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ETALE ENDOMORPHISMS OF ALGEBRAIC VARIETIES

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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1. Introduction

Let k be an algebraically closed field of characteristic zero, which we fix as the ground field throughout this article. Let f: $X \rightarrow X$ be an étale endomorphism of an algebraic variety X. Then f is, in particular, a quasi-finite morphism. We shall be concerned with the following:

PROBLEM. *Is an étale endomorphism f: $X \rightarrow X$ finite?*

If f is set-theoretically injective then f is bijective by Ax's theorem [1, 3]; hence f is an automorphism. If X is complete, f is clearly finite. In the case where X is the affine n-space $\boldsymbol{A}_k^n$, the Jacobian conjecture (cf. [2]) is equivalent to showing that f: $X \rightarrow X$ is finite. In the following *we assume that X is a nonsingular, non-complete algebraic variety.* Our results show that f is an automorphism (hence finite) for a fairly wide class of varieties X, while there are abundant examples of varieties X with non-finite étale endomorphisms.

2. Preliminary result

We recall the logarithmic ramification formula (cf. Iitaka [6]). Let f: $X \rightarrow Y$ be a dominant morphism of nonsingular algebraic varieties. Then there exist nonsingular complete varieties V and W and a dominant morphism ϕ: $V \rightarrow W$ satisfying the following conditions:

(1) X and Y are open subsets of V and W, respectively; hence V and W are nonsingular completions of X and Y, respectively;

(2) the boundaries $D := V - X$ and $\Delta := W - Y$ are the divisors with simple normal crossings; namely, all irreducible components of D (or Δ) are nonsingular subvarieties of codimension 1 intersecting each other normally at every point of intersection of D (or Δ); we denote by the symbol D (or Δ) the reduced divisor whose support is D (or Δ);

(3) the restriction of ϕ onto X coincides with f; hence $\phi^{-1}(\Delta) \subseteq D$.

Denote by K_V (or K_W) the canonical divisor of V (or W). The logarithmic ramification formula then asserts that there exists an effective divisor R_ϕ such that

$$D+K_V \sim \phi^*(\Delta+K_W)+R_\phi\,,$$

where the linear equivalence of divisors is denoted customarily by $\sim$. If the morphism $f\colon X\to Y$ is, moreover, étale, then Supp R_ϕ is contained in D.

The logarithmic ramification formula has the following two consequences.

Lemma 1. *Let X be a nonsingular curve and let $f\colon X\to X$ be a dominant morphism. Let C be the nonsingular completion of X, let g be the genus of C and let n be the number of places of C with center outside X. Let $d:=\deg f$. Then the following assertions hold:*

(1) *If $2g+n\geqq 3$ then f is an automorphism.*

(2) *If $g=0$ and $n=2$ then $X\cong A^1_*:=$the affine line A^1_k with one point* (0) *deleted off; if we identify X with the multiplicative group scheme G_m then $f=T_a\cdot\mu_d$, where T_a is the translation of G_m by a and μ_d is the "multiplication by d" morphism; hence f is finite.*

(3) *If $g=0$ and $n=1$ then $X\cong A^1_k$ and f is finite; if f is étale then f is an automorphism.*

Proof. The morphism $f\colon X\to X$ extends to an endomorphism $\phi\colon C\to C$. Let $D:=C-X$. Then, by the logarithmic ramification formula, we have

$$D+K_C\sim\phi^*(D+K_C)+R_\phi \qquad \text{with} \quad R_\phi\geqq 0\,.$$

Thence we obtain $(1-d)(n+2g-2)=\deg R_\phi\geqq 0$. The assertion (1) then follows immediately. If $g=0$ and $n=2$ then $X\cong A^1_*$, and the assertion (2) is readily verified. The first part of the assertion (3) is clear and easy to verify. If f is étale and $d\geqq 2$ then $R_\phi=(d-1)P_\infty$, where $P_\infty=C-X$. Namely, $\phi\colon C\to C$ ramifies only (and totally) over P_∞. This contradicts the Hurwitz-Riemann formula. Q.E.D.

Theorem 2 (Iitaka [6]). *Let X be a nonsingular algebraic variety with the logarithmic Kodaira dimension $\bar\kappa(X)$ equal to $\dim X$. Let $f\colon X\to X$ be a quasi-finite endomorphism. Then f is an automorphism.*

Proof. We employ the same notations as in the statement of the logarithmic ramification formula, where we set $Y=X$. Since the logarithmic plurigenus $\bar P_m(X)$ is independent of the choice of nonsingular completions $X\subset V$ and $X\subset W$, we have

$$\bar P_m(X)=\dim H^0(V, m(D+K_V))=\dim H^0(W, m(\Delta+K_W))$$

for $m>0$. Then the logarithmic ramification formula implies mR_ϕ is contained in the fixed part of the linear system $|(mD+K_V)|$, i.e.,

$$|m(D+K_V)|=|m\phi^*(\Delta+K_W)|+mR_\phi\,.$$

Let $\Phi_1\colon V\to P^N$ (or $\Phi_2\colon W\to P^N$, resp.) be the rational mapping defined by

$|m(D+K_V)|$ (or $|m(\Delta+K_W)|$, resp.), where $N=\bar{P}_m(X)-1$. If m is sufficiently large, we have then the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] & V \arrow[r, "\Phi_1"] \arrow[d, "\phi"] & \Phi_1(V) = \Phi_2(W) \subset \boldsymbol{P}^N \\ X \arrow[r, hook] & W \arrow[ur, "\Phi_2"'] & \end{tikzcd}$$

where $\Phi_1=\Phi_2\cdot\phi$, and Φ_1: $V\to\Phi_1(V)$ and Φ_2: $W\to\Phi_2(W)$ are, indeed, birational. Hence, so are ϕ and f. Since f is quasi-finite, f is an open immersion by the Zariski main theorem. Then f is an automorphism by virtue of Ax's Theorem.

REMARK. We have the following result by virtue of Iitaka [6; Th. 2]:

Let X be a nonsingular algebraic variety with $\bar{\kappa}(X)\geqq 0$. Then any dominant morphism $f: X\to X$ is an étale morphism.

3. Case where X is an affine surface

Hereafter, we shall assume, unless otherwise specified, that X is a nonsingular affine surface. In view of Lemma 2, we only consider the case where $\bar{\kappa}(X)\leqq 1$. We shall start with the following:

Lemma 3. *Suppose that $\bar{\kappa}(X)=-\infty$ and that one of the following conditions is satisfied:*

(i) *X is irrational but not elliptic ruled,*

(ii) $\Gamma(X, O_X)^*\neq k^*$ *and* rank $(\Gamma(X, O_X)^*/k^*)\geqq 2$ *if X is rational. Then an étale endomorphism f: $X\to X$ is an automorphism.*

Proof. Note that X is affine-ruled because $\bar{\kappa}(X)=-\infty$.

Case (i). Let V be a nonsingular completion of X and let α: $V\to A$ be the Albanese morphism, where $A=\mathrm{Alb}(V/k)$. Let $C=\alpha(X)$ and let $\phi: X\to C$ be the restriction of α onto X. Then C is a nonsingular curve and ϕ defines an $\boldsymbol{A}^1$-fibration on X (cf. [8]). The étale endomorphism f: $X\to X$ then induces an étale endomorphism h: $C\to C$ such that $h\cdot\phi=\phi\cdot f$. By the hypothesis and Lemma 1, h is an automorphism. Let K be the function field of C over k and let X_K be the generic fiber of ϕ which is isomorphic to $\boldsymbol{A}^1_K$. By restricting f onto the generic fiber of ϕ, we obtain an étale K-endomorphism f_K: $\boldsymbol{A}^1_K\to\boldsymbol{A}^1_K$. Lemma 1 implies that $f_K\otimes\bar{K}$: $\boldsymbol{A}^1_{\bar{K}}\to\boldsymbol{A}^1_{\bar{K}}$ is an isomorphism for an algebraic closure $\bar{K}$ of K. Hence, so is f_K. Therefore f is birational, and f becomes an automorphism by virtue of Zariski's main theorem and Ax's theorem.

Case (ii). Let $A=\Gamma(X, O_X)$. Since $\bar{\kappa}(X)=-\infty$, X contains a cylinderlike open set $U_0\times\boldsymbol{A}^1_k=\mathrm{Spec}\,B[x]$, where $U_0=\mathrm{Spec}\,B$ is an affine curve. Hence $A\subset B[x]$ and $A^*\subseteq B^*$. Let R_0 be the k-subalgebra of A generated by all elements of A^* and let R be the normalization of R_0 in A. Then we have $R\subseteq B$.

Hence R is finitely generated. Let $\bar{C}=\operatorname{Spec} R$ and let $\phi: X\to C\subseteq\bar{C}$ be the morphism induced by the canonical injection $R\subseteq A$, where $C=\phi(X)$. Since $R^*\supseteq A^*\supsetneqq k^*$, we know that $\bar{\kappa}(C)\geqq 0$. Let F be a general fiber of ϕ. By virtue of Kawamata's addition formula [7], we have $\bar{\kappa}(F)=-\infty$. Namely, ϕ defines an $\boldsymbol{A}^1$-fibration on X. Moreover, the étale endomorphism $f: X\to X$ induces an étale endomorphism $h: C\to C$, which is an automorphism possibly except the case where $C\cong\boldsymbol{A}^1_*$. But the last case is eliminated by the hypothesis. Now we can verify the assertion by repeating the same arguments as in the preceding case.

Q.E.D.

