Title	Some decomposition properties of injective and pure-injective modules
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Citation	Osaka Journal of Mathematics. 1994, 31(1), p. 95-108
Version Type	VoR
URL	https://doi.org/10.18910/12636
rights	
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Osaka J. Math.
31 (1994), 95–108

SOME DECOMPOSITION PROPERTIES OF INJECTIVE AND PURE-INJECTIVE MODULES

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(Received August 28, 1992)

It is well known that over a left Notherian ring any direct sum of injecive modules is again injective, and every injective module is a direct sum of indecomposable modules. Faith [6] introduced Σ-injective modules as modules M such that all direct sums of copies of M are injective. Cailleau [2] showed that a Σ-injective module is a direct sum of indecomposable modules. The concept of Σ-injective modules had several intersting developments and applications (see e.g. Faith [7]). Also, some generalizations of Σ-injective modules, such as Σ-quasi-injective modules (Cailleau-Renault [3]), or Σ-M-injecitve modules (Albu-Nastasescu [1]), were studied. Of special interest are Σ-pure-injective modules which were introduced and investigated extensively by W. Zimmermann and B. Zimmermann-Huisgen [22, 23, 24, 25]. These modules include, besides Σ-injective modules, also Π-projective modules (i.e., modules M such that all direct products of copies of M are projective).

Harada [12] studies Σ-injectivity in the context of Grothendieck categories and used it to characterize QF-categories. In this paper we continue the study of Σ-injectivity in the general categorical setting. One of our motivations comes from the fact that, by using the functor ring techniques of Gruson and Jensen [10, 11], serveral decomposition properties of (Σ-) pure-injective modules (over a ring with identity) can be obtained rather easily from the corresponding properties of injective obejcts in a Grothendieck category.

In Section 1 we will work in a Grothendieck category $\mathcal{A}$ with a family of finitely generated generators $\{G_\alpha/\alpha\in\Omega\}$. Our purpose is to study the basic properties of Σ-injective objects, but sometimes we require just some weakened forms of the injectivity. An object $M\in\mathcal{A}$ is called CS [4] (or extending [16, 17]) if every subobject of M is essential in a direct summand. Extending Okado [16], we show that if $M\in\mathcal{A}$ is a CS object such that for each $\alpha\in\Omega$, G_α has ACC on the subobjects $\{\mathrm{Ker} f/f\in\mathrm{Hom}(G_\alpha, M)\}$, then M is a direct sum of indecomposable objects (Proposition 1.5). Consequently, any CS subobject of a Σ-injective object in $\mathcal{A}$ is a direct sum of indecomposable objects (Corollary 1.6). Further, generalizing a result of Lawrence [14] on self-injective rings, we show that if A and M are objects of $\mathcal{A}$ and M is A-injective, and $\aleph$ is an infinite cardianl such that $|\mathrm{Hom}(A, M)| \leq \aleph$, then any well-ordered ascending chain of the subobjects of A of the form $\bigcap_{f\in K}$ kerf with $K \subseteq \mathrm{Hom}(A, M)$ has cardinality $<\aleph$. In particular, if M is injective in $\mathcal{A}$ and $|\mathrm{Hom}(G_\alpha, M)| \leq \aleph_0$ for each $\alpha \in \Omega$, then M is Σ-injective (Corollary 1.9). Next, we give a criterion of Σ-injectivity, inspired by Faith-Walker [8], that an injective object $M \in \mathcal{A}$ is Σ-injective if and only if there exists an infinite cardinal $\boldsymbol{m}$ such that the injective envelope of any direct sum of copies of M is a direct sum of $\boldsymbol{m}$-generated objects (Theorem 1.12). Finally, an application of this criterion is given for Σ-CS objects.

In Section 2, we are concerned with Σ-pure-injective modules over rings with identity. By using the functor ring techniques of Gruson and Jensen [10, 11] (cf. Wisbauer [21]) and the results of Section 1, we are able to recover (and extend) several known properties of Σ-pure-injective modules which were obtained in [22, 23, 24] by different methods.

1. Injective and CS objects in Grothendieck catgeories.

In this section, we will always assume that $\mathcal{A}$ is a locally finitely generated Grothendieck category, with a family of finitely generated generators $\{G_\alpha/\alpha \in \Omega\}$. For the basic definitions and properties concerning Grothendieck categories we refer to [19].

An injective object M in $\mathcal{A}$ is called Σ-injective if all direct sums of copies of M are injective. Let A and M be any objects in $\mathcal{A}$, then a subobject B of A is called an M-annihilator if $B = \bigcap_{f\in K} \mathrm{Ker} f$ for some subset K of $\mathrm{Hom}(A, M)$. M is called A-injective if every morphism from a subobject of A to M can be extended to a morphism from A to M. The injective envelope of an object M is denoted by $\mathrm{E}(M)$.

The following result was proved by Harada [12, Theorem 1] and generalizes the characterization of Σ-injective modules of Faith [6].

Lemma 1.1. *Let M be an injective object in $\mathcal{A}$; then the following conditions are equivalent:*

a) M is Σ-injective;
b) $M^{(N)}$ is injective;
c) For each $\alpha \in \Omega$, G_α has ACC *on M-annihilator subobjects.*

A short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ will be called a pure sequence when the induced morphism $\mathrm{Hom}(F, B) \to \mathrm{Hom}(F, C)$ is an epimorphism for every finitely presented object F of $\mathcal{A}$. In this case A is called a pure subobject of B. An object $M \in \mathcal{A}$ is called FP-injective if it has the injectivity property with respect to any short exact sequence $0 \to A \to B \to C \to 0$ with C finitely presented. We observe the following simple fact.

