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ON THE SINGULARITIES OF THE SCATTERING KERNEL FOR THE ELASTIC WAVE EQUATION IN THE CASE OF MODE-CONVERSION

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Abstract

For the elastic wave, there are waves of different modes and a remarkable phenomenon called "mode-conversion" which causes serious difficulties in the analysis of singularities of the scattering kernel. In the present paper, by considering the case of a non back-scattering, we examine singularities of the scattering kernel for the elastic wave equation in the case of mode-conversion.

1. Introduction

Let Ω be an exterior domain in $\mathbf{R}^3$ with smooth and compact boundary. We consider the isotropic elastic wave equation with the Dirichlet boundary condition

$$
(1.1)\qquad \begin{cases} (\partial_t^2 - L)u(t,x) = 0 & \text{in} \quad \mathbf{R}\times\Omega, \\ u(t,x) = 0 & \text{on} \quad \mathbf{R}\times\partial\Omega, \\ u(0,x) = f_1(x), \quad \partial_t u(0,x) = f_2(x) & \text{on} \quad \Omega, \end{cases}
$$

where $u(t,x) = {}^t(u_1,u_2,u_3)$ and $f_i(x) = {}^t(f_{i1}, f_{i2}, f_{i3})$ $(i = 1,2)$. Recall that L has the following form:

$$
L = \sum_{i,j=1}^{3} a_{ij}\partial_{x_i}\partial_{x_j},
$$

where a_{ij} are 3×3 matrices of which (p,q)-entry is expressed by a_{ipjq}. We say that the elastic medium Ω is isotropic, if a_{ipjq} is given by

$$
a_{ipjq} = \lambda\delta_{ip}\delta_{jq} + \mu\left(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp}\right),
$$

where λ,μ are Lame's constants satisfying the following inequalities:

$$
\lambda + \frac{2}{3}\mu > 0, \quad \mu > 0.
$$

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Under the assumption that the elastic medium Ω is isotropic, Yamamoto [14] and Shibata-Soga [8] have formulated the scattering theory which is analogous to the theory of Lax-Phillips [5]. Let $k_-(s,\omega)$ and $k_+(s,\omega) \in L^2(\mathbf{R}\times S^2)$ denote the incoming and outgoing translation representations of an initial data $f = {}^t(f_1, f_2)$ respectively (see [5]). Recall that the scattering operator S is the mapping

$$S\colon k_-(s,\omega) \mapsto k_+(s,\omega).$$

The scattering operator S admits a representation of the form with a distribution kernel $S(s,\theta,\omega)$ called the scattering kernel:

$$(Sk_-)(s,\theta) = \iint_{\mathbf{R}\times S^2} S(s-\tilde{s},\theta,\omega)k_-(\tilde{s},\omega)\,d\tilde{s}\,d\omega.$$

Majda [6] has obtained the representation formula of the scattering kernel $S(s,\theta,\omega)$ for the scalar-valued case. This representation formula is very effective to investigate inverse scattering problems (cf. Majda [6], Soga [9], Petkov [7]). For the elastic case, Soga [10] and Kawashita [3] have derived the representation formula of the scattering kernel.

The characteristic matrix $L(\xi)$ of the operator $L(\partial_x)$ has the eigenvalues $C_1^2|\xi|^2$ and $C_2^2|\xi|^2$, where

$$C_1 = (\lambda + 2\mu)^{1/2}, \quad C_2 = \mu^{1/2}.$$

Let $P_i(\xi)$ be the eigenprojector for the eigenvalues $C_i^2|\xi|^2$ $(i = 1,2)$, where

$$P_1(\xi) = \xi \otimes \xi, \quad P_2(\xi) = I - P_1(\xi).$$

Then $P_1(\xi)\mathbf{R}^3$ is the space spaned by ξ, and $P_2(\xi)\mathbf{R}^3$ is the orthogonal complement of $P_1(\xi)\mathbf{R}^3$. Associated with the eigenvalues $C_i^2|\xi|^2$ $(i = 1,2)$, there are waves of two different types (modes). The one propagates with the speed C_1, and the other with C_2. Furthermore their amplitudes are longitudinal and transverse to the propagation direction respectively, and therefore these waves are called longitudinal and transverse waves respectively. For elastic waves there is a remarkable phenomenon called "mode-conversion," that is, when longitudinal or transverse incident wave hits the boundary $\partial\Omega$, both longitudinal reflected wave and transverse reflected wave appear. This phenomenon causes serious difficulties in the analysis of singularities of the scattering kernel for the elastic wave equation.

In view of results concerning mode-conversion (cf. Chapter 5 of Achenbach [1] and Theorem 2.1 of Soga [12]), we can expect that corresponding phenomenon occurs for the scattering kernel $S(s,\theta,\omega)$, because in the asymptotic sense the kernel $P_i(\theta)S(C_i^{-1/2}\theta\cdot x - t,\theta,\omega)P_l(\omega)$ expresses the C_i-mode component of the scattered wave in the direction θ for the C_l-mode incident plane wave in the direction ω. In the back-scattering case (i.e. $\theta = -\omega$), by Soga [10, 11] and Yamamoto [14] we can obtain results of the same type as in Majda [6]:

(i) $\quad \operatorname{supp}[P_i(-\omega)S(\,\cdot\,, -\omega, \omega)P_l(\omega)] \subset (-\infty, -r_{il}(\omega)]$,

(ii) $\quad P_i(-\omega)S(s, -\omega, \omega)P_i(\omega)$ is singular (not C^∞) at $s = -r_{ii}(\omega)$,

where $r_{il}(\omega) = (\lambda_i(-\omega)^{-1/2} + \lambda_l(\omega)^{-1/2}) \min_{x\in\partial\Omega} x \cdot \omega$. In Soga [11], he has derived an asymptotic expansion of $P_i(-\omega)S(\,\cdot\,, -\omega, \omega)P_l(\omega)$ which is valid near the right end point of the singular support for $s \in \mathbf{R}$ (i.e. $s = -r_{il}(\omega)$):

$$
\begin{aligned}
&P_i(-\omega)S(s, -\omega, \omega)P_l(\omega) \\
&\sim c_3(i, l, \omega) \sum_{k=1}^{N} K(a_k)^{-1/2} \delta^{(1)}(-s - r_{il}(\omega)) P_i(\omega) P_l(\omega) + \cdots,
\end{aligned} \tag{1.2}
$$

where $c_3(i, l, \omega)$ is a constant, $\{x\,;\ \omega \cdot x = r(\omega)\} \cap \partial\Omega = \{a_t\}_{t=1,\dots,N}$ and $K(a_t)$ is the Gaussian curvature of $\partial\Omega$ at a_t. For the detailed proof, see Theorem 6.1 in [11]. Since the leading term of the above expansion vanishes in the mode-conversion case (i.e. $i \neq l$), in the analysis of the singularity we can use it only when $i = l$. In the mode-conversion case, by Kawashita-Soga [4], it is necessary to examine the lower term of the asymptotic expansion of the scattering kernel. However, considering the case of non-back scattering (i.e. $\theta \neq \omega$) and making more precise studies of oscillatory integrals than those in [11], we shall show that the first term of $P_i(\theta)S(s, \theta, \omega)P_l(\omega)$ does not vanish, if $i \neq l$ and $|\theta + \omega|$ is different from zero and sufficiently small.

