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THE MODULI SPACE OF YANG-MILLS CONNECTIONS OVER A KÄHLER SURFACE IS A COMPLEX MANIFOLD

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1. Introduction

Let M be a compact, connected, oriented Riemannian 4-manifold. Let P be a smooth principal G-bundle over M. For simplicity we assume that the Lie group $G=SU(n)$, $n\geqq 2$. An $SU(n)$-connection A on P is called self-dual (anti-self-dual) if curvature form $F(A)=dA-A\wedge A$ satisfies $*F(A)=\pm F(A)$. Each self-dual (anti-self-dual) connection is characterized as a connection minimizing the Yang-Mills functional $\int_M |F|^2 dv$ and then gives a solution to the Yang-Mills equation. That the second Chern class $c_2(\mathfrak{g}^C)<0(>0)$ for the adjoint bundle $\mathfrak{g}$ of P is a topological restriction to P in order to admit a self-dual (anti-self-dual) connection. The moduli space $\mathcal{M}$ of self-dual (anti-self-dual) connections, namely, the orbit space of self-dual (anti-self-dual) connections with respect to the group $\mathcal{G}$ of gauge transformations has a structure of smooth manifold ([3], [7]).

A Kähler surface M with a Kähler metric g, which is certainly a Riemannian 4-manifold, carries the canonical orientation induced from the complex structure. Relative to this orientation a connection A is anti-self-dual if and only if its curvature is a 2-form of type (1,1) which is primitive (that is, orthogonal to the Kähler form ω). Therefore, by the integrability condition ([3]) each anti-self-dual connection induces a holomorphic structure on the complex adjoint bundle $\mathfrak{g}^C$. Since gauge-equivalent anti-self-dual connections give holomorphic structures which are isomorphic with respect to automorphisms of $\mathfrak{g}^C$, we have the canonical mapping from $\mathcal{M}$ to the moduli spcae of holomorphic structures on $\mathfrak{g}^C$. Furthermore an anti-self-dual $SU(n)$-connection A naturally defines an Einstein-Hermitian structure on the associated holomorphic vector bundle $\boldsymbol{E}=P\times_{SU(n)}\boldsymbol{C}^n$. We have also the fact that $\boldsymbol{E}$ is ω-semi-stable in the sense of Mumford and Takemoto ([9]). If A is moreover irreducible, then $\boldsymbol{E}$ is ω-stable. On the other hand, over a nonsingular projective surface the moduli space of holomorphic, rank two vector bundles of fixed Chern classes is a quasi-projective variety ([12]). From these reasons together with an easy observation that the moduli space $\mathcal{M}$

has even dimension (Proposition 2.4), it is natural that $\mathcal{M}$ may possibly be a complex manifold ([1]). The aim of this paper is to show that $\mathcal{M}$ is indeed a complex manifold with singularities by using notion of holomorphic (0,1)-connections.

The singularities of $\mathcal{M}$ are described as gauge-equivalent classes $[A]$ of $\mathcal{M}$ either with non-zero 0-th cohomology H^0 or with non-zero second cohomology H^2 for a certain complex associated to the connection A. Denote by $\mathcal{K}$ the subset of $\mathcal{M}$ $\{[A]\in\mathcal{M};\ H^0\neq 0\}$. Then we obtain the following

Theorem 1. *Let M be a compact Kähler surface with a Kähler metric of positive total scalar curvature or with trivial canonical line bundle K_M. Let P be a smooth principal $SU(n)$-bundle with second Chern class $c_2(\mathfrak{g}^C)>0$. If $\mathcal{M}\backslash\mathcal{K}$ is non-empty, then it is a complex manifold of dimension $c_2(\mathfrak{g}^C)-(n^2-1)p_a(M)$, where $p_a(M)$ is arithmetic genus of M.*

We denote by $\boldsymbol{H}$ the space $H^0(M;\ \mathcal{O}(\mathfrak{g}^C\otimes K_M))$ relative to the holomorphic structure on $\mathfrak{g}^C$ induced from an anti-self-dual connection A. Theorem 1 is a direct consequence of the following theorem.

Theorem 2. *Let M be a compatc Kähler surface, P a smooth principal $SU(n)$-bundle with $c_2(\mathfrak{g}^C)>0$. If $(\mathcal{M}\backslash\mathcal{K})_0=\{[A]\in\mathcal{M}\backslash\mathcal{K};\ \boldsymbol{H}=0\}$ is non-empty, then it is a complex manifold of dimension $c_2(\mathfrak{g}^C)-(n^2-1)p_a(M)$.*

These theorems are obtained as follows. We first show in §2 that each $[A]\in(\mathcal{M}\backslash\mathcal{K})_0$ has a neighborhood in the first cohomology H^1 defining a local coordinate of $\mathcal{M}$. But such coordinate neighborhoods are not necessarily each other related holomorphically. Therefore we should verify by an indirect method that $(\mathcal{M}\backslash\mathcal{K})_0$ is in fact a complex manifold. For this purpose we define in §3 a holomorphic (0,1)-connection on the complexification P^C of P. A holomorphic (0,1)-connection is a system of local $\mathfrak{sl}(n;\boldsymbol{C})$-valued (0,1)-forms satisfying a transition condition whose curvature form vanishes. In a manner analoguous to the case of anti-self-dual $SU(n)$-connections we can define complex gauge transformations, moduli space of holomorphic (0,1)-connections and an elliptic complex which is a gauge field version of the Dolbeault complex. We obtain at §4 a canonical mapping f from $\mathcal{M}$ to the moduli space of holomorphic (0,1)-connections which is injective and open over $(\mathcal{M}\backslash\mathcal{K})_0$ and then use the Atiyah-Singer index theorem and Kuranishi's integrating method together with the moment map due to Donaldson ([6]) to verify that the open subspcae $f((\mathcal{M}\backslash\mathcal{K})_0)$ in the moduli is definitely a complex manifold of dimension $c_2(\mathfrak{g})-(n^2-1)p_a(M)$ (Proposition 5.1).

Holomorphic (0,1)-connections over a complex manifold are inseparably related to holomorphic structures on $\mathfrak{g}^C$. Then the moduli space of holomorphic connections reflects aspects and properties of the moduli of holomorphic structures on $\mathfrak{g}^C$. See Ch. 2 of [13] and [2] as references for theory of holomorphic structures on a vector bundle over a compact complex manifold.

An announcement of this article is appeared in [8]. With respect to basical references we refer to [3] and [7].

2. Moduli space of anti-self-dual connections

Let M be a compact Kähler surface with a Kähler metric g. We denote by Λ^k and $\Lambda^{(p,q)}$ the vector bundles of real k-forms and of complex (p,q)-forms on M, respectively. For a real vector bundle E and a complex vector bundle F we denote by $\Omega^k(E)$ and $\Omega^{(p,q)}(F)$ the space of all smooth k-forms with values in E and the space of all smooth (p,q)-forms with values in F. Let P be a smooth principal bundle over M with gauge group $SU(n)$. We denote by G and $\mathfrak{g}$ the associated bundles $P\times_{Ad}SU(n)$ and $P\times_{Ad}\mathfrak{su}(n)$, respectively. We call $\mathfrak{g}$ the adjoint bundle of P.

Let $\{W_\alpha\}$ be an open covering of M consisting of local trivializing neighborhoods of P.

DEFINITION 2.1. A system $A=\{A_\alpha\}$ of local smooth $\mathfrak{su}(n)$-valued 1-forms A_α defined over W_α is called an $SU(n)$-connection on P, if A satisfies the cocycle condition;

$$A_\beta = dg\cdot g^{-1}+g\cdot A_\alpha\cdot g^{-1} \tag{2.1}$$

on $W_\alpha\cap W_\beta$, where $g=g_{\alpha\beta}$ is a transiton transition function of P over $W_\alpha\cap W_\beta$.

The set $\mathcal{A}$ of all $SU(n)$-connections on P has an affine structure. That is, $\mathcal{A}$ is given by $\{A+\alpha;\ \alpha\in\Omega^1(\mathfrak{g})\}$ for a fixed $SU(n)$-connection A. We call $SU(n)$-connection A irreducible when the covariant derivative d_A; $\Omega^0(\mathfrak{g})\to\Omega^1(\mathfrak{g})$, $\psi\mapsto d\psi+[\psi, A]$ has trivial kernel. An $SU(n)$-connection is called reducible if it is not irreducible.

The complex surface M has the canonical orientation induced from the complex structure. The Hodge star operator $*$ gives an endomorphism of Λ^2 with property $*\circ*=id$. We denote by Λ^2_+ and Λ^2_- the eigenspaces of $+1$ and -1, respectively. The projection from Λ^2 onto Λ^2_+ is denoted by p_+. Over Kähler surface M we have the following ([7]). A real 2-form α belongs to Λ^2_+ if and only if (1,1)-part of α is proportional to the Kähler form ω, and a real 2-form β is in Λ^2_- if and only if β is of type (1,1) and orthogonal to ω. A 2-form in Λ^2_+(or in Λ^2_-) is called self-dual (or anti-self-dual).

