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CONNECTEDNESS OF LEVELS FOR MOMENT MAPS ON VARIOUS CLASSES OF LOOP GROUPS

Dedicated to Jost-Hinrich Eschenburg on his sixtieth birthday

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Abstract

The space $\Omega(G)$ of all based loops in a compact simply connected Lie group G has an action of the maximal torus $T \subset G$ (by pointwise conjugation) and of the circle S^1 (by rotation of loops). Let $\mu\colon \Omega(G) \to (\mathfrak{t} \times i\mathbb{R})^*$ be a moment map of the resulting $T \times S^1$ action. We show that all levels (that is, pre-images of points) of μ are connected subspaces of $\Omega(G)$ (or empty). The result holds if in the definition of $\Omega(G)$ loops are of class C^∞ or of any Sobolev class H^s, with $s \geq 1$ (for loops of class H^1 connectedness of regular levels has been proved by Harada, Holm, Jeffrey, and the author in [3]).

1. Introduction

Let G be a compact simply connected Lie group and $T \subset G$ a maximal torus. The based loop group of G is the space $\Omega(G)$ consisting of all smooth maps $\gamma\colon S^1 \to G$ with $\gamma(1) = e$. The assignments

$$T \times \Omega(G) \to \Omega(G), \quad (t, \gamma) \mapsto [S^1 \ni z \mapsto t\gamma(z)t^{-1}]$$

and

$$S^1 \times \Omega(G) \to \Omega(G), \quad (e^{i\theta}, \gamma) \mapsto [S^1 \ni z \mapsto \gamma(ze^{i\theta})\gamma(e^{i\theta})^{-1}]$$

define an action of $T \times S^1$ on $\Omega(G)$. In fact, the latter space is an infinite dimensional smooth symplectic manifold and the action of $T \times S^1$ is Hamiltonian. Let

$$\mu\colon \Omega(G) \to (\mathfrak{t} \oplus i\mathbb{R})^*$$

denote a moment map, where $\mathfrak{t} := \mathrm{Lie}(T)$ and $i\mathbb{R} = \mathrm{Lie}(S^1)$. Atiyah and Pressley [1] extended the celebrated convexity theorem of Atiyah and Guillemin–Sternberg and showed that the image of μ is the convex hull of its singular values. Their proof's idea is to determine first the image under μ of the subspace $\Omega_{\mathrm{alg}}(G) \subset \Omega(G)$ whose

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elements are restrictions of algebraic maps from $\mathbb{C}^*$ to the complexification $G^{\mathbb{C}}$ of G: they notice that $\mu(\Omega_{\text{alg}}(G))$ is a closed subspace of $(\mathfrak{t} \oplus i\mathbb{R})^*$; since $\Omega_{\text{alg}}(G)$ is dense in $\Omega(G)$ (by a theorem of Segal), they deduce from the continuity of μ that

$$\mu(\Omega(G)) = \mu(\Omega_{\text{alg}}(G)).$$

The goal of this paper is to extend to $\Omega(G)$ the well-known result which says that all levels of the moment map arising from a Hamiltonian torus action on a compact symplectic manifold are connected. That is, we will prove the following theorem.

Theorem 1.1. *For any $a \in \mu(\Omega(G))$, the pre-image $\mu^{-1}(a)$ is a connected topological subspace of $\Omega(G)$.*

REMARKS. 1. A version of Theorem 1.1 has been proved in [3]. More specifically, instead of $\Omega(G)$ the authors consider there the space $\Omega_1(G)$ of all loops $S^1 \to G$ of Sobolev class H^1. They prove that all *regular* levels of $\mu\colon \Omega_1(G) \to (\mathfrak{t} \oplus i\mathbb{R})^*$ are connected. It is not obvious how to adapt that proof for singular levels or/and for loops of class C^∞.
2. One can easily see that our proof of Theorem 1.1 works for $\mu\colon \Omega_1(G) \to (\mathfrak{t} \oplus i\mathbb{R})^*$ (and even loops of Sobolev class H^s, with $s \geq 1$) as well. In other words, we can prove that *all* levels of $\mu\colon \Omega_1(G) \to (\mathfrak{t} \oplus i\mathbb{R})^*$ are connected topological subspaces of $\Omega_1(G)$. We decided to deal here with $\Omega(G)$ (smooth loops) rather than $\Omega_1(G)$ because the former is discussed in detail in our main reference [11], and the reader can make the connections directly.

We will give here an outline of the paper. In Section 2 we present basic notions and results concerning loop groups. In Section 3 we define the key ingredient of the proof of Theorem 1.1. This is a certain geometric invariant theory (shortly G.I.T.) quotient of $\Omega(G)$ with respect to the complexification $T^{\mathbb{C}} \times \mathbb{C}^*$ of $T \times S^1$. To define this quotient, we face the difficulty that the S^1 action on $\Omega(G)$ mentioned above does not extend to a $\mathbb{C}^*$ action (only the T action extends canonically to a $T^{\mathbb{C}}$ action). However, for any $\gamma \in \Omega(G)$ there is a natural way to define the loop $u\gamma$ for any $u \in \mathbb{C}$ which is contained in the exterior of a disk with center at 0 and radius smaller than 1 (which depends on γ); if $|u| = 1$ then $u\gamma$ is given by the S^1 action on $\Omega(G)$ defined above. By putting $\gamma \sim gu\gamma$, where u is as before and $g \in T^{\mathbb{C}}$ arbitrary, we obtain an equivalence relation $\sim$ on $\Omega(G)$. The G.I.T. quotient mentioned before is $A/\sim$, where A consists of all elements of $\Omega(G)$ which are equivalent to elements of $\mu^{-1}(a)$. The main result of Section 3 is Proposition 3.5, which says that the natural map $\mu^{-1}(a)/(T \times S^1) \to A/\sim$ is bijective (the idea of the proof belongs to Kirwan, see [5, Chapter 7]). In Section 4 we note that the image of $(\mu^{-1}(a) \cap \Omega_{\text{alg}}(G))/(T \times S^1)$ under the map above is $(A \cap \Omega_{\text{alg}}(G))/\sim$. The former space is connected (by a result of [3]) and we prove that the latter is dense in $A/\sim$ (see Proposition 4.2). Consequently,

$A/\sim$ is a connected topological subspace of $\Omega(G)/\sim$. We deduce that $\mu^{-1}(a)/T \times S^1$ is connected. Hence $\mu^{-1}(a)$ is connected as well.

2. Notions of loop groups

In this section we collect results about loop groups which will be needed later. The details can be found in Pressley and Segal [11] and/or Atiyah and Pressley [1].

