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THE MODULI SPACE OF YANG-MILLS CONNECTIONS
OVER A KÄHLER SURFACE IS A COMPLEX MANIFOLD

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1. Introduction

Let $M$ be a compact, connected, oriented Riemannian 4-manifold. Let $P$ be a smooth principal $G$-bundle over $M$. For simplicity we assume that the Lie group $G=SU(n)$, $n \geq 2$. An $SU(n)$-connection $A$ on $P$ is called self-dual (anti-self-dual) if curvature form $F(A)=dA-A\wedge A$ satisfies $*F(A)=\pm F(A)$. Each self-dual (anti-self-dual) connection is characterized as a connection minimizing the Yang-Mills functional $\int_M |F|^2 \text{dv}$ and then gives a solution to the Yang-Mills equation. That the second Chern class $c_2(Q_c)<0(>0)$ for the adjoint bundle $Q_c$ of $P$ is a topological restriction to $P$ in order to admit a self-dual (anti-self-dual) connection. The moduli space $\mathcal{M}$ of self-dual (anti-self-dual) connections, namely, the orbit space of self-dual (anti-self-dual) connections with respect to the group $G$ of gauge transformations has a structure of smooth manifold ([3], [7]).

A Kähler surface $M$ with a Kähler metric $g$, which is certainly a Riemannian 4-manifold, carries the canonical orientation induced from the complex structure. Relative to this orientation a connection $A$ is anti-self-dual if and only if its curvature is a 2-form of type (1,1) which is primitive (that is, orthogonal to the Kähler form $\omega$). Therefore, by the integrability condition ([3]) each anti-self-dual connection induces a holomorphic structure on the complex adjoint bundle $Q_c$. Since gauge-equivalent anti-self-dual connections give holomorphic structures which are isomorphic with respect to automorphisms of $g^C$, we have the canonical mapping from $\mathcal{M}$ to the moduli space of holomorphic structures on $g^C$. Furthermore an anti-self-dual $SU(n)$-connection $A$ naturally defines an Einstein-Hermitian structure on the associated holomorphic vector bundle $E=P\times_{SU(n)} C^n$. We have also the fact that $E$ is $\omega$-semi-stable in the sense of Mumford and Takemoto ([9]). If $A$ is moreover irreducible, then $E$ is $\omega$-stable. On the other hand, over a nonsingular projective surface the moduli space of holomorphic, rank two vector bundles of fixed Chern classes is a quasi-projective variety ([12]). From these reasons together with an easy observation that the moduli space $\mathcal{M}$
M. Itoh has even dimension (Proposition 2.4), it is natural that \( M \) may possibly be a complex manifold ([1]). The aim of this paper is to show that \( M \) is indeed a complex manifold with singularities by using notion of holomorphic \((0,1)\)-connections.

The singularities of \( M \) are described as gauge-equivalent classes \( [A] \) of \( M \) either with non-zero 0-th cohomology \( H^0 \) or with non-zero second cohomology \( H^2 \) for a certain complex associated to the connection \( A \). Denote by \( \mathcal{K} \) the subset of \( M \) \( \{[A] \in M; H^0 \neq 0\} \). Then we obtain the following

**Theorem 1.** Let \( M \) be a compact Kähler surface with a Kähler metric of positive total scalar curvature or with trivial canonical line bundle \( K_M \). Let \( P \) be a smooth principal \( SU(n) \)-bundle with second Chern class \( c_2(\mathfrak{g}^C) > 0 \). If \( M \backslash \mathcal{K} \) is non-empty, then it is a complex manifold of dimension \( c_2(\mathfrak{g}^C) - (n^2 - 1)p_a(M) \), where \( p_a(M) \) is arithmetic genus of \( M \).

We denote by \( H \) the space \( H^0(M; \mathcal{O}(\mathfrak{g}^C \otimes K_M)) \) relative to the holomorphic structure on \( \mathfrak{g}^C \) induced from an anti-self-dual connection \( A \). Theorem 1 is a direct consequence of the following theorem.

**Theorem 2.** Let \( M \) be a compact Kähler surface, \( P \) a smooth principal \( SU(n) \)-bundle with \( c_2(\mathfrak{g}^C) > 0 \). If \( M \backslash \mathcal{K} = \{[A] \in M \backslash \mathcal{K}; H = 0\} \) is non-empty, then it is a complex manifold of dimension \( c_2(\mathfrak{g}^C) - (n^2 - 1)p_a(M) \).

