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1. Introduction

A pseudo-symmetric set is a pair \((U, \sigma)\) where \(U\) is a set and \(\sigma\) is a mapping of \(U\) into the group of permutations on \(U\) such that \(\sigma(u)\) fixes \(u\) for every element \(u\) in \(U\) and that it satisfies a fundamental identity: 
\[
\sigma(u\sigma(w)) = \sigma(v)^{-1}\sigma(u)\sigma(v)
\]
for \(u\) and \(v\) in \(U\).

In [1], a possibility of developing a structure theory of pseudo-symmetric set is indicated. In this paper, we shall establish an analogue of Jordan-Hölder theorem in group theory for pseudo-symmetric sets.

Contrary to group theory, the concept of kernels of homomorphisms is not available. Instead, a concept of a normal decomposition is introduced in [1]. It is a partition of \(U\) such that each class of the partition consists of elements that are mapped to an element by a given homomorphism. When a partition \(A\) is a refinement of a partition \(B\), we denote \(A \leq B\). The partition of \(U\) which has just one class \(U\) itself is denoted by \(\mathcal{U}\). The complete partition of \(U\) whose classes are one-point sets is denoted by \(E\). So, \(E \leq A \leq U\) for every partition \(A\). Suppose we have a sequence of normal decompositions \(P_i\) such that
\[
U = P_0 > P_1 > P_2 > \cdots > P_s = E,
\]
where there is no normal decomposition between \(P_i\) and \(P_{i+1}\). Suppose we have another sequence of normal decompositions \(Q_j\) of the same properties:
\[
U = Q_0 > Q_1 > Q_2 > \cdots > Q_m = E.
\]
We say that \(P_i/P_{i+1}\) is non-trivial if \(H(P_i/P_{i+1}) \neq 1\), where \(H(P_i/P_{i+1})\) is the group of displacements for \(P_i/P_{i+1}\). (The definition will be given in 3.) The main theorem we obtain is that between the set of non-trivial \(P_i/P_{i+1}\) and that of non-trivial \(Q_j/Q_{j+1}\) there is a one to one correspondence such that if \(P_i/P_{i+1}\) corresponds to \(Q_j/Q_{j+1}\) then \(H(P_i/P_{i+1}) \approx H(Q_j/Q_{j+1})\).

2. Partitions of a set

Let \(U\) be a (universal) set, and \(U = \bigcup A_i\) a partition of \(U\) into non-empty
disjoint classes $A_i$. We denote this partition simply by $A$ and call $A_i$ components of the partition $A$.

Let $B$ be another partition. If every $A_i$ is contained in a component $B_j$, we say that $A \leq B$. $A$ is a refinement of $B$. Let $C$ be a partition. We define a partition $A \cap C$ by taking all non-empty intersections $A_i \cap C_j$ as its components. $A \cap C$ is the cross partition of $A$ and $C$. Clearly, $A \cap C \leq A$ and $A \cap C \leq C$. If $B$ is a partition such that $B \leq A$ and $B \leq C$, then $B \leq A \cap C$.

Next, we define a partition $AB$ for partitions $A$ and $B$. A component of $AB$ is a union of $A_i$ as well as a union of $B_j$ and is minimal. Thus, a component of $AB$ is connected in a sense that if $u$ and $v$ are elements in it there exist $A_i, B_j, A_k, \ldots, B_w$ in it such that $u \in A_i$ and $v \in B_w$ and that adjacent sets in the above have non-empty intersections. Clearly, $A \leq AB$ and $B \leq AB$. If $A \leq C$ and $B \leq C$, then $AB \leq C$.

**Proposition 1.** If $A \geq B$, then $A \cap BC \geq B(A \cap C)$ for every partition $C$. Generally, the equality does not hold.

