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## On a Non-Parametric Test

By Masashi OKAMOTO

**1. Introduction.** Let  $X$  be a random variable having the distribution function (d.f.)  $F(x)$ . We want to test the hypothesis  $H_0$  that  $F(x)$  is identical with a specified continuous d.f.  $F_0(x)$ . F. N. David [1] has recently proposed the following test (though this is slightly modified in comparison with the original one):

Let  $x_1, x_2, \dots, x_N$  be  $N$  independent observations of  $X$ . As  $F_0(x)$  is continuous, there are real numbers  $\{a_i\}$ ,  $i=1, \dots, n-1$ , such that  $F_0(a_i) - F_0(a_{i-1}) = 1/n$ ,  $i=1, \dots, n$ , where  $a_0 = -\infty$ ,  $a_n = +\infty$ . Let  $C$  be the set of intervals on the real line on each of which  $F_0(x)$  is constant and  $C'$  be its complementary set. The intersection of  $(a_{i-1}, a_i]$  with  $C'$  will be called "part". Let  $v$  be the number of parts which contain no  $x$ 's and  $w$  be the number of  $x$ 's which fall in  $C$ . If either  $w$  is positive or  $v$  is too large we reject  $H_0$ .

David conjectured that under the null hypothesis  $H_0$ ,  $v$  is asymptotically normally distributed when  $n, N \rightarrow \infty$ ,  $N/n \rightarrow \text{const}$ . This can be proved by the method of B. Sherman [2]. Furthermore this test is consistent and unbiased against a rather general class of alternative hypotheses. As Lehmann [3] says, very little work has been done on the existence of unbiased tests for non-parametric problems. It is remarkable that David's test has this property.

**2. Distribution of  $v$  under  $H_0$ .** Put  $u=n-v$ , i.e.,  $u$  is the number of parts which contain at least one  $x$ . First we shall determine the distribution of  $u$  under  $H_0$ .

Denote by  $P_k$  the probability that  $N$   $x$ 's "fill"  $k$  given parts (i.e., every  $x_i$  falls in some of them and each of them contains at least one  $x$ ). The probability that  $N$   $x$ 's fall into  $k$  given parts is

$$\left(\frac{k}{n}\right)^N = \sum_{i=1}^k \binom{k}{i} P_i .$$

Therefore, for every positive integer  $v$ ,

$$\begin{aligned} \sum_{k=1}^v (-1)^{v-k} \binom{v}{k} \left(\frac{k}{n}\right)^N &= \sum_{k=1}^v (-1)^{v-k} \binom{v}{k} \sum_{i=1}^k \binom{k}{i} P_i \\ &= \sum_{i=1}^v \binom{v}{i} P_i \sum_{k=i}^v (-1)^{v-k} \binom{v-i}{k-i} \\ &= P_v , \end{aligned}$$

because

$$\sum_{i=0}^n (-1)^i \binom{n}{i} = 1 \quad \text{if } n=0, \\ = 0 \quad \text{if } n > 0.$$

Thus

$$P(u = \nu) = \binom{n}{\nu} P_\nu \\ = n^{-N} \binom{n}{\nu} \sum_{k=1}^{\nu} (-1)^{\nu-k} \binom{\nu}{k} k^N, \quad \nu = 1, \dots, n.$$

Replacing  $\nu$  by  $n-\nu$ ,

$$P(v = \nu) = n^{-N} \binom{n}{\nu} \sum_{k=1}^{n-\nu} (-1)^{n-\nu-k} \binom{n-\nu}{k} k^N, \quad \nu = 0, 1, \dots, n-1.$$

**3. Moments of  $v$  under  $H_0$ .** Define  $x^{(s)}$  for every non-negative integer  $s$  as

$$x^{(s)} = x(x-1)(x-2)\dots(x-s+1) \quad \text{if } s > 0, \\ x^{(0)} = 1.$$

Then the  $s$  th factorial moment is

$$E(v^{(s)}) = \sum_{\nu=0}^{n-1} \nu^{(s)} P(v = \nu) \\ = \sum_{\nu=s}^{n-1} \frac{\nu!}{(\nu-s)!} n^{-N} \binom{n}{\nu} \sum_{k=1}^{n-\nu} (-1)^{n-\nu-k} \binom{n-\nu}{k} k^N \\ = \frac{n!}{n^N} \sum_{k=1}^{n-s} \frac{k^N}{k!(n-s-k)!} \sum_{\nu=s}^{n-k} (-1)^{n-\nu-k} \binom{n-k-s}{\nu-s} \\ = \frac{n! (n-s)^N}{n^N (n-s)!}.$$