The main theorem is stated precisely in Section 2. The proof of our theorem is based on methods in Soga [11]. In Section 3, we derive the asymptotic expansion of $P_i(\theta)S(s, \theta, \omega)P_l(\omega)$ which is valid not only for the case $i = l$ but also for the case $i \neq l$. Using the results of Section 3, we prove our theorem in Section 4.

2. Main results

Before giving the main results in the present paper, we give a definition for stating those.

We set $r_{il}(\theta, \omega) := \min_{x\in\partial\Omega} x \cdot n_{il}(\theta, \omega)$, where $n_{il}(\theta, \omega) := -\big(C_i^{-1}\theta - C_l^{-1}\omega\big)$. Next, we denote the first hitting points at $\partial\Omega$ by $N_{il}(\theta, \omega) := \{x\,;\ n_{il}(\theta, \omega) \cdot x = r_{il}(\theta, \omega)\} \cap \partial\Omega$. Furthermore, we arbitrarily pick a point $a_t \in N_{il}(\theta, \omega)$ and choose a system of orthogonal local coordinates $y = (y', y_3)$, with $y' = (y_1, y_2)$, in $\mathbf{R}^3$ such that $y_3 = (r_{il}(\theta, \omega) - n_{il}(\theta, \omega) \cdot x)|n_{il}(\theta, \omega)|^{-1}$, and that $y = 0$ expresses the reference point a_t. Then Ω is represented by $y_3 > \psi(y')$ in a neighborhood U of a_t, where $\psi(y')$ is a C^∞ function defined in a neighborhood of $y' = 0$.

If the Hessian matrix $H_{\psi(y')}$ of $\psi(y')$ is negative definite at $y' = 0$ for every such picked point, we say that $n_{ij}(\theta, \omega)$ is a regular direction for $\partial\Omega$, which does not depend on the choice of the coordinates $y = (y', y_3)$. If $n_{il}(\theta, \omega)$ is a regular direction, the set $N_{il}(\theta, \omega)$ consists of a finite number of isolated points.

For a distribution $f(s)$ on $\mathbf{R}$ we use the notation

$$f(s) \sim f_0(s) + f_1(s) + \cdots \quad \text{at} \quad s_0,$$

which means that there exists an integer m and a C^∞ function $\varphi(s)$ with $\varphi(s_0) \neq 0$ such that for every integer $N \geq 0$

$$\varphi(s)\{f(s) - (f_0(s) + \cdots + f_N(s))\} \in H^{m+N}(\mathbf{R}).$$

Then we have

Theorem 2.1. *Let $\omega, \theta \in S^2$. Assume that $|\theta + \omega|$ is different from zero and sufficiently small, and $n_{il}(\theta, \omega)$ is a regular direction for $\partial\Omega$. Then we have*

(i) $\operatorname{supp}[P_i(\theta)S(\,\cdot\,, \theta, \omega)P_1(\omega)] \subset (-\infty, -r_{i1}(\theta, \omega)] \quad (i = 1, 2),$

(ii) $P_i(\theta)S(s, \theta, \omega)P_1(\omega)$ *is singular (not C^∞) at* $s = -r_{i1}(\theta, \omega) \quad (i = 1, 2).$

3. Asymptotic expansion of the scattering kernel

In order to examine the singularities of $P_i(\theta)S(s, \theta, \omega)P_l(\omega)$, it is useful to know the asymptotic behavior of the scattering kernel. In this section we shall derive an asymptotic expansion of the scattering kernel which plays an essential role in the proof of Theorem 2.1.

In order to derive an expansion of $P_i(\theta)S(s, \theta, \omega)P_l(\omega)$, we review some results in [11]. Let $v_l(t, x; \omega)$ be the solution of the following boundary value problem:

$$\begin{cases} \left(\partial_t^2 - L\right)v_l(t, x; \omega) = 0 & \text{in} \quad \mathbf{R} \times \Omega \\ v_l(t, x; \omega) = \left(2\sqrt{2}\pi\right)^{-2} C_l^{-3/2} \delta\left(t - C_l^{-1}\omega \cdot x\right) P_l(\omega) & \text{on} \quad \mathbf{R} \times \partial\Omega \\ v_l(t, x; \omega) = 0 & \text{for} \quad t < C_l^{-1} r(\omega) \end{cases} \tag{3.1}$$

where $r(\omega) = \min_{x \in \partial\Omega} x \cdot \omega$. Namely $v_l(t, x; \omega)$ is the scattered wave for the incident wave

$$\left(2\sqrt{2}\pi\right)^{-2} C_l^{-3/2} \delta\left(t - C_l^{-1}\omega \cdot x\right) P_l(\omega). \tag{3.2}$$

The scattering kernel is represented by means of $v_l(t, x; \omega)$:

(3.3)
$$\begin{aligned} S(s, \theta, \omega) = \sum_{i,j=1}^{2} C_i^{-3/2} \int_{\partial\Omega} \Big\{ & P_i(\theta)(\partial_t N v_j)\left(C_i^{-1}\theta \cdot x - s, x; \omega\right) \\ & - C_i^{-1} P_i(\theta)^t (N(\theta \cdot x))\left(\partial_t^2 v_j\right)\left(C_i^{-1}\theta \cdot x - s, x; \omega\right) \Big\} dS_x \end{aligned}$$

where $N = \sum_{i,j=1}^{3} a_{ij} \nu_i \partial_{x_j}$ and $\nu = (\nu_1, \nu_2, \nu_3)$ is the unit outer normal to Ω.

By the Hamilton-Jacobi method we have a real-valued C^∞ function $\varphi_l^k(x)$ $(k, l = 1, 2)$ satisfying

$$
(3.4) \qquad \begin{cases} \left|\nabla\varphi_l^k(x)\right| = \dfrac{1}{C_k} & \text{in} \quad \Omega \cap U_\epsilon, \\ \varphi_l^k(x) = \dfrac{1}{C_l}\omega \cdot x & \text{on} \quad \partial\Omega \cap U_\epsilon, \\ \dfrac{\partial\varphi_l^k}{\partial\nu}(x) < 0 & \text{on} \quad \partial\Omega \cap U_\epsilon, \end{cases}
$$

where $U_\epsilon = \{x\,;\, |n_{il}(\theta, \omega) \cdot x - r_{il}(\theta, \omega)| < \epsilon\}$ with a small $\epsilon > 0$.

We set

$$
\rho_j(t) = \begin{cases} \dfrac{t^{j-1}}{(j-1)!} & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases} \qquad \text{when} \quad j = 1, 2, \ldots,
$$

$$
\rho_j(t) = \rho'_{j+1}(t) \qquad \text{when} \quad j = 0, -1, -2, \ldots.
$$

Let us note that

$$
\rho_0(t) = \delta(t), \qquad \rho'_{j+1}(t) = \rho_j(t) \quad \text{for any integer } j.
$$

Lemma 3.1. *Assume that there exists a sufficiently small* $\delta > 0$ *such that* $\left|\nabla\varphi^k_{l\ tan}\right| < \delta$. *Then the solution* v_l *of* (3.1) *admits the following asymptotic expansion for* $t \in \mathbf{R}$ *sufficiently close to* $r_{ij}(\theta, \omega)$

$$
(3.5) \qquad v_l(t, x; \omega) \sim \sum_{k=1}^{2} \sum_{j \geq 0} \rho_j\left(t - \varphi_l^k(x)\right) u_{lj}^k(x),
$$

where $u_{lj}^k(x)$ *are some* C^∞*functions defined in* $\overline{\Omega} \cap U_\epsilon$ *and* $\nabla\varphi^k_{l\ tan}$ *denotes the tangential part to* $\partial\Omega$ *of* $\nabla\varphi_l^k$.

