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JORDAN-HÖLDER THEOREM FOR PSEUDO-SYMMETRIC SETS

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1. Introduction

A pseudo-symmetric set is a pair (U, σ) where U is a set and σ is a mapping of U into the group of permutations on U such that $\sigma(u)$ fixes u for every element u in U and that it satisfies a fundamental identity: $\sigma(u^{\sigma(v)})=\sigma(v)^{-1}\sigma(u)\sigma(v)$ for u and v in U.

In [1], a possibility of developing a structure theory of pseudo-symmetric set is indicated. In this paper, we shall establish an analogue of Jordan-Hölder theorem in group theory for pseudo-symmetric sets.

Contrary to group theory, the concept of kernels of homomorphisms is not available. Instead, a concept of a normal decomposition is introduced in [1]. It is a partition of U such that each class of the partition consists of elements that are mapped to an element by a given homomorphism. When a partition A is a refinement of a partition B, we denote $A \leq B$. The partition of U which has just one class U itself is denoted by U. The complete partition of U whose classes are one-point sets is denoted by E. So, $E \leq A \leq U$ for every partition A. Suppose we have a sequence of normal decompositions P_i such that

$$(1) \qquad U=P_0>P_1>P_2>\cdots>P_n=E$$

where there is no normal decomposition between P_i and P_{i+1}. Suppose we have another sequence of normal decompositions Q_j of the same properties:

$$(2) \qquad U=Q_0>Q_1>Q_2>\cdots>Q_m=E\,.$$

We say that P_i/P_{i+1} is non-trivial if $H(P_i/P_{i+1}) \neq 1$, where $H(P_i/P_{i+1})$ is the group of displacements for P_i/P_{i+1}. (The definition will be given in **3**.) The main theorem we obtain is that between the set of non-trivial P_i/P_{i+1} and that of non-trivial Q_j/Q_{j+1} there is a one to one correspondence such that if P_i/P_{i+1} corresponds to Q_j/Q_{j+1} then $H(P_i/P_{i+1}) \simeq H(Q_j/Q_{j+1})$.

2. Partitions of a set

Let U be a (universal) set, and $U=\cup A_i$ a partition of U into non-empty disjoint classes A_i. We denote this partition simply by A and call A_i components of the partition A.

Let B be another partition. If every A_i is contained in a component B_j, we say that $A \leq B$. A is a refinement of B. Let C be a partition. We define a partition $A \cap C$ by taking all non-empty intersections $A_i \cap C_j$ as its components. $A \cap C$ is the cross partition of A and C. Clearly, $A \cap C \leq A$ and $A \cap C \leq C$. If B is a partition such that $B \leq A$ and $B \leq C$, then $B \leq A \cap C$.

Next, we define a partition AB for partitions A and B. A component of AB is a union of A_i as well as a union of B_j and is minimal. Thus, a component of AB is connected in a sense that if u and v are elements in it there exist $A_i, B_j, A_k, \cdots, B_m$ in it such that $u \in A_i$ and $v \in B_m$ and that adjacent sets in the above have non-empty intersections. Clearly, $A \leq AB$ and $B \leq AB$. If $A \leq C$ and $B \leq C$, then $AB \leq C$.

Proposition 1. *If $A \geq B$, then $A \cap BC \geq B(A \cap C)$ for every partition C. Generally, the equality does not hold.*

Proof. Almost clear.

For a partition A, we define the quotient set U/A. U/A is the set of all components A_i of A. Let $A \leq B$. Then, B induces a partition on U/A in a natural way; for B_j, let $(B/A)_j = \{A_i \mid A_i \subseteq B_j\}$. Then, $U/A = \cup (B/A)_j$ is a partition of U/A, which we denote by B/A. Since B/A is a partition of U/A, we can consider the quotient set $(U/A)/(B/A)$. It follows from the definition that $(U/A)/(B/A)$ is bijective to U/B.

3. Normal decompositions

From now on, U stands for a pseudo-symmetric set (U, σ) for a fixed σ. Let $G(U)$ be the group generated by all $\sigma(u)$; $G(U) = \langle \sigma(u) \mid u \in U \rangle$. In the following we denote $G(U)$ by G. G is a group of automorphisms of the pseudo-symmetric set U. Now, we define a normal decomposition of U. It is a partition A of U such that $\sigma(u)$ induces a permutation on U/A for every u in U and that $\sigma(u)$ and $\sigma(v)$ induce the same permutation on U/A if u and v belong to the same component of A. In this case, $(U/A, \sigma)$ is a pseudo-symmetric set, where $\sigma(A_i)$ is the permutation of U/A induced by $\sigma(u)$ for $u \in A_i$. Clearly, the mapping $u \rightarrow A_i$ gives a homomorphism of U onto U/A.

