangle,$$

where $\omega$ denotes the symplectic form and $\langle \ , \ \rangle$ the Kähler metric on $\Omega(G)$ (see Proposition 2.1). We deduce that

$$(w_2, i\beta) . x_0 = 0$$

which, according to the claim above, implies that

$$\frac{d}{ds}\bigg|_0 \exp(isw_2)e^{-s\beta}x_0 = 0.$$

From this we deduce that

(12) $$\exp(isw_2)e^{-s\beta}x_0 = x_0$$

for all $s \le 0$ (note that for any such $s$, the pair $(e^{-s\beta}, x_0)$ is admissible, since $e^{-s\beta} \ge 1$). Indeed, by using Lemma 3.4 we deduce that for any $s_1 \le 0$ we have

$$\frac{d}{ds}\bigg|_{s_1} \exp(isw_2)e^{-s\beta}x_0 = \frac{d}{ds}\bigg|_{s_1} (\exp(is_1w_2)e^{-s_1\beta}) \exp(i(s-s_1)w_2)e^{-(s-s_1)\beta}x_0$$

$$= d(\exp(is_1w_2)e^{-s_1\beta})_{x_0}\left(\frac{d}{ds}\bigg|_0 (\exp(isw_2)e^{-s\beta}x_0)\right)$$

$$= 0.$$

Here we have used the (differential of the) map $\exp(is_1w_2)e^{-s_1\beta}: \Omega(G) \to \Omega(G)$ given by

$$\gamma \mapsto \exp(is_1w_2)e^{-s_1\beta}\gamma,$$

which is well defined, since $e^{-s_1\beta} \ge 1$.

By Lemma 3.3 (a), equation (12) implies that the pair $(e^{-s\beta}, x_0)$ is admissible for any $s \ge 0$; moreover, equation (12) holds for all $s \ge 0$ as well. We make $s = -s_0$ in (12) and deduce $x = x_0$; then we make $s = 1 - s_0$ and deduce $y = x_0$. We conclude

$$x = y$$

and the proof is finished.                                                                 □

REMARK. Let $M$ be a compact Kähler manifold acted on by a complex Lie group $G$, which is the complexification of a compact Lie group $K$, in such a way that the action of $K$ on $M$ is Hamiltonian. Kirwan has proved that if $x, y \in M$ have the same image under the moment map and are on the same $G$ orbit, then they are on the same $K$-orbit (see [5, Lemma 7.2]). We have used above the idea of her proof. Kirwan's result cannot be used directly in our context: first, $\Omega(G)$ is not a compact manifold; second, and most importantly, the $T \times S^1$ action on $\Omega(G)$ does not extend in any reasonable way to a $T^{\mathbb{C}} \times \mathbb{C}^*$ action. We are substituting this action by the equivalence relation $\sim$.

## 4. Connectedness of $A/\sim$ and of $\mu^{-1}(a)$

We start with the following proposition.

**Proposition 4.1.** (a) *If* $x \in \Omega_{\mathrm{alg}}(G)$ *then the pair* $(u, x)$ *is admissible* (*in the sense of* Definition 3.2) *for any* $u \in \mathbb{C}^*$. *The map*

$$T^{\mathbb{C}} \times \mathbb{C}^* \times \Omega_{\mathrm{alg}}(G) \to \Omega_{\mathrm{alg}}(G), \quad (g, u, x) \mapsto gux$$

*is the action of* $T^{\mathbb{C}} \times \mathbb{C}^*$ *on* $\Omega_{\mathrm{alg}}(G)$ *defined in* Section 2 (*see equation* (8)).
(b) *The image of* $(\mu^{-1}(a) \cap \Omega_{\mathrm{alg}}(G))/(T \times S^1)$ *under the map in* Proposition 3.5 *is* $(A \cap \Omega_{\mathrm{alg}}(G))/\sim$. *The latter space is a connected topological subspace of* $\Omega(G)/\sim$.

Proof. Point (a) follows from equations (7) and (8) and the remark following Definition 3.2. To prove the first assertion of (b), we only need to note that if $x \in \Omega_{\mathrm{alg}}(G)$ and $y \in \Omega(G)$ such that $x \sim y$, then $y \in \Omega_{\mathrm{alg}}(G)$. To prove the second assertion of (b), we note that the natural map

$$(\mu^{-1}(a) \cap \Omega_{\mathrm{alg}}(G))/(T \times S^1) \to \Omega(G)/\sim$$

is continuous. We use Proposition 2.3. □

The key result of this section is

**Proposition 4.2.** *The subspace* $(A \cap \Omega_{\mathrm{alg}}(G))/\sim$ *of* $A/\sim$ *is dense* (*both spaces have the topology of subspace of* $\Omega(G)/\sim$).