Lemma 1.2. *Let N be a subobject of an FP-injective object M in $\mathcal{A}$. Then N is pure in M if and only if N is* FP-*injective.*

Proof. See e.g. Wisbauer [21, 35.1].

For the next two result we require that $\mathcal{A}$ be a locally finitely presented Grothendieck category. In particular, the following proposition extends the equivalence (a)$\Leftrightarrow$(c) of [21, 28.7], while its proof follows partially the ideas in [21, 28.4].

Proposition 1.3. *Suppose that $\mathcal{A}$ has a family of finitely presented generators $\{G_\alpha/\alpha\in\Omega\}$ and that M is an* FP-*injective object of $\mathcal{A}$. Then M is Σ-injective if and only if for each $\alpha\in\Omega$, G_α has ACC on M-annihilator subobjects.*

Proof. The necessity follows from Lemma 1.1. For the converse, by Lemma 1.1 it is enough to show that M is injective, or equivalently, M is G_α-injective for each $\alpha\in\Omega$ (see [12, Lemma 1]).

Let $\alpha\in\Omega$. We want to show that any morphism f: $A\to M$ with $A\subseteq G_\alpha$ can be extended to g: $G_\alpha\to M$. First, the same argument of [21, 28.3 (i)] allows us to deduce that there exists a finitely generated subobject $Z\subseteq A$ such that for any h: $G_\alpha\to M$, we have that $Z\subseteq \operatorname{Ker} h$ implies $A\subseteq \operatorname{Ker} h$. Let us denote by $\boldsymbol{u}$: $Z\to A$ and ω: $A\to G_\alpha$ the canonical inclusions, and consider the morphism $f\circ\boldsymbol{u}$: $Z\to M$. Since G_α/Z is finitely presented and M is FP-injective, we have that there is g: $G_\alpha\to M$ such that $g\circ\omega\circ\boldsymbol{u}=f\circ\boldsymbol{u}$, that is, $\operatorname{Ker}(g\circ\omega-f)\supseteq Z$. We are going to show that $g\circ\omega=f$, so that g extends f.

Suppose on the contrary that $g\circ\omega\neq f$. Then there is some finitely generated subobject $Y\subseteq A$ such that $(g\circ\omega-f)(Y)\neq 0$. Clearly we may assume that $Z\subseteq Y$. Now, since Y is finitely generated and M is FP-injective, there is some h: $G_\alpha\to M$ such that $Y\subseteq\operatorname{Ker}(h\circ\omega-f)$. If we now consider the morphism $g-h$: $G_\alpha\to M$, we see that $Z\subseteq\operatorname{Ker}(g-h)$. By the above remark it follows that $A\subseteq\operatorname{Ker}(g-h)$, that is, $g\circ\omega=h\circ\omega$. But then $Y\subseteq\operatorname{Ker}(g\circ\omega-f)$, a contradiction which completes the proof.

Corollary 1.4. *Suppose that $\mathcal{A}$ has a family of finitely presented generators $\{G_\alpha/\alpha\in\Omega\}$, and let N be a pure subobject of a Σ-injective object M in $\mathcal{A}$. Then N is Σ-injective.*

Proof. By Lemma 1.2, N is FP-injective. By Lemma 1.1, for each $\alpha\in\Omega$, G_α has *ACC* on M-annihilator subobjects. But each N-annihilator subobject of G_α is also an M-annihilator, thus G_α has ACC on N-annihilator subobjects. It follows by Proposition 1.3 that N is Σ-injective.

An object $M\in\mathcal{A}$ will be called CS if every subobject of M is essential in a direct summand. This is the terminology of [4], while modules with this property are also called extending modules by many authors (e.g. [13, 16, 17]).

A family $\{X_i/i\in I\}$ of subobjects of an object $M\in\mathcal{A}$ is called a local direct summand of M if $\Sigma_{i\in I} X_i$ is direct and $\Sigma_{i\in F} X_i$ is a direct summand of M for any finite subset $F\subseteq I$. In the case that $\Sigma_{I\in I} X_i$ is a direct summand of M, we say that the local direct summand is a direct summand.

The next result extends a theorem of Okado [16], and our proof uses a modified version of his arguments.

Proposition 1.5. *Let M be a CS object in $\mathcal{A}$, and assume that for each $\alpha\in\Omega$, G_α has ACC on the subobjects $\{\mathrm{Ker}\, f/f\in\mathrm{Hom}(G_\alpha, M)\}$. Then every local direct summand of M is a direct summand, and M is a direct sum of indecomposable objects.*

Proof. Let $X=\bigoplus_{i\in I} X_i$ be a local direct summand of M. Since direct summands of CS objects are also CS, we may assume without loss of generality that X is essential in M.

Suppose that $X\neq M$. Let ε: $M\to M/X$ be the canonical projection, then there is some $\alpha\in\Omega$ and a morphism g: $G_\alpha\to M$ such that $\varepsilon\circ g\neq 0$. By the hypothesis, the set of kernels of the morphisms $\{g\colon G_\alpha\to M/\varepsilon\circ g\neq 0\}$ has a maximal element, say $\mathrm{Ker}\, f$.