Proposition 2. *If A and B are normal decompositions, then $A \cap B$ and AB are also normal decompositions.*

Proof. It is clear that $A \cap B$ is a normal decomposition. To show AB is a normal decomposition, let $\rho \in G$. The image of a component $(AB)_i$ by ρ is a component of the partition AB, because it is a union of A_j as well as a union of B_k and it must be connected in the previously explained sense. We must show that if u and v bbelong to the same component of AB, then $\sigma(u)$ and $\sigma(v)$ induce the same permutation on U/AB. Due to the connectedness of a component of AB, it is enough to show the above in case that u and v belong to either a component A_i or a component B_j. If u and v are in A_i, then $\sigma(u)$ and $\sigma(v)$ induce the same permutation on U/A and hence on U/AB. Similarly, if u and v are in B_j, then $\sigma(u)$ and $\sigma(v)$ induce the same permutation on U/AB, which proves Proposition 2.

From now on, A, B, C,$\cdots$ stand for normal decompositions of U. For A, the group of displacements is defined by $H(A)=\langle\sigma(u)^{-1}\sigma(v)\,|\,u$ and v belong to the same component$\rangle$. $H(A)$ is shown to be a normal subgroup of G due to the fundamental identity. If $A\leq B$, then $H(A)\subseteq H(B)$. Note also that $H(A)$ acts trivially on U/A.

Proposition 3. *$H(A\cap B)\subseteq H(A)\cap H(B)$ and $H(AB)=H(A)H(B)$.*

Proof. The first is trivial. Just note that the equality does not generally hold. For the second, it is clear that $H(AB)\supseteq H(A)H(B)$. Let u and $v\in(AB)_i$. We show that $\sigma(u)^{-1}\sigma(v)\in H(A)H(B)$. Due to the connectedness of a component of AB, there exist $u=u_0, u_1, \cdots, u_n=v$ where u_j and u_{j+1} are either in a component of A or of B. In both cases, $\sigma(u_j)^{-1}\sigma(u_{j+1})\in H(A)H(B)$. Since $\sigma(u)^{-1}\sigma(v)$ generate $H(AB)$, this proves that $H(AB)\subseteq H(A)H(B)$. So, $H(AB)=H(A)H(B)$.

For a normal subgroup N of G, we define a partition D of U by letting $D_i=\{u\,|\,\sigma(u)\equiv\sigma(u_i) \bmod N$ for a fixed element $u_i\}$. D is seen to be a normal decomposition, which we denote by $D(N)$. If N_1 and N_2 are normal subgroups of G such that $N_1\subseteq N_2$, then $D(N_1)\leq D(N_2)$. Note also that $D(N\cap M)=D(N)\cap D(M)$ for normal subgroups N and M. The following is given in [1].

Proposition 4. *$D(H(A))\geq A$, and the equality holds if and only if $A=D(N)$ for some N. $H(D(N))\subseteq N$ for any normal subgroup N, and the equality holds if and only if $N=H(A)$ for some A.*

4. Isomorphism theorems

The restriction of $G(=G(U))$ on U/A induces a homomorphsim of G onto $G(U/A)$. Denote its kernel by $K(A)$. So, $K(A)=\{\rho\,|\,\rho$ induces the identity permutation on $U/A\}$. Clearly, $H(A)\subseteq K(A)$. If $A\leq B$, then $K(A)\subseteq K(B)$. For any A and C, $K(A\cap C)=K(A)\cap K(C)$.

Let $A\leq B$. B/A is a normal decomposition of U/A, and hence $H(B/A)$ is defined and is a normal subgroup of $G(U/A)$.

Theorem 1. *$H(B/A)\cong H(B)/(K(A)\cap H(B))$.*

Proof. Consider the homomorphism $G\rightarrow G(U/A)$. $H(B)$ is mapped onto

$H(B/A)$ as we can see easily. The kernel is clearly $K(A)\cap H(B)$.

When $H(B/A)=1$, we say that B is trivial over A, or B/A is trivial (more precisely, H-trivial). This implies that $H(B)\subseteq K(A)$ or $H(B)$ acts trivially on U/A.