We next consider the case where X has an $\boldsymbol{A}^1_*$-fibration $\phi: X\to C$; see [8] for the definition and the relevant results. Given such an $\boldsymbol{A}^1_*$-fibration, we have to classify all possible types of singular fibers. This is given in the following:

Lemma 4. *Let $\phi: X\to C$ be an $\boldsymbol{A}^1_*$-fibration on an affine nonsingular surface X over a nonsingular curve C, and let S be a singular fiber of ϕ. Then S is written (as a divisor) in the form $S=\Gamma+\Delta$, where*

(1) *$\Gamma=0$, $\Gamma=\alpha\Gamma_1$ with $\alpha\geqq 1$ and $\Gamma_1\cong\boldsymbol{A}^1_*$, or $\Gamma=\alpha_1\Gamma_1+\alpha_2\Gamma_2$, where $\alpha_1\geqq 1$, $\alpha_2\geqq 1$, $\Gamma_1\cong\Gamma_2\cong\boldsymbol{A}^1_k$ and Γ_1 and Γ_2 meet each other transversally in a single point:*

(2) *$\Delta\geqq 0$, and Supp Δ is a disjoint union of connected components isomorphic to $\boldsymbol{A}^1_k$ provided $\Delta>0$.*

Proof. There exist a nonsingular projective surface V and a surjective morphism $p: V\to B$ onto a complete nonsingular curve B such that:

(i) X and C are open subsets of V and B, respectively, and ϕ is the restriction of p onto X;

(ii) p defines a $\boldsymbol{P}^1$-fibration on V.

Since ϕ defines an $\boldsymbol{A}^1_*$-fibration on X, the boundary divisor $D:=V-X$ contains two cross-sections of p, and since X is affine, D is connected. Let Σ be a singular fiber of p such that $\Sigma\cap X=S$. Then, noting that each irreducible component of Σ is a nonsingular rational curve and that the dual graph of Σ is a tree, we can readily verify the assertion (cf. [8; Chap. I, §6]). Q.E.D.

Lemma 5. *Let $\phi: X\to C$ be the same as in Lemma 4. Let $f: X\to X$ be an étale endomorphism such that $\phi\cdot f=\phi$ and that $\operatorname{codim}_X(X-f(X))\geqq 2$. Let $\tilde{X}$ be the normalization of the lower (the right) X in the function field of the upper (the left) X over k and let $\tilde{\phi}: \tilde{X}\to X$ be the normalization morphism. Then the following assertions hold:*

(1) *There exists an open immersion $\iota: X\hookrightarrow\tilde{X}$ such that $f=\tilde{\phi}\cdot\iota$;*

(2) *$\tilde{\phi}$ makes $\tilde{X}$ an étale Galois covering of X with the cyclic group G of order n as the Galois group, where $n=\deg f$;*

(3) *With the notations of Lemma 4, f is finite over the part Γ of a singular fiber S of ϕ, i.e., $f^*\Gamma$ is invariant under the action of G, and f is totally decomposable*

over the part Δ, i.e., the stabilizer subgroup of each connected component of Δ is trivial.

Proof. Let K be the function field of C over k and let X_K be the generic fiber of ϕ. Then $X_K = \operatorname{Spec} K[x, x^{-1}]$, and f induces an étale K-endomorphism $f_K: X_K \to X_K$. Clearly, f_K is given by a K-endomorphism θ_K of $K[x, x^{-1}]$; $x \mapsto ax^{\pm n}$, where $a \in K^*$ and $n = \deg f$. Let G be the group of all n-th roots of the unity in k, which is a cyclic group of order n. It is then clear that θ_K is invariant under the G-action $(x, \zeta) \mapsto x\zeta$, where $\zeta \in G$; hence f_K is invaiiant under the induced G-action, and the lower X_K is thought of as the quotient variety X_K/G. Now, let P be a closed point of C and let F be the fiber $\phi^*(P)$ of ϕ over P. Suppose F is not a singular fiber. Let $O = O_{C,P}$ and let $X_0 = X \underset{C}{\times} \operatorname{Spec} O$. Then we can choose x above so that $X_0 = \operatorname{Spec} O[x, x^{-1}]$ and the induced endomorphism $f_0: X_0 \to X_0$ is given by an O-endomorphism $x \mapsto ax^{\pm n}$ of $O[x, x^{-1}]$, where $a \in O^*$. Thus the G-action extends over X_0 and the quotient variety X_0/G is the lower X_0. Suppose $F = \phi^*(P)$ is a singular fiber $S = \Gamma + \Delta$. Then, noting that there are no nontrivial morphisms from $\boldsymbol{A}_k^1$ to $\boldsymbol{A}_*^1$ and that $\operatorname{codim}_X (X - f(X)) \geqq 2$, we can readily show that $f_*\Gamma = \Gamma$ and $f_*\Delta = \Delta$ as cycles. In particular, $f: X \to X$ is surjective.

Take the normalization $\tilde{\phi}: \tilde{X} \to X$ as in the above-mentioned fashion. Then G acts on $\tilde{X}$, and the upper X is embedded into $\tilde{X}$ as an open set. If $F = \phi^*(P)$ is a nonsingular fiber of ϕ then $\tilde{X}_0 = X_0$ as shown above. Let F be a singular fiber $S = \Gamma + \Delta$. If $\Gamma \neq 0$ then Γ is invariant under the G-action. Indeed, we cah take a nonsingular completion $p: V \to B$ as in the proof of Lemma 4 as follows. Let $\hat{X}$ be a G-equivariant resolution of singularities of $\tilde{X}$ such that X is still an open set of $\hat{X}$ and that $\hat{X} - X$ consists of nonsingular irreducible components which meet each other at worst normally. We then take a nonsingular completion $p: V \to B$ so that it extends the fibration $\hat{X} \to C$ induced by the $\boldsymbol{A}_*^1$-fibration $\phi \cdot \tilde{\phi}: \tilde{X} \to C$ and that $V - \hat{X}$ is a divisor with simple normal crossings. This is possible by virtue of Sumihiro's equivariant completion theorem [11]. Let $\Sigma = p^*(P)$. Then $\Sigma \cap X = S$ and Σ is G-invariant. If Γ were not G-invariant, then the translation $g^*\Gamma$ of Γ by some element g of G would be a divisor disjoint from Γ and Σ would therefore contain a loop. This is a contradiction. Thus Γ is G-invariant. Now, suppose $\Delta \neq 0$, and let Δ_1 be an irreducible component of Δ. Since Δ_1 and $f(\Delta_1)$ are isomorphic to $\boldsymbol{A}_k^1$ and since the restriction $f_{\Delta_1}: \Delta_1 \to f(\Delta_1)$ is an étale morphism, it is an isomorphism by Lemma 1. This implies that $g(\Delta_1) \neq \Delta_1$ for any non-unit element g of G. Note that $\tilde{\phi}: \tilde{X} \to X$ is étale at the component $g(\Delta_1)$. Since $\phi: X \to X$ is surjective, the above observations imply that $\tilde{\phi}: \tilde{X} \to X$ is étale everywhere and $\tilde{X}$ is, therefore, nonsingular. We thus verified all the assertions. Q.E.D.

We need the following:

Lemma 6. *Let $\tilde{\phi}: \tilde{X} \to X$ be an étale Galois covering of an algebraic variety X with the cyclic gorup G of order n as the Galois group. Then there exists an invertible O_X-module L such that $L^{\otimes n} \cong O_X$ and $X \cong Spec(\bigoplus_{i=0}^{n-1} L^{\otimes i})$. Moreover, we have $\tilde{\phi}^* L \cong O_{\tilde{X}}$.*

Proof. Since the assertion to be verified is of local nature on X, we may and shall assume that X is affine. So, let $X = \mathrm{Spec}\, A$ and $\tilde{X} = \mathrm{Spec}\, \tilde{A}$; we regard A as a subalgebra of $\tilde{A}$. As is well-known, the group G is written as a k-group scheme in the form:

$$G = \mathrm{Spec}\, k[t] \quad \text{with} \quad t^n = 1\,, \quad \mu(t) = t \otimes t\,,$$
$$\varepsilon(t) = 1 \quad \text{and} \quad \eta(t) = t^{-1}\,,$$

where μ, ε and η are respectively the comultiplication, the augmentation and the coinverse. The action of G on $\tilde{X}$ is translated in terms of the following coaction.

$$\Delta:\ \tilde{A} \to \tilde{A}[t]\,, \quad a \mapsto \Delta(a) = \sum_{i=0}^{n-1} \Delta_i(a) t^i\,;$$

see [4] for the relevant results. The property that Δ is a coaction is equivalent to the following properties:

(i) The mapping Δ_i defined by $a \mapsto \Delta_i(a)$ is a k-endomorphism of $\tilde{A}$;

(ii) $\Delta_i \Delta_j = \delta_{ij} \Delta_j$, where δ_{ij} is the Kronecker's delta, and $\sum_{i=0}^{n-1} \Delta_i = 1$ (=the identity);

(iii) $\Delta_i(a) \cdot \Delta_j(b) \in \Delta_{i+j}(\tilde{A})$ for $a, b \in \tilde{A}$, where we take an integer l for $i+j$ with $0 \leqq l < n$ and $l \equiv i+j \pmod n$ if $i+j \geqq n$.