**Proof.** Almost clear.

For a partition $A$, we define the quotient set $U/A$. $U/A$ is the set of all components $A_i$ of $A$. Let $A \leq B$. Then, $B$ induces a partition on $U/A$ in a natural way; for $B_j$, let $(B_jA)_j = \{A_i | A_i \subseteq B_j\}$. Then, $U/A = \bigcup (B_jA)_j$ is a partition of $U/A$, which we denote by $B/A$. Since $B/A$ is a partition of $U/A$, we can consider the quotient set $(U/A)/(B/A)$. It follows from the definition that $(U/A)/(B/A)$ is bijective to $U/B$.

### 3. Normal decompositions

From now on, $U$ stands for a pseudo-symmetric set $(U, \sigma)$ for a fixed $\sigma$. Let $G(U)$ be the group generated by all $\sigma(u)$; $G(U) = \langle \sigma(u) \mid u \in U \rangle$. In the following we denote $G(U)$ by $G$. $G$ is a group of automorphisms of the pseudo-symmetric set $U$. Now, we define a normal decomposition of $U$. It is a partition $A$ of $U$ such that $\sigma(u)$ induces a permutation on $U/A$ for every $u$ in $U$ and that $\sigma(u)$ and $\sigma(v)$ induce the same permutation on $U/A$ if $u$ and $v$ belong to the same component of $A$. In this case, $(U/A, \sigma)$ is a pseudo-symmetric set, where $\sigma(A_i)$ is the permutation of $U/A$ induced by $\sigma(u)$ for $u \in A_i$. Clearly, the mapping $u \mapsto A_i$ gives a homomorphism of $U$ onto $U/A$.

**Proposition 2.** If $A$ and $B$ are normal decompositions, then $A \cap B$ and $AB$ are also normal decompositions.

**Proof.** It is clear that $A \cap B$ is a normal decomposition. To show $AB$ is a normal decomposition, let $\rho \in G$. The image of a component $(AB)_i$ by $\rho$ is a component of the partition $AB$, because it is a union of $A_j$ as well as a union of $B_k$ and it must be connected in the previously explained sense. We must show
that if \( u \) and \( v \) belong to the same component of \( AB \), then \( \sigma(u) \) and \( \sigma(v) \) induce the same permutation on \( U/AB \). Due to the connectedness of a component of \( AB \), it is enough to show the above in case that \( u \) and \( v \) belong to either a component \( A_i \) or a component \( B_j \). If \( u \) and \( v \) are in \( A_i \), then \( \sigma(u) \) and \( \sigma(v) \) induce the same permutation on \( U/A \) and hence on \( U/AB \). Similarly, if \( u \) and \( v \) are in \( B_j \), then \( \sigma(u) \) and \( \sigma(v) \) induce the same permutation on \( U/AB \), which proves Proposition 2.

From now on, \( A, B, C, \ldots \) stand for normal decompositions of \( U \). For \( A \), the group of displacements is defined by \( H(A) = \langle \sigma(u)^{-1}\sigma(v) \mid u, v \in A \rangle \). \( H(A) \) is shown to be a normal subgroup of \( G \) due to the fundamental identity. If \( A \leq B \), then \( H(A) \subseteq H(B) \). Note also that \( H(A) \) acts trivially on \( U/A \).

**Proposition 3.** \( H(A \cap B) \leq H(A) \cap H(B) \) and \( H(AB) = H(A)H(B) \).

Proof. The first is trivial. Just note that the equality does not generally hold. For the second, it is clear that \( H(AB) \supseteq H(A)H(B) \). Let \( u, v \in (AB) \). We show that \( \sigma(u)^{-1}\sigma(v) \in H(A)H(B) \). Due to the connectedness of a component of \( AB \), there exist \( u = u_0, u_1, \ldots, u_n = v \) where \( u_i \) and \( u_{i+1} \) are either in a component of \( A \) or of \( B \). In both cases, \( \sigma(u_i)^{-1}\sigma(u_{i+1}) \in H(A)H(B) \). Since \( \sigma(u)^{-1}\sigma(v) \) generate \( H(AB) \), this proves that \( H(AB) \subseteq H(A)H(B) \). So, \( H(AB) = H(A)H(B) \).