Like in the introduction, G is a compact simply connected Lie group. We denote by $L(G)$ the space of all smooth maps $S^1 \to G$ (call them loops). The obvious multiplication makes it into a Lie group. By $\Omega(G)$ we denote the space of all loops which map $1 \in S^1$ to the unit e of G. It can be naturally identified with the homogeneous space $L(G)/G$. In fact, the presentation of $\Omega(G)$ which is most appropriate for our goals is

$$\Omega(G) = L(G^{\mathbb{C}})/L^+(G^{\mathbb{C}}). \tag{1}$$

Here $G^{\mathbb{C}}$ is the complexification of G and $L(G^{\mathbb{C}})$ the set of all (smooth) loops $\alpha\colon S^1 \to G^{\mathbb{C}}$; by $L^+(G^{\mathbb{C}})$ we denote the subgroup of $L(G^{\mathbb{C}})$ consisting of all α as above which extend holomorphically for $|\zeta| \leq 1$ (this notion is explained in detail at the beginning of the next section). Since $L(G^{\mathbb{C}})$ is a complex Lie group and $L^+(G^{\mathbb{C}})$ a complex Lie subgroup, equation (1) shows that the manifold $\Omega(G)$ has a complex structure. More precisely, the complex structure J_x at a point $x \in \Omega(G)$ is induced by the multiplication by i in the tangent space $T_\alpha L(G^{\mathbb{C}})$, where $\alpha \in L(G^{\mathbb{C}})$ is such that $x = \alpha L^+(G^{\mathbb{C}})$.

Let us embed G into some special unitary group $SU(N)$. We consider the Hilbert space $H := L^2(S^1, \mathbb{C}^N)$ and the corresponding "Grassmannian" $Gr(H)$. The latter consists of all closed vector subspaces of H which satisfy certain supplementary properties; it turns out that $Gr(H)$ can be equipped with a Kähler (Hilbert) manifold structure (the details can be found in [11, Chapter 7]). An important subspace of $Gr(H)$ is $Gr_0(H)$. For the goals of our paper it is sufficient to mention that $Gr_0(H)$ contains H_+, which is the closed vector subspace of H spanned by $S^1 \ni z \mapsto z^k v$, with $k \geq 0$ and $v \in \mathbb{C}^N$. Also, the connected component of H_+ in $Gr_0(H)$ consists of all vector subspaces W of H for which there exists $n \geq 0$ such that

$$z^n H_+ \subset W \subset z^{-n} H_+$$

and

$$\dim[(z^{-n} H_+)/W] = \dim[W/(z^n H_+)].$$

In other words, if $\mathcal{G}_n$ denotes the subspace of all W which satisfy the last two equations, then the connected component of H_+ in $Gr_0(H)$ is $\bigcup_{n\geq 0} \mathcal{G}_n$. It is important to note that via the map

$$\mathcal{G}_n \ni W \mapsto W/z^n H_+,$$

the space $\mathcal{G}_n$ can be identified with the Grassmannian $Gr_{nN}(\mathbb{C}^{2nN})$ of all vector subspaces of dimension nN in

$$\mathbb{C}^{2nN} = z^{-n}H_+/z^nH_+. \tag{2}$$

Also note that we have the chain of inclusions

$$\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots. \tag{3}$$

Less obvious is the fact that for any $n \geq 0$, the canonical symplectic structure on the Grassmannian $\mathcal{G}_n$ makes this space into a symplectic submanifold of $Gr(H)$. The role of the above construction is revealed by the following result.

Proposition 2.1. (a) *The map*

$$\Omega(G) \to Gr(H), \quad \gamma \mapsto \gamma H_+ \tag{4}$$

is an embedding, which induces on $\Omega(G)$ a structure of symplectic manifold. Together with the complex structure J defined above, this makes $\Omega(G)$ into a Kähler manifold.
(b) *The image of $\Omega_{\rm alg}(G)$ (see the introduction) under the embedding* (4) *is contained in $\bigcup_{n\geq 0}\mathcal{G}_n$.*

Based on point (b), we identify $\Omega_{\rm alg}(G)$ with a subspace of $\bigcup_{n\geq 0}\mathcal{G}_n$. The inclusions (3) induce the filtration

$$\Omega_{\rm alg}(G) = \bigcup_{n\geq 0}\Omega_n, \quad \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \cdots,$$

where

$$\Omega_n := \Omega_{\rm alg}(G) \cap \mathcal{G}_n.$$

The space Ω_n is a closed subvariety of the Grassmannian $\mathcal{G}_n$. We refer to the topology on $\Omega_{\rm alg}(G)$ induced by the filtration above as the *direct limit topology*. There is another natural topology on $\Omega_{\rm alg}(G)$, namely the subspace topology, induced by the inclusion $\Omega_{\rm alg}(G) \subset \Omega(G)$.

The following proposition can be proved with the same arguments as Proposition 2.1 of [3] (the result is also mentioned in [2, Section 2]).

Proposition 2.2. *The direct limit topology on $\Omega_{\rm alg}(G)$ is finer than the subspace topology.*

Let us consider again the $T \times S^1$ action on $\Omega(G)$ described at the beginning of the paper, and the corresponding moment map $\mu\colon \Omega(G) \to (\mathfrak{t} \oplus i\mathbb{R})^*$. This is uniquely determined up to an additive constant, which will be made more precise momentarily (a standard moment map is described explicitly in [1, Section 3], but we will not need that expression here). For the moment, we would like to deduce from Proposition 2.2 a result which will be useful later. Namely, let us take $a \in \mu(\Omega(G))$; by [3, Proposition 3.4], $\mu^{-1}(a) \cap \Omega_{\mathrm{alg}}(G)$ is a connected subspace of $\Omega_{\mathrm{alg}}(G)$ with respect to the direct limit topology. We deduce:

Proposition 2.3. *For any $a \in \mu(\Omega_{\mathrm{alg}}(G))$, the space $\mu^{-1}(a) \cap \Omega_{\mathrm{alg}}(G)$ is a connected topological subspace of $\Omega(G)$.*