Let $A_i := \Delta_i(A)$, $0 \leqq i < n$; hence $A_0 = A$, which is the G-invariant subalgebra of $\tilde{A}$. In view of the above properties, we have: $\tilde{A} = \sum_{i=0}^{n-1} A_i$, $A_i \cdot A_j \subseteq A_{i+j}$ and A_i is an A-module. Now the property that $\tilde{\phi}$ is étale implies that A_1 is a projective A-module of rank 1, $A_i \cong A_1^{\otimes i}$ $(1 \leqq i < n)$ and $A_1^{\otimes n} \cong A$. Conversely, if A_1 is a projective A-module of rank 1 such that $A_1^{\otimes n} \cong A$, then $\tilde{A} := \sum_{i=0}^{n-1} A_1^{\otimes i}$ is endowed with an A-algebra structure if an isomorphism $\theta: A_1^{\otimes n} \xrightarrow{\sim} A$ is assigned. The group G acts on $\tilde{A}$ as follows: $(\sum_{i=0}^{n-1} a_i)^\zeta = \sum_{i=0}^{n-1} a_i \zeta^i$ if $a_i \in A_1^{\otimes i}$ and ζ is an n-th root of the unity. Clearly, we have $\tilde{\phi}^* L \cong O_X$ because $A_1 \tilde{A} \cong \tilde{A}$. Q.E.D.

As a consequence of Lemmas 4, 5 and 6, we can now prove:

Theorem 7. *Let $\phi: X \to C$ be an $\boldsymbol{A}^1_*$-fibration on an affine nonsingular surface X over a nonsingular curve C and let $f: X \to X$ be an étale endomorphism such* *that $\phi\cdot f=\phi$ and $codim_X(X-f(X))\geqq 2$. Then f is an automorphism in each of the following cases*:

(1) *There exists a singular fiber $S=\Gamma+\Delta$ of ϕ such that $\Gamma=\alpha_1\Gamma_1+\alpha_2\Gamma_2$, where $\alpha_1\geqq 1$, $\alpha_2\geqq 1$, $\Gamma_1\cong\Gamma_2\cong A_k^1$ and Γ_1 and Γ_2 meet each other transversally in one point*; see Lemma 4 for the notations.

(2) $\Gamma(X, O_X)^*=\Gamma(C, O_C).^*$

(3) $\Gamma(C, O_C)=k$, i.e., *C is complete, and there exists a singular fiber $S=\Gamma+\Delta$ of ϕ such that $\Gamma=\alpha\Gamma_1$ with $\alpha\geqq 1$ and $\Gamma_1\cong A_*^1$.*

Proof. We employ the previous notations.

(1) As observed in the proof of Lemma 5, we have $\tilde{\phi}^*\Gamma=f^*\Gamma$. This implies that the fiber S contains n pairs $\Gamma^{(1)},\cdots,\Gamma^{(n)}$ which have the same form as $\Gamma=\alpha_1\Gamma_1+\alpha_2\Gamma_2$. This implies $n=1$. Hence $f: X\to X$ is an automorphism.

(2) Let $X=\operatorname{Spec} A$ and $\tilde{X}=\operatorname{Spec}\tilde{A}$. Then there exists a projective A-module A_1 of rank 1 associated with the étale Galois covering $\tilde{\phi}: \tilde{X}\to X$. Let L be the invertible O_X-module associated with A_1. Then there exists a Weil divisor $D=\sum_i n_iD_i$ (D_i: irreducible; $n_i\neq 0$) such that $L=O_X(D)$. Since the generic fiber X_K of ϕ has the trivial Picard group, we may assume that every irreducible component of D lies in a fiber of ϕ. Let D_1 be an irreducible component of D and let $P:=\phi(D_1)$. If $D_1\cong A_*^1$ we have $f^*D_1=D_1$ as divisors on the upper X (cf. Lemma 5). Suppose that the fiber $S:=\phi^*(P)$ is a singular fiber and $D_1\cong A_k^1$. If the part Γ of S is of the form $\Gamma=\alpha_1\Gamma_1+\alpha_2\Gamma_2$ as in the case (1) above, f is an automorphism. So, we may assume that the part Γ of S (and any singular fiber as well) is not of this form. Suppose that D_1 is a component of Δ. Then f^*D_1 is also a component of Δ. Hence f^* induces a permutation among the components of Δ. Therefore there exists a positive integer N such that $(f^N)^*D_1=D_1$ for any component D_1 of Δ for all possible singular fibers S of ϕ. Then $(f^N)^*D=D$. On the other hand, note that $nD\sim 0$ and $f^*D\sim 0$ (cf. Lemma 6). Hence $D=(f^N)^*D\sim 0$. This implies that $A_1=A\xi$ with $\xi\in\tilde{A}$. Since $\tilde{A}\subseteq$ the upper A, ξ is considered as an element of the upper A. We have $\xi^n=a\in A_0=$ the lower A. If $n=1$ then f is birational. Hence f is an automorphism. So, suppose $n>1$. Since $X=\operatorname{Spec} A[\xi]/(\xi^n-a)$ is an integral scheme, we have $a\notin k^*$. Suppose now $\Gamma(X, O_X)^*=\Gamma(C, O_C)^*$. Then $a\in\Gamma(C, O_C)^*$. Since $\xi\in$ the upper A and $k(C)$ is algebraically closed in $k(X)$, we have $\xi\in k(C)$. Since $\xi^n=a\in\Gamma(C, O_C)$ and C is normal, $\xi\in\Gamma(C, O_C)$. Hence $\xi\in$ the lower A. This is apparently a contradiction.

(3) Suppose next that C is complete and $n:=\deg f>1$. With the notations of Lemma 5, the generic fiber X_K of $\phi: X\to C$ is isomorphic to $\operatorname{Spec} K[x, x^{-1}]$, where we may assume $x\in A$. With the notations and the arguments in the case (2) above, we have $A_1=A\xi$ with $\xi\in$ the upper A. Write $\xi=sx^m$ with $s\in K$ and an integer m. Replacing ξ by ξ^{-1} if necessary, we may

assume $m>0$. Then s, as a rational function on X, has only poles. Hence s^{-1}, as a rational function on C, has only zeroes, i.e., $s^{-1}\in\Gamma(C, O_C)=k$. Write $s=\alpha^m$ with $\alpha\in k^*$. Then $\xi=(\alpha x)^m$. Namely, we may assume $s=1$ and $x\in A^*$. Since m is clearly prime to $n:=\deg f$, we have $A_1=A_0\xi=A_0x$, where $A_0=$the lower A. So, we can identify x with ξ. Let $R=k[\xi, \xi^{-1}]$, let $T=\operatorname{Spec} R$ and let $q: X\to T$ be the morphism induced by the inclusion $R\hookrightarrow A$. Sinc $k(X)=K(\xi)$, X is birational to a product $C\times T$. Indeed, $\psi:=\phi\times q$: $X\to C\times T$ is a birational morphism such that all irreducible components of the part Δ in a singular fiber $S=\Gamma+\Delta$ are contracted to points by ψ, for there are no nontrivial morphisms from $\boldsymbol{A}_k^1$ to $\boldsymbol{A}_*^1$. On the other hand, the given étale endomorphism $f: X\to X$ factors as a composite of an open immersion $\eta: X\hookrightarrow X\underset{P}{\times}(T, g)$ and the base change $g_X: X\underset{T}{\times}(T, g)\to X$ (by $q: X\to T$) of a morphism $g: T\to T$ defined by $\xi\mapsto\alpha\xi^n$, $\alpha\in k^*$; replacing ξ by $\beta\xi$ with $\beta\in k$ and $\beta^{n-1}=\alpha$, we may assume $\alpha=1$. Let $F=\phi^*(P)$ be a fiber of $\phi: X\to C$. If F is nonsingular, i.e., $F\cong\boldsymbol{A}_*^1$, then $f^*F=\tilde{\phi}^*F$ by Lemma 5. This implies that $q|_F: F\to T$ is an isomorphism. Hence an arbitrary fiber of q meets F transversally in one point, i.e., a section of ϕ over the point P. Suppose that F is a singular fiber $S=\Gamma+\Delta$, where $\Gamma=\Gamma\alpha_1$ with $\Gamma_1\cong\boldsymbol{A}_*^1$. Since $f^*\Gamma_1=\tilde{\phi}^*\Gamma_1$, we know that $\alpha=1$ and $q|_{\Gamma_1}: \Gamma_1\to T$ is an isomorphism, i.e., an arbitrary fiber of q is a section over the point P. Suppose that such a singular fiber $S=\Gamma+\Delta$ as above exists. Then $\Delta\neq 0$. Let Δ_1 be an irreducible component of Δ and let $Q:=q(\Delta_1)$. Then X and Δ_1 are obtained from $C\times T$ by blowing up the point (P, Q) and its infinitely near points and by deleting several exceptional curves. Hence the point $\Gamma\cap q^{-1}(Q)$ should have been deleted off. This is a contradiction. Thus, if one assumes the existence of a singular fiber S as above, f must be an automorphism.

For a later use, we continue an analysis of the morphism $q: X\to T$. If ϕ has no singular fibers, the morphism $\psi: X\to C\times T$ is an isomorphism. Then X is not affine. So, this is not the case, and at least one singular fiber of ϕ exists. Let $S_i=\phi^*(P_i)$ $(1\leqq i\leqq r)$ be all singular fibers of ϕ. If $n:=\deg f>1$, any S_i is of the form $S_i=\Delta_i$, i.e., $\Gamma_i=0$, by virtue of the case (3) above. Let $C_0:=C-\{P_1, \cdots, P_r\}$. Then $X-\bigcup_{i=1}^{r}\operatorname{Supp} S_i\cong C_0\times T$. Any singular fiber L of q is of the form $L=M+N$, where $C_0\subseteq M\subset C$ if one identifies M with an open set of C by ϕ, and where N is a disjoint union of irreducible components isomorphic to $\boldsymbol{A}_k^1$. Q.E.D.

REMARK. *With the same situations as in the proof of the case* (3) *of Theorem* 8, *every irreducible component N_1 of N meets M transversally in one point provided* $n:=\deg f>1$.