Combined with Proposition 4.1 (b), this implies

**Corollary 4.3.** *The space* $A/\sim$ *is a connected topological subspace of* $\Omega(G)/\sim$.

In turn, this implies the main result of the paper, as follows.

Proof of Theorem 1.1.   The natural map

(13) $$\mu^{-1}(a)/(T \times S^1) \to \Omega(G)/\sim$$

is continuous, one-to-one, and its image is $A/\sim$ (by Proposition 3.5). Since $A/\sim$ is connected (see the previous corollary), we deduce that $\mu^{-1}(a)/(T \times S^1)$ is connected as well. Consequently, $\mu^{-1}(a)$ is a connected topological subspace of $\Omega(G)$.    $\square$

The rest of the section is devoted to the proof of Proposition 4.2. First, if $\lambda \in \check{T}$, we say that a point $x \in \overline{C_\lambda}$ is $(\mu - a)$-*semistable* if

(14) $$\overline{(T^{\mathbb{C}} \times \mathbb{C}^*)x} \cap (\mu^{-1}(a) \cap \overline{C_\lambda}) \neq \emptyset.$$

Here the closure is taken in $\Omega_{\text{alg}}(G)$ with respect to the direct limit topology. We may assume that $\overline{C_\lambda}$ is contained in the Grassmannian $\mathcal{G}_n$ as a $T^{\mathbb{C}} \times \mathbb{C}^*$-invariant closed subvariety (see Proposition 2.6). Then $x$ is $(\mu - a)$-semistable if and only if it is $(\mu_n - a)$-semistable in the usual sense, that is, if

$$\overline{(T^{\mathbb{C}} \times \mathbb{C}^*)x} \cap (\mu_n^{-1}(a) \cap \overline{C_\lambda}) \neq \emptyset$$

(see for instance [5, Chapter 7]). This follows immediately from the fact that $\mu$ and $\mu_n$ coincide on $\overline{C_\lambda}$, by Proposition 2.4. We denote by $\overline{C_\lambda}^{ss}$ the set of all semistable points in $\overline{C_\lambda}$. We also consider the set $\mathcal{G}_n^{ss}$ of all $(\mu_n - a)$-semistable points in $\mathcal{G}_n$. We have

(15) $$\overline{C_\lambda}^{ss} = \overline{C_\lambda} \cap \mathcal{G}_n^{ss}.$$

The following description of the semistable set of $\overline{C_\lambda}$ will be needed later.

**Lemma 4.4.** *We have*

$$A \cap \overline{C_\lambda} = \overline{C_\lambda}^{ss}.$$

Proof.   By Proposition 4.1 (b), we have

$$A \cap \Omega_{\text{alg}}(G) = (T^{\mathbb{C}} \times \mathbb{C}^*)(\mu^{-1}(a) \cap \Omega_{\text{alg}}(G)).$$

Consequently, a point $x \in \Omega(G)$ is in $A \cap \overline{C_\lambda}$ if and only if $x \in [(T^{\mathbb{C}} \times \mathbb{C}^*)\mu^{-1}(a)] \cap \overline{C_\lambda}$. The latter set is obviously equal to $(T^{\mathbb{C}} \times \mathbb{C}^*)(\mu^{-1}(a) \cap \overline{C_\lambda})$, which is the same as $\overline{C_\lambda}^{ss}$ (by [5, Theorems 7.4 and 8.10], applied for the Grassmannian $\mathcal{G}_n$ which contains $\overline{C_\lambda}$ as a $T^{\mathbb{C}} \times \mathbb{C}^*$-invariant closed subvariety, as indicated above).    $\square$

We are now ready to prove Proposition 4.2.