There is also an action of $T \times S^1$ on each $\mathcal{G}_n$, $n \geq 0$, which can be described as follows. We fix a basis, say $b_1, \ldots, b_N$, of $\mathbb{C}^N$, and consider the induced basis $z^k b_j$, $-n \leq k \leq n-1$, $1 \leq j \leq N$, of $\mathbb{C}^{2nN}$ (see equation (2)). The action of T on $\mathcal{G}_n$ is induced by

$$t \,.\, (z^k b_j) := z^k (t b_j), \tag{5}$$

for any $t \in T$ and k, j as above; the action of S^1 is induced by

$$e^{i\theta} \,.\, (z^k b_j) := (e^{i\theta} z)^k b_j = z^k e^{ik\theta} b_j \tag{6}$$

for all $e^{i\theta} \in S^1$. This $T \times S^1$ action is the restriction of an obvious $T^{\mathbb{C}} \times \mathbb{C}^*$ action: namely, in equation (5) we take $t \in T^{\mathbb{C}}$ and in equation (6) we replace $e^{i\theta}$ by an arbitrary element of $\mathbb{C}^*$. The $T^{\mathbb{C}} \times \mathbb{C}^*$ action turns out to be linear with respect to the Plücker embedding of $\mathcal{G}_n$ (see [1, Section 4]). Thus, the $T \times S^1$ action is Hamiltonian. We pick

$$\mu_n \colon \mathcal{G}_n \to (\mathfrak{t} \oplus i\mathbb{R})^*$$

a moment map, which is again uniquely determined up to an additive constant. We can arrange the constants in such a way that if $m < n$ then

$$\mu_n|_{\mathcal{G}_m} = \mu_m.$$

The reason is that $\mathcal{G}_m$ is a $T \times S^1$-invariant symplectic submanifold of $\mathcal{G}_n$. We obtain the map $\tilde{\mu} \colon \bigcup_{n\geq 0} \mathcal{G}_n \to (\mathfrak{t} \oplus i\mathbb{R})^*$ such that $\tilde{\mu}|_{\mathcal{G}_n} = \mu_n$, for all $n \geq 0$. The map $\tilde{\mu}$ is uniquely determined up to an additive constant. The following proposition relates the moment maps μ and $\tilde{\mu}$.

Proposition 2.4. *We can choose μ and $\tilde{\mu}$ such that*

$$\mu|_{\Omega_{\mathrm{alg}}(G)} = \tilde{\mu}|_{\Omega_{\mathrm{alg}}(G)}.$$

Proof. The idea of the proof is that there exists a submanifold $Gr_\infty(H)$ of $Gr(H)$ acted on smoothly by $T \times S^1$ and such that

- $Gr_0(H) \subset Gr_\infty(H)$ and the inclusion is $T \times S^1$ equivariant
- there exists $\hat{\mu}\colon Gr_\infty(H) \to \mathfrak{t} \oplus i\mathbb{R}$ which is a moment map for the $T \times S^1$ action
- the image of $\Omega(G)$ under the inclusion (4) is contained in $Gr_\infty(H)$.

It is worth noticing that $Gr(H)$ does not admit a smooth action of $T \times S^1$; only its subspace $Gr_\infty(H)$ does (see [11, Section 7.6]). This is why we need to use the latter space in our proof.

We deduce that $\hat{\mu}|_{\Omega(G)}$ differs from μ by a constant; the same can be said about $\hat{\mu}|_{\mathcal{G}_n}$ and μ_n, for any $n \geq 0$. The result follows. □

Let us consider again the $T^{\mathbb{C}} \times \mathbb{C}^*$ action on $\mathcal{G}_n$ defined above. Any of the inclusions $\mathcal{G}_n \subset \mathcal{G}_{n+1}$ is equivariant. Thus, we have an action of $T^{\mathbb{C}} \times \mathbb{C}^*$ on $\bigcup_{n\geq 0} \mathcal{G}_n$. The same group acts on $\Omega_{\mathrm{alg}}(G)$, as follows. We take into account that

$$\Omega_{\mathrm{alg}}(G) = L_{\mathrm{alg}}(G^{\mathbb{C}})/L^+_{\mathrm{alg}}(G^{\mathbb{C}}) \tag{7}$$

where $L_{\mathrm{alg}}(G^{\mathbb{C}})$ is the space of all algebraic maps $\alpha\colon \mathbb{C}^* \to G^{\mathbb{C}}$ and $L^+_{\mathrm{alg}}(G^{\mathbb{C}})$ the subgroup consisting of those α which can be extended holomorphically to $\mathbb{C}$. Then the action we are referring to is

$$T^{\mathbb{C}} \times \mathbb{C}^* \times \Omega_{\mathrm{alg}}(G) \ni (g, u, \alpha L^+_{\mathrm{alg}}(G^{\mathbb{C}})) \mapsto [\mathbb{C}^* \ni \zeta \mapsto g\alpha(u\zeta)]L^+_{\mathrm{alg}}(G^{\mathbb{C}}). \tag{8}$$

The following result will be needed later.

Proposition 2.5. *The inclusion $\Omega_{\mathrm{alg}}(G) \subset \bigcup_{n\geq 0} \mathcal{G}_n$ defined in* Proposition 2.1 (b) *is $T^{\mathbb{C}} \times \mathbb{C}^*$ equivariant.*

Proof. Take $\gamma \in \Omega_{\mathrm{alg}}(G)$, which is of the form

$$S^1 \ni z \mapsto \gamma(z) = \sum_{-k_0\leq k\leq k_0} A_k z^k,$$

where $k_0 \geq 0$. Here A_k are $N \times N$ matrices with entries in $\mathbb{C}$. The subspace γH_+ of H has the property that

$$z^n H_+ \subset \gamma H_+ \subset z^{-n} H_+,$$

for some $n \geq 0$. Any element v of H_+ has a Fourier expansion of the form $v = \sum_{m\geq 0} v_m z^m$, where $v_m \in \mathbb{C}^N$, for all $m \geq 0$. Then

$$\gamma v = \sum_{m\geq 0, k\in\mathbb{Z}} (A_k v_m) z^{k+m}.$$

The corresponding element of $(\gamma H_+)/z^n H_+$ is

$$[\gamma v] = \gamma v \bmod z^n H_+ = \sum_{m\geq 0, -k_0\leq k\leq k_0, k+m\leq n-1} (A_k v_m) z^{k+m}.$$

In this sum we have $m \le n-k-1 \le n+k_0-1$. Thus, to describe all elements of $\gamma H_+/z^m H_+$, it is sufficient to take v of the form

$$v = \sum_{0\le m\le n+k_0-1} v_m z^m.$$

If $t \in T^{\mathbb{C}}$, then

$$t\,.\,[\gamma v] = \sum_{m\ge 0, -k_0\le k\le k_0, k+m\le n-1} t(A_k v_m) z^{k+m} = \sum_{m\ge 0, -k_0\le k\le k_0, k+m\le n-1} (tA_k) v_m z^{k+m}$$
$$= [(t\,.\,\gamma)v].$$

Consequently, $t\,.\,(\gamma H_+) = (t\,.\,\gamma)H_+$. If $u \in \mathbb{C}^*$, then

$$u\,.\,[\gamma v] = \sum_{m\ge 0, -k_0\le k\le k_0, k+m\le n-1} (A_k v_m)(uz)^{k+m} = \sum_{m\ge 0, -k_0\le k\le k_0, k+m\le n-1} (u^k A_k u^m v_m) z^{k+m}.$$