Proof. Let $L_1, \cdots, L_e$ be all singular fibers of q and let $Q_j:=q(L_j)$. Suppose

$\xi = c_j \in k$ at the point Q_j. As seen above, $g\colon T \to T$ is defined by $g^*(\xi) = \xi^n$. Since $f\colon X \to X$ is surjective, it is easily ascertained that $f_*(L_j) = L_{\sigma(j)}$ (as cycles) for $1 \leqq j \leqq e$, where σ is a permutation on the set $\{Q_1, \cdots, Q_e\}$. Replacing f by a suitable power f^N $(N > 0)$ if necessary, we may assume that $g(Q_j) = Q_j$ for $1 \leqq j \leqq e$. Then $c_j^n = c_j$, i.e., c_j is an $(n-1)$-*st* root of the unity. Let L be one of L_j's, and write $L = M + \nu_1 N_1 + \cdots + \nu_b N_b$, where $N_i \cong \boldsymbol{A}_k^1$ $(1 \leqq i \leqq b)$. Since $f_* L = L$ and $f_* M = M$, f_* induces a permutation on the set $\{N_1, \cdots, N_b\}$. Replacing f again by a suitable power of f, we may assume $f_* N_i = N_i$ for $1 \leqq i \leqq b$.

Suppose that a singular fiber L of q has an irreducible component N_1 such that $M \cap N_1 = \phi$. We shall show that this assumption together with the hypothesis $n := \deg f > 1$ leads to a contradiction. Let $P := \phi(N_1)$ and let $Q := q(N_1)$, where $\xi = c \in k$. The component N_1 is produced by blowing up the point (P, Q) of $C \times T$ and its infinitely near points and by throwing off several exceptional curves. Let x be a local parameter of C at the point P. Then $(\xi - c, x)$ is a system of local parameters of $C \times T$ at the point (P, Q). Since X is affine and $M \cap N_1 = \phi$, we find, in the course of blowing-ups to obtain N_1, an exceptional curve $E \cong \boldsymbol{P}^1$ with an inhcmogeneous coordinate $t := (\xi - c)^\alpha / x^\beta$ (α, β: positive integers) and a point $t = \gamma \in k^*$ to be blown up further.

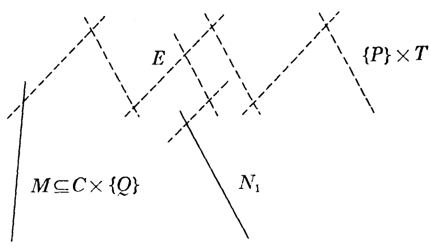

Figure 1

On the surface X_0 (=the lower X), we have the same situation. Namely, there exist an exceptional curve E_0 with an inhomogeneous coordinate $t_0 := (\xi_0 - c)^\alpha / x^\beta$ and a point $t_0 = \gamma$ on E_0. Let $\theta\colon k(X_0) \to k(X)$ be the homomorphism induced by f, i.e., $\theta(x) = x$ and $\theta(\xi_0) = \xi^n$. Then we have

$$\theta(t_0) = (\xi^n - c)^\alpha / x^\beta = t(\xi^{n-1} + \xi^{n-2} c + \cdots + \xi c^{n-2} + c^{n-1})^\alpha .$$

The rational mapping f induces a rational mapping $\sigma\colon E_0 \to E$ which is defined by the assignment

$$\tau := \sigma^*(t_0) = \theta(t_0) \pmod{x} = t(nc^{n-1})^\alpha = n^\alpha t , \quad \text{where} \quad c^n = c ,$$

and which is regular at the point $t = \gamma$. Hence the point $t = \gamma$ is sent to the point $t_0 = n^\alpha \gamma$ under σ. Since f sends the upper N_1 to the lower N_1, the point $t_0 = n^\alpha \gamma$ must coincide with the point $t_0 = \gamma$, which implies $n = 1$. This is a contradiction. Q.E.D.

Let $\phi: X \to C$ be anew as urjective morphism from a nonsingular affine surface X onto a nonsingular curve C. We say that ϕ defines a twisted $\boldsymbol{A}^1_*$-fibration on X if the generic fiber X_K of ϕ is a nontrivial K-form of $\boldsymbol{A}^1_*$, where $K=k(C)$. Then there exists a quadratic extension $\tilde{K}$ of K scuh that $X_K \underset{K}{\otimes} \tilde{K} \cong \boldsymbol{A}^1_{*,\tilde{K}}$. Let $\rho: \tilde{C} \to C$ be the normalization of C in $\tilde{K}$ and let $\nu: \tilde{X} \to X \underset{C}{\times} \tilde{C}$ be the normalization of $X \underset{C}{\times} \tilde{C}$ in the function field $\tilde{K}(X)$. Let $\tilde{\phi}: \tilde{X} \to \tilde{C}$ be the composite of ν and the projection $X \underset{C}{\times} \tilde{C}$ onto $\tilde{C}$. Let F be a closed fiber of ϕ. If F is reduced and isomorphic to $\boldsymbol{A}^1_*$, F is said to be nonsingular. Otherwise, F is called singular. We shall then show the following:

Lemma 8. *With the above notations, the following assertions hold true*:

(1) *$\tilde{X}$ is a nonsingular affine surface and $\tilde{\phi}: \tilde{X} \to \tilde{C}$ defines an $\boldsymbol{A}^1_*$-fibration on $\tilde{X}$.*

(2) *Let f be an étale endomorphism such that $\phi=\phi \cdot f$ and $\mathrm{codim}_X (X-f(X)) \geqq 2$. Then f extends uniquely to an étale endomorphism $\tilde{f}: \tilde{X} \to \tilde{X}$ such that $\tilde{\phi}=\tilde{\phi} \cdot \tilde{f}$ and $\mathrm{codim}_{\tilde{X}} (\tilde{X}-\tilde{f}(\tilde{X})) \geqq 2$. Conversely, if $\tilde{f}: \tilde{X} \to \tilde{X}$ is an étale endomorphism such that $\tilde{\phi}=\tilde{\phi} \cdot \tilde{f}$, $\mathrm{codim}_{\tilde{X}} (\tilde{X}-\tilde{f}(\tilde{X})) \geqq 2$ and $\iota \cdot \tilde{f}=\tilde{f} \cdot \iota$, where $\iota: \tilde{X} \to \tilde{X}$ is the canonical involution associated with the double covering $\theta: \tilde{X} \to X$, then there exists an étale endomorphism $f: X \to X$ such that $\phi=\phi \cdot f$, $\mathrm{codim}_X (X-f(X)) \geqq 2$ and f extends uniquely to $\tilde{f}$.*

Proof. As in the case of an $\boldsymbol{A}^1_*$-fibration, a twisted $\boldsymbol{A}^1_*$-fibration is induced by a $\boldsymbol{P}^1$-fibration on a suitable nonsingular completion of X. In view of this fact, we can show that a singular fiber S of a twisted $\boldsymbol{A}^1_*$-fibration is written in the same form $S=\Gamma+\Delta$ as in Lemma 4. Let $P := \phi(S)$. If $\rho: \tilde{C} \to C$ is not ramified over P, then $\rho^{-1}(P)=\{\tilde{P}_1, \tilde{P}_2\}$ and $\tilde{\phi}^*(\tilde{P}_i)$ $(i=1, 2)$ has the same form as S as cycles. If P is a branch point of ρ then $S=\Delta=\sum_{i=1}^{r} a_i \Delta_i$, where $\Delta_i \cong \boldsymbol{A}^1_k$ and $a_i>0$. Indeed, there exist a nonsingular projective surface V and a surjective morphism $p: V \to B$ onto a nonsingular complete curve B such that X and C are open subsets of V and B, respectively, that $\phi=p|_X$ and that p defines a $\boldsymbol{P}^1$-fibration on V. Furthermore, we may assume that the boundary divisor $D := V-X$ is a divisor with simple normal crossings. Since $\phi: X \to C$ is a twisted $\boldsymbol{A}^1_*$-fibration, there exists an irreducible component D_1 of D such that $\tilde{C} \cong p^{-1}(C) \cap D_1$ and ϕ induces the double covering $\rho: \tilde{C} \to C$. Since P is a branch point of ρ, the fiber $p^{-1}(P)$ touches D_1 at the point $\tilde{P} := D_1 \cap p^{-1}(P)$. If $\Gamma \neq 0$, then there would be two connected chains of irreducible components Σ_1 and Σ_2 in the fiber $p^{-1}(P)$ which connect the point P with two (missing) end points at infinity of Γ_{red}. Thus $p^{-1}(P)$ would contain a loop, which is a contradiction.