Take any $X'=\bigoplus_{i\in F} X_i$ with F finite. We claim that $X'\cap\mathrm{Im}\, f=0$. Let p: $M\to M/X'$ and π: $M/X'\to M/X$ be the canonical projections. It is clear that $\pi\circ p=\varepsilon$ and p splits, say $p\circ\omega=1$. Hence $\varepsilon\circ\omega\circ p=\pi\circ p\circ\omega\circ p=\pi\circ p=\varepsilon$, and if we put h: $G_\alpha\to M$ as $h=\omega\circ p\circ f$, then $\mathrm{Ker}\, f\subseteq\mathrm{Ker}\, h$ and $\varepsilon\circ h=\varepsilon\circ\omega\circ p\circ f=\varepsilon\circ f\neq 0$, so that $\mathrm{Ker}\, f=\mathrm{Ker}\, h$. But $\mathrm{Ker}\, h=f^{-1}(\mathrm{Ker}(\omega\circ p))=f^{-1}(X')$, so $f^{-1}(X')=\mathrm{Ker}\, f$. Thus clearly $X'\cap\mathrm{Im}\, f=0$.

But X is the sum of all the summands of the form $\bigoplus_{i\in F} X_i$ with F finite. Therefore we have $X\cap\mathrm{Im}\, f=0$ which contradicts the fact that X is essential in M. Thus we have $X=M$.

Now we use the just proved fact to show that M is a direct sum of indecomposable objects. First we show that every nonzero direct summand D of M contains a nonzero indecomposable direct summand. Without loss of generality we may assume that $D=M$. Let X be any nonzero finitely generated subobject of M. By Zorn's lemma, there exists a local direct summand $\{A_j/j\in J\}$ maximal with respect to the property that $\bigoplus_{j\in J} A_j$ does not contain X. Put $A=\bigoplus_{j\in J} A_j$, then $M=A\oplus N$ for some subobject N. Suppose that N is not indecomposable, then $N=N_1\oplus N_2$, N_1 and N_2 are nonzero. Now the maximality of $\{A_j/j\in J\}$ implies that $X\subseteq N_1\oplus A$ and $X\subseteq N_2\oplus A$, hence $X\subseteq A$, a contradiction. Now, again by Zorn's lemma, there exists a maximal local direct summand consisting of indecomposable objects $\{U_\lambda/\lambda\in\Lambda\}$, and clearly $M=\bigoplus_{\lambda\in\Lambda} U_\lambda$.

Our following corollary extends Cailleau [2].

Corollary 1.6. *Let M be a Σ-injective object in $\mathcal{A}$ and N a* CS *subobject of M. Then N is a direct sum of indecomposable objects. In particular, M is a direct sum of indecomposable objects.*

Proof. By Lemma 1.1, G_α has ACC on M-annihilator subobjects, for each $\alpha \in \Omega$. Thus each G_α has ACC on the subobjects $\{\operatorname{Ker} f / f \in \operatorname{Hom}(G_\alpha, N)\}$. Then apply Proposition 1.5.

For the next corollary we need to recall some definitions. An object $A \in \mathcal{A}$ is said to have finite rank if there are no infinite independent families of subobjects of A (see e.g. [19, p. 126]). An object $M \in \mathcal{A}$ is called nonsingular if for any object $N \in \mathcal{A}$ and any nonzero morphism $f: N \to M$, $\operatorname{Ker} f$ is not essential in N. M is called singular if there exists an epimorphism $g: N \to M$ with $\operatorname{Ker} g$ essential in N.

Corollary 1.7. *Let M be a nonsingular CS object in $\mathcal{A}$ and suppose that G_α has finite rank for each $\alpha \in \Omega$. Then M is a direct sum of indecomposable objects.*

Proof. Consider any morphism $f: G_\alpha \to M$, and we claim that $\operatorname{Ker} f$ is essentially closed in G_α, i.e. $\operatorname{Ker} f$ has no proper essential extensions in G_α. Suppose it is not so, then $\operatorname{Ker} f$ is essential in a subobject A of G_α with $A \neq \operatorname{Ker} f$. Then there exists G_λ, $\lambda \in \Omega$, and a morphism $g: G_\lambda \to A$ such that $f(\operatorname{Im} g) \neq 0$. Let e be the inclusion of A in G_α, and put $h = f \circ e \circ g$, $h: G_\lambda \to M$. Then clearly $h \neq 0$ and $\operatorname{Ker} h$ is essential in G_λ, which contradicts the nonsingularity of M. Hence $\operatorname{Ker} f$ is essentially closed in G_α.

Suppose that there exists an infinite strictly ascending chain

$$\operatorname{Ker} f_1 \subset \operatorname{Ker} f_2 \subset \cdots \subset \operatorname{Ker} f_n \subset \cdots$$

where $f_i: G_\alpha \to M$. Then for each n, there is a nonzero subobject C_n of $\operatorname{Ker} f_{n+1}$ such that $C_n \cap \operatorname{Ker} f_n = 0$. It is clear that the family $\{C_n / n = 1, 2, \cdots\}$ is independent which is a contradiction because G_α has finite rank. Now it follows by Proposition 1.5 that M is a direct sum of indecomposable objects.

Our next theorem generalizes a result of Lawrence [14] about the length of well-ordered ascending chains of left annihilators in a left self-injective ring. The proof below essentially follows Lawrence's idea, but our arguments allow to drop the regularity hypothesis of the cardinal which was in fact needed for Lawrence's proof of [14, Theorem 4] (see [15, Remark, p.9]).

If X is a set, then $|X|$ denotes the cardinality of X.

Theorem 1.8. *Let A and M be any objects in $\mathcal{A}$ such that M is A-injective, and let $\aleph$ be an infinite cardinal such that $|\operatorname{Hom}(A, M)| \leq \aleph$. Then any well-*

ordered ascending chain of M-annihilator subobjects of A has cardinality less than $\aleph$.