This is the same as $[(u\,.\,\gamma)\tilde{v}]$, where

$$\tilde{v} = \sum_{0\le m\le n+k_0-1} u^m v_m z^m.$$

Consequently, $u\,.\,(\gamma H_+) = (u\,.\,\gamma)H_+$. □

Finally, let us pick $B \subset G^{\mathbb{C}}$ a Borel subgroup with $T \subset B$. The presentation (7) of $\Omega_{\mathrm{alg}}(G)$ allows us to define on the latter space a natural action of the group

$$\mathcal{B}_+ := \{\alpha \in L^+_{\mathrm{alg}}(G^{\mathbb{C}}) \colon \alpha(0) \in B\}$$

on $\Omega_{\mathrm{alg}}(G)$. The orbit decomposition is

$$\Omega_{\mathrm{alg}}(G) = \bigcup_{\lambda \in \check{T}} C_\lambda,$$

where the union is disjoint and

$$C_\lambda := \mathcal{B}_+ \lambda$$

is called a *Bruhat cell*. Here $\check{T}$ denotes the lattice of group homomorphisms $S^1 \to T$. The space C_λ is really a (finite dimensional) cell, being homeomorphic to $\mathbb{C}^r$ for some r. In this paper, by $\overline{C_\lambda}$ we will always mean the closure of C_λ in the direct limit topology (see above). The following property of the Bruhat cells will be needed later.

Proposition 2.6. *For any $\lambda \in \check{T}$, there exists $n \ge 1$ such that $\overline{C_\lambda}$ is contained in $\mathcal{G}_n$ as a $T^{\mathbb{C}} \times \mathbb{C}^*$-invariant closed subvariety.*

This can be proved as follows. There exists $n \geq 1$ such that $C_\lambda \subset \mathcal{G}_n$, because $\mathcal{B}_+$ leaves each Ω_k, $k \geq 0$, invariant (see [6, Lemma 3.3.2]). The space C_λ is a locally Zariski closed subspace of $\Omega_{\rm alg}(G)$ (see [8, Proposition 2.13 and Theorem 3.1]), thus also of Ω_n and of $\mathcal{G}_n$. Consequently, the closures of C_λ in the Zariski, respectively differential topology of $\mathcal{G}_n$ are equal.

Another result concerning the Bruhat cells is the following proposition (cf. [1, Section 1], see also [3, Proof of Proposition 3.4]).

Proposition 2.7. *For any $\lambda_1, \lambda_2 \in \check{T}$ there exists $\lambda \in \check{T}$ such that*

$$C_{\lambda_1} \subset \overline{C_\lambda}$$

and

$$C_{\lambda_2} \subset \overline{C_\lambda}.$$

Consequently, for any $x, y \in \Omega_{\rm alg}(G)$ there exists $\lambda \in \check{T}$ such that both x and y are in $\overline{C_\lambda}$.

3. The equivalence relation $\sim$

We begin with the following definition. Take $0 < r \leq 1$. We say that a free loop $S^1 \to G^{\mathbb{C}}$ *extends holomorphically for* $|\zeta| \geq r$ if it is the restriction of a map

$$\alpha\colon \{\zeta \in \mathbb{C} \cup \infty\colon |\zeta| \geq r\} \to G^{\mathbb{C}}$$

which is continuous, holomorphic on $\{\zeta \in \mathbb{C} \cup \infty\colon |\zeta| > r\}$ and smooth on $\{\zeta \in \mathbb{C}\colon |\zeta| = r\}$; the same terminology is adopted if we take $r \geq 1$ and replace "$\geq$" and "$>$" by "$\leq$", respectively "$<$" (and also $\mathbb{C} \cup \infty$ by $\mathbb{C}$).

Let $L^-(G^{\mathbb{C}})$ denote the subspace of $L(G^{\mathbb{C}})$ consisting of those α which extend holomorphically for $|\zeta| \geq 1$ in the sense of the definition above. One knows that any $\alpha \in L(G^{\mathbb{C}})$ can be written as

$$\alpha = \alpha_- \lambda \alpha_+,$$

where $\alpha_- \in L^-(G^{\mathbb{C}})$, $\alpha_+ \in L^+(G^{\mathbb{C}})$, and λ is a group homomorphism $S^1 \to T$ (see [11, Theorem 8.1.2]). By using the presentation (1), the elements of $\Omega(G)$ are cosets of the form $\alpha_- \lambda L^+(G^{\mathbb{C}})$, where α_- and λ are as above. The following lemma will be used later.

Lemma 3.1. *Take α_-, β_- in $L^-(G^{\mathbb{C}})$ and $\lambda, \mu\colon S^1 \to T$ group homomorphisms such that*

$$\alpha_- \lambda L^+(G^{\mathbb{C}}) = \beta_- \mu L^+(G^{\mathbb{C}}).$$

Let r be a strictly positive real number.

(a) *Assume that* $r < 1$. *If* α_- *extends holomorphically for* $|\zeta| \geq r$ *then* β_- *extends holomorphically for* $|\zeta| \geq r$ *as well.*
(b) *Assume that* $r \geq 1$ *or* $r < 1$ *and* α_- *extends holomorphically for* $|\zeta| \geq r$. *For any* $u \in \mathbb{C}^*$ *with* $|u| \geq r$ *we have*

$$[S^1 \ni z \mapsto \alpha_-(uz)]\lambda L^+(G^{\mathbb{C}}) = [S^1 \ni z \mapsto \beta_-(uz)]\mu L^+(G^{\mathbb{C}}).$$

Proof. We have

$$\alpha_-\lambda = \beta_-\mu\alpha_+, \tag{9}$$

where $\alpha_+ \in L^+(G^{\mathbb{C}})$.

(a) The loops λ and μ are one-parameter subgroups in T, thus they have obvious (holomorphic) extensions to group homomorphisms $\mathbb{C}^* \to T^{\mathbb{C}}$. From (9) we deduce that β_- is the restriction of a function holomorphic on the annulus

$$\{\zeta \in \mathbb{C} : r < |\zeta| < 1\}$$

and continuous on the closure of this space. Consequently, the map $\xi \mapsto \beta_-(1/\xi)$ extends holomorphically for $|\xi| \leq 1/r$, that is, β_- extends holomorphically for $|\zeta| \geq r$. Indeed, let us consider again the embedding $G \subset SU(N)$, as in Section 2. The resulting embedding $G^{\mathbb{C}} \subset \mathrm{Mat}^{N\times N}(\mathbb{C})$ is holomorphic. We use the following claim:

Claim. *If* $f : \{\xi \in \mathbb{C} : |\xi| \leq 1/r\} \to \mathbb{C}$ *is a continuous function which is holomorphic on* $\{\xi \in \mathbb{C} : |\xi| < 1/r,\ |\xi| \neq 1\}$, *then* f *is holomorphic on* $\{\xi \in \mathbb{C} : |\xi| < 1/r\}$.