For a normal subgroup \( N \) of \( G \), we define a partition \( D \) of \( U \) by letting \( D = \{ M \mid \sigma(u) = \sigma(v) \mod N \) for a fixed element \( u \} \). \( D \) is seen to be a normal decomposition, which we denote by \( D(N) \). If \( N_1 \) and \( N_2 \) are normal subgroups of \( G \) such that \( N_1 \subseteq N_2 \), then \( D(N_1) \leq D(N_2) \). Note also that \( D(N \cap M) = D(N) \cap D(M) \) for normal subgroups \( N \) and \( M \). The following is given in [1].

**Proposition 4.** \( D(H(A)) \geq A \), and the equality holds if and only if \( A = D(N) \) for some \( N \). \( H(D(N)) \subseteq N \) for any normal subgroup \( N \), and the equality holds if and only if \( N = H(A) \) for some \( A \).

### 4. Isomorphism theorems

The restriction of \( G(=G(U)) \) on \( U/A \) induces a homomorphism of \( G \) onto \( G(U/A) \). Denote its kernel by \( K(A) \). So, \( K(A) = \{ \rho \mid \rho \) induces the identity permutation on \( U/A \} \). Clearly, \( H(A) \subseteq K(A) \). If \( A \leq B \), then \( K(A) \subseteq K(B) \). For any \( A \) and \( C \), \( K(A \cap C) = K(A) \cap K(C) \).

Let \( A \leq B \). \( B/A \) is a normal decomposition of \( U/A \), and hence \( H(B/A) \) is defined and is a normal subgroup of \( G(U/A) \).

**Theorem 1.** \( H(B/A) \approx H(B)/(K(A) \cap H(B)) \).

Proof. Consider the homomorphism \( G \to G(U/A) \). \( H(B) \) is mapped onto
$H(B/A)$ as we can see easily. The kernel is clearly $K(A) \cap H(B)$.

When $H(B/A)=1$, we say that $B$ is trivial over $A$, or $B/A$ is trivial (more precisely, $H$-trivial). This implies that $H(B) \subseteq K(A)$ or $H(B)$ acts trivially on $U/A$.

**Proposition 5.** Let $A \geq B$. Then, $A \cap BC$ is trivial over $B(A \cap C)$ for any $C$.

Proof. First note that $A \cap BC \geq B(A \cap C)$ by Proposition 1. Now, $H(A \cap BC) \subseteq H(A) \cap H(BC) = H(A) \cap H(B)H(C) = H(B)[H(A) \cap H(C)]$, as $H(B)$ is a normal subgroup of $H(A)$. Clearly, $H(B) \subseteq K(B(A \cap C))$. Also, $H(A) \cap H(C) \subseteq K(A) \cap K(C) = K(A \cap C) \subseteq K(B(A \cap C))$. Therefore, $H(A \cap BC) \subseteq H(B)[H(A) \cap H(C)] \subseteq K(B(A \cap C))$, which proves that $A \cap BC$ is trivial over $B(A \cap C)$.

**Theorem 2.** $H(AB/B) \cong H(A/(A \cap B))$.

Proof. $H(AB/B) \cong H(AB)/(K(B) \cap H(AB))$ by Theorem 1. But, $H(AB) = H(A)H(B) = H(A)[K(B) \cap H(AB)]$, as $H(B) \subseteq K(B) \cap H(AB) \subseteq H(AB)$. Therefore, $H(AB/B) = H(A)[K(B) \cap H(AB)]/(K(B) \cap H(AB)) \cong H(A)/(H(A) \cap K(B) \cap H(AB)) = H(A)/(H(A) \cap K(B))$. It is easy to see that $H(A) \cap K(B) = K(A \cap B) \cap H(A)$. Thus, $H(A)/(H(A) \cap K(B)) = H(A)/(K(A \cap B) \cap H(A))$, which is isomorphic with $H((A \cap B))$ by Theorem 1. So, $H(AB/B) \cong H(A/(A \cap B))$.