DEFINITION 2.2. An $SU(n)$-connection A is called anti-self-dual if the curvature form $F(A)=dA-A\wedge A$ which belongs to $\Omega^2(\mathfrak{g})$ satisfies $*F(A)=-F(A)$, namely $p_+F(A)=0$.

The group $\mathcal{G}=\Gamma(M;G)$ of all smooth gauge transformations of P acts on $\mathcal{A}$

as $g(A)=dg\cdot g^{-1}+g\cdot A\cdot g^{-1}$, $g\in\mathcal{G}$, $A\in\mathcal{A}$. Let Z be the center of $SU(n)$. Each element of Z defines a gauge transformation which commutes with all g's of $\mathcal{G}$. It is easily seen that the center $Z(\mathcal{G})$ of $\mathcal{G}$ coincides with Z. The center $Z=Z(\mathcal{G})$ acts trivially on $\mathcal{A}$. Let A be an irreducible connection on P. Then the isotropy subgroup $\Gamma_A=\{g\in\mathcal{G}; g(A)=A\}$ is just Z. This fact is observed by the following. The endomorphism bundle $\mathrm{End}(\boldsymbol{E})$ of the associated vector bundle $\boldsymbol{E}=P\times_\rho \boldsymbol{C}^n$, which is written as $\mathrm{End}(\boldsymbol{E})=P\times_{Ad}\mathfrak{gl}(n;\boldsymbol{C})$, decomposes into $\mathrm{End}(\boldsymbol{E})=\mathbf{1}\oplus\mathfrak{g}\oplus\sqrt{-1}\ \mathfrak{g}$ as an $SU(n)$-vector bundle, where $\mathbf{1}$ is a one-dimensional trivial bundle. The bundle $G=P\times_{Ad} SU(n)$ is considered as a subbundle of $\mathrm{End}(\boldsymbol{E})$ with fibers consisting of $SU(n)$. Then a gauge transformation g is in Γ_A if and only if $g(A)-A=(dg+[g,A])\cdot g^{-1}=d_Ag\cdot g^{-1}=0$, that is, g is a parallel section of $\mathrm{End}(\boldsymbol{E})$. By the irreducibility of A g must be a constant multiple of identity transformation 1_E, hence $g\in Z$ since g takes values in $SU(n)$. As a consequence the quotient group $\tilde{\mathcal{G}}=\mathcal{G}/Z$ acts effectively on $\mathcal{A}$ and freely on the subset of irreducible connections.

Denote by $\mathcal{B}$ the quotient space $\mathcal{A}/\tilde{\mathcal{G}}$ and by π the projection of $\mathcal{A}$ onto $\mathcal{B}$. The equivalence class $\pi(A)$ is denoted by $[A]$. Since $F(g(A))=g\cdot F(A)\cdot g^{-1}$, $g\in\tilde{\mathcal{G}}$, $g(A)$ is anti-self-dual for every anti-self-dual connection A. The subset $\mathcal{M}$ in $\mathcal{B}$ given by {anti-self-dual connections on P}$/\tilde{\mathcal{G}}$ is called the moduli space of anti-self-dual connections on P.

In order to introduce a local coordinate neighborhood for each $[A]$ of $\mathcal{M}$ we define suitable topologies on $\mathcal{B}$. On the spaces $\Omega^p(\mathfrak{g})$ the inner product $\langle\cdot,\cdot\rangle_M$ is defined by $\langle\phi,\psi\rangle_M=\int_M\langle\phi,\psi\rangle(x)dv$, $\langle\phi,\psi\rangle(x)dv=Tr\{\phi(x)\wedge *{}^t\overline{\psi(x)}\}$, $p\geqq 0$. By using a partition of unity we also define the Sobolev's norm $|\cdot|_k$ on $\Omega^p(\mathfrak{g})$ for a positive integer k. In the completion $L_k^2(\Omega^p(\mathfrak{g}))$ of $\Omega^p(\mathfrak{g})$ relative to $|\cdot|_k$ the subspace $\Omega^p(\mathfrak{g})$ of all smooth sections is dense. Note that norms $|\cdot|_0$ and $|\cdot|_M=\langle\cdot,\cdot\rangle_M^{1/2}$ are equivalent. Now we complete the space $\mathcal{A}$ and the group $\mathcal{G}$. Namely, let $\mathcal{A}$ be the space $\{A_0+\alpha; \alpha\in L_k^2(\Omega^1(\mathfrak{g}))\}$ for a fixed smooth connection A_0 and $\mathcal{G}$ the subset $\{g\in L_{k+1}^2(\Gamma(M; \mathrm{End}(\boldsymbol{E})); g$ takes values in $SU(n)\}$. Then $\mathcal{G}$, and hence $\tilde{\mathcal{G}}$ acts on $\mathcal{A}$ and we get the quotient topology on the space $\mathcal{B}=\mathcal{A}/\tilde{\mathcal{G}}$. In the following we assume that k is sufficiently large relative to the dimension of the base space M in order to apply Sobolev's imbedding theorem.

For a connection A a subset U_A of $\mathcal{A}\{A+\alpha; \alpha\in L_k^2(\Omega^1(\mathfrak{g})), d_A^*\alpha=0\}$ is said to be a slice at A. Here $d_A^*; \Omega^1(\mathfrak{g})\to\Omega^0(\mathfrak{g})$ is the formal adjoint of d_A relative to the inner product $\langle\cdot,\cdot\rangle_M$.

Proposition 2.1. *Let A be an irreducible connection. Then there is a positive ε such that $U_{A,\varepsilon}=\{A+\alpha; |\alpha|_k<\varepsilon, d_A^*\alpha=0\}\subset\mathcal{A}$ is homeomorphic to its image $\pi(U_{A,\varepsilon})$ through the restriction of π to $U_{A,\varepsilon}$ and $\pi(U_{A,\varepsilon})$ gives a neighborhood of $[A]$ in $\mathcal{B}$.*

Proof. This proposition is shown in the proof of Theorem 6 in [5]. Then we give here a sketch of the proof. We define a mapping S; $U_{A,\varepsilon}\times \mathcal{G}/Z\to\mathcal{A}$, $S(A+\alpha, g)=g(A+\alpha)$. Then S is smooth relative to the L_k^2-topologies and its derivative at $\alpha=0$ and $g=$the identity is given by

$$DS;\ \mathrm{Ker}\ d_A^*\times\Omega^0(\mathfrak{g})\to\Omega^1(\mathfrak{g})\ ,$$
$$(\alpha,\ \phi)\mapsto\alpha+d_A\phi\ ,$$

which is an isomorphism since $\mathrm{Ker}\ d_A=0$ and $\Omega^1(\mathfrak{g})=\mathrm{Im}\ d_A\oplus\mathrm{Ker}\ d_A^*$. Then S gives a local diffeomorphism. Thus for a sufficiently small ε there is a neighborhood Q of A in $\mathcal{A}$ which is written as $S(U_{A,\varepsilon}\times W)$, where W is a neighborhood in $\tilde{\mathcal{G}}$. Namely, each A_1 in Q has a unique form $A_1=g(A+\beta)$, $\beta\in U_{A,\varepsilon}$, $g\in W$. By the aid of the semi-continuity of dim Ker d_A we can assume here that each connection of Q is irreducible. The proof is completed if we use the argument given at p. 448, 449 of [3].

Let $\mathcal{K}$ be the subset of $\mathcal{B}$ given by $\{[A]\in\mathcal{B};\ A$ is reducible$\}$. Since $F(A+\alpha)=F(A)+d_A\alpha-\alpha\wedge\alpha$, a slice neighborhood $\mathcal{U}_{[A]}$ of $[A]\in\mathcal{M}\backslash\mathcal{K}$ in $\mathcal{M}$ can be given by an ε-neighborhood of a slice

$$\{A+\alpha;\ |\alpha|_k<\varepsilon,\ d_A^*\alpha=0,\ d_A^+\alpha=\alpha\#\alpha\}\ , \tag{2.2}$$

where $d_A^+=p_+\circ d_A$ and $\#$; $\Omega^1(\mathfrak{g})\times\Omega^1(\mathfrak{g})\to\Omega_+^2(\mathfrak{g})=\Gamma(M;\Lambda_+^2\otimes\mathfrak{g})$ is defined by $\alpha\#\beta=(1/2)p_+(\alpha\wedge\beta+\beta\wedge\alpha)$.

To analyze more exactly the structure of neighborhoods of the moduli space $\mathcal{M}$ we need notion of an elliptic complex and also the integrating method due to Kuranishi ([11]).

For any anti-self-dual $SU(n)$-connection A the following sequence presents an elliptic complex ([3, p. 444], [7, Proposition 2.4])

$$0\to\Omega^0(\mathfrak{g})\overset{d_A}{\to}\Omega^1(\mathfrak{g})\overset{d_A^+}{\to}\Omega_+^2(\mathfrak{g})\to 0\ . \tag{2.3}$$

If the connection A is irreducible, then 0-th cohomology group H_A^0 vanishes. With respect to the second cohomology group H_A^2 we have the following two propositions.

