

Title	A note on Γ -G-spaces
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Citation	Osaka Journal of Mathematics. 28(2) p.223-p.228
Issue Date	1991
oaire:version	VoR
URL	https://doi.org/10.18910/12634
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A NOTE ON Γ_G -SPACES

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(Received April 24, 1990)

Introduction. In [3], I introduced the notion of a special Γ_G -space and showed that every special Γ_G -space A functorially determines a G -spectrum $S_G A$ such that the associated infinite loop G -space $\Omega^\infty S_G A$ is an equivariant group completion of the G -space $A(1)$. On the other hand, Hauschild, May and Waner [1] established an equivariant infinite loop space machine based on the notion of a (special) Γ - G -space. The purpose of this note is to show that these two notions are canonically equivalent, although their definitions appear to be rather different.

1. For a finite group G , let Γ_G denote the category of based finite G -sets and based maps. We endow Γ_G with the standard G -action

$$(g, f) \mapsto {}^g f = gfg^{-1} \quad \text{for } g \in G, f \in \text{mor } \Gamma_G.$$

For the trivial group $G=e$, Γ_e is equivalent to the opposite of Segal's Γ [2], and so we denote $\Gamma = \Gamma_e$. Then Γ can be regarded as the full subcategory of Γ_G consisting of trivial G -sets. In fact, we have $\Gamma \subset (\Gamma_G)^G$ because every based map between trivial G -sets is automatically G -equivariant. As usual we denote by \mathbf{n} the trivial G -set $\{0, 1, \dots, n\} \in \Gamma$.

DEFINITION. A Γ - G -space is a functor from Γ to the category of based G -spaces and based G -maps. A Γ_G -space is a G -equivariant functor from Γ_G to the category of based G -sets and based maps equipped with the standard G -action.

Let us denote by

Γ_G -**Space** (resp. Γ - **G -Space**)

the category of Γ_G -spaces (resp. Γ - G -spaces) with G -equivariant natural transformations (resp. natural transformations) as morphisms. We will show that there is an adjoint equivalence between Γ_G -**Space** and Γ - **G -Space**.

If A is a Γ_G -space then its restriction to Γ becomes a Γ -space because every morphism $f: \mathbf{m} \rightarrow \mathbf{n}$ in Γ induces a G -map $A(f): A(\mathbf{m}) \rightarrow A(\mathbf{n})$. Thus we have a functor

$$R: \Gamma_G\text{-Space} \rightarrow \Gamma\text{-}G\text{-Space}$$

induced by the inclusion $\Gamma \subset \Gamma_G$.

On the other hand, there is a functor

$$E: \Gamma\text{-}G\text{-Space} \rightarrow \Gamma_G\text{-Space}$$

which takes each Γ - G -space X to the functor

$$EX: S \mapsto S \otimes_{\Gamma} X = \coprod_{\mathbf{n}} \text{Map}_0(\mathbf{n}, S) \times X(\mathbf{n}) / \sim.$$

Here we identify $(sf, x) \in \text{Map}_0(\mathbf{m}, S) \times X(\mathbf{m})$ with $(s, X(f)(x)) \in \text{Map}_0(\mathbf{n}, S) \times X(\mathbf{n})$ for every $f: \mathbf{m} \rightarrow \mathbf{n}$ in Γ , and define a G -action on $S \otimes_{\Gamma} X$ by

$$g[s, x] = [{}^g s, gx] = [gs, gx] \quad \text{for } g \in G, [s, x] \in S \otimes_{\Gamma} X.$$

(The second equality follows from the fact that the objects of Γ are trivial G -sets.)

Then for any $g \in G$ and $f: S \rightarrow T$ in Γ_G we have $EX({}^g f) = {}^g EX(f)$ because

$$\begin{aligned} EX({}^g f)[s, x] &= [{}^g fs, x] \\ &= [gfg^{-1}s, x] \\ &= g[fg^{-1}s, g^{-1}x] \\ &= gEX(f)[g^{-1}s, g^{-1}x] \\ &= gEX(f)(g^{-1}[s, x]) \\ &= {}^g EX(f)[s, x]. \end{aligned}$$

Thus EX becomes a Γ_G -space.

It is evident that the G -homeomorphisms

$$X(\mathbf{n}) \rightarrow \mathbf{n} \otimes_{\Gamma} X = REX(\mathbf{n}), \quad x \mapsto [\text{id}_{\mathbf{n}}, x]$$

define a natural isomorphism $\text{Id} \rightarrow RE$. On the other hand, there is a natural isomorphism $ER \rightarrow \text{Id}$ given by the G -homeomorphisms

$$ERA(S) = S \otimes_{\Gamma} RA \rightarrow A(S)$$

which takes the class of $(s, a) \in \text{Map}_0(\mathbf{n}, S) \times A(\mathbf{n})$ to $A(s)(a) \in A(S)$. This shows that E is a left adjoint of R . Hence