These theorems are obtained as follows. We first show in §2 that each \( [A] \in (M \backslash \mathcal{K})_0 \) has a neighborhood in the first cohomology \( H^1 \) defining a local coordinate of \( M \). But such coordinate neighborhoods are not necessarily each other related holomorphically. Therefore we should verify by an indirect method that \( (M \backslash \mathcal{K})_0 \) is in fact a complex manifold. For this purpose we define in §3 a holomorphic \((0,1)\)-connection on the complexification \( P^C \) of \( P \). A holomorphic \((0,1)\)-connection is a system of local \( \mathfrak{sl}(n; \mathbb{C}) \)-valued \((0,1)\)-forms satisfying a transition condition whose curvature form vanishes. In a manner analogous to the case of anti-self-dual \( SU(n) \)-connections we can define complex gauge transformations, moduli space of holomorphic \((0,1)\)-connections and an elliptic complex which is a gauge field version of the Dolbeault complex. We obtain at §4 a canonical mapping \( f \) from \( M \) to the moduli space of holomorphic \((0,1)\)-connections which is injective and open over \( (M \backslash \mathcal{K})_0 \) and then use the Atiyah-Singer index theorem and Kuranishi's integrating method together with the moment map due to Donaldson ([6]) to verify that the open subspace \( f((M \backslash \mathcal{K})_0) \) in the moduli is definitely a complex manifold of dimension \( c_2(\mathfrak{g}) - (n^2 - 1)p_a(M) \) (Proposition 5.1).

Holomorphic \((0,1)\)-connections over a complex manifold are inseparably related to holomorphic structures on \( \mathfrak{g}^C \). Then the moduli space of holomorphic connections reflects aspects and properties of the moduli of holomorphic struc-
2. Moduli space of anti-self-dual connections

Let $M$ be a compact Kähler surface with a Kähler metric $g$. We denote by $\Lambda^k$ and $\Lambda^{(p,q)}$ the vector bundles of real $k$-forms and of complex $(p,q)$-forms on $M$, respectively. For a real vector bundle $E$ and a complex vector bundle $F$ we denote by $\Omega^k(E)$ and $\Omega^{(p,q)}(F)$ the space of all smooth $k$-forms with values in $E$ and the space of all smooth $(p,q)$-forms with values in $F$. Let $P$ be a smooth principal bundle over $M$ with gauge group $SU(n)$. We denote by $G$ and $g$ the associated bundles $P \times_{Ad} SU(n)$ and $P \times_{Ad} SU(n)$, respectively. We call $g$ the adjoint bundle of $P$.

Let $\{W_\alpha\}$ be an open covering of $M$ consisting of local trivializing neighborhoods of $P$.

**Definition 2.1.** A system $A=\{A_\alpha\}$ of local smooth $\mathfrak{su}(n)$-valued 1-forms $A_\alpha$ defined over $W_\alpha$ is called an $SU(n)$-connection on $P$, if $A$ satisfies the cocycle condition:

$$A_\beta = dg \cdot g^{-1} + g \cdot A_\alpha \cdot g^{-1}$$  \hspace{1cm} (2.1)

on $W_\alpha \cap W_\beta$, where $g=g_{\alpha \beta}$ is a transition transition function of $P$ over $W_\alpha \cap W_\beta$.

The set $\mathcal{A}$ of all $SU(n)$-connections on $P$ has an affine structure. That is, $\mathcal{A}$ is given by $\{A+\alpha; \alpha \in \Omega^1(g)\}$ for a fixed $SU(n)$-connection $A$. We call $SU(n)$-connection $A$ irreducible when the covariant derivative $d_A; \Omega^0(g) \rightarrow \Omega^1(g)$, $\psi \mapsto d\psi + [\psi, A]$ has trivial kernel. An $SU(n)$-connection is called reducible if it is not irreducible.

The complex surface $M$ has the canonical orientation induced from the complex structure. The Hodge star operator $\ast$ gives an endomorphism of $\Lambda^2$ with property $\ast \ast = id$. We denote by $\Lambda^2_+ \subset \Lambda^2_-$ the eigenspaces of +1 and -1, respectively. The projection from $\Lambda^2$ onto $\Lambda^2_\pm$ is denoted by $p_{\pm}$. Over Kähler surface $M$ we have the following ([7]). A real 2-form $\alpha$ belongs to $\Lambda^2_\pm$ if and only if (1,1)-part of $\alpha$ is proportional to the Kähler form $\omega$, and a real 2-form $\beta$ is in $\Lambda^2_\pm$ if and only if $\beta$ is of type (1,1) and orthogonal to $\omega$. A 2-form in $\Lambda^2_\pm$ (or in $\Lambda^2_\mp$) is called self-dual (or anti-self-dual).

**Definition 2.2.** An $SU(n)$-connection $A$ is called anti-self-dual if the curvature form $F(A)=dA - A \wedge A$ which belongs to $\Omega^2(g)$ satisfies $\ast F(A) = -F(A)$, namely $p_+ F(A) = 0$.