Putting  $s = 1, 2$ ,

$$E(v) = n(n-1)^N n^{-N}, \\ E(v-1) = n(n-1)(n-2)^N n^{-N},$$

whence, if  $N = nr$  ( $r$  is a constant),

$$E(v/n) = e^{-r}(1-r/2n) + O(n^{-2}), \\ D^2(v/n) = e^{-2r}(e^r-1-r)n^{-1} + O(n^{-2}).$$

#### 4. Asymptotic normality of $v$ under $H_0$ .

**Theorem 1.**  $v/n$  is asymptotically normally distributed with mean  $e^{-r}$  and variance  $e^{-2r}(e^r-1-r)n^{-1}$ , where  $r = N/n = \text{const.}$

As the proof is almost parallel to that of Theorem 2 of B. Sherman [2], we shall only sketch it.

It is sufficient to prove that moments of  $(n/c)^{\frac{1}{2}}(v/n - e^{-r})$  tend

to the moments of the standard normal distribution, where  $c = e^{-2r}(e^r - 1 - r)$ . It can be shown that if the limiting moments of even order exist the limiting moments of odd order are zero. Thus we may restrict ourselves to even order moments.

Denoting by  $B_r^{(n)}$  the Bernoulli's number of order  $n$  and degree  $r$ ,

$$v^k = \sum_{q=0}^k \binom{k}{q} B_q^{(q-k)} v^{(k-q)} .$$

Then

$$\begin{aligned} E\left[\left(\frac{n}{c}\right)^{\frac{1}{2}}\left(\frac{v}{n} - e^{-r}\right)\right]^{2m} &= \left(\frac{n}{c}\right)^m \sum_{k=0}^{2m} \binom{2m}{k} (-e^{-r})^{2m-k} E\left(\frac{v}{n}\right)^k \\ &= \left(\frac{n}{c}\right)^m \sum_{k=0}^{2m} \binom{2m}{k} (-e^{-r})^{2m-k} n^{-k} \sum_{q=0}^k \binom{k}{q} B_q^{(q-k)} E(v^{(k-q)}) \\ &= \frac{n^m (2m)!}{(e^r - 1 - r)^m} \sum_{q=0}^{2m} \frac{1}{q!} \sum_{k=q}^{2m} \frac{(-1)^k e^{kr}}{(2m-k)!(k-q)!} B_q^{(q-k)} \frac{n! (n-k+q)^N}{n^{N+k} (n-k+q)!} \\ &\equiv \frac{n^m (2m)!}{(e^r - 1 - r)^m} \left[ a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \dots + \frac{a_m}{n^m} + \dots \right] . \end{aligned}$$

We have to show that  $a_i = 0$ ,  $i = 0, 1, 2, \dots, m-1$ . Then  $\lim_{n \rightarrow \infty} \left[ \left( \frac{n}{c} \right)^{\frac{1}{2}} \left( \frac{v}{n} - e^{-r} \right) \right]^{2m} = a_m (2m)! (e^r - 1 - r)^{-m}$ . If we denote by  $a_{iq}$  the coefficient of  $n^{-i}$  in the expansion in powers of  $n^{-1}$  of

$$\sum_{k=q}^{2m} \frac{(-1)^k e^{kr}}{(2m-k)!(k-q)!} B_q^{(q-k)} \frac{n! (n-k+q)^N}{n^{N+k} (n-k+q)!} ,$$

we have

$$a_i = \sum_{q=0}^i a_{iq} / q! , \quad i = 0, 1, \dots, m .$$

Now

$$\begin{aligned} \frac{n! (n-k+q)^N}{n^{N+k} (n-k+q)!} &= \frac{1}{n^k} \left(1 - \frac{k-q}{n}\right)^N n(n-1) \dots (n-k+q+1) \\ &= x^q [1 - (k-q)x]^{r/x} (1-x)(1-2x) \dots (1-(k-q-1)x) \\ &\equiv x^q F(x) , \end{aligned}$$

where  $x = 1/n$  and

$$F(x) = [1 - (k-q)x]^{r/x} \prod_{j=1}^{k-q-1} (1-jx) .$$

Expanding this, we have

$$F(x) = a_{kq0} + a_{kq1}x + a_{kq2}x^2 + \dots ,$$

where

$$a_{kq0} = e^{-(k-q)r} ,$$

$$a_{kqp} = \frac{1}{p!} \left. \frac{d^p F(x)}{dx^p} \right|_{x=0} , \quad p = 1, \dots, i-q .$$