This can be proved by comparing the Laurent series of f on $\{\xi \in \mathbb{C} : |\xi| < 1\}$, respectively $\{\xi \in \mathbb{C} : 1 < |\xi| < 1/r\}$. The series are equal, since the coefficients of both of them are equal to $(1/(2\pi i)) \int_{|\xi|=1} f(\xi)/\xi^k d\xi$, $k \in \mathbb{Z}$ (by a uniform continuity argument). Thus, the radius of convergence of the first of the two series (which is actually the Taylor series of f around 0) is at least equal to $1/r$. The claim is proved.

(b) From equation (9) we deduce that α_+ extends holomorphically to $\mathbb{C}$. The reason is that the entries of the $N \times N$ matrix $\alpha_+ = \mu^{-1}\beta_-^{-1}\alpha_-\lambda$ are $\mathbb{C}$-valued functions which are continuous on $\mathbb{C}$ and holomorphic on $\mathbb{C} \setminus \{\xi \in \mathbb{C} : |\xi| = 1\}$; by the same argument as in the claim above, they are holomorphic on the whole $\mathbb{C}$. Again from equation (9), we deduce that

$$\alpha_-(uz)\lambda(uz) = \beta_-(uz)\mu(uz)\alpha_+(uz),$$

for all $z \in S^1$. The map $S^1 \ni z \mapsto \alpha_+(uz)$ is in $L^+(G^{\mathbb{C}})$. We only need to notice that

$$\lambda(uz) = \lambda(z)\lambda(u), \quad \mu(uz) = \mu(z)\mu(u). \qquad \square$$

DEFINITION 3.2. (a) Take $x \in \Omega(G)$ and $u \in \mathbb{C}^*$. We say that the pair (u, x) is *admissible* if

- $|u| \geq 1$

or

- $|u| < 1$ and $x = \alpha_-\lambda L^+(G^{\mathbb{C}})$, where $\alpha_- \in L^-(G^{\mathbb{C}})$ extends holomorphically for $|\zeta| \geq |u|$ and $\lambda\colon S^1 \to T$ is a group homomorphism.

If (u, x) is as above and $g \in T^{\mathbb{C}}$, then

$$gux := g[S^1 \ni z \mapsto \alpha_-(uz)]\lambda L^+(G^{\mathbb{C}})$$

is an element of $\Omega(G)$.
(b) Take $x, y \in \Omega(G)$. We say that

$$x \sim y$$

if there exist $u \in \mathbb{C}^*$ and $g \in T^{\mathbb{C}}$ such that (u, x) is an admissible pair and $y = gux$.

REMARK. We can also express gux as

$$gux := g[S^1 \ni z \mapsto (\alpha_-\lambda)(uz)]L^+(G^{\mathbb{C}}),$$

because λ is a group homomorphism $\mathbb{C}^* \to T^{\mathbb{C}}$.

Note that by Lemma 3.1, the definition of gux in part (a) is independent of the choice of the representative $\alpha_-\lambda$ of $x \in L(G^{\mathbb{C}})/L^+(G^{\mathbb{C}})$. The following lemma shows that $\sim$ is an equivalence relation.

Lemma 3.3. (a) *If $x \in \Omega(G)$, $u \in \mathbb{C}^*$ and $g \in T^{\mathbb{C}}$ such that (u, x) is admissible, then (u^{-1}, gux) is admissible and we have*

$$g^{-1}u^{-1}(gux) = x.$$

(b) *If $x \in \Omega(G)$, $u_1, u_2 \in \mathbb{C}^*$, and $g_1, g_2 \in T^{\mathbb{C}}$ such that (u_1, x) and (u_2, g_1u_1x) are admissible, then (u_1u_2, x) is admissible and*

$$(g_1g_2)(u_1u_2)x = g_2u_2(g_1u_1x).$$

Proof. (a) We can assume that $g = 1$. We write $x = \alpha_-\lambda L^+(G^{\mathbb{C}})$. Assume first that $|u| \geq 1$. The loop $S^1 \ni z \mapsto \alpha_-(uz)$ extends holomorphically for $|\zeta| \geq 1/|u|$ by $\zeta \mapsto \alpha_-(u\zeta)$. The case $|u| < 1$ is even easier to analyze. Verifying that $u^{-1}(ux) = x$ is equally easy.

(b) We can assume that $g_1 = g_2 = 1$. Again we write $x = \alpha_-\lambda L^+(G^{\mathbb{C}})$. It is sufficient to analyze the case when $|u_1u_2| < 1$. Thus, at least one of the numbers $|u_1|$ and $|u_2|$ is strictly less than 1. We distinguish the following two cases.

CASE 1. $|u_2| < 1$. The loop $S^1 \ni z \mapsto \alpha_-(u_1 z)$ is well defined and extends holomorphically for $|\xi| \geq |u_2|$. Let $\tilde{\alpha}: \{\xi \in \mathbb{C} \cup \infty : |\xi| \geq |u_2|\} \to G^{\mathbb{C}}$ be an extension of this loop. The map $\hat{\alpha}: \{\zeta \in \mathbb{C} \cup \infty : |\zeta| \geq |u_1 u_2|\} \to G^{\mathbb{C}}$ given by

$$\hat{\alpha}(\zeta) = \begin{cases} \tilde{\alpha}(u_1^{-1}\zeta), & \text{if} \quad |u_1 u_2| \leq |\zeta| \leq |u_1|, \\ \alpha_-(\zeta), & \text{if} \quad |u_1| \leq |\zeta| \end{cases}$$

is the desired extension of α_- for $|\zeta| \geq |u_1 u_2|$ (note that $\hat{\alpha}$ is holomorphic on $|\zeta| > |u_1 u_2|$, since it is continuous and is holomorphic on the complement of the circle $\{\zeta \in \mathbb{C} : |\zeta| = |u_1|\}$).

CASE 2. $|u_2| \geq 1$. This implies $|u_1| < 1$. We notice that the pair $(u_1, u_2 x)$ is admissible: indeed, by hypothesis, the loop $S^1 \ni z \mapsto \alpha_-(u_2 z)$ extends holomorphically for $|u_2 \zeta| \geq |u_1|$, hence also for $|\zeta| \geq |u_1|$. The pair (u_2, x) is admissible too. From the result proved in Case 1 we deduce that $(u_1 u_2, x)$ is admissible.

The equation $u_2(u_1 x) = (u_1 u_2) x$ is straightforward. □

The following result relates the equivalence relation $\sim$ to the $T \times S^1$ action on $\Omega(G)$ (see Section 1).

Lemma 3.4. *Take $\gamma \in \Omega(G)$. If $\theta \in \mathbb{R}$, then the pair $(e^{i\theta}, \gamma)$ is admissible. If $t \in T$, then the loop $te^{i\theta}\gamma$ given by* Definition 3.2 (b) *can be expressed as*

$$te^{i\theta}\gamma = t\gamma^{\theta}t^{-1}.$$

Here the right-hand side is given by

$$(t\gamma^{\theta}t^{-1})(z) = t\gamma(ze^{i\theta})\gamma(e^{i\theta})^{-1}t^{-1},$$

for all $z \in S^1$.