Proposition 2.2. *Let A be an anti-self-dual connection. Then for each* $\Phi=\Phi^{2,0}+\Phi^{0,2}+\Phi^0\otimes\omega\in\Omega_+^2(\mathfrak{g})$

$$|d_A^{+*}\Phi|_M^2=(1/2)\{|\tilde{\nabla}_A\Phi^{2,0}|_M^2+|\tilde{\nabla}_A\Phi^{0,2}|_M^2\}+|d_A\Phi^0|_M^2$$
$$+(1/4)\int_M \mathrm{Scal}(g)\{|\Phi^{2,0}|^2+|\Phi^{0,2}|^2\}dv\ . \tag{2.4}$$

Here $\tilde{\nabla}_A$ denotes the covariant derivative with respect to A together with the

Levi-Civita connection of the metric g and $\mathrm{Scal}(g)$ is the scalar curvature of g. Notice that since each Φ in $\Omega^2_+(\mathfrak{g})$ takes values in $\mathfrak{su}(n)$, Φ satisfies the reality condition, that is, $\Phi^0 \in \Omega^0(\mathfrak{g})$ and $\Phi^{0,2} = -{}^t(\overline{\Phi^{2,0}})$.

Proposition 2.3. *If an $SU(n)$-connection A is anti-self-dual, then the second cohomology H^2_A is $\boldsymbol{R}$-isomorphic to $H^0_A \oplus \boldsymbol{H}$, Where $\boldsymbol{H}$ denotse the space of global holomorphic sections $H^0(M; \mathcal{O}(\mathfrak{g}^C \otimes K_M))$ with respect to the holomorphic structure $\mathfrak{g}^C$ on canonically induced from the A.*

Proof of Proposition 2.2. It suffices to show the following Bochner-Weitzenböck formula with respect to a general connection A;

$$\begin{aligned} |d_A^{+*}\Phi|^2_M = {} & (1/2)\{|\tilde{\nabla}_A\Phi^{2,0}|^2_M + |\tilde{\nabla}_A\Phi^{0,2}|^2_M\} + |d_A\Phi^0|^2_M \\ & + (1/4)\int_M \mathrm{Scal}(g)\{|\Phi^{2,0}|^2| + |\Phi^{0,2}|^2\} dv \\ & + 4\int_M Re\langle[\Phi^0, \sqrt{-1}\, F^{2,0}], \Phi^{2,0}\rangle dv \\ & - 2\int_M Re\langle[\Phi^{2,0}, \sqrt{-1}\, F^0], \Phi^{2,0}\rangle dv \end{aligned} \tag{2.5}$$

for $\Phi \in \Omega^2_+(\mathfrak{g})$ and $F_+(A) = p_+F(A) = F^{2,0} + F^{0,2} + F^0 \otimes \omega$.
Since

$$\begin{aligned} d_A^+(\Phi^{1,0} + \Phi^{0,1}) = {} & \partial_A\Phi^{1,0} + \bar\partial_A\Phi^{0,1} \\ & + (1/2)\langle\bar\partial_A\Phi^{1,0} + \partial_A\Phi^{0,1}, \omega\rangle \otimes \omega \end{aligned} \tag{2.6}$$

and we have

$$d_A^{+*}(\Phi^{2,0} + \Phi^{0,2}) = \partial_A^*\Phi^{2,0} + \bar\partial_A^*\Phi^{0,2}, \tag{2.7}$$

and

$$d_A^{+*}(\Phi^0 \otimes \omega) = \sqrt{-1}\,(\partial_A\Phi^0 - \bar\partial_A\Phi^0), \tag{2.8}$$

we obtain the following

$$\begin{aligned} d_A^+ d_A^{+*}(\Phi^{2,0} + \Phi^{0,2}) = {} & \partial_A\partial_A^*\Phi^{2,0} + \bar\partial_A\bar\partial_A^*\Phi^{0,2} \\ & + (1/2)\langle\bar\partial_A\partial_A^*\Phi^{2,0} + \partial_A\bar\partial_A^*\Phi^{0,2}, \omega\rangle \otimes \omega \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} d_A^+ d_A^{+*}(\Phi^0 \otimes \omega) = {} & \sqrt{-1}\{\partial_A\partial_A\Phi^0 - \bar\partial_A\bar\partial_A\Phi^0 \\ & + (1/2)\langle\bar\partial_A\partial_A\Phi^0 - \partial_A\bar\partial_A\Phi^0, \omega\rangle \otimes \omega\}\,. \end{aligned} \tag{2.10}$$

Since $d_A d_A\Phi^0 = [\Phi^0, F(A)]$, (2.10) reduces to

$$\begin{aligned} d_A^+ d_A^{+*}(\Phi^0 \otimes \omega) = {} & \sqrt{-1}\{[\Phi^0, F^{2,0}] - [\Phi^0, F^{0,2}]\} \\ & + (1/2)\,(\Box_A\Phi^0) \otimes \omega\,. \end{aligned} \tag{2.11}$$

Here we denote by $\Box_A$ the rough Laplacian $-\sum g^{\sigma\bar{\tau}}\tilde{\nabla}_\sigma\tilde{\nabla}_{\bar{\tau}}$. Hence the inner product $\langle d_A^+ d_A^{+*}(\Phi^0\otimes\omega), \Phi\rangle_M$ is given by

$$\langle d_A^+ d_A^{+*}(\Phi^0\otimes\omega), \Phi\rangle_M = \int_M 2\,Re\langle[\Phi^0, \sqrt{-1}\,F^{2,0}], \Phi^{2,0}\rangle dv + \langle\Box_A\Phi^0, \Phi^0\rangle_M . \tag{2.12}$$

On the other hand we have by an argument similar to [7, Lemma 3.3]

$$\partial_A\partial_A^*\Phi^{2,0} = (1/2)\Box_A\Phi^{2,0} + (1/4)\,\mathrm{Scal}(g)\,\Phi^{2,0} - (1/2)\,[\Phi^{2,0}, 2\sqrt{-1}\,F^0] . \tag{2.13}$$

By using the Ricci formula we obtain further

$$\langle\partial_A\partial_A^*\Phi^{2,0}, \omega\rangle = \sqrt{-1}\sum g^{\mu\bar{\nu}}(\bar{\partial}_A\partial_A^*\Phi^{2,0})_{\mu\bar{\nu}} + (\sqrt{-1}/2)\sum g^{\sigma\bar{\tau}}g^{\mu\bar{\nu}}[\Phi_{\sigma\mu}, F_{\bar{\tau}\bar{\nu}}] . \tag{2.14}$$

Therefore (2.5) is derived from these formulas.

Proof of Proposition 2.3. Since the curvature form $F(A)$ is of type (1,1), the connection A induces a holomorphic structure on the complex adjoint bundle $\mathfrak{g}^C$. Namely a smooth section Φ of $\mathfrak{g}^C$ satisfies $\bar{\partial}_A\Phi=0$ if and only if Φ is holomorphic relative to the holomorphic structure. Then the space $\{\Phi\in\Omega^{0,2}(\mathfrak{g}^C); \bar{\partial}_A\bar{\partial}_A^*\Phi=0\}$ is isomorphic with the second cohomology $H^2(M; \mathcal{O}(\mathfrak{g}^C))$ from Theorem 4.1, ch. 3 in [10].

Moreover it is isomorphic with the space $\boldsymbol{H}$ by the aid of Serre's duality theorem and the self-duality of $\mathfrak{g}^C$ as a vector bundle. In the course of the proof of Proposition 2.2 we can also verify that

$$|\bar{\partial}_A^*\Phi^{0,2}|_M^2 = (1/2)|\tilde{\nabla}_A\Phi^{0,2}|_M^2 + (1/4)\int_M \mathrm{Scal}(g)|\Phi^{0,2}|^2 dv \tag{2.15}$$

for $\Phi^{0,2}\in\Omega^{0,2}(\mathfrak{g}^C)$. Thus we have

$$|d_A^{+*}\Phi|_M^2 = |\partial_A^*\Phi^{2,0}|_M^2 + |\bar{\partial}_A^*\Phi^{0,2}|_M^2 + |d_A\Phi^0|_M^2 \tag{2.16}$$

from which the proposition follows easily.

REMARK 2.1. If the canonical line bundle K_M is trivial, then $\boldsymbol{H}$ is $\boldsymbol{C}$-isomorphic to $(H_A^0)^C$. On the other hand, if the metric g is of positive total scalar curvature, i.e., $\int_M \mathrm{Scal}(g)\,dv>0$, then $\boldsymbol{H}$ vanishes.

By applying the Atiyah-Singer index theorem to complex (2.4), we have ([7]) $h^0-h^1+h^2 = -2c_2(\mathfrak{g}^C)+2\dim SU(n)\cdot p_a(M)$, where $p_a(M)$ denotes the arithmetic genus of M and $h^i=\dim_{\boldsymbol{R}} H_A^i$, $i=0,1,2$. If both H^0 and H^2 vanish, then H^1 has even dimension.