$a_{kqp}$  can be written as

$$a_{kqp} = \frac{1}{p} \sum_{s=0}^{p-1} a_{kq(p-s-1)} b_{kqs} ,$$

where  $b_{kqp}$  is a polynomial in  $k$  of degree  $s+2$  and particularly  $b_{q0} = -(1+r)k^2/2 + Ak + B$ ,  $A, B$  depending on  $q$  only. Hence

$$a_{kq(i-q)} = e^{-(k-q)r} B_{kqi} ,$$

where

$$\begin{aligned} B_{kqi} &= [(i-q)!]^{-1} (b_{q0})^{i-q} + \text{terms of lower degree in } k \\ &= [(i-q)!]^{-1} (-(1+r)/2)^{i-q} k^{2(i-q)} + \sum_{j=0}^{2(i-q)-1} A_j k^j . \end{aligned}$$

Now

$$\begin{aligned} a_{iq} &= \sum_{k=q}^{2m} \frac{(-1)^k e^{kr}}{(2m-k)!(k-q)!} B_q^{(q-k)} a_{q(i-q)} \\ &= e^{qr} \sum_{k=q}^{2m} \frac{(-1)^k}{(2m-k)!(k-q)!} B_q^{(q-k)} B_{kqi} . \end{aligned}$$

As  $B_q^{(q-k)}$  is the polynomial in  $k$  of degree  $q$  with  $2^{-q}$  as the coefficient of the term of the highest degree and as

$$\sum_{k=q}^{2m} \frac{(-1)^k}{(2m-k)!(k-q)!} k^l = \begin{cases} 0 & \text{if } l < 2m-q , \\ 1 & \text{if } l = 2m-q , \end{cases}$$

we obtain

$$\begin{aligned} a_{iq} &= 0 \quad \text{if } i < m , \\ a_{mq} &= e^{qr} 2^{-q} [(m-q)!]^{-1} (-(1+r)/2)^{m-q} , \end{aligned}$$

and

$$\begin{aligned} a_i &= 0 \quad \text{if } i < m , \\ a_m &= \sum_{q=0}^m a_{mq}/q! = 2^{-m} (e^r - 1 - r)^m / m! . \end{aligned}$$

Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[ \left( \frac{n}{c} \right)^{\frac{1}{2}} \left( \frac{v}{n} - e^{-r} \right) \right]^{2m} &= a_m (2m)! (e^r - 1 - r)^{-m} \\ &= 2^{-m} (2m)! / m! , \end{aligned}$$

which is the  $2m$ th moment of the standard normal distribution.

**5. Computation of the power.** The power of the test with respect to the alternative hypothesis  $H_1$  is, for a given integer  $l (\leq n)$ ,

$$P = P(w > 0 | H_1) + P(w = 0, u \leq l | H_1) .$$

Denoting by  $p_0, p_1, \dots, p_n$  the probability of  $C$  and  $n$  parts under  $H_1$ , it is readily seen that

$$P(w > 0 | H_1) = 1 - (1 - p_0)^N .$$

Denote by  $P_{i_1 \dots i_k}$  the probability that  $w=0$  and  $N$   $x$ 's fall (see the section 3)  $i_1$ th,  $\dots$ ,  $i_k$ th parts. The probability that  $w=0$  and  $Nx$ 's fall in the union of the  $i_1$ th,  $\dots$ ,  $i_j$ th parts is

$$(p_{i_1} + \dots + p_{i_j})^N = P_{i_1 \dots i_j} + \sum_{(t_1, \dots, t_{j-1})}^j P_{i(t_1) \dots i(t_{j-1})} + \dots + \sum_{(t_1)}^j P_{i(t_1)} ,$$

where  $\sum_{(t_1, \dots, t_l)}$  denotes the summation over all combinations  $(t_1, \dots, t_l)$  drawn from  $(1, 2, \dots, j)$ . In another form

$$(p_{i_1} + \dots + p_{i_j})^N = \sum_{i=1}^j \sum_{(t_1 \dots t_i)} P_{i(t_1) \dots i(t_i)} .$$