Proof. We may assume without loss of generality that $\mathrm{Hom}(A, M)=\{\varphi_\delta/\delta\in\Lambda\}$, where Λ is an infinite ordinal and $|\Lambda|=\aleph$. Suppose now that there exists a well-ordered strictly ascending chain $\{L_\delta/\delta\in\Lambda\}$ of M-annihilator subobjects L_δ of A.

Let $X_\delta=\{f\in\mathrm{Hom}(A, M)/L_\delta\subseteq\mathrm{Ker}f\}$. Then for each δ, $L_\delta=\bigcap_{f\in X_\delta}\mathrm{Ker}f$, and clearly $\{X_\delta/\delta\in\Lambda\}$ forms a strictly descending chain of subgroups of $\mathrm{Hom}(A, M)$. Now we claim that for each ordinal $\delta\in\Lambda$, there exists a morphism $f_\delta\in\mathrm{Hom}(A, M)$ satisfying the following properties

(1) If $\beta<\delta$, then $L_{\beta+1}\subseteq\mathrm{Ker}(f_\delta-f_\beta)$;

(2) $L_{\delta+1}$ is not contained in $\mathrm{Ker}(f_\delta-\varphi_\delta)$.

We shall prove our claim by transfinite induction. For $\delta=0$, we know that X_0 cannot be included in φ_0+X_1, and hence we may choose $f_0\in X_0$ such that $f_0-\varphi_0\notin X_1$, thus f_0 satisfies (1) and (2). Now suppose that we have proved our claim for all ordinals less than δ, for some $\delta\in\Lambda$. Define $K=\bigcup_{\beta<\delta}L_{\beta+1}\subseteq L_\delta$. For each $\beta<\delta$, let us denote by $\bar{f}_\beta$ the restriction of f_β to $L_{\beta+1}$. Since the L_β's from a direct system and $\bar{f}_\beta$: $L_{\beta+1}\to M$ is a system of compatible morphisms, there exists a morphism f': $K\to M$ which extends all the $\bar{f}_\beta$ with $\beta<\delta$ (note that if δ is a successor ordinal, say $\delta=\gamma+1$, then $K=L_\delta$ and f' coincides with f_γ on L_δ). Now, by the A-injectivity of M, f' can be extended to a morphism $\bar{f}$: $A\to M$. Since $X_\delta/X_{\delta+1}\neq0$, we have that X_δ is not included in $(\varphi_\delta-\bar{f})+X_{\delta+1}$, so there exists $g\in X_\delta$ such that $g\notin(\varphi_\delta-\bar{f})+X_{\delta+1}$, hence $g+\bar{f}-\varphi_\delta\notin X_{\delta+1}$. If we put $f_\delta=g+\bar{f}$, then $(f_\delta-\varphi_\delta)(L_{\delta+1})\neq0$ which shows condition (2) of the claim. Since $g(L_\delta)=0, f_\delta$ coincides with $\bar{f}$ on L_δ and therefore with f_β on $L_{\beta+1}$ for any $\beta<\delta$ (in particular, if $\delta=\gamma+1, f_\delta$ coincides with f_γ on L_δ). Thus f_δ satisfies condition (1), which proves our claim.

Let $B=\bigcup_{\delta\in\Lambda}L_\delta$. As above, we can define a morphism h': $B\to M$ such that h' coincides with f_β on $L_{\beta+1}$ for every $\beta\in\Lambda$. Since M is A-injective, h' can be extended to a morphism h: $A\to M$. But we have $h=\varphi_\delta$ for some $\delta\in\Lambda$. Since h coincides with f_δ on $L_{\delta+1}$, it follows that $(f_\delta-\varphi_\delta)(L_{\delta+1})=0$ which is a contradiction. This completes our proof.

Corollary 1.9. *Let M be an injective object in $\mathcal{A}$, and suppose that $|\mathrm{Hom}(G_\beta, M)|\leq\aleph_0$ for each $\alpha\in\Omega$. Then M is Σ-injective.*

Proof. It follows by Theorem 1.8 that G_α has ACC on M-annihilator subobjects for each $\alpha\in\Omega$. Thus M is Σ-injective by Lemma 1.1.

Recall that an object $M\in\mathcal{A}$ is called a cogenerator of $\mathcal{A}$ if every object A of $\mathcal{A}$ can be embedded into a product of copies of M. It is easy to see that if M and K are objects of $\mathcal{A}$ and L is a subobject of K, then K/L is embedded into a product of copies of M iff L is an M-annihilator subobject of K.

Corollary 1.10. *Suppose that $\mathcal{A}$ has an injective cogenerator M such that $|\mathrm{Hom}(G_\alpha, M)| \leq \aleph_0$ for each $\alpha \in \Omega$. Then $\mathcal{A}$ is locally Noetheran.*

Proof. By theorem 1.8, G_α has ACC on M-annihilator subobjects. Since M is a cogenerator, each subobject of G_α is an M-annihilator, so G_α is Noetherian for each $\alpha \in \Omega$. Thus $\mathcal{A}$ is locally Noetherian.

Now we are going to prove a criterion of Σ-injectivity which is essentially motivated by Faith-Walker [8]. We first give a categorical definition of $\boldsymbol{m}$-generated objects, for an infinite cardinal $\boldsymbol{m}$. Let M be an object of $\mathcal{A}$ and $\boldsymbol{m}$ an infinite cardinal, then we say that M is $\boldsymbol{m}$-generated if there exists an exact sequence $\oplus_{i\in I} A_i \to M \to 0$, where each A_i is finitely generated and $|I| \leq \boldsymbol{m}$. Osofsky [18] defined the concept of $\boldsymbol{m}$-generated objects for any Abelian category with exact direct limits, and it is easy to ckeck that her definition coincides with the above definition in the case of locally finitely generated Grothendieck categories.