Proof. Combining Theorem 2.1 in Soga [11] and Theorem 1.1 in [12], we can derive the above asymptotic expansion in this case. $\square$

Let $v_l(t, x; \omega)$ be the solution of (3.1). Then $u = v_l P_l(\omega)$ satisfies the equation $\left(\partial_t^2 - L\right)u = 0$ in $\mathbf{R} \times \Omega$ and verifying the same boundary condition as v_l. Hence, by the uniqueness of the solutions, we obtain that $v_l(t, x; \omega) = v_l(t, x; \omega)P_l(\omega)$. Moreover combining the representation of the scattering kernel (3.3) and the asymptotic expansion (3.5), we have

(3.6)

$$
\begin{aligned}
&P_i(\theta)S(s,\theta,\omega)P_l(\omega) \\
&\sim \sum_{k=1}^{2} C_i^{-3/2}\Bigg[\sum_{j\geq -1}\int_{\partial\Omega}\rho_{j-1}(-s-n_{il}(\theta,\omega)\cdot x) \\
&\qquad\qquad \times P_i(\theta)\sum_{p,q=1}^{3} a_{pq}\nu_p(x)\big\{\big(-\partial_{x_q}\varphi_l^k(x)\big)u_{lj+1}^k(x)+\partial_{x_q}u_{lj}^k(x)\big\}P_l(\omega)\,dS_x \\
&\qquad - C_i^{-1}\sum_{j\geq 0}\int_{\partial\Omega}\rho_{j-2}(-s-n_{il}(\theta,\omega)\cdot x)P_i(\theta)\sum_{p,q=1}^{3} {}^t a_{pq}\nu_p(x)\theta_q u_{lj}^k(x)P_l(\omega)\,dS_x\Bigg].
\end{aligned}
$$

For a regular direction $n_{il}(\theta,\omega)$ we have $N_{il}(\theta,\omega) = \{a_1,\dots,a_M\}$. By using a partition of unity, it is enough to examine the terms whose integrands are supported on a small neighborhood of the reference point $a_t \in N_{il}(\theta,\omega)$. Then we can rewrite the above integrals (3.6) as $\sum_{t=1}^{M} I_t(\theta,\omega)$. Since the analysis of above integrals near each point a_t is same, it is sufficient to study the leading term in $I_t(\theta,\omega)$ for only one a_t, where we may assume $a_t = 0$.

We take an orthonormal frame $\{p_1, p_2, p_3\}$ where $p_3 = -n_{il}(\theta,\omega)|n_{il}(\theta,\omega)|^{-1}$, and choose the local coordinate system $y = (y_1, y_2, y_3)$ such that $x = y_1p_1+y_2p_2+y_3p_3$. Let us denote by T the 3×3 orthogonal matrix $T = (t_{pq})$ such that $T(e_j) = p_j$ $(j = 1, 2, 3)$, where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbf{R}^3$. Then $\partial\Omega$ is represented by $y_3 = \psi(y')$ near 0. Since the equation is isotropic, we have the following result.

Lemma 3.2. *Assume that the elastic medium Ω is isotropic, then we have*

$$
TL({}^tT\xi)\,{}^tT = L(\xi) \quad \textit{and} \quad T\sum_{p,q=1}^{3} a_{pq}t_{rp}t_{sq}\,{}^tT = a_{rs}.
$$

Proof. By the isotropicity of the equation,

$$
TL({}^tT\xi)\,{}^tT = T\big\{(\lambda+\mu)\,{}^tT\xi\otimes{}^tT\xi+\mu|T\xi|^2I\big\}\,{}^tT = (\lambda+\mu)\xi\otimes\xi+\mu|\xi|^2I = \sum_{r,s=1}^{3} a_{rs}\xi_r\xi_s.
$$

On the other hand, a direct computation shows

$$
\sum_{p,q,m,n=1}^{3} t_{im}a_{pmqn}t_{rp}t_{sq}t_{jn} = \lambda\delta_{ri}\delta_{qj}+\mu(\delta_{rs}\delta_{ij}+\delta_{rj}\delta_{si}) = a_{risj}.
$$

Thus the proof is complete. □

By Lemma 3.2 and an easy computation, we have the following identities:

$$L(\partial_y)u|_{y={}^tTx} = {}^tTL(\partial_x)Tu({}^tTx) \quad \text{for any} \quad x \in \Omega, \tag{3.7}$$

$$(N_y u)({}^tTx) = {}^tTN_xTu({}^tTx) \quad \text{for any} \quad x \in \partial\Omega, \tag{3.8}$$

where $N_x = \sum_{pq=1}^3 a_{pq}\nu_p\partial_{x_q}$, $N_y = \sum_{pq=1}^3 a_{pq}\nu_p^*\partial_{y_q}$ and $\nu^*(y) = {}^tT\nu(Ty)$. Then from (3.7), it follows that $\tilde{v}_l(t,y;\tilde{\omega}) := {}^tTv_l(t,Ty;\omega)T$ satisfies the same boundary value problem (3.1) in $\tilde{\Omega} = {}^tT\Omega$ where ω is replaced by $\tilde{\omega} = {}^tT\omega$. Moreover $\tilde{v}_l(t,y;\tilde{\omega})$ admits the following asymptotic expansion:

$$\tilde{v}_l(t,y;\tilde{\omega}) \sim \sum_{k=1}^2 \sum_{j\ge 0} \rho_j\big(t - \tilde{\varphi}_l^k(y)\big)\tilde{u}_{lj}^k(y). \tag{3.9}$$

Here $\tilde{u}_{lj}^k(y) := {}^tTu_{lj}^k(Ty)T$ and $\tilde{\varphi}_l^k(y) := \varphi_l^k(Ty)$ which satisfies

$$\begin{cases} \big|\nabla\tilde{\varphi}_l^k(y)\big| = \dfrac{1}{C_k} & \text{in} \quad \tilde{\Omega}\cap\tilde{U}_\epsilon, \\ \tilde{\varphi}_l^k(y)\big|_{y_3=\psi(y')} = \dfrac{1}{C_l}\tilde{\omega}\cdot y & \text{on} \quad \partial\tilde{\Omega}\cap\tilde{U}_\epsilon, \\ \dfrac{\partial\tilde{\varphi}_l^k}{\partial\nu^*}(y)\bigg|_{y_3=\psi(y')} < 0 & \text{on} \quad \partial\tilde{\Omega}\cap\tilde{U}_\epsilon, \end{cases} \tag{3.10}$$

where $\tilde{U}_\epsilon = \big\{y\,;\,\big|n_{il}\big(\tilde{\theta},\tilde{\omega}\big)\cdot y - r_{il}\big(\tilde{\theta},\tilde{\omega}\big)\big| < \epsilon\big\}$ with a small $\epsilon > 0$.