**Proposition 6.** Let $D \leq C$. Then, $H((A \cap C)/(A \cap D))$ is isomorphic to a subgroup of $H(C/D)$.

Proof. Restrict the homomorphism $H(C) \to H(C/D)$ to $H(A \cap C)$ which is a subgroup of $H(C)$, and we have a homomorphism $H(A \cap C) \to H(C/D)$. Its kernel is $K(D) \cap H(A \cap C)$. But, $K(D) \cap H(A \cap C) = K(A \cap D) \cap H(A \cap C)$, as $H(A \cap C) = H(A \cap C) \cap K(A)$ and $K(D) \cap K(A) = K(A \cap D)$. So, $H(A \cap C)/(K(A \cap D) \cap H(A \cap C))$ is isomorphic to a subgroup of $H(C/D)$. Lastly note that $H((A \cap C)/(K(A \cap D) \cap H(A \cap C))$ is isomorphic with $H((A \cap C)/(A \cap D))$ by Theorem 1, which proves Proposition 6.

**Proposition 7.** Let $D \leq C$. Then, $H(C/D)$ is homomorphic onto $H(CB/DB)$ for any $B$.

Proof. $H(C/D) \cong H(C)/(K(D) \cap H(C))$, and the latter is homomorphic onto $H(C)/H(B)/[K(D) \cap H(C)]H(B)$ as we can see easily. But, $[K(D) \cap H(C)]H(B) \subseteq K(DB) \cap H(CB)$. Thus, $H(C/D)$ is homomorphic onto $H(CB)/(K(DB) \cap H(CB)) \cong H(CB/DB)$.

**Theorem 3.** Let $D \leq C$. Then, $H(C/D)$ contains a subgroup $N$ such that $N$ is homomorphic onto $H((C \cap A)B/(D \cap A)B)$. 


The following is a basic theorem, which is a generalization of the "simplicity" theorem. ([1], Corollary 2) When $A \geq B$, $H(A/B)$ is a normal subgroup of $G(U/B)$ and hence a $G(U/B)$-group. As there is the homomorphism from $G$ onto $G(U/B)$, we can consider $H(A/B)$ as a $G$-group.

**Theorem 4.** Let $A > B$. If there is no normal decomposition between $A$ and $B$, then $H(A/B)$ is $G$-simple.

Proof. $H(A/B) \cong H(A)/(K(B) \cap H(A))$. So, it is enough to show that if $N$ is a normal subgroup of $G$ such that $K(B) \cap H(A) \subseteq N \subseteq H(A)$, then $N=K(B) \cap H(A)$. Let $D=D(N)$ for such normal subgroup $N$. Then, $A \leq D$. For, if $A \leq D$, then $H(A) \subseteq H(D) \subseteq N$ by Proposition 4, which is a contradiction. Next, we show $A \cap D=B$. For, $B \leq D(H(B)) \leq D(N)=D$ and hence $B \leq A \cap D < A$. So, $A \cap D=B$ by the assumption in Theorem 4. Since $N$ acts trivially on $D(N)=D$ as is seen from the definition of $D(N)$, $N \subseteq K(D)$. Clearly, $N \subseteq H(A) \subseteq K(A)$. Therefore, $N \subseteq K(A \cap D)$. As we have shown $A \cap D=B$ in the above, we have $N \subseteq K(B)$. Thus, $N \subseteq K(B) \cap H(A)$, which implies that $N=K(B) \cap H(A)$. This proves Theorem 4. Note that in the above, "G-simple" means either $H(A/B)=1$ or else $H(A/B)$ does not contain a proper $G$-subgroup.