We observe some elementary properties of $\boldsymbol{m}$-generated objects.

Lemma 1.11. *Let $\boldsymbol{m}$ be an infinite cardinal, then the following hold:*

1) *If M is $\boldsymbol{m}$-generated in $\mathcal{A}$ and $M \subseteq \oplus_{i\in I} N_i$, then there exists $I' \subseteq I$ with $|I'| \leq \boldsymbol{m}$ such that $M \subseteq \oplus_{i\in I'} N_i$.*

2) *There exists a cardinal c (depending on $\boldsymbol{m}$) such that for any $\boldsymbol{m}$-generated object M of $\mathcal{A}$, the set of all the subobjects of M has cardinality $\leq c$.*

Proof. 1) Straighforward.

2) Let $G = \oplus_{\alpha\in\Omega} G_\alpha$, then G is a generator of $\mathcal{A}$. If M is any $\boldsymbol{m}$-generated object of $\mathcal{A}$, then there is an exact sequence $G^{(\boldsymbol{m})} \to M \to 0$. Let c be the cardinality of the set of all the subobjects of $G^{(\boldsymbol{m})}$, then it is clear that the set of all the subobjects of M has cardinality $\leq c$.

We are now able to prove the following result.

Theorem 1.12. *Let M be an injective object in $\mathcal{A}$. Then M is Σ-injective if and only if there exists an infinite cardinal $\boldsymbol{m}$ such that the injective envelope of any direct sum of copies of M is a direct sum of $\boldsymbol{m}$-generated objects.*

Proof. Suppose first that M is Σ-injective. By Corollary 1.6, M and hence all direct sums of copies of M are direct sums of indecomposable injective objects. But an indecomposable injective object L is the injective envelope of any nonzero finitely generated subobject of L. Thus the class of all indecomposable injective objects of $\mathcal{A}$ is a set. Therefore there exists a cardinal $\boldsymbol{m}$ such that each indecomposable injective object in $\mathcal{A}$ is $\boldsymbol{m}$-generated which proves the necessity.

For the converse, by Lemma 1.1 it is enough to show that $M^{(N)}$ is injective.

Without loss of generality, we may assume that M is $\boldsymbol{m}$-generated. By Lemma 1.11, there is a cardinal c such that the set of all the subobjects of any $\boldsymbol{m}$-generated object in $\mathcal{A}$ has cardinality $\leq c$. Let $M_i \simeq M$ for all $i \in I$, $|I| = 2^c > c$, and let $E = \mathrm{E}(\oplus_{i \in I} M_i)$. By the hypothesis, we have $E = \oplus_{j \in J} E_j$, where each E_j is $\boldsymbol{m}$-generated. Take any $i_1 \in I$ and consider M_{i_1}. By Lemma 1.11, there exists $J_1 \subset J$ with $|J_1| \leq \boldsymbol{m}$ such that $M_{i_1} \subseteq \oplus_{j \in J_1} E_j$. Then clearly $\oplus_{j \in J_1} E_j$ is $\boldsymbol{m}$-generated, so $\oplus_{j \in J_1} E_j$ has $\leq c$ subobjects. Suppose that $M_i \cap (\oplus_{j \in J_1} E_j) \neq 0$ for all $i \in I$, then $\oplus_{j \in J_1} E_j$ must contain 2^c independent subobjects, a contradiction. Thus there is $i_2 \in I$ such that $M_{i_2} \cap (\oplus_{j \in J_1} E_j) = 0$. Then M_{i_2} is isomorphic to a subobject M'_{i_2} of $\oplus_{j \in J \setminus J_1} E_j$. Repeating the above argument, we have $M'_{i_2} \subseteq \oplus_{j \in J_2} E_j$ with $J_2 \subseteq J \setminus J_1$ and $|J_2| \leq \boldsymbol{m}$. Similarly, there exists $i_3 \in I$ with $M_{i_3} \cap (\oplus_{j \in J_1 \cup J_2} E_j) = 0$.

By induction, we get an infinite sequence $J_1, J_2, \cdots, J_n, \cdots$ of subsets of J such that $J_m \cap J_n = 0$ for $m \neq n$, and $\oplus_{j \in J_n} E_j$ contains an isomorphic copy of M. Thus $M^{(N)}$ is isomorphic to a direct summand of $\oplus_{j \in K} E_j$, where $K = J_1 \cup J_2 \cup \cdots$, hence $M^{(N)}$ is injective, so M is Σ-injective.

We proceed now with an application of Theorem 1.12. Since CS objects generalize injective objects, it is natural to introduce the following concept.

DEFINITION. An object M of $\mathcal{A}$ is called Σ-CS if all direct sums of copies of M are CS.

There are some reasons which motivate our interest towards the Σ-CS concept. Goodearl [9] studied the left nonsingular rings whose nonsingular left modules are projective, and it is easily seen that these are precisely the left nonsingular rings R such that ${}_R R$ is Σ-CS (in fact these are Artinian serial hereditary rings). Arbitrary Σ-CS rings were studied by Oshiro (e.g. [17]), and they form an interesting class of Artinian rings (they are also called Harada rings). However, very little seems to be known about the structure of Σ-CS objects (or Σ-CS modules) in the general case. The next result relates Σ-CS objects to Σ-injectivity in some particular cases.