Since $P_l(\omega) = TP_l(\tilde{\omega})\,{}^tT$, $P_i(\theta) = TP_i\big(\tilde{\theta}\big)\,{}^tT$, by Lemma 3.2, $I_t(\theta,\omega)$ takes the following form:

$$\begin{aligned} &\sum_{k=1}^2 C_i^{-3/2}P_i(\theta)\int_{\mathbf{R}^2}\rho_{-2}(-s+|n_{il}(\theta,\omega)|\psi(y')-r_{il}(\theta,\omega)) \\ &\qquad\qquad \times\left\{\sum_{p,q=1}^3 a_{pq}\nu_p(Ty)\big(-\partial_{x_q}\varphi_l^k(Ty)\big) - C_i^{-1}\sum_{p,q=1}^3 {}^ta_{pq}\nu_p(Ty)\theta_q\right\} \\ &\qquad\qquad \times u_{l0}^k(Ty)\beta_{-2}(y')P_l(\omega)\,dy' \\ &+\sum_{j\ge -1}\int \rho_j(-s+|n_{il}(\theta,\omega)|\psi(y')-r_{il}(\theta,\omega))\beta_j(y')\,dy' \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{2} C_i^{-3/2} \int_{\mathbf{R}^2} \rho_{-2}\bigl(-s + \bigl|n_{il}\bigl(\tilde{\theta}, \tilde{\omega}\bigr)\bigr|\psi(y') - r_{il}\bigl(\tilde{\theta}, \tilde{\omega}\bigr)\bigr) \\
&\qquad\qquad \times T \left\{ P_i\bigl(\tilde{\theta}\bigr) \sum_{r,s=1}^{3} \left\{ {}^tT \sum_{p,q=1}^{3} a_{pq} t_{pr} t_{qs} T \right\} \nu_r^*(y)\bigl(-\partial_{y_s}\tilde{\varphi}_l^k(y)\bigr) \right. \\
&\qquad\qquad\qquad \left. - C_i^{-1} \sum_{r,s=1}^{3} \left\{ {}^tT \sum_{p,q=1}^{3} {}^t a_{pq} t_{pr} t_{qs} T \right\} \nu_r^*(y)\tilde{\theta}_s \right\} \\
&\qquad\qquad \times \left\{ {}^tT u_{l0}^k(Ty)T \right\} \beta_{-2}(y') P_l(\tilde{\omega})\, {}^tT \, dy' \\
&\quad + \sum_{j \geq -1} \int \rho_j\bigl(-s + \bigl|n_{il}\bigl(\tilde{\theta}, \tilde{\omega}\bigr)\bigr|\psi(y') - r_{il}\bigl(\tilde{\theta}, \tilde{\omega}\bigr)\bigr)\beta_j(y')\, dy' \\
&= \sum_{k=1}^{2} C_i^{-3/2} \int_{\mathbf{R}^2} \rho_{-2}\bigl(-s + \bigl|n_{il}\bigl(\tilde{\theta}, \tilde{\omega}\bigr)\bigr|\psi(y') - r_{il}\bigl(\tilde{\theta}, \tilde{\omega}\bigr)\bigr) \\
&\qquad\qquad \times T \left[P_i\bigl(\tilde{\theta}\bigr) \left\{ \sum_{r,s=1}^{3} a_{rs}\nu_r^*(y)\bigl(-\partial_{y_s}\tilde{\varphi}_l^k(y)\bigr) - C_i^{-1} \sum_{r,s=1}^{3} {}^t a_{rs}\nu_r^*(y)\tilde{\theta}_s \right\} \right. \\
&\qquad\qquad\qquad \left. \times \tilde{u}_{l0}^k(y)\beta_{-2}(y')P_l(\tilde{\omega}) \right] {}^tT \, dy' \\
&\quad + \sum_{j \geq -1} \int \rho_j\bigl(-s + \bigl|n_{il}\bigl(\tilde{\theta}, \tilde{\omega}\bigr)\bigr|\psi(y') - r_{il}\bigl(\tilde{\theta}, \tilde{\omega}\bigr)\bigr)\beta_j(y')\, dy',
\end{aligned}
$$

where $\beta_j(y')$ are some C^∞ functions supported near $y' = 0$ and $\beta_{-2}(0) = 1$.

Since $n_{il}(\theta, \omega)$ is a regular direction, by the Morse lemma we can take a new system of local coordinates $\tilde{y}'$ so that $\tilde{y}' = 0$ means $y' = 0$ and that

$$
\psi\bigl(y'(\tilde{y}')\bigr) = -\frac{1}{2}\bigl|\tilde{y}'\bigr|^2,
$$

$$
\det \frac{\partial \tilde{y}'}{\partial y'}(0) = K(a_t)^{-1/2}.
$$

We can determine the phase functions $\tilde{\varphi}_l^k$ and the amplitudes $\tilde{u}_{l0}^k$ by the methods in Kawashita [2]. Applying the Taylor expansions to $\nu^*(\tilde{y})$, $\nabla\tilde{\varphi}_l^k(\tilde{y})\big|_{y_3=\psi(\tilde{y}')}$, $\tilde{u}_{l0}^k(\tilde{y})\big|_{y_3=\psi(\tilde{y}')}$:

$$
\begin{aligned}
\nu^*(\tilde{y}) &= \bigl|n_{il}\bigl(\tilde{\theta}, \tilde{\omega}\bigr)\bigr|^{-1} n_{il}\bigl(\tilde{\theta}, \tilde{\omega}\bigr) + \cdots, \\
\nabla\tilde{\varphi}_l^k(\tilde{y})\big|_{y_3=\psi(\tilde{y}')} &= C_l^{-1\,t}(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\kappa}_{kl}) + \cdots, \\
\tilde{u}_{l0}^k(\tilde{y})\big|_{y_3=\psi(\tilde{y}')} &= \bigl(2\sqrt{2}\pi\bigr)^{-2} C_l^{-3/2}\, \tilde{P}_{k,0}^l P_l(\tilde{\omega}) + \cdots,
\end{aligned}
$$

where $\tilde{y} = (\tilde{y}', y_3)$, $\tilde{\kappa}_{kl} = \sqrt{\tilde{\omega}_3^2 + C_l^2 \cdot C_k^{-2} - 1}$ and $\left(\nabla\tilde{\varphi}_l^1\big|_{y_3=\psi(\tilde{y}')} \cdot \nabla\tilde{\varphi}_l^2\big|_{y_3=\psi(\tilde{y}')}\right)^{-1} \cdot$

$\nabla\tilde{\varphi}_l^1\big|_{y_3=\psi(\tilde{y}')} \otimes \nabla\tilde{\varphi}_l^2\big|_{y_3=\psi(\tilde{y}')} = \tilde{P}_{1,0}^l + O(|\tilde{y}'|)$ and $I - \left(\nabla\tilde{\varphi}_l^1\big|_{y_3=\psi(\tilde{y}')} \cdot \nabla\tilde{\varphi}_l^2\big|_{y_3=\psi(\tilde{y}')}\right)^{-1} \cdot \nabla\tilde{\varphi}_l^1\big|_{y_3=\psi(\tilde{y}')} \otimes \nabla\tilde{\varphi}_l^2\big|_{y_3=\psi(\tilde{y}')} = \tilde{P}_{2,0}^l + O(|\tilde{y}'|)$. We can rewrite the integrals $I_t(\theta,\omega)$ in the following way:

(3.11)

$$
\begin{aligned}
&\left(2\sqrt{2}\pi\right)^{-2} C_i^{-3/2} C_l^{-3/2} \\
&\times \int_{\mathbf{R}^2} \rho_{-2}\left(-s - \left|n_{il}\left(\tilde{\theta},\tilde{\omega}\right)\right| \frac{|\tilde{y}'|^2}{2} - r_{il}\left(\tilde{\theta},\tilde{\omega}\right)\right) \\
&\qquad \times T\left[P_i\left(\tilde{\theta}\right)\sum_{k=1}^{2}\left\{\sum_{p=1}^{3}\sum_{q=1}^{2} a_{pq} n_{il}\left(\tilde{\theta},\tilde{\omega}\right)_p\left(-C_l^{-1}\tilde{\omega}_q\right) + \sum_{p=1}^{3} a_{p3} n_{il}\left(\tilde{\theta},\tilde{\omega}\right)_p\left(-C_l^{-1}\tilde{\kappa}_{kl}\right)\right.\right. \\
&\qquad\qquad \left.\left. + \sum_{p,q=1}^{3} {}^t a_{pq} n_{il}\left(\tilde{\theta},\tilde{\omega}\right)_p\left(-C_i^{-1}\tilde{\theta}_q\right)\right\}\left|n_{il}\left(\tilde{\theta},\tilde{\omega}\right)\right|^{-1}\tilde{P}_{k,0}^l P_l(\tilde{\omega})\right] \\
&\qquad \times {}^t T K(a_t)^{-1/2}\tilde{\beta}_{-2}\left(\tilde{y}'\right) d\tilde{y}' \\
&+ \sum_{j+|\alpha|\geq -1} \int_{\mathbf{R}^2} \rho_j\left(-s - \left|n_{il}\left(\tilde{\theta},\tilde{\omega}\right)\right| \frac{|\tilde{y}'|^2}{2} - r_{il}\left(\tilde{\theta},\tilde{\omega}\right)\right)\tilde{\beta}_j^{\alpha}\left(\tilde{y}'\right)\tilde{y}'^{\alpha}\, d\tilde{y}',
\end{aligned}
$$

where $\tilde{\beta}_j\left(\tilde{y}'\right)$ are some C^∞ functions supported near $\tilde{y}' = 0$ and $\tilde{\beta}_{-2}(0) = 1$. By using Lemma 6.3 and Lemma 6.4 in Soga [11], we show that the leading term of (3.11) is the following form:

$$
\begin{aligned}
&\left(2\sqrt{2}\pi\right)^{-2} C_i^{-3/2} C_l^{-3/2} \left|n_{il}\left(\tilde{\theta},\tilde{\omega}\right)\right|^{-2} \delta^{(1)}\left(-s - r_{il}\left(\tilde{\theta},\tilde{\omega}\right)\right) K(a_t)^{-1/2} \left|S^1\right| \\
&\times T\left[P_i\left(\tilde{\theta}\right)\sum_{k=1}^{2}\left\{\sum_{q=1}^{2}\left\{a_{3q}\left(C_l^{-1}\tilde{\omega}_q\right) + {}^t a_{3q}\left(C_i^{-1}\tilde{\theta}_q\right)\right\} + a_{33}\left(C_l^{-1}\tilde{\kappa}_{kl} + C_i^{-1}\tilde{\theta}_3\right)\right\}\tilde{P}_{k,0}^l P_l(\tilde{\omega})\right]{}^t T.
\end{aligned}
$$

Summing over all points a_t, we arrive at the following proposition.

Proposition 3.3. *Let $\omega, \theta \in S^2$. Assume that $|\theta + \omega|$ is sufficiently small, and $n_{il}(\theta,\omega)$ is a regular direction for $\partial\Omega$. Then we have*

$$
\begin{aligned}
&P_i(\theta) S(s,\theta,\omega) P_l(\omega) \\
&\sim \left(2\sqrt{2}\pi\right)^{-2} C_i^{-3/2} C_l^{-3/2} \left|n_{il}\left(\tilde{\theta},\tilde{\omega}\right)\right|^{-2} \delta^{(1)}\left(-s - r_{il}\left(\tilde{\theta},\tilde{\omega}\right)\right) \sum_{t=1}^{M} K(a_t)^{-1/2} \left|S^1\right| \\
&\quad \times T\left[P_i\left(\tilde{\theta}\right)\sum_{k=1}^{2}\left\{\sum_{q=1}^{2}\left\{a_{3q}\left(C_l^{-1}\tilde{\omega}_q\right) + {}^t a_{3q}\left(C_i^{-1}\tilde{\theta}_q\right)\right\} + a_{33}\left(C_l^{-1}\tilde{\kappa}_{kl} + C_i^{-1}\tilde{\theta}_3\right)\right\}\tilde{P}_{k,0}^l P_l\left(\tilde{\omega}\right)\right]{}^t T \\
&\quad + \cdots,
\end{aligned}
$$

where $T = (t_{pq})$ *is* 3×3 *orthogonal matrix and* $x = Ty$, $\tilde{\omega} = {}^t T\omega$, $\tilde{\theta} = {}^t T\theta$ *and* $\left(\nabla\tilde{\varphi}_l^1\big|_{y_3=\psi(\tilde{y}')} \cdot \nabla\tilde{\varphi}_l^2\big|_{y_3=\psi(\tilde{y}')}\right)^{-1} \cdot \nabla\tilde{\varphi}_l^1\big|_{y_3=\psi(\tilde{y}')} \otimes \nabla\tilde{\varphi}_l^2\big|_{y_3=\psi(\tilde{y}')} = \tilde{P}_{1,0}^l + O(\tilde{y}'|)$ *and* $I - \left(\nabla\tilde{\varphi}_l^1\big|_{y_3=\psi(\tilde{y}')} \cdot \nabla\tilde{\varphi}_l^2\big|_{y_3=\psi(\tilde{y}')}\right)^{-1} \cdot \nabla\tilde{\varphi}_l^1\big|_{y_3=\psi(\tilde{y}')} \otimes \nabla\tilde{\varphi}_l^2\big|_{y_3=\psi(\tilde{y}')} = \tilde{P}_{2,0}^l + O(|\tilde{y}'|)$.

4. Proof of Theorem 2.1

In this section, using Proposition 3.3, we investigate the support and singularities of the scattering kernel $S(s, \theta, \omega)$ for the non-back scattering, and prove Theorem 2.1.

Proof of Theorem 2.1. (i) By using the method of the proof in the first half in Proposition 3.3, we obtain

$$\begin{aligned} &P_i(\theta)S(s,\theta,\omega)P_1(\omega) \\ &= C_i^{-3/2}P_i(\theta)\int_{\partial\Omega}\partial_t N v_1\left(C_i^{-1}\theta\cdot x - s, x;\omega\right) \\ &\quad - C_i^{-5/2}P_i(\theta)\int_{\partial\Omega}{}^t(N\theta\cdot x)\partial_t^2 v_1\left(C_i^{-1}\theta\cdot x - s, x;\omega\right)dS_x. \end{aligned} \tag{4.1}$$

Since due to the finite propagation speed for solutions to the isotropic elastic wave equation $v_1(t, x; \omega) = 0$ if $t < C_1^{-1}\omega \cdot x$, it follows that $v_1\left(C_i^{-1}\theta \cdot x - s, x; \omega\right) = 0$ if $C_1^{-1}\omega \cdot x > C_i^{-1}\theta \cdot x - s$. Therefore since the right-hand side of (4.1) is equal to 0 if $s > -\left(C_1^{-1}\omega \cdot x - C_i^{-1}\theta \cdot x\right)$, by taking a supremum with respect to $x \in \partial\Omega$, which proves (i) of Theorem 2.1.