Theorem 1. *The restriction $R: \Gamma_G\text{-Space} \rightarrow \Gamma\text{-}G\text{-Space}$ is an equivalence of categories.*

Theorem 1 implies that for a Γ_G -space A , the G -space $A(S)$ ($S \in \Gamma_G$) can be reconstructed from those $A(\mathbf{n})$ ($\mathbf{n} \in \Gamma$). To see this, let us choose a bijection $f: S \rightarrow \mathbf{n}$ with $S \in \Gamma_G$ and $\mathbf{n} \in \Gamma$. Then f determines a homomorphism $\rho: G \rightarrow \text{Map}_0(\mathbf{n}, \mathbf{n}) = \Sigma_{\mathbf{n}}$ such that

$$\begin{array}{ccc}
 S & \xrightarrow{f} & \mathbf{n} \\
 g \downarrow & & \downarrow \rho(g) \\
 S & \xrightarrow{f} & \mathbf{n}
 \end{array}$$

commutes for every $g \in G$. Let $A(\mathbf{n})_g$ denote the based G -space having the underlying space $A(\mathbf{n})$ and equipped with the G -action

$$(g, a) \mapsto A(\rho(g))(ga) \quad \text{for } g \in G, a \in A(\mathbf{n}).$$

(This formula in fact gives a G -action because $A(\rho(g)): A(\mathbf{n}) \rightarrow A(\mathbf{n})$ are G -maps.) Then we have

Proposition 2. $A(f): A(S) \rightarrow A(\mathbf{n})_g$ is a G -homeomorphism.

Proof. First note that ${}^g f = fg^{-1}$ holds for any $g \in G$ because \mathbf{n} has the trivial G -action. Now, for every $g \in G$ and $a \in A(\mathbf{n})$ we have

$$\begin{aligned}
 A(f)(ga) &= A(\rho(g)fg^{-1})(ga) \\
 &= A(\rho(g))A({}^g f)(ga) \\
 &= A(\rho(g)){}^g A(f)(ga) \\
 &= A(\rho(g))(gA(f)(g^{-1}ga)) \\
 &= A(\rho(g))(gA(f)(a)).
 \end{aligned}$$

This shows that $A(f): A(S) \rightarrow A(\mathbf{n})_g$ is a G -map. Since $A(f)$ has the inverse $A(f^{-1})$, we conclude that $A(f)$ is a G -homeomorphism.

2. Proposition 2 enables us to restate the definition of a special Γ_G -space in terms of the associated Γ - G -space, and so, to compare with the definition of a special Γ - G -space given by Hauschild, May and Waner [1].

First recall the definition of a special Γ_G -space. Let A be a Γ_G -space such that for all $S \in \Gamma_G$, $A(S)$ has the G -homotopy type of a based G -CW complex. For each based G -set $S \in \Gamma_G$ let us consider the based map

$$P_s: A(S) \rightarrow \text{Map}_0(S, A(\mathbf{1})) = A(\mathbf{1})^{S-}, \quad a \mapsto \{A(p_s)(a)\}$$

where $S_- = S - \{\text{point}\}$ and for every $s \in S_-$, p_s denotes the based map $|S \rightarrow \mathbf{1} = \{0, 1\}$ such that $p_s^{-1}(1) = s$. Then it is easily observed that P_s becomes a G -map, although each p_s is not necessarily G -equivariant.

DEFINITION. A is called a special Γ_G -space if

(1) for every $S \in \Gamma_G$ the based G -map $P_s: A(S) \rightarrow A(\mathbf{1})^{S-}$ is a G -homotopy equivalence.

Notice that if we take $S = \text{point}$ then the condition (1) says that $A(\text{point})$ is

G -contractible. (Thus the condition (a) of [3, Definition 1.3] can be regarded as a special case of the condition (b).)

For every homomorphism $\rho: G \rightarrow \Sigma_n$ let us denote $A(\mathbf{1})_\rho^n = \text{Map}_0(\mathbf{n}_\rho, A(\mathbf{1}))$; that is, the n -fold product $A(\mathbf{1})^n$ equipped with G -action

$$(g, \{a_j\}) \mapsto \{ga_{\rho(g^{-1})(j)}\} \quad \text{for } g \in G, \{a_j\} \in A(\mathbf{1})^n.$$

Then, by Proposition 2, we have

Proposition 3. *Let A be a Γ_G -space such that $A(S)$ has the G -homotopy type of a based G -CW complex for every $S \in \Gamma_G$. Then A is special if and only if (2) for every $n \geq 0$ and every homomorphism $\rho: G \rightarrow \Sigma_n$ the based G -map $P_n: A(\mathbf{n})_\rho \rightarrow A(\mathbf{1})_\rho^n$ is a G -homotopy equivalence.*

We now turn to the definition of a special Γ - G -space [1]. Let X be a Γ - G -space. For each n , we endow $X(\mathbf{n})$ with the $G \times \Sigma_n$ -action

$$((g, \sigma), x) \mapsto X(\sigma)(gx) \quad \text{for } (g, \sigma) \in G \times \Sigma_n, x \in X(\mathbf{n}).$$

Then the canonical map $P_n: X(\mathbf{n}) \rightarrow X(\mathbf{1})^n$ can be regarded as a $G \times \Sigma_n$ -map.