Proof of Proposition 4.2.   We show that in any open subset $V$ of $A/\sim$ there exists an element of $(A \cap \Omega_{\mathrm{alg}}(G))/\sim$. Since $A/\sim$ is equipped with the topology of subspace of $\Omega(G)/\sim$, we can write

$$V = (A/\sim) \cap (U/\sim) = (A \cap U)/\sim.$$

Here $U$ is an open subspace of $\Omega(G)$ with the property that for any $x \in U$, we have

$$\{y \in \Omega(G) \colon y \sim x\} \subset U.$$

The subspace $U \cap \Omega_{\mathrm{alg}}(G)$ of $\Omega_{\mathrm{alg}}(G)$ is open in the direct limit topology (because the direct limit topology on $\Omega_{\mathrm{alg}}(G)$ is finer than the subspace topology, see Proposition 2.2) and non-empty (because $\Omega_{\mathrm{alg}}(G)$ is dense in $\Omega(G)$, see [11, Section 3.5]). For any $x \in U \cap \Omega_{\mathrm{alg}}(G)$ we have

(16) $$(T^{\mathbb{C}} \times \mathbb{C}^*)x = \{y \in \Omega_{\mathrm{alg}}(G) \colon y \sim x\} \subset U \cap \Omega_{\mathrm{alg}}(G),$$

which follows from Proposition 4.1 (a). There exists $\lambda \in \check{T}$ such that $\overline{C_\lambda} \cap U \neq \emptyset$ and $\mu^{-1}(a) \cap \overline{C_\lambda} \neq \emptyset$. Indeed, we can pick $x \in \Omega_{\mathrm{alg}}(G) \cap U$ (the intersection is non-empty, see above) and $y \in \Omega_{\mathrm{alg}}(G) \cap \mu^{-1}(a)$ (the intersection is non-empty, since $a \in \mu(\Omega(G)) = \mu(\Omega_{\mathrm{alg}}(G))$); by Proposition 2.7, there exists $\lambda \in \check{T}$ such that $x$ and $y$ are both in $\overline{C_\lambda}$.

**Claim.**   *If $\lambda \in \check{T}$ is as above, then $\overline{C_\lambda}^{ss}$ is a dense subspace of $\overline{C_\lambda}$ (here $\overline{C_\lambda}$ is equipped with the direct limit topology it inherits from $\Omega_{\mathrm{alg}}(G)$).*

To prove the claim, we consider again a Grassmannian $\mathcal{G}_n$ which contains $\overline{C_\lambda}$ as a $T^{\mathbb{C}} \times \mathbb{C}^*$-invariant closed subvariety. By the main theorem of [4], there exists on $\mathcal{G}_n$ a $T^{\mathbb{C}} \times \mathbb{C}^*$-invariant very ample line bundle $L$ such that $\mathcal{G}_n^{ss} = \mathcal{G}_n^{ss}(L)$. The latter space consists of all $L$-semistable points in $\mathcal{G}_n$, that is points $x \in \mathcal{G}_n$ such that there exists $k \geq 1$ and $s \colon X \to L^{\otimes k}$ equivariant holomorphic section with $s(x) \neq 0$ (cf. e.g. [10]). Consequently, $\mathcal{G}_n^{ss}$ is a Zariski open subspace of $\mathcal{G}_n$. Since $\overline{C_\lambda}^{ss} = \mathcal{G}_n^{ss} \cap \overline{C_\lambda}$, we deduce that $\overline{C_\lambda}^{ss}$ is a Zariski open subspace of $\overline{C_\lambda}$. The space $\overline{C_\lambda}^{ss}$ is non-empty, since $\mu^{-1}(a) \cap \overline{C_\lambda} \subset \overline{C_\lambda}^{ss}$. Thus $\overline{C_\lambda}^{ss}$ is dense in $\overline{C_\lambda}$ with respect to the usual differential topology on the latter space: this can be deduced by using [9, Theorem 2.33] for $\overline{C_\lambda}$, which is an irreducible projective variety (cf. [8, p. 360]).

From the claim we deduce that the intersection $\overline{C_\lambda}^{ss} \cap U$ is non-empty (since $\overline{C_\lambda} \cap U$ is a non-empty subspace of $\overline{C_\lambda}$ which is open with respect to the direct limit topology). By Lemma 4.4 we have

$$\overline{C_\lambda}^{ss} \cap U = A \cap \overline{C_\lambda} \cap U,$$

thus

$$U \cap A \cap \Omega_{\mathrm{alg}}(G) \neq \emptyset.$$

By equation (16), the quotient $(U \cap A \cap \Omega_{\mathrm{alg}}(G))/\sim$ is a (non-empty) subspace of $\Omega(G)/\sim$. It is contained in both $V = (U \cap A)/\sim$ and $(A \cap \Omega_{\mathrm{alg}}(G))/\sim$. Consequently, the intersection $V \cap [(A \cap \Omega_{\mathrm{alg}}(G))/\sim]$ is non-empty. This finishes the proof.  $\square$

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Department of Mathematics and Statistics
University of Regina
Regina SK, S4S 0A2
Canada
e-mail: mareal@math.uregina.ca