Proposition 1.13. *Let M be a Σ-CS object in $\mathcal{A}$. Suppose that M satisfies one of the following conditions:*

a) M is nonsingular and M generates the injective envelope of any direct sum of copies of M.

b) M is projective and M generates the injective envelope of any direct sum of copies of M.

Then $\mathrm{E}(M)$ is Σ-injective and M is a direct sum of indecomposable objects.

Proof. a) We observe that the class of all the nonsingular objects of $\mathcal{A}$ is closed under taking essential extensions and direct sums. Suppose that M is $\boldsymbol{m}$-generated, for some infinite cardinal $\boldsymbol{m}$. Let N be the injective envelope of a direct sum of copies of $\mathrm{E}(M)$, then N is nonsingular, Also, it is clear that N is the injective envelope of a direct sum of copies of M, so N is generated by M by [21, 17.9(2)]. Thus there is a short exact sequence

$$0 \to A \to \oplus_{i\in I} M_i \to N \to 0$$

where $M_i \simeq M$ for all $i \in I$. But $\oplus_{i\in I} M_i$ is CS, so A is essential in a direct summand D of $\oplus_{i\in I} M_i$. Thus D/A is isomorphic to a subobject of N which implies that $D/A=0$, i.e. $D=A$. Thus N is isomorphic to a direct summand of $\oplus_{i\in I} M_i$. By the generalized Kaplansky's theorem, N is a direct sum of $\boldsymbol{m}$-generated objects. Thus by Theorem 1.12, $\mathrm{E}(M)$ is Σ-injective. By Corollary 1.6, M is a direct sum of indecomposable objects.

b) Similarly as in the proof of (a), it is enough to show that if L is a direct sum of copies of M, and $N=\mathrm{E}(L)$, then N is a direct summand of a direct sum of copies of M. Again by [21, 17.9(2)], there is a short exact sequence

$$0 \to A \to \oplus_{i\in I} M_i \to N \to 0$$

with $M_i \simeq M$ for all $i \in I$. Then $N=N_1 \oplus B$, where N_1 is a direct summand of $\oplus_{i\in I} M_i$ and B is singular. Let $\pi: N_1 \oplus B \to N_1$ be the canonical projection. Let $\bar{\pi}$ be the restriction of π to L. Since L is CS, $\operatorname{Ker} \bar{\pi}$ is essential in a direct summand of L, hence $L=L_1 \oplus C$ such that $\operatorname{Ker} \bar{\pi}$ is essential in C. Then $\pi(L)=\pi(L_1)\oplus\pi(C)$, and $\pi(C)$ is singular. It follows that $C \subseteq \pi(C)\oplus B$. But C is projective and $\pi(C)\oplus B$ is singular, so clearly $C=0$, hence $\operatorname{Ker} \bar{\pi}=0$. But L is essential in N, so it implies that $\operatorname{Ker} \pi=0$, i.e. $B=0$ which proves (b).

We end this section by the following open question which arises naturally in view of Corollary 1.6 and Proposition 1.13.

QUESTION. Let M be any Σ-CS object of $\mathcal{A}$. Is M a direct sum of indecomposable objects?

2. Σ-pure-injective modules.

Throughout this section, unless otherwise stated, we consider associative rings with identity and unitary left modules.

For a ring R, pure exact sequences are defined as in Section 1. A module M is called pure-injective if for any module A and any pure submodule B of A, any homomorphism $f: B \to M$ extends to a homomorphism $g: A \to M$. If N is a pure submodule of M, then M is a pure-essential extension of N if there are no nonzero submodules $X \subseteq M$ with $X \cap N=0$ and $(N+X)/X$ pure in M/X. A module M is a pure-injective envelope of N if N is a pure submodule of M, N is pure-essential in M and M is pure-injective. By [20, Proposition 6], pure-injective envelopes exist and are unique up to isomorphism. We denotes the pure-injective envelope of M by $\mathrm{PE}(M)$.

A module M is called Σ-pure-injective if all direct sums of copies of M are pure-injective. Σ-pure-injective modules were studied by W. Zimmermann and B. Zimmermann-Huisgen ([22, 23, 24, 25]), by using the equivalence of pure-injectivity and algebraic compactness (see [20, Theorem 2]) and matrix subgroup techniques. In this section, by applying Gruson and Jensen [10, 11] and the results in Section 1, we are able to recover and extend several decomposition properties of Σ-pure-injective modules which were obtained in the above-mentioned works.

Let us recall briefly the functor ring techniques of Gruson and Jensen [10, 11]. Let R be a ring and $\{U_\lambda/\lambda\in\Lambda\}$ a set containing one isomorphic copy of each finitely presented right R-module. Set $U_R=\oplus_{\lambda\in\Lambda}U_\lambda$ and define

$S=\{f\colon U_R\to U_R/f(U_\lambda)=0$ for all but a finite number of $\lambda\in\Lambda\}$.

Then S is a ring with enough idempotents, i.e. there is a set of orthogonal idempotents $\{e_i/i\in I\}$ in S such that $S=\oplus_{i\in I}Se_i=\oplus_{i\in I}e_iS$.

Let S-Mod be the category of all the unitary left S-modules (note that ${}_SM$ is unitary iff for every $x\in M$ there exists $e=e^2\in S$ such that $x=ex$). Then ${}_SS$ is a projective generator of S-Mod, and clearly S-Mod is a locally finitely presented Grothendieck category.