Proposition 2.4. *The first cohomology group H_A^1 is $\boldsymbol{R}$-isomorphic to the com-*

plex vector space $\mathcal{H}^1=\{\alpha^{(0,1)}\in\Omega^{(0,1)}(\mathfrak{g}^C),\ \bar\partial_A\alpha^{(0,1)}=0,\ \bar\partial_A^*\alpha^{(0,1)}=0\}$.

Proof. Each $\mathfrak{g}$-valued 1-form α splits into

$$\alpha=\alpha^{(1,0)}+\alpha^{(0,1)},\ \alpha^{(1,0)}=\sum_\mu \alpha_\mu dz^\mu\in\Omega^{(1,0)}(\mathfrak{g}^C),$$
$$\alpha^{(0,1)}=\sum_\mu \alpha_{\bar\mu}dz^{\bar\mu}\in\Omega^{(0,1)}(\mathfrak{g}^C)\ \text{ with } {}^t(\overline{\alpha^{(1,0)}})=-\alpha^{(0,1)}.$$

We define a mapping $h;\Omega^1(\mathfrak{g})\to\Omega^{(0,1)}(\mathfrak{g}^C)$ by assigning $\alpha^{(0,1)}$ to α. We show that $h_{|H^1}$ gives an isomorphism of H^1 to $\mathcal{H}^1$. By an argument given in [7] we see that $d_A^*\alpha=0$ if and only if

$$\sum g^{\mu\bar\nu}\nabla_{\bar\nu}\alpha_\mu+\sum g^{\mu\bar\nu}\nabla_\mu\alpha_{\bar\nu}=0 \tag{2.17}$$

and that $d_A^+\alpha=0$ if and only if

$$\begin{cases}\partial_A\alpha^{(1,0)}=0, & \bar\partial_A\alpha^{(0,1)}=0,\\ \sum g^{\mu\bar\nu}(\nabla_{\bar\nu}\alpha_\mu-\nabla_\mu\alpha_{\bar\nu})=0. \end{cases} \tag{2.18}$$

Hence, if α is in H^1, then $\bar\partial_A\alpha^{(0,1)}=0$ and $\bar\partial_A^*\alpha^{(0,1)}=-\sum g^{\mu\bar\nu}\nabla_\mu\alpha_{\bar\nu}=0$. Since ${}^t(\overline{\alpha^{(1,0)}})=-\alpha^{(0,1)}$, the inverse implication is easily derived.

Remark 2.2. Proposition 2.4 is also established for a connection which is not necessarily anti-self-dual.

Now we define for each $[A]$ in the moduli space $\mathcal{M}\backslash\mathcal{K}$ a mapping $\Phi=\Phi_A$; $\Omega^1(\mathfrak{g})\to\Omega^1(\mathfrak{g})$ by $\Phi(\alpha)=\alpha-d_A^{+*}(G_A(\alpha\#\alpha))$ ([2], [4]). Here G_A is the Green operator of the Laplace operator $d_A^+\circ d_A^{+*}$. Relative to the norms $|\cdot|_k$ we have

$$|d_A\alpha|_{k-1}\leqq c_k|\alpha|_k, \tag{2.19}$$

$$|G_A\Psi|_{k+2}\leqq c_k|\Psi|_k \tag{2.20}$$

and

$$|\alpha\#\beta|_k\leqq c_k|\alpha|_k|\beta|_k \tag{2.21}$$

for $\alpha,\beta\in L_k^2(\Omega^1(\mathfrak{g}))$, $\Psi\in L_k^2(\Omega_+^2(\mathfrak{g}))$, where c_k is a constant depending only on the manifold M(Ch. 4 of [10], [11]). Therefore the mapping Φ_A; $L_k^2(\Omega^1(\mathfrak{g}))\to L_k^2(\Omega^1(\mathfrak{g}))$ is differentiable. Suppose that $H_A^2=0$. Then we have on $\Omega_+^2(\mathfrak{g})$ $d_A^+\circ d_A^{+*}\circ G_A=\mathrm{id}$. Hence a slice neighborhood $U_{A,\varepsilon}$, identified with $\mathcal{V}_{[A]}$ of $[A]$ is mapped by the Φ into H_A^1. Since the derivative of Φ at $\alpha=0$ is identity, it has an inverse on a sufficiently small neighborhood $U_\varepsilon=\{\beta\in H_A^1;\ |\beta|_M<\varepsilon\}$.

Notice that by using a prior estimates of elliptic differential operators each β in $L_k^2(\Omega^1(\mathfrak{g}))$ satisfying $(d_A d_A^*+d_A^{+*}d_A^+)\beta=0$ is a smooth section and norms $|\beta|_k$ and $|\beta|_M$ are equivalent.

As a consequence of these propositions we obtain

Proposition 2.5. *Let M be a compact Kähler surface with a Kähler metric g and P a principal $SU(n)$-bundle with $c_2(\mathfrak{g}^C)>0$. Suppose that either the canonical line bundle K_M is trivial or the metric is with positive total scalar curvature. Then, if the moduli space $\mathcal{M}\backslash\mathcal{K}$ of irreducible anti-self-dual connections on P is not empty, it is a smooth manifold of dimension $2\,c_2(\mathfrak{g}^C)-2(n^2-1)\cdot p_a(M)$.*

REMARK 2.3. On the subset $\mathcal{B}\backslash\mathcal{K}=\{[A]\in\mathcal{B};\ A$ is irreducible$\}$ we define a metric function σ (see for the precise discussion p. 448 in [3]); $\sigma([A],[A_1])=\inf_{g\in\mathcal{G}}|A-g(A_1)|_M$. Since σ is continuous relative to the L^2_k-topology, $\mathcal{B}\backslash\mathcal{K}$ is a Hausdorff space. Therefore the moduli space $\mathcal{M}\backslash\mathcal{K}$, a closed subset of $\mathcal{B}\backslash\mathcal{K}$, is also Hausdorff with respect to the relative topology.

3. (0,1)-connections and moduli space of holomorphic (0,1)-connections

We denote by P^C a smooth principal $SL(n;\boldsymbol{C})$-bundle given by extending the transition functions of the bundle P to $SL(n;\boldsymbol{C})$. The complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ clearly coincides with $P^C\times_{Ad}\mathfrak{sl}(n;\boldsymbol{C})$. Now we define on P^C a (0,1)-connection and a holomorphic (0,1)-connection as follows.

DEFINITION 3.1. Let $\{W_\alpha\}$ be the open covering of M consisting of local trivializing neighborhoods of P. A system $A=\{A_\alpha\}$, where each A_α is a smooth $\mathfrak{sl}(n;\boldsymbol{C})$-valued (0,1)-form defined over W_α, is called a (0,1)-connection on P^C, when it satisfies the cocycle condition

$$A_\beta=\bar{\partial}g\cdot g^{-1}+g\cdot A_\alpha\cdot g^{-1} \tag{3.1}$$

on $W_\alpha\cap W_\beta$, where $g=g_{\alpha\beta}$ is the transition function of P.

The set $\mathcal{A}^{(0,1)}$ of all (0,1)-connections on P^C has a structure of affine space. The group of complex gauge transformations $\mathcal{G}^C=\Gamma(M;\,P^C\times_{Ad}SL(n;\boldsymbol{C}))$ acts on $\mathcal{A}^{(0,1)}$ in the form

$$g(A)=\bar{\partial}g\cdot g^{-1}+g\cdot A\cdot g^{-1}\,, \tag{3.2}$$

$g\in\mathcal{G}^C$, $A\in\mathcal{A}^{(0,1)}$. We denote by $\mathcal{B}^{(0,1)}$ the quotient space $\mathcal{A}^{(0,1)}/\mathcal{G}^C$.

REMARK 3.1. By its definition, each (0,1)-connection is not a connection by itself. But we have a mapping $h;\mathcal{A}\to\mathcal{A}^{(0,1)};\ A\mapsto A^{(0,1)}$, where $A^{(0,1)}$ is the (0,1)-component of A. Then h is one-to-one and onto, because for every (0,1)-connection $A=\{A_\alpha\}$ on P^C a system $\tilde{A}=\{\tilde{A}_\alpha\}$ given by $\tilde{A}_\alpha=A_\alpha-{}^t(\overline{A_\alpha})$ satisfies (2.1) from (3.1) and it takes values in $\mathfrak{su}(n)$, and hence it gives an $SU(n)$-connection on P and $h(\tilde{A})=A$.

A (0,1)-connection A is called irreducible, if $\bar{\partial}_A;\,\Omega^0(\mathfrak{g}^C)\to\Omega^{(0,1)}(\mathfrak{g}^C);\ \Psi\mapsto\bar{\partial}\Psi+[\Psi,A]$ has trivial kernel. We call a (0,1)-connection reducible when it is not irreducible.

For each $A\in\mathcal{A}^{(0,1)}$ the curvature form $F(A)=\bar{\partial}A-A\wedge A$ is defined. The curvature form $F(A)$ belongs to $\Omega^{(0,2)}(\mathfrak{g}^C)$.