Then for every positive integer  $k$ ,

$$\begin{aligned} & \sum_{j=1}^k (-1)^{k-j} \binom{n-j}{k-j} \sum_{(i_1 \dots i_j)}^n (p_{i_1} + \dots + p_{i_j})^N \\ &= \sum_{j=1}^k (-1)^{k-j} \binom{n-j}{k-j} \sum_{(i_1 \dots i_j)}^n \sum_{i=1}^j \sum_{(t_1 \dots t_i)} P_{i(t_1) \dots i(t_i)} \\ &= \sum_{i=1}^k \sum_{(m_1 \dots m_i)}^n P_{m_1 \dots m_i} \sum_{j=i}^k (-1)^{k-j} \binom{n-j}{k-j} \binom{n-\nu}{j-\nu} \\ &= \sum \sum P_{m_1 \dots m_i} (-1)^{k-\nu} \binom{n-\nu}{n-k} \sum_{j=i}^k (-1)^{j-\nu} \binom{k-\nu}{j-\nu} \\ &= \sum_{(m_1 \dots m_k)}^n P_{m_1 \dots m_k} = P(w = 0, u = k | H_1) . \end{aligned}$$

Thus

$$P(w = 0, u = k | H_1) = \sum_{i=1}^k (-1)^{k-i} \binom{n-i}{k-i} \sum_{(i_1 \dots i_j)}^n (p_{i_1} + \dots + p_{i_j})^N , \quad k = 1, \dots, n .$$

Therefore

$$\begin{aligned} P(w = 0, u \leq l | H_1) &= \sum_{k=1}^l P(w = 0, u = k | H_1) \\ &= \sum_{j=1}^l \sum_{k=j}^l (-1)^{k-j} \binom{n-j}{k-j} \sum_{(i_1 \dots i_j)}^n (p_{i_1} + \dots + p_{i_j})^N \\ &= \sum_{j=1}^l \sum_{k=j}^l (-1)^{k-j} \binom{n-j}{k-j} \sum_{(i_1 \dots i_j)}^n (p_{i_1} + \dots + p_{i_j})^N \\ &= \sum_{j=1}^l (-1)^{l-j} \binom{n-j-1}{l-j} \sum_{(i_1 \dots i_j)}^n (p_{i_1} + \dots + p_{i_j})^N . \end{aligned}$$

Thus

$$P = 1 - (1 - p_0)^N + \sum_{j=1}^l (-1)^{l-j} \binom{n-j-1}{l-j} \sum_{(i_1 \dots i_j)}^n (p_{i_1} + \dots + p_{i_j})^N .$$

**6. Moments of  $v$  under  $H_1$  with  $p_0=0$ .** If  $p_0=0$ , so  $P(w=0)=1$ . Then, from the result of the preceding section

$$\begin{aligned} P(u=k|H_1) &= P(w=0, u=k|H_1) \\ &= \sum_{j=1}^k (-1)^{k-j} \binom{n-j}{k-j} \sum_{(i_1 \dots i_j)} (p_{i_1} + \dots + p_{i_j})^N. \end{aligned}$$

Hence

$$\begin{aligned} E(v^{(s)}|H_1) &= \sum_{k=s}^{n-1} k^{(s)} P(v=k|H_1) = \sum_{k=s}^{n-1} k^{(s)} P(u=n-k|H_1) \\ &= \sum_{k=s}^{n-1} \frac{k!}{(k-s)!} \sum_{j=1}^{n-k} (-1)^{n-k-j} \binom{n-j}{k} \sum_{(i_1 \dots i_j)} (p_{i_1} + \dots + p_{i_j})^N \\ &= \sum_{j=1}^{n-s} \frac{(n-s)!}{(n-s-j)!} \sum_{(i_1 \dots i_j)} (p_{i_1} + \dots + p_{i_j})^N \sum_{k=s}^{n-j} (-1)^{n-k-j} \binom{n-s-j}{k-s} \\ &= s! \sum_{(i_1 \dots i_{n-s})} (p_{i_1} + \dots + p_{i_{n-s}})^N \\ &= s! \sum_{(i_1 \dots i_s)} (1-p_{i_1} - \dots - p_{i_s})^N. \end{aligned}$$

Putting  $s=1, 2$ , we get

$$\begin{aligned} E(v|H_1) &= \sum_{i=1}^n (1-p_i)^N, \\ E(v(v-1)|H_1) &= \sum_{i \neq j} (1-p_i-p_j)^N. \end{aligned}$$

### 7. Consistency.

**Theorem 2.** *The test based on  $v$  is consistent against the class of alternative hypothesis  $H_1$  as far as we are concerned with absolute continuous d.f.'s whose density functions are differentiable.*

**Proof.** If  $p_0 > 0$ ,  $(1-p_0)^N$  tends to zero and so  $P$  tends to 1 when  $N \rightarrow \infty$ , that is, the test is consistent against this  $H_1$ .