We can readily show that $\tilde{X}$ is a nonsingular affine surface and that, for a singular fiber $S=\sum_{i=1}^{r} a_i\Delta_i$ over a branch point P of ρ,

$$\theta^*(\Delta_i)=\begin{cases} 2\tilde{\Delta}_i & \text{if } a_i\equiv 1 \pmod 2 \\ \Delta_i^{(1)}+\Delta_i^{(2)} & \text{if } a_i\equiv 0 \pmod 2 \end{cases}$$

and $\tilde{\Delta}:=\tilde{\phi}^*(\tilde{P})=\sum_{a_i\equiv 0(2)}\frac{a_i}{2}(\Delta_i^{(1)}+\Delta_i^{(2)})+\sum_{a_i\equiv 1(2)}a_i\tilde{\Delta}_i$, where $\tilde{\Delta}_i\cong \boldsymbol{A}_k^1$ and $\Delta_i^{(j)}\cong\boldsymbol{A}_k^1$ $(j=1, 2)$. Let f be an étale endomorphism of X such that $\phi=\phi\cdot f$ and $\mathrm{codim}_X(X-f(X))\geqq 2$. Then $\Delta=f^*\Delta$ for Δ as above, and since $\mathrm{codim}_X(X-f(X))\geqq 2$, $f^*\Delta_i=\Delta_{\sigma(i)}$ with a permutation σ on $\{1, 2, \cdots, r\}$ and $a_i=a_{\sigma(i)}$. Since $\phi=\phi\cdot f$, f extends to an endomorphism $\tilde{f}$ of $\tilde{X}$ such that $\tilde{\phi}=\tilde{\phi}\cdot\tilde{f}$ and $\theta\cdot\tilde{f}=f\cdot\theta$. Then we have

$$\begin{aligned} \tilde{f}^*(\tilde{\Delta}_i) &= \tilde{\Delta}_{\sigma(i)} && \text{if } a_i\equiv 1 \pmod 2 \\ \tilde{f}^*(\Delta_i^{(1)}+\Delta_i^{(2)}) &= \Delta_{\sigma(i)}^{(1)}+\Delta_{\sigma(i)}^{(2)} && \text{if } a_i\equiv 0 \pmod 2 . \end{aligned}$$

Let $\iota\colon \tilde{X}\to\tilde{X}$ be the involution of the double covering $\theta\colon\tilde{X}\to X$ which is induced by the involution ι of $\rho\colon\tilde{C}\to C$. Then $\iota\cdot\tilde{f}=\tilde{f}\cdot\iota$, and $\iota^*\Delta_i^{(1)}=\Delta_i^{(2)}$ if $a_i\equiv 0 \pmod 2$. Hence, by exchanging $\Delta_{\sigma(i)}^{(1)}$ and $\Delta_{\sigma(i)}^{(2)}$ if necessary, we may assume that $\tilde{f}^*(\Delta_i^{(j)})=\Delta_{\sigma(i)}^{(j)}$ $(j=1, 2)$ if $a_i\equiv 0 \pmod 2$. This implies that $\tilde{f}$ is étale and that $\mathrm{codim}_{\tilde{X}}(\tilde{X}-\tilde{f}(\tilde{X}))\geqq 2$.

Conversely, suppose that an étale endomorphism $\tilde{f}\colon\tilde{X}\to\tilde{X}$ is given as stated above. Since X is the quotient variety of $\tilde{X}$ with respect to the involution ι, $\tilde{f}$ descends down to an endomorphism $f\colon X\to X$ such that $\phi=\phi\cdot f$ and $\mathrm{codim}_X(X-f(X))\geqq 2$. It is easy to verify that f is étale. Q.E.D.

Corollary 9. *Let $\phi\colon X\to C$ be the same as in Lemma 8 and let $f\colon X\to X$ be an étale endomorphism such that $\phi=\phi\cdot f$ and $\mathrm{codim}_X(X-f(X))\geqq 2$. Suppose that C is complete and that ϕ has a singular fiber $S=\Gamma+\Delta$ with $\Gamma\neq 0$. Then f is an automorphism.*

Proof. Take $\tilde{\phi}\colon\tilde{X}\to\tilde{C}$ and $\tilde{f}\colon\tilde{X}\to\tilde{X}$ as in Lemma 8. Then $\tilde{C}$ is complete. As shown in the above proof, the point $P:=\phi(S)$ is not a branch point of ρ. Thus $\tilde{\phi}$ has two singular fibers of the same form as S. By Theorem 7, $\tilde{f}$ is an automorphism. Hence, so is f. Q.E.D.

REMARK. Let X be a nonsingular affine surface. Suppose that either $\bar{\kappa}(X)=1$ or $\bar{\kappa}(X)=0$ and X is irrational. Then, as shown in [8; Chap. II, §5], X has a surjective morphism $\phi\colon X\to C$ onto a nonsingular curve which defines either an $\boldsymbol{A}_*^1$-fibration or a twisted $\boldsymbol{A}_*^1$-fibration on X. Let $f\colon X\to X$ be an étale endomorphism such that $\mathrm{codim}_X(X-f(X))\geqq 2$. Then, as in the proof of Lemma 2, we can show that $\phi\cdot f=\phi$. We are interested in determining in which cases f becomes an automorphism. However, as Theorem 7 and Lemma 8 show, this is not an easy task. One obstacle is the existence of singular fibers $S=\Gamma+\Delta$ of ϕ with $\Gamma=0$.

4. Counterexamples

EXAMPLE 1. Let C be a complete nonsingular curve of genus $g(C)$, and let $T=\operatorname{Spec} k[\xi, \xi^{-1}]$, which is isomorphic to A^1_*. Let Q_1 and Q_2 be respectively the points of T defined by $\xi=1$ and $\xi=-1$. Choose two distinct points P_1 and P_2 of C. Let $C_i:=C\times\{Q_i\}$ and $T_i:=\{P_i\}\times T\,(i=1, 2)$ which are the curves on the product $Y:=C\times T$. Let $\sigma: Z\to Y$ be the blowing-up with centers (P_1, Q_1) and (P_2, Q_2), and let $E_i=\sigma^{-1}((P_i, Q_i))$ $(i=1, 2)$. Let $X:=Z-\sigma'T_1-\sigma'T_2$, where $\sigma'T_i$ $(i=1, 2)$ is the proper transform of T_i by σ.

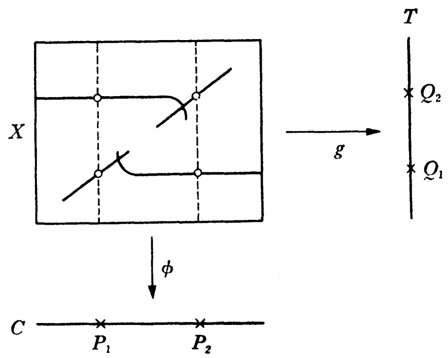

Figure 2

As shown in the above figure, let $\phi: X\to C$ and $q: X\to T$ be the morphisms induced naturally by the projections from $C\times T$ onto C and T, respectively. Then ϕ defines an A^1_*-fibration for which $\phi^*(P_1)=\Delta_1$ and $\phi^*(P_2)=\Delta_2$ exhaust the singular fibers, where $\Delta_i:=E_i-E_i\cap\sigma'T_i\cong A^1_k$.

On the other hand, let $g: T\to T$ be the endomorphism defined by $g^*(\xi)=\xi^3$, and let $\tilde{X}:=X\times_T(T, g)$, the base change of $q: X\to T$ by $g: T\to T$. Let $\tilde{q}: \tilde{X}\to T$ be the canonical projection. Then $\tilde{q}$ has 6 singular fibers $L_{1j}=\tilde{q}^*(Q_{1j})$ and $L_{2j}=\tilde{q}^*(Q_{2j})$ $(j=1, 2, 3)$, where Q_{1j} $(j=1, 2, 3)$ is defined by $\xi=\omega^{j-1}$ and Q_{2j} $(j=1, 2, 3)$ is defined by $\xi=-\omega^{j-1}$; ω is a primitive cubic root of the unity. The fibers L_{1j} and L_{2j} have the same forms as the fibers $L_1:=q^*(Q_1)$ and $L_2:=q^*(Q_2)$, respectively. Write $L_{1j}=M_{1j}+\Delta_{1j}$ and $L_{2j}=M_{2j}+\Delta_{2j}$, where $\Delta_{1j}\cong\Delta_{2j}\cong A^1_k$ and M_{1j} and M_{2j} are considered as open sets of C.

It is then easy to verify that X is affine, that $X_1:=\tilde{X}-\sum_{i=1}^{2}(\Delta_{i2}+\Delta_{i3})$ is isomorphic to X, and that the composite of an open immersion $X_1\hookrightarrow\tilde{X}$ and the canonical projection $\tilde{X}\to X$ is an étale endomorphism of X with degree 3 which is surjective but not finite. Moreover, $\bar{\kappa}(X)=1$ if $g(C)>0$ and $\bar{\kappa}(X)=0$ if $g(C)=0$. In fact, if $g(C)=0$ then $X\cong F_0-D_1\cup D_2$, where $F_0\cong \boldsymbol{P}_k^1\times\boldsymbol{P}_k^1$ and $D_1\sim D_2\sim M+l$, M and l being fibers of two distinct $\boldsymbol{P}^1$-fibrations on F_0.

EXAMPLE 2. In the example 1, assume that C is rational. Choose an inhomogeneous coordinate η on C so that $\eta=0$ at P_1 and $\eta=\infty$ at P_2. Let $\iota\colon Y\to Y$ be the involution defined by $\iota^*(\xi)=\xi^{-1}$ and $\iota^*(\eta)=-\eta$. Since $\iota((P_i, Q_i))=(P_i, Q_i)$ $(i=1, 2)$, ι lifts up to an involution $\iota\colon X\to X$ such that $\iota\cdot f=f\cdot\iota$. Let $\hat{X}$ be the quotient variety of X with respect to ι. Then $\hat{X}$ is a nonsingular affine surface endowed with the twisted $\boldsymbol{A}_*^1$-fibration $\hat{\phi}\colon \hat{X}\to\boldsymbol{P}_k^1$, which is induced by the $\boldsymbol{A}_*^1$-fibration $\phi\colon X\to C$. By Lemma 8, the étale endomorphism $f\colon X\to X$ induces an étale endomorphism $\hat{f}\colon \hat{X}\to\hat{X}$ of degree 3 such that $\theta\cdot f=\hat{f}\cdot\theta$, where $\theta\colon X\to\hat{X}$ is the quotient morphism; $\hat{f}$ is surjective but not finite. The surface $\hat{X}$ is, indeed, constructed in the following way. Let D be an irreducible curve on $F_0\cong\boldsymbol{P}_k^1\times\boldsymbol{P}_k^1$ such that $D\sim 2M+l$. Let $p=\Phi_{|l|}\colon F_0\to\boldsymbol{P}_k^1$ be the projection along the fibers l. Then D is a nonsingular rational curve, and $p|_D\colon D\to\boldsymbol{P}_k^1$ is a double covering. Then $\hat{X}$ is isomorphic to F_0-D, and $\hat{\phi}\colon \hat{X}\to\boldsymbol{P}_k^1$ coincides with the restriction of p onto $\hat{X}$; see the following figure:

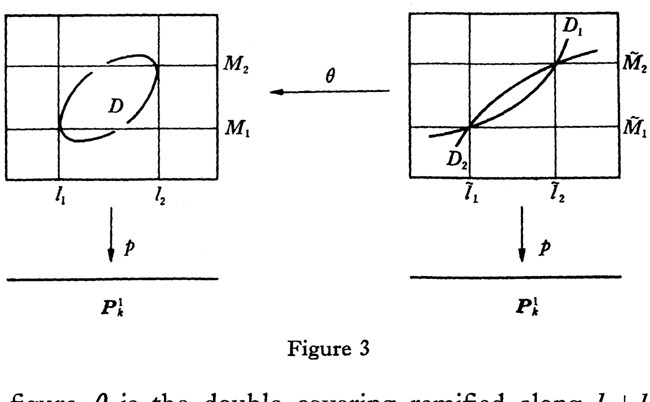

Figure 3

In the above figure, θ is the double covering ramified along l_1+l_2; $\theta^*(l_i)=2\tilde{l}_i$, $\theta^*(M_i)=\tilde{M}_i$ $(i=1, 2)$ and $\theta^*(D)=D_1+D_2$, where $D_1\sim D_2\sim\tilde{M}_1+\tilde{l}_1$. The logarithmic Kodaira dimension $\bar{\kappa}(\hat{X})$ is $-\infty$, $\mathrm{Pic}(\hat{X})\cong\boldsymbol{Z}$ and $\Gamma(\hat{X}, O_{\hat{X}})^*=k^*$.

EXAMPLE 3. Let C be a nonsingular cubic curve on $\boldsymbol{P}_k^2$ and let $X:=\boldsymbol{P}_k^2-C$. Then $\bar{\kappa}(X)=0$ and $\mathrm{Pic}(X)\cong\boldsymbol{Z}/3\boldsymbol{Z}$. Furthermore, X has no $\boldsymbol{A}_*^1$-fibrations nor twisted $\boldsymbol{A}_*^1$-fibrations. We shall show that X has a surjective, non-finite, étale endomorphism $f\colon X\to X$ of degree 3.

Let $\pi\colon W\to\boldsymbol{P}_k^2$ be a triple cyclic covering of $\boldsymbol{P}_k^2$ which ramifies totally over C. This is constructed as follows: Let $\boldsymbol{L}$ be the line bundle $O_{\boldsymbol{P}}(1)$. Choose an open covering $\{U_\alpha\}$ of $\boldsymbol{P}_k^2$ such that $\boldsymbol{L}|_{U_\alpha}=\mathrm{Spec}\, O_{U_\alpha}[\zeta_\alpha]$ with a fiber-coordinate ζ_α and that $C\in|O_{\boldsymbol{P}}(3)|$ is defined by $a_\alpha=0$, where $a_\alpha\in\Gamma(U_\alpha, O_{\boldsymbol{P}})$.

Then $\zeta_\beta=\zeta_\alpha f_{\alpha\beta}$ and $a_\beta=a_\alpha\zeta_{\alpha\beta}^3$ with transition functions $\{f_{\alpha\beta}\}$. Define a subvariety W in $\boldsymbol{L}$ locally over U_α by the equation $\zeta_\alpha^3=a_\alpha$; local data then patch together. Let W_0 be the zero section of $\boldsymbol{L}$, and complete $\boldsymbol{L}$ to a $\boldsymbol{P}^1$-bundle V over $\boldsymbol{P}_k^2$ by adding the infinity section W_∞. Then we have:

$$K_V\sim p^*(K_P)-W_0-W_\infty\,,\quad p^*\boldsymbol{H}\sim W_0-W_\infty$$
$$W\sim 3W_0 \quad\text{and}\quad K_W\sim(K_V+W)_W\,,$$

where $p\colon V\to\boldsymbol{P}_k^2$ is the canonical projection and H is a hyperplane on $\boldsymbol{P}_k^2$. Since $W_0\cap W_\infty=\phi$, we have $K_W\sim -W_0|_W$; we denote $W_0|_W$ by the same letter W_0. Thus $K_W\sim -W_0$. Apparently, $\pi:=p|_W\colon W\to\boldsymbol{P}_k^2$ is a cyclic covering of order 3 which ramifies totally over C. Hence $\pi^*(C)=3W_0$. Since $X:=\boldsymbol{P}^2-C$ is affine, $\pi\colon W-W_0\to X$ is a finite étale covering and $W-W_0$ is affine. Hence W_0 is ample on W. This implies that W is a del Pezzo surface of degree $(K_W^2)=(W_0^2)=3$. Therefore W is a cubic hypersurface in $\boldsymbol{P}_k^3$ and W_0 is a hyperplane section.

As is well-known, W is obtained from $\boldsymbol{P}_k^2$ by blowing up 6 points $P_1,\cdots,P_6$ in general position. Let $\sigma\colon W\to\boldsymbol{P}_k^2$ be the blowing-up of these six points, and let $E_i=\sigma^{-1}(P_i)$. Then $(E_i\cdot W_0)=1$, i.e., E_i is a line of $\boldsymbol{P}_k^3$. Let $C':=\sigma(W_0)$. Then the points $P_1,\cdots,P_6$ lie on C', and C' is isomorphic to W_0, hence to C. Let $X':=\boldsymbol{P}_k^2-C'$. Then X' is isomorphic to $W-(W_0+E_1+\cdots+E_6)$ under σ. Let $f\colon X'\to X$ be the composite

$$f\colon X'\xrightarrow{\sigma^{-1}} W-(W_0+E_1+\cdots+E_6)\xrightarrow{\pi} X\,.$$

Then f is a non-finite étale morphism. Since $C\cong C'$, it is well-known that C is isomorphic to C' by a linear transformation of $\boldsymbol{P}_k^2$. Hence X' is isomorphic to X.

So, it remains to show that f is surjective. Note that C has 9 flexes $Q_1,\cdots,Q_9$. Let l_j $(1\leqq j\leqq 9)$ be the tangent line to C at Q_j, and let R_j be the unique point of W_0 lying over Q_j. Then, for each $1\leqq i\leqq 9$, $\pi^*(l_j)=E_{j1}+E_{j2}+E_{j3}$, where E_{ji} $(1\leqq i\leqq 3)$ is an exceptional curve of the first kind such that $E_{j1}\cap E_{j2}\cap E_{j3}=\{R_j\}$ and $(E_{j1}\cdot E_{j2})=(E_{j2}\cdot E_{j3})=(E_{j3}\cdot E_{j1})=1$. Thus, W contains 27 exceptional curves. The exceptional curves $E_1,\cdots,E_6$ are disjoint from each other. Hence at most one of $E_1,\cdots,E_6$ is contained in $\{E_{j1},E_{j2},E_{j3}\}$ for each $1\leqq j\leqq 9$. This implies that f is surjective.

5. Finite étale endomorphisms

We shall prove the following:

Theorem 10. *Let X be a nonsingular affine surface with an étale endomorphism $f\colon X\to X$. Suppose $n:=\deg f>1$ and $\operatorname{codim}_X(X-f(X))\geqq 2$. Let $\tilde{X}$ be* *the normalization of the lower X in the function field of the upper X over the field k. Suppose $\tilde{X}$ is nonsingular. When we regard the upper X as an open subset of $\tilde{X}$, then $\tilde{X}-X$ is a disjoint union of irreducible curves which are isomorphic to $\boldsymbol{A}_k^1$.*

Proof. By virtue of Theorem 2, X has the logarithmic Kodaira dimension $\leqq 1$. We consider each of the following cases separately: $\bar{\kappa}(X)=-\infty$, 0 and 1.

(I) *Case* $\bar{\kappa}(X)=1$. As in the proof of Theorem 2, we consider nonsingular completions, the upper $X\subset V$ and the lower $X\subset W$, such that $D:=V-X$ and $\Delta:=W-X$ are divisors with simple normal crossings and that $f: X\to X$ extends to a morphism $\psi: V\to W$. By the logarithmic ramification formula, we have, for every $m>0$