Since U is a End(U_R)-R-bimodule, if ${}_RM$ is any left R-module then $U\otimes_R M$ is a left End(U_R)-module and hence a unitary S-module. Thus we have a functor

$$U\otimes_R -\colon R\text{-Mod}\to S\text{-Mod}$$

which has a right adjoint

$$\mathrm{Hom}_S(U,-)\colon S\text{-Mod}\to R\text{-Mod}.$$

We collect in the following lemma some basic properties of these functors (for more details see e.g. Wisbauer [21]).

Lemma 2.1. *Let R be a ring. With the above notations, the following statements hold:*

a) For any ${}_RM\in R$-Mod, ${}_RM$ is pure-injective if and only if $U\otimes_R M$ is injective in S-Mod.

b) $U\otimes_R$- and $\mathrm{Hom}_S(U,-)$ preserve direct sums.

c) $U\otimes_R$- preserves pure short exact sequences.

d) If ${}_RM\in R$-Mod, then ${}_RM$ is indecomposable if and only if $U\otimes_R M$ is indecomposable in S-Mod.

e) Let ${}_RM$ be $\mathbf{m}$-generated for some cardinal $\mathbf{m}\geq\aleph_0$. Then $U\otimes_R M$ is $\mathbf{m}$-generated in S-Mod.

f) For any ${}_RM\in R$-Mod, $U\otimes_R \mathrm{PE}(M)\cong\mathrm{E}(U\otimes_R M)$.

Proof. a) and d) are [21, 52.3 (6) and 52.3 (7)], while c) follows also from

[21, 52.3 (3)].

b) is clear, because ${}_SU$ is finitely generated and projective; in fact, $U \cong Se$ for an idempotent $e \in S$.

e) Clearly, if ${}_RM$ is $\boldsymbol{m}$-generated, there exists an epimorphism of $\boldsymbol{S}$-Mod $U^{(m)} \to U \otimes_R M$. Since ${}_SU$ is finitely generated, $U \otimes_R M$ is $\boldsymbol{m}$-generated.

f) follows from [5, p. 385].

Corollary 2.2. (W. Zimmermann [22]). *Let R be a ring and M a left R-module. Then M is Σ-pure-injective if and only if $M^{(N)}$ is pure-injective. In this case, M is a direct sum of indecomposable modules and every pure submodule of M is also Σ-pure-injective.*

Proof. By Lemma 2.1, M is Σ-pure-injective iff $U \otimes_R M$ is Σ-injective in $\boldsymbol{S}$-Mod. Thus, the first assertion follows from Lemma 1.1. That M is a direct sum of indecomposable modules follows by Corollary 1.6 and Lemma 2.1. Now, suppose that M is Σ-pure-injective and N is a pure submodule of M. By Lemma 2.1, $U \otimes_R N$ is a pure subobject of $U \otimes_R M$, and $U \otimes_R M$ is Σ-injective. It follows by Corollary 1.4 that $U \otimes_R N$ is Σ-injective in $\boldsymbol{S}$-Mod, hence N is Σ-pure-injective.

Corollary 2.3. (W. Zimmermann [23, Proposition 3]). *Let R be any ring and M a countable pure-injective left R-module. Then M is Σ-pure-injective.*

Proof. As above, let $\{U_\lambda / \lambda \in \Delta\}$ be a set of representatives of the isomorphism classes of finitely presented right R-modules, $U = \oplus_{\lambda \in \Lambda} U_\lambda$, and let e_λ denote the canonical projection of U onto U_λ, $e_\lambda \in S$. Then $\{Se_\lambda / \lambda \in \Lambda\}$ is a family of finitely presented generators of the category $\boldsymbol{S}$-Mod. Now, we have for each $\lambda \in \Lambda$ an epimorphism of Mod-R $R^n \to U_\lambda$, and an epimorphism of abelian groups $M^n \to U_\lambda \otimes_R M$. Since M is countable, we deduce that $|U_\lambda \otimes_R M| \leq \aleph_0$. Consider now, for any $\alpha \in \Lambda$, the set $A_\alpha := \mathrm{Hom}_S(Se_\alpha, U \otimes_R M)$. We have isomorphisms of abelian groups $A_\alpha \cong e_\beta(U \otimes_R M) \cong \oplus_{\lambda \in \Lambda} e_\alpha(U_\lambda \otimes_R M)$. But for $\lambda \neq \alpha$, e_α annihilates U_λ and so we have $A_\alpha \cong e_\alpha(U_\alpha \otimes_R M)$, from which it follows that $|A_\alpha| \leq |U_\alpha \otimes_R M| \leq \aleph_0$. Therefore, each A_α is at most countable, and by Corollary 1.9, $U \otimes_R M$ is Σ-injective in $\boldsymbol{S}$-Mod. By Lemma 2.1, ${}_RM$ is Σ-pure-injective.

Theorem 2.4. *Let R be a ring and M a pure-injective left R-module. Then M is Σ-pure-injective if and only if there exists an infinite cardinal $\boldsymbol{m}$ such that the pure-injective envelope of any direct sum of copies of M is a direct sum of $\boldsymbol{m}$-generated modules.*

Proof. Suppose first that M is Σ-pure-injective. By Corollary 2.2, M and hence all direct sums of copies of M are direct sums of indecomposable pure-injective modules. Let L be any indecomposable pure-injective left R-module.