If  $p_0=0$ ,  $F_1(x)$  is absolutely continuous with respect to  $F_0(x)$  and its relative density is differentiable. Putting  $Y=F_0(X)$ ,  $Y$  is distributed uniformly over  $[0, 1]$  under  $H_0$  and according to d.f.  $G(y)=F_1(F_0^{-1}(y))$  under  $H_1$ . As  $p_0=0$ ,  $G(y)$  is defined uniquely with differentiable derivative  $g(y)$  such that

$$p^i = \int_{(i-1)/n}^{i/n} g(y) dy, \quad i = 1, \dots, n.$$

By Taylor expansion

$$p_i = \frac{1}{n} g\left(\frac{i}{n}\right) - \frac{1}{2n^2} g'\left(\frac{i}{n}\right) + O(n^{-3}),$$

$$(1-p_i)^N = e^{-r g(i/n)} \left[ 1 + \frac{r}{2n} \left\{ g'\left(\frac{i}{n}\right) - g^2\left(\frac{i}{n}\right) \right\} + O(n^{-2}) \right],$$

whence

$$\begin{aligned} E(v/n) &= \sum_{i=1}^n (1-p_i)^N/n \\ &= \int_0^1 e^{-rg(y)} dy - \frac{r}{2n} \int_0^1 g^2(y) e^{-rg(y)} dy + O(n^{-2}) . \end{aligned}$$

In the same manner

$$D^2(v/n) = \frac{1}{n} \left[ \int_0^1 (e^{-rg} - e^{-2rg}) dy - r \left( \int_0^1 g e^{-rg} dy \right)^2 \right] + O(n^{-2}) .$$

As

$$\int_0^1 e^{-rg(y)} dy \geq e^{-r}$$

with the equality if and only if  $g(y) \equiv 1$ , the test is, as in Sherman's case, consistent against  $H_1$ .

### 8. Unbiasedness.

**Theorem 3.** *The test based on  $v$  is unbiased against the class of all alternative hypotheses.*

**Proof.** We need only prove that for any integer  $l (l \leq n)$  the following relation holds :

$$P \geq P(u \leq l | H_0) ,$$

where  $P$  is the power. This is trivial for  $l=0$  or  $l=n$ .

As  $P$  attains its minimum with respect to  $p_0$  at  $p_0=0$ , we have only to prove that, for  $l=1, \dots, n-1$ ,

$$P_0 \equiv \sum_{j=1}^l (-1)^{i-j} \binom{n-j-1}{l-j} \sum_{(i_1 \dots i_j)}^n (p_{i_1} + \dots + p_{i_j})^N$$

attains its minimum with respect to  $(p_1, \dots, p_n)$  at  $p_1 = \dots = p_n = 1/n$ .

As the case  $l=1$  is simple, we consider the cases  $2 \leq l \leq n-1$ .

If all  $p_i$  are not equal, there are  $i, j (i=1, \dots, n)$  such that  $p_i < p_j$ . We can assume without any loss of generality that  $p_{n-1} < p_n$ . Put  $p_n - p_{n-1} = 2\epsilon$  and

$$\begin{aligned} p'_i &= p_i, \quad i = 1, \dots, n-2 ; \\ p'_{n-1} &= p_{n-1} + x, \quad p'_n = p_n - x ; \\ P' &= \sum_{j=1}^l A_j \sum_{(i_1 \dots i_j)}^n (p'_{i_1} + \dots + p'_{i_j})^N , \end{aligned}$$

where

$$A_j = (-1)^{i-j} \binom{n-j-1}{l-j} .$$

$P'$  is a function of  $x$  and we have only to show  $dP'/dx \leq 0$  for  $0 \leq x \leq \epsilon$  in order to complete the proof of the theorem. Now