Proposition 5. *Let $A\geq B$. Then, $A\cap BC$ is trivial over $B(A\cap C)$ for any C.*

Proof. First note that $A\cap BC\geq B(A\cap C)$ by Proposition 1. Now, $H(A\cap BC)\subseteq H(A)\cap H(BC)=H(A)\cap H(B)H(C)=H(B)[H(A)\cap H(C)]$, as $H(B)$ is a normal subgroup of $H(A)$. Clearly, $H(B)\subseteq K(B(A\cap C))$. Also, $H(A)\cap H(C)\subseteq K(A)\cap K(C)=K(A\cap C)\subseteq K(B(A\cap C))$. Therefore, $H(A\cap BC)\subseteq H(B)[H(A)\cap H(C)]\subseteq K(B(A\cap C))$, which proves that $A\cap BC$ is trivial over $B(A\cap C)$.

Theorem 2. $H(AB/B)\cong H(A/(A\cap B))$.

Proof. $H(AB/B)\cong H(AB)/(K(B)\cap H(AB))$ by Theorem 1. But, $H(AB)=H(A)H(B)=H(A)[K(B)\cap H(AB)]$, as $H(B)\subseteq K(B)\cap H(AB)\subseteq H(AB)$. Therefore, $H(AB/B)\cong H(A)[K(B)\cap H(AB)]/(K(B)\cap H(AB))\cong H(A)/(H(A)\cap K(B)\cap H(AB))=H(A)/(H(A)\cap K(B))$. It is easy to see that $H(A)\cap K(B)=K(A\cap B)\cap H(A)$. Thus, $H(A)/(H(A)\cap K(B))=H(A)/(K(A\cap B)\cap H(A))$, which is isomorphic with $H(A/(A\cap B))$ by Theorem 1. So, $H(AB/B)\cong H(A/(A\cap B))$.

Proposition 6. *Let $D\leq C$. Then, $H((A\cap C)/(A\cap D))$ is isomorphic to a subgroup of $H(C/D)$.*

Proof. Restrict the homomorphism $H(C)\to H(C/D)$ to $H(A\cap C)$ which is a subgroup of $H(C)$, and we have a homomorphism $H(A\cap C)\to H(C/D)$. Its kernel is $K(D)\cap H(A\cap C)$. But, $K(D)\cap H(A\cap C)=K(A\cap D)\cap H(A\cap C)$, as $H(A\cap C)=H(A\cap C)\cap K(A)$ and $K(D)\cap K(A)=K(A\cap D)$. So, $H(A\cap C)/(K(A\cap D)\cap H(A\cap C))$ is isomorphic to a subgroup of $H(C/D)$. Lastly note that $H(A\cap C)/(K(A\cap D)\cap H(A\cap C))$ is isomorphic with $H((A\cap C)/(A\cap D))$ by Theorem 1, which proves Proposition 6.

Proposition 7. *Let $D\leq C$. Then, $H(C/D)$ is homomorphic onto $H(CB/DB)$ for any B.*

Proof. $H(C/D)\cong H(C)/(K(D)\cap H(C))$, and the latter is homomorphic onto $H(C)H(B)/[K(D)\cap H(C)]H(B)$ as we can see easily. But, $[K(D)\cap H(C)]H(B)\subseteq K(DB)\cap H(CB)$. Thus, $H(C/D)$ is homomorphic onto $H(CB)/(K(DB)\cap H(CB))\cong H(CB/DB)$.

Theorem 3. *Let $D\leq C$. Then, $H(C/D)$ contains a subgroup N such that N is homomorphic onto $H((C\cap A)B/(D\cap A)B)$.*

Proof. Simply apply Propositions 6 & 7.

The following is a basic theorem, which is a generalization of the "simplicity" theorem. ([**1**], Corollary 2) When $A \geq B$, $H(A/B)$ is a normal subgroup of $G(U/B)$ and hence a $G(U/B)$-group. As there is the homomorphism from G onto $G(U/B)$, we can consider $H(A/B)$ as a G-group.