DEFINITION 3.2. A (0,1)-connection A is called holomorphic if $F(A)=0$.

REMARK 3.2. Since the curvature form of a (0,1)-connection A coincides with the (0,2)-component of the curvature form of the $SU(n)$-connection $\tilde{A}$ induced from A, there exists for each holomorphic (0,1)-connection A a holomorphic structure $J=J_A$ on $\mathfrak{g}^C$ relative to which $\tilde{A}$ gives a hermitian holomorphic connection on $\mathfrak{g}^C$ in the usual sense ([4]). Namely, there exist smooth mappings h_α; $W_\alpha\to SL(n;\boldsymbol{C})$ with properties that (i) $h_{\alpha\beta}=h_\alpha\cdot g_{\alpha\beta}\cdot h_\beta^{-1}$; $W_\alpha\cap W_\beta\to SL(n;\boldsymbol{C})$ is holomorphic for each α and β and (ii) $\tilde{A}_\alpha$ is transformed into a (1,0)-form $h_\alpha(\tilde{A}_\alpha)=dh_\alpha\cdot h_\alpha^{-1}+h_\alpha\cdot\tilde{A}_\alpha\cdot h_\alpha^{-1}$ by h_α.

Proposition 3.1. *Let A be a holomorphic connection. Then the following sequence gives an elliptic complex;*

$$0\to\Omega^0(\mathfrak{g}^C)\xrightarrow{\bar{\partial}_A}\Omega^{(0,1)}(\mathfrak{g}^C)\xrightarrow{\bar{\partial}_A}\Omega^{(0,2)}(\mathfrak{g}^C)\to 0 \tag{3.3}$$

Proof. Since $\bar{\partial}_A\bar{\partial}_A\Psi=[\Psi, F(A)]$ for $\psi\in\Omega^0(\mathfrak{g}^C)$, the above sequence gives a complex. It is easily verified that the symbol sequence of the above is exact.

On the spaces $\Omega^{(0,p)}(\mathfrak{g}^C)$ we define inner products $\langle\cdot,\cdot\rangle_M$ by $\langle\Phi, \Psi\rangle_M=\int_M Tr(\Phi\wedge *^t(\bar{\Psi}))$, $p=0,1,2$. Notice that these products are not $\mathfrak{g}^C$-invariant.

We set the subspaces $\mathcal{H}^p=\operatorname{Ker}\Delta^p$ of $\Omega^{(0,p)}(\mathfrak{g}^C)$ by the aid of the complex Laplacians Δ^p, $p=0,1,2$ associated to the above complex. Then by using the Atiyah-Singer index theorem we have the index of the complex (3.3) as

$$\begin{aligned} h^0-h^1+h^2 = {} & ch(\mathfrak{g}^C)\{ch(\Lambda^{0C})-ch(\Lambda^{(0,1)})+ch(\Lambda^{(0,2)})\}\times \\ & e(TM)^{-1}\cdot\mathcal{I}(TM^C)\,[M] \end{aligned} \tag{3.4}$$

where $h^p=\dim_{\boldsymbol{C}}\mathcal{H}^p$. By a simple computation the index equals to $-c_2(\mathfrak{g}^C)+(n^2-1)\cdot p_a(M)$.

Since the group $\mathcal{G}^C$ leaves the set of holomorphic (0,1)-connections invariant, we obtain its quotient space $\mathcal{M}_h$, called the moduli space of holomorphic (0,1)-connections.

The center of $SL(n;\boldsymbol{C})$ which coincides with the center of $SU(n)$ gives complex gauge transformations commuting with each g of $\mathcal{G}^C$. In the same way as the case of $SU(n)$ the center $Z(\mathcal{G}^C)$ of $\mathcal{G}^C$ is just the center Z and it acts trivially on $\mathcal{A}^{(0,1)}$. Since $\mathcal{G}^C$ is a subset of $\Gamma(M;\operatorname{End}\boldsymbol{E})=\Gamma(M;\boldsymbol{1})\oplus\Gamma(M;\mathfrak{g}^C)$ the isotropy subgroup $\Gamma_A^{\mathcal{G}}$ of each irreducible (0,1)-connection A reduces to Z. Thus the quotient group $\tilde{\mathcal{G}}^C=\mathcal{G}^C/Z$ acts effectively on $\mathcal{A}^{(0,1)}$ and its action is free on the subset $\{A\in\mathcal{A}^{(0,1)}; A \text{ is irreducible}\}$. Besides the inner product $\langle\cdot,\cdot\rangle_M$

we define on $\Omega^{(0,p)}(\mathfrak{g}^C)$ the Sobolev's norms $|\cdot|_k$ and let $\mathcal{A}^{(0,1)}$ be $\{A_0+\alpha;\ \alpha\in L_k^2(\Omega^{(0,1)}(\mathfrak{g}^C))\}$ for a fixed smooth (0,1)-connection A_0. In L_{k+1}^2-topology $\mathcal{G}^C$ and hence $\tilde{\mathcal{G}}^C$ acts smoothly on $\mathcal{A}^{(0,1)}$. The quotient space $\mathcal{B}^{(0,1)}=\mathcal{A}^{(0,1)}/\tilde{\mathcal{G}}^C$ gets the canonical quotient topology by the projection π'; $\mathcal{A}^{(0,1}\to\mathcal{B}^{(0,1)}$. We denote by $\mathcal{K}^{(0,1)}\{[A]\in\mathcal{B}^{(0,1)};\ A$ is reducible$\}$, the subset of $\mathcal{B}^{(0,1)}$.

Like an $SU(n)$-connection we call a subset V_A of $\mathcal{A}^{(0,1)}\{A+\alpha;\ \alpha\in L_k^2(\Omega^{(0,1)}(\mathfrak{g}^C)),\ \bar{\partial}_A^*\alpha=0\}$ a slice at A.

Lemma 3.2. *Let A be an irreducible* (0,1)*-connection on P^C. Then there exists for a sufficiently small $\varepsilon>0$ a slice neighborhood $V_{A,\varepsilon}=\{A+\alpha\in V_A;\ |\alpha|_k<\varepsilon\}$ whose image $\pi'(V_{A,\varepsilon})$ gives a neighborhood of $[A]$ in $\mathcal{B}^{(0,1)}$.*

Proof. Define a mapping $T;\ V_{A,\varepsilon}\times\mathcal{G}^C/Z\to\mathcal{A}^{(0,1)};\ T(A+\alpha,g)=g(A+\alpha)$. Then in a manner similar to the case of $SU(n)$-connections, T is smooth relative to the L_k^2-topologies and its derivative at $\alpha=0$ and $g=$identity is written by

$$DT;\ \mathrm{Ker}\ \bar{\partial}_A^*\times\Omega^0(\mathfrak{g}^C)\to\Omega^{(0,1)}(\mathfrak{g}^C)$$
$$(\alpha,\psi)\mapsto\alpha+\bar{\partial}_A\psi\,.$$

Since Ker $\bar{\partial}_A=0$ and $\Omega^{(0,1)}(\mathfrak{g}^C)=\mathrm{Im}\ \bar{\partial}_A\oplus\mathrm{Ker}\ \bar{\partial}_A^*$ T is a local diffeomorphism. Therefore by using the argument which was used at the proof of Proposition 2.1 we obtain the lemma.

Proposition 3.3. *Each irreducible $[A]\in\mathcal{M}_h$ has a neighborhood $\mathcal{V}_{[A]}$ which is given by the image of $V_{A,\varepsilon}=\{A+\alpha;\ \alpha\in\Omega^{(0,1)}(\mathfrak{g}^C),\ |\alpha|_k<\varepsilon,\ \bar{\partial}_A^*\alpha=0,\ \bar{\partial}_A\alpha=\alpha\wedge\alpha\}$.*

Proof. Since $F(A+\alpha)=F(A)+\bar{\partial}_A\alpha-\alpha\wedge\alpha$, this is a direct consequence of the above lemma.

Let $\Psi=\Psi_A$ be a mapping from $L_k^2(\Omega^{(0,1)}(\mathfrak{g}^C))$ to itself defined by $\Psi(\alpha)=\alpha-(\bar{\partial}_A^*)(G_A(\alpha\wedge\alpha))$. Here G_A denotes the Green operator of Δ_A^2. Assume now that the second cohomology group $\mathcal{H}^2$ vanishes. Then we see that $\bar{\partial}_A^*\alpha=0$ and $\bar{\partial}_A\alpha=\alpha\wedge\alpha$ if and only if $\Psi(\alpha)\in\mathcal{H}^1$. Thus the slice neighborhood $V_{A,\varepsilon}$ is mapped through Ψ into $\mathcal{H}^1$. Because over $L_k^2(\Omega^{(0,1}(\mathfrak{g}^C))$ the derivative $D\Psi$ at $\alpha=0$ is identity, $\Psi_{|V_{A,\varepsilon}}$ has an inverse over a small ε-neighborhood V_ε of $\mathcal{H}^1$. We remark that $\Psi^{-1}{}_{|V_\varepsilon}$ is holomorphic as a mapping from an open subset of a Banach space to a Banach space, since Ψ is quadratic over the completed Banach space $L_k^2(\Omega^{(0,1)}(\mathfrak{g}^C))$ ([11]).