$$\begin{aligned}
P' &= A_1 \sum_{i=1}^n p_i'^N + \sum_{j=2}^l A_j \sum_{(i_1 \dots i_j)} (p_{i_1}' + \dots + p_{i_j}')^N \\
&= A_1 \left\{ \sum_{i=1}^{n-2} p_i^N + (p_{n-1} + x)^N + (p_n - x)^N \right\} \\
&\quad + \sum_{j=2}^l A_j \left\{ \sum_{(i_1 \dots i_j)} (p_{i_1} + \dots + p_{i_j})^N + \sum_{(i_1 \dots i_{j-1})} (p_{i_1} + \dots + p_{i_{j-1}} + p_{n-1} + x)^N \right. \\
&\quad \left. + \sum_{(i_1 \dots i_{j-1})} (p_{i_1} + \dots + p_{i_{j-1}} + p_n - x)^N + \sum_{(i_1 \dots i_{j-2})} (p_{i_1} + \dots + p_{i_{j-2}} + p_{n-1} + p_n)^N \right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{N} \frac{dP'}{dx} &= A_1 \left\{ (p_{n-1} + x)^{N-1} - (p_n - x)^{N-1} \right\} \\
&\quad + \sum_{j=2}^l A_j \sum_{(i_1 \dots i_{j-1})} \left\{ (p_{i_1} + \dots + p_{i_{j-1}} + p_{n-1} + x)^{N-1} - (p_{i_1} + \dots + p_{i_{j-1}} + p_n - x)^{N-1} \right\}.
\end{aligned}$$

In order to prove  $dP'/dx \leq 0$  for  $0 \leq x < \varepsilon$ , we have only to show that

$$A_1 p^{N-1} + \sum_{j=2}^l A_j \sum_{(i_1 \dots i_{j-1})} (p + p_{i_1} + \dots + p_{i_{j-1}})^{N-1}, \quad p \geq 0,$$

is a monotone increasing function of  $p$ . This is now a polynomial of  $p$  and the coefficients  $B_s$  of  $p^{N-1-s}$  are

$$\begin{aligned}
B_0 &= A_1 + \sum_{j=2}^l A_j \binom{n-2}{j-1}, \\
B_s &= \binom{N-1}{s} \sum_{j=2}^l A_j \sum_{(i_1 \dots i_{j-1})} (p_{i_1} + \dots + p_{i_{j-1}})^s, \quad s = 1, \dots, N-1.
\end{aligned}$$

It suffices to show that  $B_s \geq 0$ ,  $s = 0, 1, \dots, N-1$ .

Now

$$\begin{aligned}
B_0 &= \sum_{j=1}^l A_j \binom{n-2}{j-1} = \sum_{j=1}^l (-1)^{l-j} \binom{n-j-1}{l-j} \binom{n-2}{j-1} \\
&= \binom{n-2}{l-1} \sum_{j=1}^l (-1)^{l-j} \binom{l-1}{l-j} = 0,
\end{aligned}$$

as  $l > 1$ . On the other hand

$$\begin{aligned}
B_s &= \binom{N-1}{s} \sum_{j=1}^{l-1} A_{j+1} \sum_{(i_1 \dots i_j)}^{\frac{n-2}{s}} (p_{i_1} + \dots + p_{i_j})^s \\
&= \binom{N-1}{s} \sum_{j=1}^{l-1} (-1)^{l-1-j} \binom{n-2-j}{l-1-j} \sum_{(i_1 \dots i_j)}^{\frac{n-2}{s}} (p_{i_1} + \dots + p_{i_j})^s.
\end{aligned}$$

By the result of section 5

$$P(w=0, u=k | H_1) = \sum_{j=1}^k (-1)^{k-j} \binom{n-j}{k-j} \sum_{(i_1 \dots i_j)}^{\frac{n-2}{s}} (p_{i_1} + \dots + p_{i_j})^N.$$

Comparing these two equations, we obtain

$$B_s \geq 0,$$

which completes the proof.

**9. Remark.** There is another type of the non-parametric problem. Given two random samples independently from two populations, it is asked whether they have the same d.f.. The run test of Wald and Wolfowitz [4] occupies the same position in this problem as David's test in our problem. Wolfowitz [5] recently proved that their test statistic is asymptotically normally distributed even under the alternative hypothesis. Therefore we may perhaps expect so in our case, though we have not yet succeeded in proving it.

It is a pleasure to express my hearty thanks to Professor J. Ogawa for his valuable suggestions and instructions.

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