(ii) Note that $P_1(\xi) = \xi \otimes \xi$, $P_2(\xi) = I - \xi \otimes \xi$ and each $\tilde{P}_{k,0}^1 P_1(\tilde{\omega})$ $(k = 1, 2)$ takes the following form:

$$\tilde{P}_{1,0}^1 P_1(\tilde{\omega}) = \frac{A(\tilde{\omega})}{A(\tilde{\omega})}\begin{pmatrix} \tilde{\omega}_1^2 & \tilde{\omega}_1\tilde{\omega}_2 & \tilde{\omega}_1\tilde{\omega}_3 \\ \tilde{\omega}_2\tilde{\omega}_1 & \tilde{\omega}_2^2 & \tilde{\omega}_2\tilde{\omega}_3 \\ |\tilde{\omega}_3|\tilde{\omega}_1 & |\tilde{\omega}_3|\tilde{\omega}_2 & |\tilde{\omega}_3|\tilde{\omega}_3 \end{pmatrix}, \tag{4.2}$$

$$\tilde{P}_{2,0}^1 P_1(\tilde{\omega}) = \frac{-2\tilde{\omega}_3\tilde{\kappa}_{21}}{\overline{A}(\tilde{\omega})}\begin{pmatrix} \tilde{\omega}_1^2 & \tilde{\omega}_1\tilde{\omega}_2 & \tilde{\omega}_1\tilde{\omega}_3 \\ \tilde{\omega}_2\tilde{\omega}_1 & \tilde{\omega}_2^2 & \tilde{\omega}_2\tilde{\omega}_3 \\ |\tilde{\omega}_3|\tilde{\omega}_1 z_{\tilde{\omega}_3} & |\tilde{\omega}_3|\tilde{\omega}_2 z_{\tilde{\omega}_3} & |\tilde{\omega}_3|\tilde{\omega}_3 z_{\tilde{\omega}_3} \end{pmatrix}, \tag{4.3}$$

where $A(\tilde{\omega}) = \tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3\tilde{\kappa}_{21}$, $\overline{A}(\tilde{\omega}) = \tilde{\omega}_1^2 + \tilde{\omega}_2^2 + |\tilde{\omega}_3|\tilde{\kappa}_{21}$ and $z_{\tilde{\omega}_3} = \left(1 - \tilde{\omega}_3^2\right)/\tilde{\omega}_3\tilde{\kappa}_{21}$. Recall that $n_{i1}\left(\tilde{\theta}, \tilde{\omega}\right)/\left|n_{i1}\left(\tilde{\theta}, \tilde{\omega}\right)\right| = (0, 0, -1)$, we can rewrite $P_i\left(\tilde{\theta}\right)\sum_{k=1}^2\left[\sum_{q=1}^2\left\{a_{3q}\left(C_1^{-1}\tilde{\omega}_q\right) + {}^t a_{3q}\left(C_i^{-1}\tilde{\theta}_q\right)\right\} + a_{33}\left(C_1^{-1}\tilde{\kappa}_{k1} + C_i^{-1}\tilde{\theta}_3\right)\right]\tilde{P}_{k,0}^1 P_1(\tilde{\omega})$ in the following form:

$$P_i\left(\tilde{\theta}\right)\sum_{k=1}^2\{(a_{31} + {}^t a_{31})\tilde{\omega}_1 + (a_{32} + {}^t a_{32})\tilde{\omega}_2 + a_{33}(\tilde{\omega}_3 + \tilde{\kappa}_{k1} + C_1|\tilde{n}_{i1}|)\}\frac{\tilde{P}_{k,0}^1 P_1(\tilde{\omega})}{C_1^{-1}}. \tag{4.4}$$

Then, calculating each term in (4.4) more carefully, we can obtain

$$
\begin{aligned}
&P_i\left(\tilde{\theta}\right)\sum_{k=1}^{2}\begin{pmatrix} 0 & 0 & \lambda+\mu \\ 0 & 0 & 0 \\ \lambda+\mu & 0 & 0 \end{pmatrix}\tilde{\omega}_1\tilde{P}^1_{k,0}P_1(\tilde{\omega}) \\
&= (\lambda+\mu)\tilde{\omega}_1\frac{A(\tilde{\omega})(a\otimes\tilde{\omega})+2|\tilde{\omega}_3|\tilde{\kappa}_{21}(\overline{a}\otimes\tilde{\omega})}{A(\tilde{\omega})},
\end{aligned}
$$

$$
\begin{aligned}
&P_i\left(\tilde{\theta}\right)\sum_{k=1}^{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda+\mu \\ 0 & \lambda+\mu & 0 \end{pmatrix}\tilde{\omega}_2\tilde{P}^1_{k,0}P_1(\tilde{\omega}) \\
&= (\lambda+\mu)\tilde{\omega}_2\frac{A(\tilde{\omega})(b\otimes\tilde{\omega})+2|\tilde{\omega}_3|\tilde{\kappa}_{21}\left(\overline{b}\otimes\tilde{\omega}\right)}{\overline{A}(\tilde{\omega})},
\end{aligned}
$$

$$
\begin{aligned}
&P_i\left(\tilde{\theta}\right)\sum_{k=1}^{2}\begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda+2\mu \end{pmatrix}(\tilde{\omega}_3+\tilde{\kappa}_{k1}+C_1|\tilde{n}_{i1}|)\tilde{P}^1_{k,0}P_1(\tilde{\omega}) \\
&= \frac{C_1|\tilde{n}_{i1}|A(\tilde{\omega})(c\otimes\tilde{\omega})+2|\tilde{\omega}_3|\tilde{\kappa}_{21}(\tilde{\omega}_3+\tilde{\kappa}_{21}+C_1|\tilde{n}_{i1}|)(\overline{c}\otimes\tilde{\omega})}{\overline{A}(\tilde{\omega})},
\end{aligned}
$$

where

$$a={}^t\!\left(\tilde{\omega}_1 p_i^{(13)}\left(\tilde{\theta}\right)+|\tilde{\omega}_3|\,p_i^{(11)}\left(\tilde{\theta}\right),\tilde{\omega}_1 p_i^{(23)}\left(\tilde{\theta}\right)+|\tilde{\omega}_3|\,p_i^{(21)}\left(\tilde{\theta}\right),\tilde{\omega}_1 p_i^{(33)}\left(\tilde{\theta}\right)+|\tilde{\omega}_3|\,p_i^{(31)}\left(\tilde{\theta}\right)\right),$$

$$\overline{a}={}^t\!\left(\tilde{\omega}_1 p_i^{(13)}\left(\tilde{\theta}\right)+|\tilde{\omega}_3|z_{\tilde{\omega}_3}p_i^{(11)}\left(\tilde{\theta}\right),\tilde{\omega}_1 p_i^{(23)}\left(\tilde{\theta}\right)+|\tilde{\omega}_3|z_{\tilde{\omega}_3}p_i^{(21)}\left(\tilde{\theta}\right),\tilde{\omega}_1 p_i^{(33)}\left(\tilde{\theta}\right)+|\tilde{\omega}_3|z_{\tilde{\omega}_3}p_i^{(31)}\left(\tilde{\theta}\right)\right),$$

$$b={}^t\!\left(\tilde{\omega}_2 p_i^{(13)}\left(\tilde{\theta}\right)+|\tilde{\omega}_3|\,p_i^{(12)}\left(\tilde{\theta}\right),\tilde{\omega}_2 p_i^{(23)}\left(\tilde{\theta}\right)+|\tilde{\omega}_3|\,p_i^{(22)}\left(\tilde{\theta}\right),\tilde{\omega}_2 p_i^{(33)}\left(\tilde{\theta}\right)+|\tilde{\omega}_3|\,p_i^{(32)}\left(\tilde{\theta}\right)\right),$$