$$|m(D+K_V)|=|m\psi^*(\Delta+K_W)|+mR_\psi$$

with an effective divisor R_ψ. Let $\Phi_1:=\Phi_{|m(D+K_V)|}$ and $\Phi_2:=\Phi_{|m(\Delta+K_W|)}$. Then $\Phi_1=\Phi_2\cdot\psi$, and both Φ_1 and Φ_2 are morphisms because $\bar{C}:=\Phi_1(V)=\Phi_2(W)$ is a nonsingular complete curve for a sufficiently large $m>0$. Moreover, Φ_1 (the upper X)$=\Phi_2$ (the lower X), which we denote by C, because $\operatorname{codim}_X(X-f(X))\geqq 2$ and $\Phi_1=\Phi_2\cdot\psi$. By Iitaka [6], we have $\Phi_1|_X=\Phi_2|_X$, i.e., it is independent of the choice of nonsingular completions. So, denoting $\Phi_1|_X: X\to C$ by ϕ, we have $\phi=\phi\cdot f$. By virtue of [8; Chap. II, §5], $\phi: X\to C$ defines either an $\boldsymbol{A}_*^1$-fibration or a twisted $\boldsymbol{A}_*^1$-fibration. Suppose $\phi: X\to C$ is an $\boldsymbol{A}_*^1$-fibration. We have then the same situation as considered in Lemmas 4 and 5. We already observed that $\tilde{X}-X$ is a disjoint union of irreducible curves which are isomorphic to $\boldsymbol{A}_k^1$. Suppose $\phi: X\to C$ is a twisted $\boldsymbol{A}_*^1$-fibration. Then there exists a double covering $\rho: C_1\to C$ such that $\phi_1: X_1\to C_1$ is an $\boldsymbol{A}_*^1$-fibration, where X_1 is the normalization of $X\underset{C}{\times}C_1$ and ϕ_1 is the composite of the normalization morphism $X_1\to X\underset{C}{\times}C_1$ and the projection $X\underset{C}{\times}C_1\to C_1$; see Lemma 8, where the notations differ slightly from the present notations. The endomorphism $f: X\to X$ induces an étale endomorphism $f_1: X_1\to X_1$ such that $\deg f_1=\deg f$ and $\operatorname{codim}_{X_1}(X_1-f_1(X_1))\geqq 2$. Let $\tilde{X}_1$ be the normalization of the lower X_1 in the function field of the upper X_1 over k. Then it is readily verified that $\tilde{X}_1$ is the normalization of $\tilde{X}\underset{C}{\times}C_1$ in its function field over k, where $\tilde{X}$ is the normalization of the lower X in the function field of the upper X over k. More precisely, $\tilde{X}_1$ has an involution $\iota: \tilde{X}_1\to\tilde{X}_1$ induced by the involution of the double covering $\rho: C_1\to C$, and $\tilde{X}$ is the quotient variety of $\tilde{X}_1$ with respect to ι. As shown above, the complement $\tilde{X}_1-X_1$ is a disjoint union of irreducible curves which are isomorphic to $\boldsymbol{A}_k^1$. Therefore, the complement $\tilde{X}-X$ is a disjoint union of the affine lines as well; see the proof of Lemma 8.

(iII) *Case where* $\bar{\kappa}(X)=0$ *and X is irrational.* As in the proof of Lemma 3, let $\phi: X\to C$ be a surjective morphism onto a nonsingular curve C which is defined by the Albanese morphism $\alpha: V\to\operatorname{Alb}(V/k)$, where V is a nonsingular completion of X. The morphism $\phi: X\to C$ defines either an A^1_*-fibration or a twisted A^1_*-fibration (cf. [8; Chap. II, §5]). Then the endomorphism $f: X\to X$ induces an étale endomorphism $h: C\to C$ such that $h\cdot\phi=\phi\cdot f$. Let $X_0=X\times_C(C, h)$ be the fiber product of $\phi: X\to C$ and $h: C\to C$, and let $\phi_0: X_0\to C$ be the projection onto the second factor. Then f induces an étale morphism $g: X\to X_0$ such that $\phi=\phi_0\cdot g$ and f is the composite of g and the projection $X_0\to X$. Let X_1 be the normalization of X_0 in the function field of the upper X over k. Since h is finite by Lemma 1, X_1 coincides with $\tilde{X}$. Now we look at the C-morphism $g: X\to X_0$ which preserves the A^1_*-fibrations (or the twisted A^1_*-fibrations) on X and X_0 over C. Let $m:=\deg g$ and let H be the group of all m-th roots of the unity in k. As in Lemma 5 and its proof, H acts on X_1 and X_0 is the quotient variety X_1/H. Let $\phi_1: X_1\to C$ be the composite of the normalization morphism $X_1\to X_0$ and ϕ_0. Then ϕ_1 defines an A^1_*-fibration or a twisted A^1_*-fibration on X_1 such that $\phi_1|_X=\phi$. Let S be a singular fiber of X_0 over a point P of C. Write $S=\Gamma+\Delta$ as in Lemma 4. Then we can readily show that $S_1:=\phi_1^*(P)$ is a singular fiber on X_1, $S_1=\Gamma+\Delta_1$ and Γ is stable under the action of H, where $\operatorname{Supp}\Delta_1$ is a disjoint union of irreducible curves isomorphic to A^1_k. This implies that $\tilde{X}-X=X_1-X$ is a disjoint union of irreducible curves which are isomorphic to A^1_k.

(III) *Case where* $\bar{\kappa}(X)=0$ *and* X *is rational.* We note that $\bar{\kappa}(X)=\bar{\kappa}(\tilde{X})$. Indeed, by the logarithmic ramification formula applied to the normalization morphism $\tilde{X}\to X$, we have $\bar{\kappa}(X)\leqq\bar{\kappa}(\tilde{X})$. Since the upper X is an open set of $\tilde{X}$, we have $\bar{\kappa}(\tilde{X})\leqq\bar{\kappa}(X)$. Hence $\bar{\kappa}(X)=\bar{\kappa}(\tilde{X})$. Let V be anew a nonsingular completion of $\tilde{X}$ such that $D:=V-\tilde{X}$ is a divisor with simple normal crossings. Since $\bar{\kappa}(\tilde{X})=0$ as shown above, $\bar{P}_m(\tilde{X})\leqq 1$ for every $m\geqq 0$. Let $C_1, \cdots, C_r$ exhaust all irreducible components of V such that $C_i\cap X=\phi$ and $C_i\not\subset\operatorname{Supp}(D)$ for $1\leqq i\leqq r$. We may assume that C_i is nonsingular at the points $C_i-C_i\cap\tilde{X}$ for $1\leqq i\leqq r$, and that $D+C_1+\cdots+C_r$ has only normal crossings as singularities at every point of $V-\tilde{X}$. Let $\pi: V^*\to V$ be a succession of blowing-ups with centers at $\bigcup_{i=1}^r\operatorname{Sing}(C_i)$ and their infinitely near points such that $(\pi^*(D+C_1+\cdots+C_r))_{\text{red}}$ is a divisor with simple normal crossings. Let $D^*=(\pi^*(D+C_1+\cdots+C_r))_{\text{red}}$. Since we have $K_{V^*}=\pi^*(K_V)+R_\pi$ with an effective divisor R_π, we have

$$\pi^*(D+K_V)+(\pi^*(C_1+\cdots+C_r))_{\text{red}}\leqq D^*+K_{V^*}\,.$$

Since $\bar{\kappa}(\tilde{X})=\kappa(D+K_V, V)=0$, we have, by the κ-calculus (cf. [5]):

$$\begin{aligned}&\kappa(\pi^*(D+K_V)+(\pi^*(C_1+\cdots+C_r))_{\text{red}}, V^*)=\kappa(D^*+K_{V^*}, V^*)\\&\kappa(\pi^*(D+K_V)+(\pi^*(C_1+\cdots+C_r))_{\text{red}}, V^*)=\kappa(C_1+\cdots+C_r+D+K_V, V)\\&\text{and}\quad\kappa(D^*+K_{V^*}, V^*)=\bar{\kappa}(X)=0\,.\end{aligned}$$

Therefore we have $\kappa(C_1+\cdots+C_r+D+K_V, V)=0$. Let C now be one of C_i's, $1\leqq i\leqq r$. Then $|C+K_V|=\phi$. Indeed, suppose $|C+K_V|\neq\phi$. Since $\tilde{X}$ is affine, we have $mD\geqq$ an ample divisor for a sufficiently large $m>0$. Since $|m(C+D+K_V)|\supseteq|m(C+K_V)|+|mD|$, we have $\kappa(C+D+K_V, V)=2$, which is a contradiction. Hence $|C+K_V|=\phi$. This implies that C is a nonsingular rational curve (cf. [8; Chap. I, Lemma 2.1.3]). Consider an exact sequence

$$0\to O_V(D+K_V)\to O_V(C+D+K_V)\to O_C((C\cdot D)-2)\to 0\,.$$

Thence we have an exact sequence

$$\begin{aligned}0\to H^0(V,\,D+K_V)\to H^0(V,\,C+D+K_V)\to H^0(C,\,O_C((C\cdot D)-2))\\ \to H^1(V,D+K_V)\,,\end{aligned}$$

where $\dim H^1(V, D+K_V)=\dim H^1(C, -D)=0$ because V is rational and D is 1-connected. We claim that $(C\cdot D)=1$. Suppose $|C+D+K_V|=\phi$. Then, by the above exact sequence, $(C\cdot D)\leqq 1$, while $(C\cdot D)>0$ because $\tilde{X}=V-D$ is affine. Now suppose $|C+D+K_V|\neq\phi$. If $C\not\subseteq Bs|C+D+K_V|$, there exists a member $M\in|C+D+K_V|$ such that C is not a component of M. Then, for a large integer $m<0$ with $|m(D+K_V)|\neq\phi$, $|m(C+D+K_V)|$ contains mM and $mC+|m(D+K_V)|$. Hence $\kappa(C+D+K_V, V)>0$, which is a contradiction. Hence $C\subseteq Bs|C+D+K_V|$. This implies $|D+K_V|\neq\phi$. The above exact sequence then implies $(C\cdot D)\leqq 1$, hence $(C\cdot D)=1$. This is the case for C_1. It is then readily seen that the above argument applies even if C and D replaced by C_2 and C_1+D. Thus we can show that $(C_2\cdot C_1)=0$ and $(C_2\cdot D)=1$. We apply the above argument for C_i and $C_1+\cdots+C_{i-1}+D+K_V$, $1\leqq i\leqq r$, to conclude that $(C_i\cdot C_j)=0$ for $i\neq j$ and $(C_i\cdot D)=1$. This implies that $\tilde{X}-X$ is a disjoint union of irreducible curves isomorphic to $\boldsymbol{A}^1_k$.