5. Jordan-Hölder Theorem

**Proposition 8.** Let $A > B$ and $C > D$. Suppose that $H(A/B) \neq 1$ and that there is no normal decomposition between $C$ and $D$. If $A=(C \cap A)B$ and $B=(D \cap A)B$, then $C=(A \cap C)D$ and $D=(B \cap C)D$.

Proof. Clearly, $C \geq (A \cap C)C \geq (B \cap C)C \geq D$. If we show that $(A \cap C)D \neq (B \cap C)D$, then Proposition 8 follows due to the assumption on $C$ and $D$. So, assume that $(A \cap C)D=(B \cap C)D$, and we are going to derive a contradiction. $A \cap C=(A \cap C) \cap (A \cap C)D=(A \cap C) \cap (B \cap C)D$. Apply Proposition 5 for $A \cap C$ and $B \cap C$ in place of $A$ and $B$, and we obtain that $(A \cap C) \cap (B \cap C)D$ is trivial over $(B \cap C)(A \cap C \cap D)=(B \cap C)(A \cap D)$, or that $A \cap C$ is trivial over $(B \cap C)(A \cap D)$. Hence, $H(A \cap C) \subseteq K[(B \cap C)(A \cap D)]$. Next, we show that $B \cap C=(B \cap C)(A \cap D)$. Since $C \geq (B \cap C)(A \cap D)$ and $B=(D \cap A)B \geq (A \cap D) \cdot (B \cap C)$, we have $B \cap C \geq (B \cap C)(A \cap D)$, or $B \cap C=(B \cap C)(A \cap D)$. We have obtained that $H(A \cap C) \subseteq K(B \cap C)$. Now, $H(A/B)=H(\{(C \cap A)B\}/B)=H(\{(C \cap A)(C \cap B)\})$ (by Theorem 2). Since $H(C \cap A) \subseteq K(B \cap C)$, we have that $H(\{(C \cap A)(C \cap B)\})=1$, or $H(A/B)=1$, which contradicts the assumption that $H(A/B) \neq 1$.

Now we prove the Jordan-Hölder Theorem for pseudo-symmetric sets.

**Theorem 5.** Let $U=P_0 \geq P_1 \geq P_2 \geq \cdots \geq P_n=E$ and $U=Q_0 \geq Q_1 \geq Q_2 \geq \cdots$
> \mathcal{Q}_m = E be sequences of normal decompositions such that between \( P_i \) and \( P_{i+1} \) and between \( Q_j \) and \( Q_{j+1} \) there is no normal decomposition. Let \( X \) be the set of all non-trivial \( P_i/P_{i+1} \) and \( Y \) that of all non-trivial \( Q_j/Q_{j+1} \). Then, there is a bijection between \( X \) and \( Y \) such that if \( P_i/P_{i+1} \) corresponds to \( Q_j/Q_{j+1} \), then \( H(P_i/P_{i+1}) \cong H(Q_j/Q_{j+1}) \).

Proof. Let \( P_i/P_{i+1} \in X \). Let \( A = P_i \) and \( B = P_{i+1} \). Put \( R_k = (Q_k \cap A)B \) for \( 0 \leq k \leq m \). Then, \( R_k \geq R_{k+1} \), \( R_0 = A \) and \( R_m = B \). So, there is \( j \) such that \( R_j = A \) and \( R_{j+1} = B \). Let \( C = Q_j \) and \( D = Q_{j+1} \). We show that \( C/D \in Y \) and that \( H(A/B) \cong H(C/D) \). By Theorem 3, \( H(C/D) \) contains a subgroup which is homomorphic onto \( H(A/B) \). Since \( H(A/B) \neq 1 \), this implies that \( H(C/D) \neq 1 \). So, \( C/D \in Y \). Clearly, \( H(C/D) \cong H(A/B) \), as \( H(C/D) \) is G-simple by Theorem 4. We have established a mapping from \( X \) to \( Y \). To show that it is a bijection, construct a mapping from \( Y \) to \( X \) in a similar manner. By Proposition 8, these mappings are inverse each other.

Reference