Proof. There exist $\alpha_- \in L^-(G^{\mathbb{C}})$ and $\lambda : S^1 \to T$ a group homorphism such that the image of γ under the isomorphism (1) is $\alpha_-\lambda L^+(G^{\mathbb{C}})$. This means that

$$\alpha_-\lambda = \gamma\alpha_+,$$

for some $\alpha_+ \in L^+(G^{\mathbb{C}})$. We deduce that for any $z \in S^1$ we have

$$[t\alpha_-(ze^{i\theta})\lambda(z)]\lambda(e^{i\theta}) = [t\gamma^{\theta}(z)t^{-1}]t\gamma(e^{i\theta})\alpha_+(ze^{i\theta}).$$

In other words, via the isomorphism (1), to $t\gamma^{\theta}t^{-1}$ corresponds the coset of

$$t[S^1 \ni z \mapsto \alpha_-(ze^{i\theta})]\lambda,$$

which is the same as $te^{i\theta}(\alpha_-\lambda L^+(G^{\mathbb{C}}))$. □

We now denote by A the set of all $x \in \Omega(G)$ with $x \sim y$, for some $y \in \mu^{-1}(a)$. We are interested in the quotient space $A/\sim$ and the (natural) map $\mu^{-1}(a)/(T \times S^1) \to A/\sim$ which assigns to the coset of $x \in \mu^{-1}(a)$ the equivalence class of x. By Lemma 3.4, this map is well defined.

Proposition 3.5. *The natural map*

$$\mu^{-1}(a)/(T \times S^1) \to A/\sim$$

is bijective.

Proof. Only the injectivity has to be proved. We have to show that if $x, y \in \mu^{-1}(a)$ with $x \sim y$, then $y = te^{i\theta}x$, where $(t, e^{i\theta}) \in T \times S^1$. By Definition 3.2 we have

$$y = gux$$

for some $u \in \mathbb{C}^*$ and $g \in T^{\mathbb{C}}$. We write $g = \exp(w_1)\exp(iw_2)$ and $u = e^{i\alpha}e^{-\beta}$, where $w_1, w_2 \in \mathfrak{t}$ and $\alpha, \beta \in \mathbb{R}$ (here we see $-\beta$ as $i(i\beta)$). Since the pair (u, x) is admissible and $|u| = |e^{-\beta}|$, the pair $(e^{-\beta}, x)$ is admissible too. By Lemma 3.3 (b) we have

$$y = \exp(w_1)e^{i\alpha}(\exp(iw_2)e^{-\beta}x).$$

Thus, it is sufficient to assume that

$$y = \exp(iw_2)e^{-\beta}x.$$

Moreover, without loss of generality we assume that

$$\beta \geq 0,$$

because if contrary we write $x = \exp(-iw_2)e^{\beta}y$ (by Lemma 3.3 (a)). Let us consider the function $h\colon [0, 1] \to \mathbb{R}$,

$$h(s) = [\mu(\exp(isw_2)e^{-s\beta}x) - a](w_2, i\beta),$$

where $0 \leq s \leq 1$. Notice that $h(s)$ is well defined for any s with $0 \leq s \leq 1$: indeed, the pair $(e^{-\beta}, x)$ is admissible hence, because $e^{-s\beta} \geq e^{-\beta}$, the pair $(e^{-s\beta}, x)$ is admissible too. Since $\mu(x) = \mu(y)$, we have $h(0) = h(1) = 0$. Consequently, there exists s_0 in the interval $(0, 1)$ such that $h'(s_0) = 0$. We denote

$$x_0 := \exp(is_0w_2)e^{-s_0\beta}x. \tag{10}$$

Claim. *We have*

$$\left.\frac{d}{ds}\right|_{s_0} \exp(isw_2)e^{-s\beta}x = J_{x_0}((w_2, i\beta) \,.\, x_0),$$

where J_{x_0} *is the complex structure at* x_0 (*see* Section 2) *and*

$$(w_2, i\beta) . x_0 := \frac{d}{ds}\bigg|_0 [\exp(sw_2)e^{is\beta}x_0]$$

arising from the infinitesimal action of $T \times S^1$ *on* $\Omega(G)$.

The claim can be proved as follows. Write $x_0 = \alpha_-\lambda L^+(G^{\mathbb{C}})$, where $\alpha_- \in L^-(G^{\mathbb{C}})$ and $\lambda\colon S^1 \to T$ is a group homomorphism. By using Lemma 3.4 and the remark following Definition 3.2, we have

$$\exp(isw_2)e^{-s\beta}x = \exp(i(s-s_0)w_2)e^{-(s-s_0)\beta}x_0 = \exp(i(s-s_0)w_2)(\alpha_-\lambda)_{-(s-s_0)}L^+(G^{\mathbb{C}}).$$

Here we have denoted

$$(\alpha_-\lambda)_{-(s-s_0)}(z) := (\alpha_-\lambda)(e^{-(s-s_0)\beta}z)$$

for all s in the interval $(0, 1)$ and all $z \in S^1$. By the definition of the complex structure J (see Section 2), it is sufficient to prove that

$$\frac{d}{ds}\bigg|_{s_0} [\exp(i(s-s_0)w_2)(\alpha_-\lambda)_{-(s-s_0)}] = i\frac{d}{ds}\bigg|_0 [\exp(sw_2)(\alpha_-\lambda)_{is}], \tag{11}$$

where

$$(\alpha_-\lambda)_{is}(z) := (\alpha_-\lambda)(e^{is\beta}z)$$

for all $s \in \mathbb{R}$ and all $z \in S^1$. By using the Leibniz rule, the left-hand side of (11) is

$$\frac{d}{ds}\bigg|_0 [\exp(isw_2)(\alpha_-\lambda)_{-s}] = i\frac{d}{ds}\bigg|_0 [\exp(sw_2)](\alpha_-\lambda) + i\frac{d}{ds}\bigg|_0 [(\alpha_-\lambda)_{is}].$$

Here we have used that

$$\frac{d}{ds}\bigg|_0 [\exp(isw_2)] = iw_2 = i\frac{d}{ds}\bigg|_0 [\exp(sw_2)]$$

and also that

$$\frac{d}{ds}\bigg|_0 (\alpha_-\lambda)(e^{-s\beta}z) = i\frac{d}{ds}\bigg|_0 (\alpha_-\lambda)(e^{is\beta}z),$$

for all $z \in S^1$ (the last equation follows from the fact that $\alpha_-\lambda$ is holomorphic on the exterior of a closed disk with center at 0 and radius strictly smaller than 1). The claim is proved.