$$\overline{b}={}^t\!\left(\tilde{\omega}_2 p_i^{(13)}\left(\tilde{\theta}\right)+|\tilde{\omega}_3|z_{\tilde{\omega}_3}p_i^{(12)}\left(\tilde{\theta}\right),\tilde{\omega}_2 p_i^{(23)}\left(\tilde{\theta}\right)+|\tilde{\omega}_3|z_{\tilde{\omega}_3}p_i^{(22)}\left(\tilde{\theta}\right),\tilde{\omega}_2 p_i^{(33)}\left(\tilde{\theta}\right)+|\tilde{\omega}_3|z_{\tilde{\omega}_3}p_i^{(32)}\left(\tilde{\theta}\right)\right),$$

$$
\begin{aligned}
c={}^t\!\Big(&\mu\left\{p_i^{(11)}\left(\tilde{\theta}\right)\tilde{\omega}_1+p_i^{(12)}\left(\tilde{\theta}\right)\tilde{\omega}_2\right\}+(\lambda+2\mu)p_i^{(13)}\left(\tilde{\theta}\right)|\tilde{\omega}_3|,\\
&\mu\left\{p_i^{(21)}\left(\tilde{\theta}\right)\tilde{\omega}_1+p_i^{(22)}\left(\tilde{\theta}\right)\tilde{\omega}_2\right\}+(\lambda+2\mu)p_i^{(23)}\left(\tilde{\theta}\right)|\tilde{\omega}_3|,\\
&\mu\left\{p_i^{(31)}\left(\tilde{\theta}\right)\tilde{\omega}_1+p_i^{(32)}\left(\tilde{\theta}\right)\tilde{\omega}_2\right\}+(\lambda+2\mu)p_i^{(33)}\left(\tilde{\theta}\right)|\tilde{\omega}_3|\Big),
\end{aligned}
$$

$$
\begin{aligned}
\overline{c}={}^t\!\Big(&\mu\left\{p_i^{(11)}\left(\tilde{\theta}\right)\tilde{\omega}_1+p_i^{(12)}\left(\tilde{\theta}\right)\tilde{\omega}_2\right\}+(\lambda+2\mu)|\tilde{\omega}_3|z_{\tilde{\omega}_3}p_i^{(13)}\left(\tilde{\theta}\right),\\
&\mu\left\{p_i^{(21)}\left(\tilde{\theta}\right)\tilde{\omega}_1+p_i^{(22)}\left(\tilde{\theta}\right)\tilde{\omega}_2\right\}+(\lambda+2\mu)|\tilde{\omega}_3|z_{\tilde{\omega}_3}p_i^{(23)}\left(\tilde{\theta}\right),\\
&\mu\left\{p_i^{(31)}\left(\tilde{\theta}\right)\tilde{\omega}_1+p_i^{(32)}\left(\tilde{\theta}\right)\tilde{\omega}_2\right\}+(\lambda+2\mu)|\tilde{\omega}_3|z_{\tilde{\omega}_3}p_i^{(33)}\left(\tilde{\theta}\right)\Big),
\end{aligned}
$$

each $p_i^{(pq)}\left(\tilde{\theta}\right)$ denotes (p,q)-entry of $P_i\left(\tilde{\theta}\right)$ and $\tilde{n}_{i1} = n_{i1}\left(\tilde{\theta},\tilde{\omega}\right)$. Hence, applying the asymptotic expansion derived in the Proposition 3.3, we obtain

(4.5)
$$
\begin{aligned}
&P_i(\theta)S(s,\theta,\omega)P_1(\omega)\\
&\sim \left(2\sqrt{2}\pi\right)^{-2}C_1^{-5/2}C_i^{-3/2}\delta^{(1)}\left(-s-r_{i1}\left(\tilde{\theta},\tilde{\omega}\right)\right)\sum_{t=1}^{M}K(a_t)^{-1/2}\left|S^1\right|T\,M_3\left(\tilde{\theta},\tilde{\omega}\right){}^tT+\cdots,
\end{aligned}
$$

where $M_3\left(\tilde{\theta},\tilde{\omega}\right)$ is a 3×3-matrix whose (p,q)-entry is expressed by $m_{pq}\left(\tilde{\theta},\tilde{\omega}\right)$. As shown above, it is represented in the following form:

$$
\begin{aligned}
&M_3\left(\tilde{\theta},\tilde{\omega}\right)\\
&=\big[(\lambda+\mu)\tilde{\omega}_1\{A(\tilde{\omega})(a\otimes\tilde{\omega})+2|\tilde{\omega}_3|\tilde{\kappa}_{21}(\overline{a}\otimes\tilde{\omega})\}\\
&\quad+(\lambda+\mu)\tilde{\omega}_2\left\{A(\tilde{\omega})(b\otimes\tilde{\omega})+2|\tilde{\omega}_3|\tilde{\kappa}_{21}\left(\overline{b}\otimes\tilde{\omega}\right)\right\}\\
&\quad+\{C_1|\tilde{n}_{i1}|A(\tilde{\omega})(c\otimes\tilde{\omega})+2|\tilde{\omega}_3|\tilde{\kappa}_{21}(\tilde{\omega}_3+\tilde{\kappa}_{21}+C_1|\tilde{n}_{i1}|)(\overline{c}\otimes\tilde{\omega})\}\big]\big/\overline{A}(\tilde{\omega}).
\end{aligned}\tag{4.6}
$$

To show that the leading term of the right-hand side of (4.5) does not vanish, we shall prove that $m_{33}\left(\tilde{\theta},\tilde{\omega}\right)\neq 0$. According to (4.6), $m_{33}\left(\tilde{\theta},\tilde{\omega}\right)$ is expressed as follows:

$$
\begin{aligned}
&m_{33}\left(\tilde{\theta},\tilde{\omega}\right)\\
&=\Big[(\lambda+\mu)\tilde{\omega}_1\left\{A(\tilde{\omega})\left(\tilde{\omega}_1\tilde{p}_i^{(33)}+|\tilde{\omega}_3|\tilde{p}_i^{(31)}\right)+2|\tilde{\omega}_3|\tilde{\kappa}_{21}\left(\tilde{\omega}_1\tilde{p}_i^{(33)}+|\tilde{\omega}_3|z_{\tilde{\omega}_3}\tilde{p}_i^{(31)}\right)\right\}\\
&\quad+(\lambda+\mu)\tilde{\omega}_2\left\{A(\tilde{\omega})\left(\tilde{\omega}_2\tilde{p}_i^{(33)}+|\tilde{\omega}_3|\tilde{p}_i^{(32)}\right)+2|\tilde{\omega}_3|\tilde{\kappa}_{21}\left(\tilde{\omega}_2\tilde{p}_i^{(33)}+|\tilde{\omega}_3|z_{\tilde{\omega}_3}\tilde{p}_i^{(32)}\right)\right\}\Big]\tilde{\omega}_3\big/\overline{A}(\tilde{\omega})\\
&\quad+\Big[C_1|\tilde{n}_{i1}|A(\tilde{\omega})\left\{\mu\left(\tilde{p}_i^{(31)}\tilde{\omega}_1+\tilde{p}_i^{(32)}\tilde{\omega}_2\right)+(\lambda+2\mu)\tilde{p}_i^{(33)}|\tilde{\omega}_3|\right\}\\
&\qquad+2|\tilde{\omega}_3|\tilde{\kappa}_{21}(\tilde{\omega}_3+\tilde{\kappa}_{21}+C_1|\tilde{n}_{i1}|)\left\{\mu\left(\tilde{p}_i^{(31)}\tilde{\omega}_1+\tilde{p}_i^{(32)}\tilde{\omega}_2\right)+(\lambda+2\mu)|\tilde{\omega}_3|z_{\tilde{\omega}_3}\tilde{p}_i^{(33)}\right\}\Big]\tilde{\omega}_3\big/\overline{A}(\tilde{\omega})\\
&=\Big[(\lambda+\mu)\left(\tilde{\omega}_1^2\tilde{p}_i^{(33)}+\tilde{\omega}_2^2\tilde{p}_i^{(33)}\right)\left(A(\tilde{\omega})+2|\tilde{\omega}_3|\tilde{\kappa}_{21}\right)\\
&\quad+(\lambda+\mu)|\tilde{\omega}_3|\left(\tilde{p}_i^{(31)}\tilde{\omega}_1+\tilde{p}_i^{(32)}\tilde{\omega}_2\right)\left(A(\tilde{\omega})+2|\tilde{\omega}_3|\tilde{\kappa}_{21}z_{\tilde{\omega}_3}\right)\\
&\quad+\mu\left(\tilde{p}_i^{(31)}\tilde{\omega}_1+\tilde{p}_i^{(32)}\tilde{\omega}_2\right)\{C_1|\tilde{n}_{i1}|A(\tilde{\omega})+2|\tilde{\omega}_3|\tilde{\kappa}_{21}(\tilde{\omega}_3+\tilde{\kappa}_{21}+C_1|\tilde{n}_{i1}|)\}\\
&\quad+(\lambda+2\mu)\tilde{p}_i^{(33)}|\tilde{\omega}_3|\{C_1|\tilde{n}_{i1}|A(\tilde{\omega})+2|\tilde{\omega}_3|\tilde{\kappa}_{21}(\tilde{\omega}_3+\tilde{\kappa}_{21}+C_1|\tilde{n}_{i1}|)z_{\tilde{\omega}_3}\}\Big]\tilde{\omega}_3\big/\overline{A}(\tilde{\omega}),
\end{aligned}
$$