4. Canonical imbedding of $\mathcal{M}\backslash\mathcal{K}$ into $\mathcal{M}_h\backslash\mathcal{K}^{(0,1)}$

Let A be an $SU(n)$-connection on the bundle P. Then the (0,1)-component $A^{(0,1)}$ of A certainly defines a (0,1)-connection on the complexified bundle P^C and the curvature $F(A^{(0,1)})$ is given by the (0,2)-component of $F(A)$. If A

is anti-self-dual, then $F(A)$ is of type (1,1), and hence $A^{(0,1)}$ is holomorphic. Because $\mathcal{G}\subset\mathcal{G}^C$, to each $[A]$ of $\mathcal{M}$ we can assign $[A^{(0,1)}]$ of $\mathcal{M}_h$. We denote this assignment by f.

Proposition 4.1. *If an anti-self-dual connection A is irreducible, then $A^{(0,1)}$ is also irreducible.*

Proof. Since A is anti-self-dual we have the formula $\sum g^{\mu\bar{\nu}}F_{\mu\bar{\nu}}(A)=0$ ([7, Proposition 2.2]). Then we obtain for a nonzero ψ of $\Omega^0(\mathfrak{g}^C)$ satisfying $\bar{\partial}_A\psi=0$ that

$$\begin{aligned}\sum g^{\mu\bar{\nu}}\nabla_{\bar{\nu}}\nabla_{\mu}Tr(\psi\cdot{}^t\bar{\psi}) &= \sum g^{\mu\bar{\nu}}Tr(\nabla_{\mu}\psi\cdot{}^t\nabla_{\nu}\psi) \\ \sum g^{\mu\bar{\nu}}Tr([\psi, F(A)_{\mu\bar{\nu}}]\cdot{}^t\bar{\psi}) &= |\partial_A\psi|^2 .\end{aligned} \tag{4.1}$$

We integrate this over M to get $\partial_A\psi=0$, that is, $d_A\psi=0$. The sections ϕ and ϕ' of the adjoint bundle $\mathfrak{g}$ given by $\phi=\psi-{}^t\bar{\psi}$ and $\phi'=(1/\sqrt{-1})(\psi+{}^t\bar{\psi})$, respectively, are parallel with respect to d_A.

From this proposition we have $f(\mathcal{M}\backslash\mathcal{K})\subset\mathcal{M}_h\backslash\mathcal{K}^{(0,1)}$.

Now we show the following

Proposition 4.2. *The mapping f restricted to $\mathcal{M}\backslash\mathcal{K}$ is injective.*

Proof. It suffices to verify that if there is for irreducible anti-self-dual connections A and A_1 $g\in\mathcal{G}^C$ satisfying $(A_1)^{(0,1)}=g(A^{(0,1)})$, then g must lie in $\mathcal{G}$.

By the way $SL(n; \boldsymbol{C})$ has the following decomposition; $SL(n; \boldsymbol{C})=H_0^+(n)\cdot SU(n)$, where $H_0^+(n)$ means the set of all positive definite Hermitian matrices with determinant 1. This decomposition is invariant under the adjoint representation of $SU(n)$, namely, if $X\in SL(n;\boldsymbol{C})$ splits into $X=X^h\cdot X^u$, $X^u\in SU(n)$, $X^h\in H_0^+(n)$, then $Y\cdot X\cdot Y^{-1}=(Y\cdot X^h\cdot Y^{-1})(Y\cdot X^u\cdot Y^{-1})$, $Y\in SU(n)$ gives the decomposition of $Y\cdot X\cdot Y^{-1}$. Therefore the complex gauge transformation g splits into $g=g_1\cdot g^u$, $g^u\in\mathcal{G}$, $g_1\in\Gamma(M; P\times_{SU(n)}H_0^+(n))$. Then we have $(A_1)^{(0,1)}=g_1(g^u(A^{(0,1)}))$. Moreover $g^u(A^{(0,1)})=(g^uA)^{(0,1)}$ and $g^u(A)$ is anti-self-dual since g^u is unitary.

Because the exponential map $\exp; H_0(n)\to H_0^+(n); X\mapsto \exp X$ is a diffeomorphism, here $H_0(n)$ is the set of all Hermitian matrices of trace zero, we can lift exp to a bundle map $\exp; P\times_{SU(n)}H_0(n)\to P\times_{SU(n)}H_0^+(n)$. From the fact $H_0(n)=\sqrt{-1}\,\mathfrak{su}(n)$ we induce a canonical mapping from $\mathfrak{g}$ to $P\times_{SU(n)}H_0^+(n)$ by $\phi\mapsto\exp\sqrt{-1}\,\phi$. Then there is a $\psi\in\Omega^0(\mathfrak{g})$ such that $g_1=\exp\sqrt{-1}\,\psi$. A one-parameter subgroup $g_t=\exp(t\sqrt{-1}\,\psi)$, $t\in\boldsymbol{R}$, of $\mathcal{G}^C$ yields a one-parameter family of (0,1)-connections $\{\hat{A}_t\}$ by $\hat{A}_t=g_t((A_0)^{(0,1)})$, where $A_0=g^u(A)$. Further the family $\{\hat{A}_t\}$ defines a family of connections $\{A_t\}$ of P by $A_t=\hat{A}_t-{}^t(\overline{\hat{A}_t})$. The curvature F_t of A_t is certainly of type (1,1).

Now we apply the method of moment map developed at [6, p. 11]. Define for $\{A_t\}$ a function $m; \boldsymbol{R}\to\boldsymbol{R}$ by

$$m(t) = \int_M R_2(t) \wedge \omega \,, \tag{4.2}$$

where $R_2(t)$ is a 2-form of type (1,1) over M modulo Im ∂+Im $\bar\partial$ satisfying

$$\sqrt{-1}\,\bar\partial\partial R_2(t) = -TrF_t \wedge F_t - (-TrF_0 \wedge F_0)\,. \tag{4.3}$$

Then we have the following facts (Proposition 8 of [6]). Since A_0 is anti-self-dual, $d/dt\,|_{t=0} m(t)=0$ and

$$d^2/dt^2\, m(t) = |d_{A_t}\psi|^2_M \geqq 0\,. \tag{4.4}$$

Because $m(t)$ is critical at also $t=1$, $d^2/dt^2\, m(t)=0$ identically, hence $d_{A_t}\psi=0$. Using the irreducibility of A_0 we have $\psi=0$ and hence g_1=identity, that is, $g\in\mathcal{G}$.

We define open subsets $(\mathcal{M}\backslash\mathcal{K})_0$ and $(\mathcal{M}_h\backslash\mathcal{K}^{(0,1)})_0$ of $\mathcal{M}\backslash\mathcal{K}$ and $\mathcal{M}_h\backslash\mathcal{K}^{(0,1)}$, respectively, by $(\mathcal{M}\backslash\mathcal{K})_0=\{[A]\in\mathcal{M}\backslash\mathcal{K};\ \boldsymbol{H}_A=0\}$ and $(\mathcal{M}_h\backslash\mathcal{K}^{(0,1)})_0=\{[A']\in \mathcal{M}_h\backslash\mathcal{K}^{(0,1)};\ \mathcal{H}^2_{A'}=0\}$. Since from Proposition 2.3 $\mathcal{H}^2_{A(0,1)}\cong\boldsymbol{H}_A$ for the (0,1)-component $A^{(0,1)}$ of an anti-self-dual connection A we have $f((\mathcal{M}\backslash\mathcal{K})_0)\subset(\mathcal{M}_h\backslash \mathcal{K}^{(0,1)})_0$.

Proposition 4.3. $f|_{(\mathcal{M}\backslash\mathcal{K})_0};\ (\mathcal{M}\backslash\mathcal{K})_0\to(\mathcal{M}_h\backslash\mathcal{K}^{(0,1)})_0$ *is an open mapping.*

Proof. Let $\mathcal{U}_{[A]}$ be a neighborhood of $[A]\in(\mathcal{M}\backslash\mathcal{K})_0$, identified with a slice neighborhood $U_{A,\varepsilon}=\{A+\alpha;\ |\alpha|_k<\varepsilon,\ d_A^*\alpha=0,\ d_A^+\alpha=\alpha\#\alpha\}$. We notice that if α is such a one-form its (0,1)-component $\alpha^{(0,1)}$, denoted by $h(\alpha)$ in §2, satisfies $\bar\partial_{A'}\alpha^{(0,1)}=\alpha^{(0,1)}\wedge\alpha^{(0,1)}$ but does not necessarily satisfy $(\bar\partial^*_{A'})\alpha^{(0,1)}=0$ for $A'=A^{(0,1)}\in\mathcal{A}^{(0,1)}$. Let $\mathcal{V}_{[A']}$ be a neighborhood of $[A']$ in $(\mathcal{M}_h\backslash\mathcal{K}^{(0,1)})_0$, written in the form of the image of a slice neighborhood $V_{A',\varepsilon'}=\{A'+\gamma^{(0,1)};\ |\gamma^{(0,1)}|_k<\varepsilon',\ (\bar\partial^*_{A'})\gamma^{(0,1)}=0,\ \bar\partial_{A'}\gamma^{(0,1)}=\gamma^{(0,1)}\wedge\gamma^{(0,1)}\}$.