From the claim we deduce as follows:

$$
\begin{aligned}
h'(s_0) &= (d\mu)_{x_0}\left(\frac{d}{ds}\bigg|_{s_0}(\exp(isw_2)e^{-s\beta}x)\right)(w_2, i\beta)\\
&= \omega_{x_0}\left(\frac{d}{ds}\bigg|_{s_0}(\exp(isw_2)e^{-s\beta}x), (w_2, i\beta) . x_0\right)\\
&= \omega_{x_0}(J_{x_0}((w_2, i\beta) . x_0), (w_2, i\beta) . x_0)\\
&= \langle (w_2, i\beta) . x_0, (w_2, i\beta) . x_0\rangle,
\end{aligned}
$$

where ω denotes the symplectic form and $\langle\ ,\ \rangle$ the Kähler metric on $\Omega(G)$ (see Proposition 2.1). We deduce that

$$(w_2, i\beta) . x_0 = 0$$

which, according to the claim above, implies that

$$\frac{d}{ds}\bigg|_0 \exp(isw_2)e^{-s\beta}x_0 = 0.$$

From this we deduce that

$$\exp(isw_2)e^{-s\beta}x_0 = x_0 \tag{12}$$

for all $s \le 0$ (note that for any such s, the pair $(e^{-s\beta}, x_0)$ is admissible, since $e^{-s\beta} \ge 1$). Indeed, by using Lemma 3.4 we deduce that for any $s_1 \le 0$ we have

$$
\begin{aligned}
\frac{d}{ds}\bigg|_{s_1} \exp(isw_2)e^{-s\beta}x_0 &= \frac{d}{ds}\bigg|_{s_1}(\exp(is_1w_2)e^{-s_1\beta})\exp(i(s-s_1)w_2)e^{-(s-s_1)\beta}x_0\\
&= d(\exp(is_1w_2)e^{-s_1\beta})_{x_0}\left(\frac{d}{ds}\bigg|_0(\exp(isw_2)e^{-s\beta}x_0)\right)\\
&= 0.
\end{aligned}
$$

Here we have used the (differential of the) map $\exp(is_1w_2)e^{-s_1\beta}\colon \Omega(G) \to \Omega(G)$ given by

$$\gamma \mapsto \exp(is_1w_2)e^{-s_1\beta}\gamma,$$

which is well defined, since $e^{-s_1\beta} \ge 1$.

By Lemma 3.3 (a), equation (12) implies that the pair $(e^{-s\beta}, x_0)$ is admissible for any $s \ge 0$; moreover, equation (12) holds for all $s \ge 0$ as well. We make $s = -s_0$ in (12) and deduce $x = x_0$; then we make $s = 1 - s_0$ and deduce $y = x_0$. We conclude

$$x = y$$

and the proof is finished. □

REMARK. Let M be a compact Kähler manifold acted on by a complex Lie group G, which is the complexification of a compact Lie group K, in such a way that the action of K on M is Hamiltonian. Kirwan has proved that if $x, y \in M$ have the same image under the moment map and are on the same G orbit, then they are on the same K-orbit (see [5, Lemma 7.2]). We have used above the idea of her proof. Kirwan's result cannot be used directly in our context: first, $\Omega(G)$ is not a compact manifold; second, and most importantly, the $T \times S^1$ action on $\Omega(G)$ does not extend in any reasonable way to a $T^{\mathbb{C}} \times \mathbb{C}^*$ action. We are substituting this action by the equivalence relation $\sim$.

4. Connectedness of $A/\sim$ and of $\mu^{-1}(a)$

We start with the following proposition.

Proposition 4.1. (a) *If $x \in \Omega_{\mathrm{alg}}(G)$ then the pair (u, x) is admissible (in the sense of* Definition 3.2) *for any $u \in \mathbb{C}^*$. The map*

$$T^{\mathbb{C}} \times \mathbb{C}^* \times \Omega_{\mathrm{alg}}(G) \to \Omega_{\mathrm{alg}}(G), \quad (g, u, x) \mapsto gux$$

is the action of $T^{\mathbb{C}} \times \mathbb{C}^*$ on $\Omega_{\mathrm{alg}}(G)$ defined in Section 2 (*see equation* (8)).
(b) *The image of $(\mu^{-1}(a) \cap \Omega_{\mathrm{alg}}(G))/(T \times S^1)$ under the map in* Proposition 3.5 *is $(A \cap \Omega_{\mathrm{alg}}(G))/\sim$. The latter space is a connected topological subspace of $\Omega(G)/\sim$.*

Proof. Point (a) follows from equations (7) and (8) and the remark following Definition 3.2. To prove the first assertion of (b), we only need to note that if $x \in \Omega_{\mathrm{alg}}(G)$ and $y \in \Omega(G)$ such that $x \sim y$, then $y \in \Omega_{\mathrm{alg}}(G)$. To prove the second assertion of (b), we note that the natural map

$$(\mu^{-1}(a) \cap \Omega_{\mathrm{alg}}(G))/(T \times S^1) \to \Omega(G)/\sim$$

is continuous. We use Proposition 2.3. □

The key result of this section is

Proposition 4.2. *The subspace $(A \cap \Omega_{\mathrm{alg}}(G))/\sim$ of $A/\sim$ is dense (both spaces have the topology of subspace of $\Omega(G)/\sim$).*

Combined with Proposition 4.1 (b), this implies

Corollary 4.3. *The space $A/\sim$ is a connected topological subspace of $\Omega(G)/\sim$.*

In turn, this implies the main result of the paper, as follows.

Proof of Theorem 1.1. The natural map

$$\mu^{-1}(a)/(T \times S^1) \to \Omega(G)/\sim \tag{13}$$

is continuous, one-to-one, and its image is $A/\sim$ (by Proposition 3.5). Since $A/\sim$ is connected (see the previous corollary), we deduce that $\mu^{-1}(a)/(T \times S^1)$ is connected as well. Consequently, $\mu^{-1}(a)$ is a connected topological subspace of $\Omega(G)$. □

The rest of the section is devoted to the proof of Proposition 4.2. First, if $\lambda \in \check{T}$, we say that a point $x \in \overline{C_\lambda}$ is $(\mu - a)$-*semistable* if

$$\overline{(T^{\mathbb{C}} \times \mathbb{C}^*)x} \cap (\mu^{-1}(a) \cap \overline{C_\lambda}) \neq \emptyset. \tag{14}$$

Here the closure is taken in $\Omega_{\mathrm{alg}}(G)$ with respect to the direct limit topology. We may assume that $\overline{C_\lambda}$ is contained in the Grassmannian $\mathcal{G}_n$ as a $T^{\mathbb{C}} \times \mathbb{C}^*$-invariant closed subvariety (see Proposition 2.6). Then x is $(\mu - a)$-semistable if and only if it is $(\mu_n - a)$-semistable in the usual sense, that is, if

$$\overline{(T^{\mathbb{C}} \times \mathbb{C}^*)x} \cap (\mu_n^{-1}(a) \cap \overline{C_\lambda}) \neq \emptyset$$

(see for instance [5, Chapter 7]). This follows immediately from the fact that μ and μ_n coincide on $\overline{C_\lambda}$, by Proposition 2.4. We denote by $\overline{C_\lambda}^{ss}$ the set of all semistable points in $\overline{C_\lambda}$. We also consider the set $\mathcal{G}_n^{ss}$ of all $(\mu_n - a)$-semistable points in $\mathcal{G}_n$. We have

$$\overline{C_\lambda}^{ss} = \overline{C_\lambda} \cap \mathcal{G}_n^{ss}. \tag{15}$$

The following description of the semistable set of $\overline{C_\lambda}$ will be needed later.