where $\tilde{p}_i^{(pq)}=p_i^{(pq)}\left(\tilde{\theta}\right)$ and $\tilde{n}_{i1}=n_{i1}\left(\tilde{\theta},\tilde{\omega}\right)$.

By Lemma 4.1 below, we can prove $m_{33}\left(\tilde{\theta},\tilde{\omega}\right)\neq 0$, that is, we show that the leading term of the right-hand side of (4.5) does not vanish. Thus the proof is completed. $\square$

Lemma 4.1. *Assume that $\left|\tilde{\theta}+\tilde{\omega}\right|$ is different from zero and sufficiently small. Then we have $m_{33}(\theta,\omega)\neq 0$.*

Proof. (i) Let $i=1$. Since, in the case of back-scattering, $\tilde{\omega}=(0,0,-1)$ and $\tilde{\theta}=(0,0,1)$, we can derive that $m_{33}\left(\tilde{\theta},\tilde{\omega}\right)=2(\lambda+2\mu)+O\left(\left|\tilde{\theta}+\tilde{\omega}\right|\right)$. Therefore, by using our assumption that $\left|\tilde{\theta}+\tilde{\omega}\right|$ is sufficiently small, we can prove that $m_{33}\left(\tilde{\theta},\tilde{\omega}\right)\neq 0$.

(ii) Let $i = 2$. By $\tilde{n}_{21}(\tilde{\theta}, \tilde{\omega}) / |\tilde{n}_{21}(\tilde{\theta}, \tilde{\omega})| = (0, 0, -1)$, that is,

$$C_1^{-1}\tilde{\omega}_p = C_2^{-1}\tilde{\theta}_p \quad (p = 1, 2), \quad C_1^{-1}\tilde{\omega}_3 = C_2^{-1}\tilde{\theta}_3 - |\tilde{n}_{21}(\tilde{\theta}, \tilde{\omega})| \quad \text{and} \quad |\tilde{\theta}| = 1,$$

we can express $m_{33}(\tilde{\theta}, \tilde{\omega})$ as a function in $(\tilde{\theta}_1, \tilde{\theta}_2)$:

$$m_{33}(\tilde{\theta}_1, \tilde{\theta}_2) = \frac{F(\tilde{\theta}_1, \tilde{\theta}_2)\tilde{\omega}_3(\tilde{\theta}_1, \tilde{\theta}_2)}{\bar{A}(\tilde{\omega}(\tilde{\theta}_1, \tilde{\theta}_2))},$$

where

$$\begin{aligned} F(\tilde{\theta}_1, \tilde{\theta}_2) &= (\lambda + \mu)(\tilde{\theta}_1^2 + \tilde{\theta}_2^2)(\tilde{\omega}_1^2(\tilde{\theta}) + \tilde{\omega}_2^2(\tilde{\theta}))\bar{A}(\tilde{\omega}(\tilde{\theta})) \\ &\quad + (\lambda + \mu)C_1C_2^{-1}(\tilde{\theta}_1^2 + \tilde{\theta}_2^2)|\tilde{\omega}_3(\tilde{\theta})|\tilde{\theta}_3\bar{A}(\tilde{\omega}(\tilde{\theta})) \\ &\quad - \mu C_1C_2^{-1}(\tilde{\theta}_1^2 + \tilde{\theta}_2^2)\tilde{\theta}_3\{C_1|\tilde{n}_{21}|\bar{A}(\tilde{\omega}(\tilde{\theta})) + 2|\tilde{\omega}_3(\tilde{\theta})|\tilde{\kappa}_{21}(\tilde{\omega}_3(\tilde{\theta}) + \tilde{\kappa}_{21})\} \\ &\quad - (\lambda + 2\mu)(\tilde{\theta}_1^2 + \tilde{\theta}_2^2)|\tilde{\omega}_3(\tilde{\theta})|\{C_1|\tilde{n}_{21}|\bar{A}(\tilde{\omega}(\tilde{\theta})) + 2(\tilde{\omega}_1^2(\tilde{\theta}) + \tilde{\omega}_2^2(\tilde{\theta}))(\tilde{\omega}_3(\tilde{\theta}) + \tilde{\kappa}_{21})\} \\ &= (\tilde{\theta}_1^2 + \tilde{\theta}_2^2)\tilde{F}(\tilde{\theta}_1, \tilde{\theta}_2). \end{aligned}$$

Here we note that $|\tilde{\theta} + \tilde{\omega}| \neq 0$ is equivalent to $(\tilde{\theta}_1, \tilde{\theta}_2) \neq (0, 0)$. In order to show $m_{33}(\tilde{\theta}, \tilde{\omega}) \neq 0$, it suffices to show that $\tilde{F}(\tilde{\theta}_1, \tilde{\theta}_2) \neq 0$.

Since $\tilde{F}(\tilde{\theta}_1, \tilde{\theta}_2)$ is a C^∞ function near $(\tilde{\theta}_1, \tilde{\theta}_2) = (0, 0)$ and

$$\tilde{F}(0, 0) = -\{(\lambda + 2\mu)\tilde{\kappa}_{21} + \mu C_1C_2^{-1}\tilde{\kappa}_{21}(C_1C_2^{-1} + 2\tilde{\kappa}_{21})\} < 0,$$

we can obtain that $\tilde{F}(\tilde{\theta}_1, \tilde{\theta}_2) \neq 0$ provided $|\tilde{\theta} + \tilde{\omega}|$ is different from zero and sufficiently small.

Thus the proof is completed. □

REMARK 4.2. If $\tilde{\theta} = -\tilde{\omega}$ (i.e. back-scattering case), then $m_{33}(\tilde{\theta}, \tilde{\omega}) = 0$.

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