(IV) *Case* $\bar{\kappa}(X)=-\infty$. The assertion was verified in [9]. Q.E.D.

Hereafter, we assume that the ground field k is the complex number field $\boldsymbol{C}$. Let X be a nonsingular affine surface defined over $\boldsymbol{C}$ and let V be a nonsingular completion of X such that $D:=V-X$ is a divisor with simple normal crossings. Let $e(X)$, $e(V)$ and $e(D)$ be the Euler numbers of X, V and D, respectively. If $D=D_1+\cdots+D_r$ be the decomposition into irreducible components, the Euler number $e(D)$ is given as

$$e(D)=\sum_{i=1}^{r}\{2-2g(D_i)\}-\sum_{i<j}(D_i\cdot D_j)\,,$$

where $g(D_i)$ is the genus of a nonsingular curve D_i (cf. [10]). Suppose X has a *finite* étale endomorphism $f: X\to X$ of degree $n>1$. Then $e(X)=ne(X)$ by virtue of the well-known formula of the Euler numbers for a finite étale covering. Hence we have $e(X)=0$. This condition provides a strong restriction on the structure of X. More precisely, we have the following:

Theorem 11. *Let X be a nonsingular affine surface defined over $\boldsymbol{C}$, which is endowed with a finite étale endomorphism $f: X \to X$ of degree $n>1$. Then X is one of the following:*

(1) *Case $\bar{\kappa}(X)=-\infty$. X is either $\boldsymbol{A}_C^1 \times \boldsymbol{A}_*^1$ or a relatively minimal elliptic ruled surface with one cross-section deleted off.*

(2) *Case $\bar{\kappa}(X)=0$ or 1. We have then either*

(i) *X is a rational surface with $\bar{\kappa}(X)=0$ such that, if (V, D) is any nonsingular completion of X with the boundary divisor D of simple normal crossings and if $D=D_1+\cdots+D_r$ is the decomposition into irreducible components, any component D_i is rational and $(K_V^2) \leqq 12-r$, or*

(ii) *there exists a surjective morphism $\phi: X \to C$ onto a nonsingular curve which defines an $\boldsymbol{A}_*^1$-fibration or a twisted $\boldsymbol{A}_*^1$-fibration and which has no singular fibers except those of the type $S=\alpha\Gamma_1$ with $\Gamma_1 \cong \boldsymbol{A}_*^1$.*

Proof. Note that if Y_1 is an open set in a nonsingular affine surface Y such that $Y-Y_1$ is isomorphic to $\boldsymbol{A}_C^1$, then $e(Y_1)=e(Y)-1$.

(1) Suppose $\bar{\kappa}(X)=-\infty$. Consider, first of all, the case where X is irrational or $\Gamma(X, \mathcal{O}_X)^* \neq \boldsymbol{C}^*$. By Lemma 3, either X is elliptic-ruled or $\operatorname{rank}(\Gamma(X, \mathcal{O}_X)^*/\boldsymbol{C}^*)=1$. As shown in the proof of Lemma 3, there exist a surjective morphism $\phi: X \to C$ onto a nonsingular curve C and a finite étale endomorphism $h: C \to C$ such that $\phi \cdot f = h \cdot \phi$, where C is a complete elliptic curve or isomorphic to $\boldsymbol{A}_*^1$. Let S_i $(1 \leqq i \leqq t)$ exhaust all singular fibers of ϕ, which defines an $\boldsymbol{A}^1$-fibration and let δ_i be the number of irreducible components of S_i. If we note that every component of S_i is isomorphic to $\boldsymbol{A}_C^1$, we know, by the above remark, that $e(X)=\sum_{i=1}^{t}(\delta_i-1)$. Hence $\delta_i=1$ for $1 \leqq i \leqq t$, and $S_i=\alpha_i\Delta_i$ with $\alpha_i>1$ and $\Delta_i \cong \boldsymbol{A}_C^1$. Note, on the other hand, that X is isomorphic to the fiber product of ϕ: $X \to C$ and $h: C \to C$; see the proof of Lemma 3. Hence $\deg h = \deg f = n > 1$. Then, for any singular fiber S_i, $f^*(S_i)$ consists of n singular fibers. Indeed, if $P_i=\phi(S_i)$ and $h^{-1}(P_i)=\{Q_{i1}, \cdots, Q_{in}\}$, then $\phi^*(Q_{ij})$ $(1 \leqq j \leqq n)$ is a singular fiber. Thus we have $nt=t$. This implies that ϕ has no singular fibers. If $C \cong \boldsymbol{A}_*^1$, then X is isomorphic to $\boldsymbol{A}_C^1 \times \boldsymbol{A}_*^1$.

Consider, next, the case where X is rational and $\Gamma(X, \mathcal{O}_X)^*=\boldsymbol{C}^*$. Since X is an affine surface with $\bar{\kappa}(X)=-\infty$, there exists a surjective morphism $\phi: X \to C$ which defines an $\boldsymbol{A}^1$-fibration, where $C \cong \boldsymbol{A}_C^1$ or $\boldsymbol{P}_C^1$ (cf. [8; Chap. I]). Let S_i $(1 \leqq i \leqq t)$ exhaust all singular fibers of ϕ and let δ_i be the number of irreducible components of S_i. Then we have

$$e(X) = 1+\varepsilon+\sum_{i=1}^{t}(\delta_i-1),$$

where $\varepsilon=0$ or 1 according as $C \cong \boldsymbol{A}_C^1$ or $\boldsymbol{P}_C^1$. Hence $e(X)>0$, which contradicts the hypothesis that f is a finite étale endomorphism with $\deg f>1$.

(2) Suppose $\bar{\kappa}(X)=0$ or 1. Consider the case where either $\bar{\kappa}(X)=1$ or $\bar{\kappa}(X)=0$ and X is irrational. As in the proof of Theorem 10, there exists a surjective morphism $\phi: X\to C$ onto a nonsingular curve which defines either an A^1_*-fibration or a twisted A^1_*-fibration on X. Let S_i $(1\leqq i\leqq t)$ exhaust all singular fibers of ϕ, let $P_i=\phi(S_i)$ and let $C_0=C-\{P_1, \cdots, P_t\}$. Then $e(X)=e(\phi^{-1}(C_0))+\sum_{i=1}^{t}\gamma_i$, where $e(\phi^{-1}(C_0))=e(C_0)\cdot e(A^1_*)=0$ and γ_i is the contribution of S_i as described below. Let S be one of S_i's, and write $S=\Gamma+\Delta$ (cf. Lemma 4). Let γ be the contribution of S to $e(X)$. If $\Gamma=0$ then $\gamma=\delta:=$ the number of irreducible components of Δ. If $\Gamma=\alpha\Gamma_1$ with $\Gamma_1\cong A^1_*$ then $\gamma=\delta$ as in the preceding case. If $\Gamma=\alpha_1\Gamma_1+\alpha_2\Gamma_2$ with $\Gamma_1\cong\Gamma_2\cong A^1_C$ then $\gamma=\delta+1$. Since $e(X)=0$, we conclude that any singular fiber S of ϕ is of the form $S=\alpha\Gamma_1$ with $\alpha>1$ and $\Gamma_1\cong A^1_*$. This verifies the assertion in the present case.

The remaining case is the one where X is a rational surface with $\bar{\kappa}(X)=0$. Let V be a nonsingular completion of X such that the boundary divisor $D:=V-X$ is a divisor with simple normal crossings. With the same notations as in [8; Chap. II, 5], let (V_m, D_m) be a relatively minimal model of (V_m, D_m) and let $X_m=V_m$-$\mathrm{Supp}(D_m)$. Then X_m is an affine open set of X, and $X-X_m$ is a disjoint union of irreducible curves isomorphic to A^1_C. Hence $e(X_m)\leqq 0$. Suppose now that D contains an irrational irreducible component. By virtue of [8; Chap. II, Lemma 5.5], we know that $e(X_m)=3$ or 4, which is a contradiction. Therefore every irreducible component of D is a nonsingular rational curve. Let $D=D_1+\cdots+D_r$ be the decomposition into irreducible components. Note that $\dim H^0(V, D+K_V)\leqq 1$ because $\bar{\kappa}(X)=0$, that $H^2(V, D+K_V)=0$ and that $H^1(V, D+K_V)=0$ because V is rational and D is 1-connected. Hence, by virtue of the Riemann-Roch theorem.

$$\dim H^0(V,\, D+K_V) = \frac{1}{2}(D\cdot D+K_V)+1\leqq 1\,,$$

where

$$(D\cdot D+K_V) = -2r+2\sum_{i<j}(D_i\cdot D_j)\,.$$

Hence we have

$$e(V) = e(D) = 2r-\sum_{i<j}(D_i\cdot D_j)\geqq r\,.$$

By virtue of Noether's formula $12\chi(O_V)=(K_V^2)+e(V)$, we obtain

$$12-(K_V^2) = e(V)\geqq r\,. \qquad \text{Q.E.D.}$$

REMARK. The following results in the complete case show that there are good similarities between the affine and complete cases. Let V be a nonsingular projective surface with an étale endomorphism $f\colon V\to V$ of degree >1. Then V is relatively minimal, i.e., there are no exceptional curves of the first kind on V, V has the Euler number $e(V)=0$, and V is, indeed, one of the following:

(1) Case $\kappa(V)=-\infty$. Then V is a ruled surface over an elliptic curve.

(2) Case $\kappa(V)=0$. Then V is either an abelian surface or a hyperelliptic surface.

(3) Case $\kappa(V)=1$. Then V is an elliptic surface $\phi: V \rightarrow C$ whose singular fiber, if any, is a multiple of a nonsingular elliptic curve.

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