Lemma 4.4. *We have*

$$A \cap \overline{C_\lambda} = \overline{C_\lambda}^{ss}.$$

Proof. By Proposition 4.1 (b), we have

$$A \cap \Omega_{\mathrm{alg}}(G) = (T^{\mathbb{C}} \times \mathbb{C}^*)(\mu^{-1}(a) \cap \Omega_{\mathrm{alg}}(G)).$$

Consequently, a point $x \in \Omega(G)$ is in $A \cap \overline{C_\lambda}$ if and only if $x \in [(T^{\mathbb{C}} \times \mathbb{C}^*)\mu^{-1}(a)] \cap \overline{C_\lambda}$. The latter set is obviously equal to $(T^{\mathbb{C}} \times \mathbb{C}^*)(\mu^{-1}(a) \cap \overline{C_\lambda})$, which is the same as $\overline{C_\lambda}^{ss}$ (by [5, Theorems 7.4 and 8.10], applied for the Grassmannian $\mathcal{G}_n$ which contains $\overline{C_\lambda}$ as a $T^{\mathbb{C}} \times \mathbb{C}^*$-invariant closed subvariety, as indicated above). □

We are now ready to prove Proposition 4.2.

Proof of Proposition 4.2. We show that in any open subset V of $A/\sim$ there exists an element of $(A\cap\Omega_{\rm alg}(G))/\sim$. Since $A/\sim$ is equipped with the topology of subspace of $\Omega(G)/\sim$, we can write

$$V=(A/\sim)\cap(U/\sim)=(A\cap U)/\sim.$$

Here U is an open subspace of $\Omega(G)$ with the property that for any $x\in U$, we have

$$\{y\in\Omega(G)\colon y\sim x\}\subset U.$$

The subspace $U\cap\Omega_{\rm alg}(G)$ of $\Omega_{\rm alg}(G)$ is open in the direct limit topology (because the direct limit topology on $\Omega_{\rm alg}(G)$ is finer than the subspace topology, see Proposition 2.2) and non-empty (because $\Omega_{\rm alg}(G)$ is dense in $\Omega(G)$, see [11, Section 3.5]). For any $x\in U\cap\Omega_{\rm alg}(G)$ we have

$$(T^{\mathbb{C}}\times\mathbb{C}^*)x=\{y\in\Omega_{\rm alg}(G)\colon y\sim x\}\subset U\cap\Omega_{\rm alg}(G), \tag{16}$$

which follows from Proposition 4.1 (a). There exists $\lambda\in\check{T}$ such that $\overline{C_\lambda}\cap U\neq\emptyset$ and $\mu^{-1}(a)\cap\overline{C_\lambda}\neq\emptyset$. Indeed, we can pick $x\in\Omega_{\rm alg}(G)\cap U$ (the intersection is non-empty, see above) and $y\in\Omega_{\rm alg}(G)\cap\mu^{-1}(a)$ (the intersection is non-empty, since $a\in\mu(\Omega(G))=\mu(\Omega_{\rm alg}(G))$); by Proposition 2.7, there exists $\lambda\in\check{T}$ such that x and y are both in $\overline{C_\lambda}$.

Claim. *If $\lambda\in\check{T}$ is as above, then $\overline{C_\lambda}^{ss}$ is a dense subspace of $\overline{C_\lambda}$ (here $\overline{C_\lambda}$ is equipped with the direct limit topology it inherits from $\Omega_{\rm alg}(G)$).*

To prove the claim, we consider again a Grassmannian $\mathcal{G}_n$ which contains $\overline{C_\lambda}$ as a $T^{\mathbb{C}}\times\mathbb{C}^*$-invariant closed subvariety. By the main theorem of [4], there exists on $\mathcal{G}_n$ a $T^{\mathbb{C}}\times\mathbb{C}^*$-invariant very ample line bundle L such that $\mathcal{G}_n^{ss}=\mathcal{G}_n^{ss}(L)$. The latter space consists of all L-semistable points in $\mathcal{G}_n$, that is points $x\in\mathcal{G}_n$ such that there exists $k\geq 1$ and $s\colon X\to L^{\otimes k}$ equivariant holomorphic section with $s(x)\neq 0$ (cf. e.g. [10]). Consequently, $\mathcal{G}_n^{ss}$ is a Zariski open subspace of $\mathcal{G}_n$. Since $\overline{C_\lambda}^{ss}=\mathcal{G}_n^{ss}\cap\overline{C_\lambda}$, we deduce that $\overline{C_\lambda}^{ss}$ is a Zariski open subspace of $\overline{C_\lambda}$. The space $\overline{C_\lambda}^{ss}$ is non-empty, since $\mu^{-1}(a)\cap\overline{C_\lambda}\subset\overline{C_\lambda}^{ss}$. Thus $\overline{C_\lambda}^{ss}$ is dense in $\overline{C_\lambda}$ with respect to the usual differential topology on the latter space: this can be deduced by using [9, Theorem 2.33] for $\overline{C_\lambda}$, which is an irreducible projective variety (cf. [8, p. 360]).

From the claim we deduce that the intersection $\overline{C_\lambda}^{ss}\cap U$ is non-empty (since $\overline{C_\lambda}\cap U$ is a non-empty subspace of $\overline{C_\lambda}$ which is open with respect to the direct limit topology). By Lemma 4.4 we have

$$\overline{C_\lambda}^{ss}\cap U=A\cap\overline{C_\lambda}\cap U,$$

thus

$$U\cap A\cap\Omega_{\rm alg}(G)\neq\emptyset.$$

By equation (16), the quotient $(U \cap A \cap \Omega_{\mathrm{alg}}(G))/\sim$ is a (non-empty) subspace of $\Omega(G)/\sim$. It is contained in both $V = (U \cap A)/\sim$ and $(A \cap \Omega_{\mathrm{alg}}(G))/\sim$. Consequently, the intersection $V \cap [(A \cap \Omega_{\mathrm{alg}}(G))/\sim]$ is non-empty. This finishes the proof. □

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