Assertion. *If we choose a sufficiently small ε, then for any $A+\alpha$ in $U_{A,\varepsilon}$ there is a unique $g=g_\alpha$ in $\mathcal{G}^C$ close to the identity so that $g(A'+h(\alpha))$ belongs to $V_{A',\varepsilon'}$.*

This assertion is shown as follows. Since $g(A'+h(\alpha))=(\bar\partial_{A'}g)\cdot g^{-1}+g\cdot h(\alpha)\cdot g^{-1}+A'$, the (0,1)-form γ' defined by $A'+\gamma'=g(A'+h(\alpha))$ is represented by $\gamma'=(\bar\partial_{A'}g)\cdot g^{-1}+g\cdot h(\alpha)\cdot g^{-1}$. The (0,1)-connection $A'+\gamma'$ is indeed holomorphic and satisfies $\bar\partial_{A'}\gamma'-\gamma'\wedge\gamma'=0$. Then γ' lies in $V_{A',\varepsilon'}$ if and only if for $\bar\partial_A=\bar\partial_{A'}$

$$(\bar\partial^*_A)\,\{(\bar\partial_A g)\cdot g^{-1}+g\cdot h(\alpha)\cdot g^{-1}\} = 0 \tag{4.5}$$

If we set $g=\exp\psi$, $\psi\in\Omega^0(\mathfrak{g}^C)$, then we reduce (4.5) to

$$\begin{aligned}\bar\partial^*_A\bar\partial_A\psi+\bar\partial^*_A h(\alpha)-\langle[\partial_A\psi,\ h(\alpha)]\rangle+[\psi,\ \bar\partial^*_A h(\alpha)]&\\ +\bar\partial^*_A R(\psi,\ h(\alpha)) = 0\,,&\end{aligned} \tag{4.6}$$

here $R(\psi, h(\alpha))$ is the remainder term of order not less than two. We operate the Green operator $G_{A'}$ of $\Delta^0_{A'}$ to (4.6) to deduce

$$\psi+G_{A'}(\bar\partial^*_A h(\alpha))-G_{A'}\langle[\partial_A\psi, h(\alpha)]\rangle+G_{A'}[\psi, \bar\partial^*_A h(\alpha)] \\ +G_A(\bar\partial^*_A R)=0\,. \qquad (4.7)$$

We remark that since $\alpha=\alpha^{(1,0)}+\alpha^{(0,1)}=\sum(\alpha_\mu dz^\mu+\alpha_{\bar\mu}dz^{\bar\mu})$ satisfies $d^*_A\alpha=0$ and $d^+_A\alpha=\alpha\sharp\alpha$,

$$\bar\partial^*_A h(\alpha)=-(\sqrt{-1}/2)\sum g^{\mu\bar\nu}[\alpha_\mu, \alpha_{\bar\nu}] \qquad (4.8)$$

and hence the $|\cdot|_k$-norm of $\bar\partial^*_A h(\alpha)$ is estimated by $|\alpha|_k$.

By using the arguments of Section 3 in Ch. 4 of [10] and also of [3], [11] we obtain for a sufficiently small $|\alpha|_k$ a unique smooth solution $\psi=\psi(\alpha)$ to (4.7) in a neighborhood of $0\in\Omega^0(\mathfrak{g}^C)$. We see easily that ψ depends smoothly on α and $g_\alpha(A'+h(\alpha))\in V_{A',\varepsilon'}$ for $g_\alpha=\exp\psi(\alpha)$.

We remark that $\psi(0)=0$ and from an implicit function theorem we have $(d\psi(\alpha)/d\alpha)|_{\alpha=0}=0$ and hence $(dg_\alpha/d\alpha)|_{\alpha=0}=\mathrm{id}$.

From the above assertion the mapping $\tilde f$; $U_{A,\varepsilon}\to V_{A',\varepsilon'}$ defined by $A+\alpha\mapsto g_\alpha(A'+h(\alpha))$ is smooth. We show now that the composition of the following mappings

$$U_\varepsilon(\subset H^1_A)\xrightarrow{\Phi_A^{-1}}U_{A,\varepsilon}\xrightarrow{\tilde f}V_{A',\varepsilon'}\xrightarrow{\Psi_{A'}}V_{\varepsilon'}(\subset\mathcal{H}^1_{A'})$$

is of maximal rank at $\beta=0$ in H^1_A. Since $(d\Phi_A/d\beta)|_{\beta=0}$ is the identity mapping of H^1_A and also $(d\Psi_{A'}/d\beta')|_{\beta'=0}$ gives the identity mapping of $\mathcal{H}^1_{A'}$ and further $(d\tilde f/d\alpha)|_{\alpha=0}(\gamma)=\lim_{t\to0}\{g_{t\gamma}(A'+h(t\gamma)-A')\}/t=h(\gamma)$ for each $\gamma\in H^1_A$, the derivative of the mapping at $\beta=0$ coincides from Porposition 2.4 with h; $H^1_A\to\mathcal{H}^1_{A'}$. Because h is $\boldsymbol{R}$-isomorphic, it gives a local diffeomorphism at $\alpha=0$ and then $\tilde f$; $U_{A,\varepsilon}\to V_{A',\varepsilon'}$ is open. Since $\tilde f$ is a lift of $f|\mathcal{U}_{[A]}$;

$$\begin{array}{ccc} U_{A,\varepsilon} & \xrightarrow{\tilde f} & V_{A',\varepsilon'} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} \\ \mathcal{U}_{[A]}(\subset(\mathcal{M}\backslash\mathcal{K})_0) & \xrightarrow{f} & \mathcal{V}_{[A']}(\subset(\mathcal{M}_h\backslash\mathcal{K}^{(0,1)})_0)\,, \end{array}$$

f is also open from the fact that π; $U_{A,\varepsilon}\to\mathcal{U}_{[A]}$ is a homeomorphism and π'; $V_{A',\varepsilon'}\to\mathcal{V}_{[A']}$ is open.

Remark 4.1. (1) The image $f((\mathcal{M}\backslash\mathcal{K})_0)$ is an open subspace in $\mathcal{M}_h\backslash\mathcal{K}^{(0,1)}$, identified with $(\mathcal{M}\backslash\mathcal{K})_0$. (2) Although $(\mathcal{M}_h\backslash\mathcal{K}^{(0,1)})_0$ may not necessarily be Hausdorff, $f((\mathcal{M}\backslash\mathcal{K})_0)$ is surely a Hausdorff space because $(\mathcal{M}\backslash\mathcal{K})_0$ is Hausdorff from Remark 2.3. (3) Since the mapping $\tilde f$; $U_{A,\varepsilon}\to V_{A',\varepsilon'}$ provided in the above proof is locally diffeomorphic, we can choose sufficiently small ε', if necessary, so that $\pi'|_{V_{A',\varepsilon}}$ gives a homeomorphism of $V_{A',\varepsilon'}$ onto a neighborhood $\mathcal{V}_{[A']}$ of

$f((\mathcal{M}\backslash\mathcal{K})_0)$.

5. Complex structure of the moduli space

The aim of this section is to pove the following.

Proposition 5.1. *The moduli space $f((\mathcal{M}\backslash\mathcal{K})_0)$ is a complex manifold of dimension $c_2(\mathfrak{g}^C)-(n^2-1)p_a(M)$, if it is not empty.*

Proof. By Propositions 4.2 and 4.3 and also from (3) of Remark 4.1 we can assume that for each $[A]\in f((\mathcal{M}\backslash\mathcal{K})_0)$ and for a sufficiently small $V_A=V_{A,\varepsilon}$ that the mapping Ψ_A; $V_A\to V_\varepsilon=\{\beta\in\mathcal{H}_A^1;\ |\beta|_M<\varepsilon\}$ defines a coordinate system for $f((\mathcal{M}\backslash\mathcal{K})_0)$.

Fix points $[A]$ and $[A']$ in $f((\mathcal{M}\backslash\mathcal{K})_0)$ with $\pi'(V_A)\cap\pi'(V_{A'})\neq\phi$. We define subsets $B\subset V_A$ and $B'\subset V_{A'}$ by $B=\{A+\alpha\in V_A;\ \pi'(A+\alpha)\in\pi'(V_{A'})\}$ and $B'=\{A'+\alpha'\in V_{A'};\ \pi'(A'+\alpha')\in\pi'(V_A)\}$, respectively. Then for each $A+\alpha$ in B there is a g in $\mathcal{G}^C$ with $g(A+\alpha)\in B'$. Since the isotrpoy subgroup $\Gamma_A^{\mathcal{G}}$ is finite, we can choose such a $g=g_\alpha$ uniquely in $\mathcal{G}^C$for $A+\alpha$.