DEFINITION. X is called a *special Γ - G -space* if

(3) for each n , P_n induces an ordinary weak homotopy equivalence on passage to K -fixed points for those subgroups K of $G \times \Sigma_n$ whose intersection with Σ_n is the trivial group; that is, $K = \{(h, \rho(h)) \mid h \in H\}$ for some subgroup H of G and homomorphism $\rho: H \rightarrow \Sigma_n$.

In other words, X is a special Γ - G -space if and only if $P_n: X(\mathbf{n})_\rho \rightarrow X(\mathbf{1})_\rho^n$ is a weak H -equivalence for every subgroup H and every homomorphism $\rho: H \rightarrow \Sigma_n$. Thus (3) implies, in particular,

(4) for every n and every homomorphism $\rho: G \rightarrow \Sigma_n$, $P_n: X(\mathbf{n})_\rho \rightarrow X(\mathbf{1})_\rho^n$ is a weak G -equivalence;

or equivalently,

(5) for every $S \in \Gamma_G$, $P_S: EX(S) \rightarrow EX(\mathbf{1})^S$ is a weak G -equivalence.

Conversely we can prove that (3) follows from the weaker condition (4) in the following way. By Proposition 2 again, it suffices to show that if X satisfies (5) then for every based finite H -set U , $P_U: EX(U) = U \otimes_{\mathbb{R}} X \rightarrow EX(\mathbf{1})^U$ is a weak H -equivalence. Let $S \in \Gamma_G$ be a based G -set which contains U as an H -invariant subset (e.g., $S = G_+ \wedge_H U$). Then S can be written as the union $S = U \vee V$ of based H -sets U and $V = S - U$, and we have a commutative diagram of based H -spaces

$$\begin{array}{ccc} EX(S) & \xrightarrow{(EX(p), EX(q))} & EX(U) \times EX(V) \\ P_S \downarrow & & \downarrow P_U \times P_V \\ EX(\mathbf{1})^S & \xlongequal{\hspace{2cm}} & EX(\mathbf{1})^U \times EX(\mathbf{1})^V \end{array}$$

where p and q denote the projections $U \vee V \rightarrow U$ and $U \vee V \rightarrow V$ respectively.

We will show that $(EX(p), EX(q))$ is a weak H -equivalence; that is

$$(EX(p), EX(q))_*: \pi_*^K EX(S) \rightarrow \pi_*^K EX(U) \oplus \pi_*^K EX(V)$$

is an isomorphism for every subgroup K of H . Since P_S is a weak G -equivalence, this implies that $P_U \times P_V$ is a weak H -equivalence, and hence P_U becomes a weak H -equivalence for any U .

Let i and j be the inclusions $U \rightarrow U \vee V$ and $V \rightarrow U \vee V$ respectively, and let us consider the commutative diagram

$$\begin{array}{ccc} EX(U \vee V) & \xrightarrow{(EX(p), EX(q))} & EX(U) \times EX(V) \\ EX(i \vee j) \downarrow & & \downarrow EX(i) \times EX(j) \\ EX(S \vee S) & \xrightarrow{(EX(pr_1), EX(pr_2))} & EX(S) \times EX(S) \\ EX(p \vee q) \downarrow & & \downarrow EX(p) \times EX(q) \\ EX(U \vee V) & \xrightarrow{(EX(p), EX(q))} & EX(U) \times EX(V) \end{array}$$

Then $(EX(pr_1), EX(pr_2))$ is a weak G -equivalence by the assumption, and $EX(i \vee j)$ (resp. $EX(i) \times EX(j)$) is a section of $EX(p \vee q)$ (resp. $EX(p) \times EX(q)$). It is now easy to see that the composite

$$EX(p \vee q)_*(EX(pr_1), EX(pr_2))_*^{-1}(EX(i)_* \oplus EX(j)_*)$$

gives the inverse of $(EX(p), EX(q))_*$. This proves that (4) implies (3).

Especially, we have

Corollary. *Let X be a Γ - G -space such that $X(\mathbf{n})_\rho$ has the G -homotopy type of a based G -CW complex for every $\mathbf{n} \in \Gamma$ and $\rho: G \rightarrow \Sigma_{\mathbf{n}}$. Then X is a special Γ - G -space in the sense of Hauschild, May and Waner [1] if and only if X is the restriction of some special Γ_G -space.*

In view of the equivalence $\Gamma_G\text{-Space} \approx \Gamma\text{-}G\text{-Space}$, this corollary says that the notion of special Γ_G -space is essentially the same with the notion of special Γ - G -Space. (The only difference lies in the fact that we impose the restriction that special Γ_G -spaces have values in the G -spaces having the G -homotopy types of based G -CW complexes.)

References

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