By Lemma 2.1, $U\otimes_R L$ is indecomposable injective in S-Mod. Also, if $U\otimes_R L\cong U\otimes_R L'$ for some left R-module L', then $L\cong \mathrm{Hom}_S(U, U\otimes_R L)\cong \mathrm{Hom}_S(U, U\otimes_R L')\cong L'$. Since the isomorphism classes of indecomposable injective objects in S-Mod constitute a set, we have that the indecomposable pure-injective left R-modules form also a set. Thus, there exists a cardinal $\boldsymbol{m}\geq\aleph_0$ such that each indecomposable pure-injective left R-module is $\boldsymbol{m}$-generated.

Conversely, let us put $A:=U\otimes_R M$. Then A is injective in S-Mod by Lemma 2.1. Let I be any index set and $A_i\cong A$, $M_i\cong M$, for all $i\in I$. By the hypothesis, $\mathrm{PE}(\oplus_{i\in I}M_i)=\oplus_{j\in J}N_j$, where each N_j is $\boldsymbol{m}$-generated. Now, by Lemma 2.1 we have $\mathrm{E}(\oplus_{i\in I}A_i)\cong U\otimes_R \mathrm{PE}(\oplus_{i\in I}M_i)\cong U\otimes_R(\oplus_{j\in J}N_j)\cong\oplus_{j\in J}(U\otimes_R N_j)$. By Lemma 2.1 (*e*), $U\otimes_R N_j$ is $\boldsymbol{m}$-generated in S-Mod. Thus, by Theorem 1.12, A is Σ-injective in S-Mod, which implies that M is Σ-pure-injective in R-Mod.

Corollary 2.5. (Zimmermann-Huisgen [24, Corollary 1]). *Let R be a ring and M a pure-injective left R-module. Then M is Σ-pure-injective if and only if any direct product of copies of M is a direct sum of indecomposable modules.*

Proof. Suppose that M is Σ-pure-injective. Then, any direct product of copies of M is again Σ-pure-injective, so the condition follows from Corollary 2.2. Conversely, let I be any index set and take $M_i\cong M$ for all $i\in I$. It is well-known that $\oplus_{i\in I}M_i$ is a pure submodule of $\Pi_{i\in I}M_i$ (see, e.g. [2], 33.9]), so by [20, Proposition 6] $\mathrm{PE}(\oplus_{i\in I}M_i)$ is isomorphic to a direct summand of $\Pi_{i\in I}M_i$. Since the class of indecomposable pure-injective modules -up to isomorphism- is a set, there exists an infinite cardinal $\boldsymbol{m}$ (not depending on I) such that $\Pi_{i\in I}M_i$ is a direct sum of $\boldsymbol{m}$-generated modules. By Kaplansky's theorem, $\mathrm{PE}(\oplus_{i\in I}M_i)$ is a direct sum of $\boldsymbol{m}$-generated modules. It follows from Theorem 2.4 that M is Σ-pure-injective.

It is clear that, for a ring R, every projective left R-module is pure-injective if and only if ${}_RR$ is Σ-pure-injective. These rings are semiprimary with ACC on left annihilators; we refer to [22, 23, 24, 25] for more informations. As an application of Theorem 2.4, we prove the following result.

Corollary 2.6. *Let R be a ring such that the pure-injective envelope of every projective left R-module is projective. Then* $\mathrm{PE}({}_RR)$ *is Σ-pure-injective and R has* ACC *on left annihilators.*

Proof. Since $\mathrm{PE}({}_RR)$ is projective, any direct sum of copies of $\mathrm{PE}({}_RR)$ is also projective, so that its pure-injective envelope is again projective by the hypothesis. By Kaplansky's theorem, every projective module is a direct sum of countably generated modules. Thus $\mathrm{PE}({}_RR)$ is Σ-pure-injective by Theorem 2.4. The latter assertion follows by [22, Satz 6.2].

Remarks. (1) Similar arguments to those used in the proofs of Theorems

1.12 and 2.4 yield the following generalization of Theorem 2.4: Let $\{M_i | i \in I\}$ be a family of pure-injective modules. Then $M = \bigoplus_{i \in I} M_i$ is Σ-pure-injective if and only if there exists an infinite cardinal $\boldsymbol{m}$ such that the pure-injective envelope of any direct sum of copies of M is a direct sum of $\boldsymbol{m}$-generated modules.

(2) Corollaries 2.2 and 2.5 were crucial for Zimmermann-Huisgen's proof in [24] of the important fact that if all the left R-modules are direct sums of indecomposable modules then all left R-modules are pure-injective (or, equivalently, all left R-modules are direct sums of finitely generated modules). Indeed, let M be any left R-module; then M is isomorphic to a pure submodule of some pure-injective module N (see [20, Corollary 6]). By Corollary 2.5 and the hypothesis, N must be Σ-pure-injective; thus it follows from Corollary 2.2 that M is pure-injective.

ACKNOWLEDGEMENTS. This paper was written during a visit of the second author to the University of Murcia. He would like to thank the Department of Mathematics for kind hospitality and the Spanish Ministry of Education and Science for the financial support. The first author acknowledges support from the DGICYT of Spain (PB90–0300–C02–02).

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Note added in proof.

After having completed this paper, Professor R. Wisbauer pointed out to us that a stronger form of Proposition 1.13 does hold. Namely, one would have the following result:

Let M be a Σ-CS object in A. Suppose that M satisfies one of the following conditions:
a) M is nonsingular in the category σ[M] of all the objects of the category A which are subgenerated by M; or
b) M is projective in σ[M].

Then the injective envelope of M in the category σ[M] is Σ-injective, and M is a direct sum of indecomposable objects.

In particular, if M is a Σ-CS object which is either nonsingular or projective, then M is a direct sum of indecomposable objects.

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