Let $\{\beta_1,\cdots,\beta_m\}$ and $\{\beta_1',\cdots,\beta_m'\}$ be orthonormal bases of $\mathcal{H}_A^1$ and $\mathcal{H}_{A'}^1$, respectively, where m is the dimension of $\mathcal{H}^1$, which is by assumption independent of A. Because Ψ_A^{-1}; $V_\varepsilon\to V_A$ is holomorphic, for $\beta(t)=\sum_{\nu=1}^m t_\nu\beta_\nu\in V_\varepsilon$, $t=(t_1,\cdots,t_m)\in\boldsymbol{C}^m(|t|=\sqrt{\sum_\nu|t_\nu|^2}<\varepsilon)$ $\alpha(t)=\Psi_A^{-1}(\beta(t))$ is holomorphic in t. Therefore, if we can show that $g_t=g_{\alpha(t)}$ is holomorphic in t, then the composition of the mappings

$$\Psi_A(B)\,(\subset V_\varepsilon)\xrightarrow{\Psi_A^{-1}}B(\subset V_A)\xrightarrow{\text{the action of } g_t}B'(\subset V_{A'})$$
$$\xrightarrow{\Psi_{A'}}\Psi_{A'}(B')\,(\subset V_{\varepsilon'})$$

is also holomorphic in t, since $\Psi_{A'}(\alpha')$ is the harmonic part of α', $\sum_{\nu=1}^m\langle\alpha',\beta_\nu'\rangle_M\beta_\nu'$.

We now verify the following assertion.

Assertion. *The complex gauge transformations g_t depend holomorphically on t.*

It suffices for this prupose to prove that for any fixed $A+\alpha(t_0)\in B$ g_t is holomrophic with respect to $A+\alpha(t)$ close to $A+\alpha(t_0)$. We set $\gamma(z)=\alpha(t_0+z)-\alpha(t_0)$ and $h_z=g_{(t_0+z)}\cdot(g_{t_0})^{-1}$. Then $\gamma(0)=0$ and $h_0=\mathrm{id}$. If we define α_0' and $\sigma(z)$ in $\Omega^{(0,1)}(\mathfrak{g}^C)$ respectively by $A'+\alpha_0'=g_{t_0}(A+\alpha(t_0))$ and $\sigma(z)=g_{t_0}\cdot\gamma(z)\cdot(g_{t_0})^{-1}$, then for $t=t_0+z$ $g_t(A+\alpha(t))=(h_z\cdot g_{t_0})\ (A+\alpha(t_0)+\gamma(t))$ is written by

$$g_t(A+\alpha(t))=A'+\alpha_0'+(\bar{\partial}_{(A'+\alpha_0')}h_z)\cdot(h_z)^{-1}+h_z\cdot\sigma(z)\cdot(h_z)^{-1}\,. \tag{5.1}$$

Since h_z is close to id in $\mathcal{G}^C$, there exists a unique $\psi(z)\in\Omega^0(\mathfrak{g}^C)$ with $\psi(0)=0$

and $h_z=\exp\psi(z)$. Then (5.1) reduces to

$$g_t(A+\alpha(t)) = \bar{\partial}_{A''}\psi+A''+\sigma(z)+R(\psi,\sigma(z)) \tag{5.2}$$

for $A''=A'+\alpha_0'$, where the remainder term $R(\psi,\sigma)$ is given by

$$R(\psi,\sigma) = (\bar{\partial}_{A''}\exp\psi)\cdot\exp(-\psi)-\bar{\partial}_{A''}\psi+\exp\psi\cdot\sigma\cdot\exp(-\psi)-\sigma\,. \tag{5.3}$$

Notice that the remainder term indeed including $\bar{\partial}_{A''}\psi$ and σ as linear terms can be represented more exactly by

$$R(\psi,\sigma) = (1/2)\,[\psi,\bar{\partial}_{A''}\psi]+[\psi,\sigma]+R_1(\psi,\bar{\partial}_{A''}\psi)+R_2(\psi,\sigma)\,, \tag{5.4}$$

where R_1 and R_2 are written as matrix-power series of order not less than 3 with respect to ψ and σ.

Since $\bar{\partial}^*_{A'}\alpha_0'=0$, we see that $(\bar{\partial}^*_{A'})\,(g_t(A+\alpha(t))-A')=0$, namely $g_t(A+\alpha(t))-A'$ belongs to the slice, if and only if from (5.2)

$$(\bar{\partial}^*_{A'})\bar{\partial}_{A''}\psi+(\bar{\partial}^*_{A'})\sigma+(\bar{\partial}^*_{A'})R(\psi,\sigma)=0\,. \tag{5.5}$$

Because $G_{A''}\circ\Delta^2_{A''}=\mathrm{id}$ on $\Omega^0(\mathfrak{g}^C)$, the above reduces to

$$\psi+G_{A''}\langle[\partial_{\tilde{A}''}\psi,\alpha_0']\rangle+G_{A''}(\bar{\partial}^*_{A'})\sigma+G_{A''}(\bar{\partial}^*_{A'})R(\psi,\sigma)=0\,, \tag{5.6}$$

here $\partial_{\tilde{A}''}\psi$ is the (1,0)-component of $d_{\tilde{A}''}\psi$ with respect to the $SU(n)$-connection $\tilde{A}''$ induced canonically from A''. Then by using the way quite similar to one to solve (4.7) we have a solution $\psi=\psi(z)$ to (5.6) depending smoothly on z. We operate on (5.6) $\bar{\partial}_z$ relative to the parameter z to obtain

$$\bar{\partial}_z\psi+G_{A''}\langle[\partial_{\tilde{A}''}(\bar{\partial}_z\psi),\alpha_0']\rangle+G_{A''}(\bar{\partial}^*_{A''})\bar{\partial}_zR(\psi,\sigma)=0 \tag{5.7}$$

since $\bar{\partial}_z\sigma(z)=0$ and $\bar{\partial}_z$ commutes with $G_{A''}$ and with $d_{\tilde{A}''}$. The term $\bar{\partial}_zR(\psi,\sigma)$ is obviously linear with respect to $\bar{\partial}_z\psi$. Define a linear operator $L=L_{\alpha_0}$ by $L(\Theta)=\Theta+G_{A''}\langle[\partial_{\tilde{A}''}\Theta,\alpha_0']\rangle$, $\Theta\in L^2_{k+2}(\Omega^0(\mathfrak{g}^C))$. Then L satisfies

$$(1-c|\alpha_0'|_k)|\Theta|_{k+2}\leqq|L(\Theta)|_{k+2}\leqq(1+c|\alpha_0'|_k)|\Theta|_{k+2} \tag{5.8}$$

for a constant $c>0$, independent of α_0. For each α_0' in a sufficiently small slice V_A, L_{α_0} gives a bounded linear operator from (5.8). On the other hand by the remark on $R(\psi,\sigma)$ the norm $|\bar{\partial}_zR(\psi,\sigma)|_{k+1}$ is estimated by

$$|\bar{\partial}_zR(\psi,\sigma)|_{k+1}\leqq c_1|\bar{\partial}_z\psi|_{k+1}\{|\sigma|_{k+1}T_1(|\psi|_{k+1})+|\psi|_{k+2}T_2(|\psi|_{k+1})\} \tag{5.9}$$

for some constant c_1, where $T_1(s)$ and $T_2(s)$ are power series of s with convergence radius ∞.

Since $|\sigma(z)|_{k+1}$ is sufficiently small for small $|z|$, we can let $|\psi(z)|_{k+2}$ be also sufficiently small from (5.5). Thus by the aid of the lower estimate of L $|\bar{\partial}_z\psi|_{k+2}\leqq c_2|\bar{\partial}_z\psi|_{k+1}\leqq c_2|\bar{\partial}_z\psi|_{k+2}$, where $c_2<1$ for sufficiently small $|z|$, therefore (5.7) admits only a trivial solution $\bar{\partial}_z\psi=0$, that is, $\psi=\psi(z)$ and consequently $g_t=(\exp \psi(z))\cdot g_{t_0}$, $t=t_0+z$, is holomorphic.

Proposition 5.1 follows from this assertion since $\dim_C \mathcal{H}^1=c_2(\mathfrak{g}^C)-(n^2-1)\cdot p_a(M)$.

The proof of Theorem 2 is now completed if we pull back to $(\mathcal{M}\backslash\mathcal{K})_0$ the complex structure of $f((\mathcal{M}\backslash\mathcal{K})_0)$ through the f. Theorem 1 is a direct consequence of Theorem 2 from Remark 2.1 because $H_A^2\cong H_A^0\oplus \boldsymbol{H}$ vanishes for every irreducible anti-self-dual connection A over a Kähler surface M which either admits a Kähler metric of positive total scalar curvature or is endowed with trivial canonical line bundle.

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