Theorem 4. *Let $A > B$. If there is no normal decomposition between A and B, then $H(A/B)$ is G-simple.*

Proof. $H(A/B) \cong H(A)/(K(B) \cap H(A))$. So, it is enough to show that if N is a normal subgroup of G such that $K(B) \cap H(A) \subseteq N \subset H(A)$, then $N = K(B) \cap H(A)$. Let $D = D(N)$ for such normal subgroup N. Then, $A \not\leq D$. For, if $A \leq D$, then $H(A) \subseteq H(D) \subseteq N$ by Proposition 4, which is a contradiction. Next, we show $A \cap D = B$. For, $B \leq D(H(B)) \leq D(N) = D$ and hence $B \leq A \cap D < A$, So, $A \cap D = B$ by the assumption in Theorem 4. Since N acts trivially on $D(N) = D$ as is seen from the definition of $D(N)$, $N \subseteq K(D)$. Clearly, $N \subset H(A) \subseteq K(A)$. Therefore, $N \subseteq K(A \cap D)$. As we have shown $A \cap D = B$ in the above, we have $N \subseteq K(B)$. Thus, $N \subseteq K(B) \cap H(A)$, which implies that $N = K(B) \cap H(A)$. This proves Theorem 4. Note that in the above, "G-simple" means either $H(A/B) = 1$ or else $H(A/B)$ does not contain a proper G-subgroup.

5. Jordan-Hölder Theorem

Proposition 8. *Let $A > B$ and $C > D$. Suppose that $H(A/B) \neq 1$ and that there is no normal decomposition between C and D. If $A = (C \cap A)B$ and $B = (D \cap A)B$, then $C = (A \cap C)D$ and $D = (B \cap C)D$.*

Proof. Clearly, $C \geq (A \cap C)D \geq (B \cap C)D \geq D$. If we show that $(A \cap C)D \neq (B \cap C)D$, then Proposition 8 follows due to the assumption on C and D. So, assume that $(A \cap C)D = (B \cap C)D$, and we are going to derive a contradiction. $A \cap C = (A \cap C) \cap (A \cap C)D = (A \cap C) \cap (B \cap C)D$. Apply Proposition 5 for $A \cap C$ and $B \cap C$ in place of A and B, and we obtain that $(A \cap C) \cap (B \cap C)D$ is trivial over $(B \cap C)(A \cap C \cap D) = (B \cap C)(A \cap D)$, or that $A \cap C$ is trivial over $(B \cap C)(A \cap D)$. Hence, $H(A \cap C) \subseteq K[(B \cap C)(A \cap D)]$. Next, we show that $B \cap C = (B \cap C)(A \cap D)$. Since $C \geq (B \cap C)(A \cap D)$ and $B = (D \cap A)B \geq (A \cap D) \cdot (B \cap C)$, we have $B \cap C \geq (B \cap C)(A \cap D)$, or $B \cap C = (B \cap C)(A \cap D)$. We have obtained that $H(A \cap C) \subseteq K(B \cap C)$. Now, $H(A/B) = H([(C \cap A)B]/B) \cong H((C \cap A)/(C \cap A \cap B))$ (by Theorem 2) $= H((C \cap A)/(C \cap B))$. Since $H(C \cap A) \subseteq K(B \cap C)$, we have that $H((C \cap A)/(C \cap B)) = 1$, or $H(A/B) = 1$, which contradicts the assumption that $H(A/B) \neq 1$.

Now we prove the Jordan-Hölder Theorem for pseudo-symmetric sets.

Theorem 5. *Let $U = P_0 > P_1 > P_2 > \cdots > P_n = E$ and $U = Q_0 > Q_1 > Q_2 > \cdots$*

$>Q_m=E$ be sequences of normal decompositions such that between P_i and P_{i+1} and between Q_j and Q_{j+1} there is no normal decomposition. Let X be the set of all non-trivial P_i/P_{i+1} and Y that of all non-trivial Q_j/Q_{j+1}. Then, there is a bijection between X and Y such that if P_i/P_{i+1} corresponds to Q_j/Q_{j+1}, then $H(P_i/P_{i+1})\cong H(Q_j/Q_{j+1})$.

Proof. Let $P_i/P_{i+1}\in X$. Let $A=P_i$ and $B=P_{i+1}$. Put $R_k=(Q_k\cap A)B$ for $0\leq k\leq m$. Then, $R_k\geq R_{k+1}$, $R_0=A$ and $R_m=B$. So, there is j such that $R_j=A$ and $R_{j+1}=B$. Let $C=Q_j$ and $D=Q_{j+1}$. We show that $C/D\in Y$ and that $H(A/B)\cong H(C/D)$. By Theorem 3, $H(C/D)$ contains a subgroup which is homomorphic onto $H(A/B)$. Since $H(A/B)\neq 1$, this implies that $H(C/D)\neq 1$. So, $C/D\in Y$. Clearly, $H(C/D)\cong H(A/B)$, as $H(C/D)$ is G-simple by Theorem 4. We have established a mapping from X to Y. To show that it is a bijection, construct a mapping from Y to X in a similar manner. By Proposition 8, these mappings are inverse each other.

Reference

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