The group $G=\Gamma(M; G)$ of all smooth gauge transformations of $P$ acts on $\mathcal{A}$.
as \( g(A) = dg \cdot g^{-1} + g \cdot A \cdot g^{-1}, g \in \mathcal{G}, A \in \mathcal{A} \). Let \( Z \) be the center of \( SU(n) \). Each element of \( Z \) defines a gauge transformation which commutes with all \( g \)'s of \( \mathcal{G} \). It is easily seen that the center \( Z(\mathcal{G}) \) of \( \mathcal{G} \) coincides with \( Z \). The center \( Z = Z(\mathcal{G}) \) acts trivially on \( \mathcal{A} \). Let \( A \) be an irreducible connection on \( P \). Then the isotropy subgroup \( \Gamma_A = \{g \in \mathcal{G}; g(A) = A\} \) is just \( Z \). This fact is observed by the following. The endomorphism bundle \( \text{End}(E) \) of the associated vector bundle \( E = P \times_S \mathcal{C}^n \), which is written as \( \text{End}(E) = P \times_S \mathfrak{gl}(n; \mathbb{C}) \), decomposes into \( \text{End}(E) = 1 \oplus \mathfrak{g} \oplus \sqrt{-1} \mathfrak{g} \) as an \( SU(n) \)-vector bundle, where \( 1 \) is a one-dimensional trivial bundle. The bundle \( G = P \times_S SU(n) \) is considered as a subbundle of \( \text{End}(E) \) with fibers consisting of \( SU(n) \). Then a gauge transformation \( g \) is in \( \tilde{\mathcal{G}} = \mathcal{G}/Z \) if and only if \( g(A) = A = (dg + [g, A]) \cdot g^{-1} = d_A g \cdot g^{-1} = 0 \), that is, \( g \) is a parallel section of \( \text{End}(E) \). By the irreducibility of \( A \) \( g \) must be a constant multiple of identity transformation \( 1_E \), hence \( g \in Z \) since \( g \) takes values in \( SU(n) \). As a consequence the quotient group \( \tilde{\mathcal{G}} = \mathcal{G}/Z \) acts effectively on \( \mathcal{A} \) and freely on the subset of irreducible connections.

Denote by \( \mathcal{B} \) the quotient space \( \mathcal{A}/\tilde{\mathcal{G}} \) and by \( \pi \) the projection of \( \mathcal{A} \) onto \( \mathcal{B} \). The equivalence class \( \pi(A) \) is denoted by \([A]\). Since \( F(g(A)) = g \cdot F(A) \cdot g^{-1}, g \in \tilde{\mathcal{G}}, g(A) \) is anti-self-dual for every anti-self-dual connection \( A \). The subset \( \mathcal{M} \) in \( \mathcal{B} \) given by \{anti-self-dual connections on \( P \)\} is called the moduli space of anti-self-dual connections on \( P \).

In order to introduce a local coordinate neighborhood for each \([A]\) of \( \mathcal{M} \) we define suitable topologies on \( \mathcal{B} \). On the spaces \( \Omega'(g) \) the inner product \( \langle \cdot, \cdot \rangle_M \) is defined by \( \langle \phi, \psi \rangle_M = \int_M \langle \phi, \psi \rangle(x) dv, \langle \phi, \psi \rangle(x) dv = \text{Tr} \{\phi(x) \wedge \ast^1 \psi(x)\} \}, p \geq 0 \). By using a partition of unity we also define the Sobolev's norm \( |\cdot|_k \) on \( \Omega'(g) \) for a positive integer \( k \). In the completion \( L^2_0(\Omega'(g)) \) of \( \Omega'(g) \) relative to \( |\cdot|_k \), the subspace \( \Omega'(g) \) of all smooth sections is dense. Note that norms \( |\cdot|_1 \) and \( |\cdot|_1 = \langle \cdot, \cdot \rangle_M^{1/2} \) are equivalent. Now we complete the space \( \mathcal{A} \) and the group \( \mathcal{G} \). Namely, let \( \tilde{\mathcal{A}} \) be the space \{\( A_0 + \alpha; \alpha \in L^2_0(\Omega'(g)) \)\} for a fixed smooth connection \( A_0 \) and \( \mathcal{G} \) the subset \{\( g \in L^2_0(\Gamma(M; \text{End}(E)); g \) takes values in \( SU(n) \)\}. Then \( \mathcal{G} \) and hence \( \tilde{\mathcal{G}} \) acts on \( \tilde{\mathcal{A}} \) and we get the quotient topology on the space \( \mathcal{B} = \mathcal{A}/\tilde{\mathcal{G}} \). In the following we assume that \( k \) is sufficiently large relative to the dimension of the base space \( M \) in order to apply Sobolev's imbedding theorem.

For a connection \( A \) a subset \( U_A \) of \( \mathcal{A} \{A + \alpha; \alpha \in L^2_0(\Omega'(g)), d_A^\alpha \alpha = 0\} \) is said to be a slice at \( A \). Here \( d^\alpha_A \Omega'(g) \rightarrow \Omega'(g) \) is the formal adjoint of \( d_A \) relative to the inner product \( \langle \cdot, \cdot \rangle_M \).

**Proposition 2.1.** Let \( A \) be an irreducible connection. Then there is a positive \( \varepsilon \) such that \( U_{A,\varepsilon} = \{A + \alpha; |\alpha|_k < \varepsilon, d^\alpha_A \alpha = 0\} \subset \mathcal{A} \) is homeomorphic to its image \( \pi(U_{A,\varepsilon}) \) through the restriction of \( \pi \) to \( U_{A,\varepsilon} \) and \( \pi(U_{A,\varepsilon}) \) gives a neighborhood of \([A]\) in \( \mathcal{B} \).
Proof. This proposition is shown in the proof of Theorem 6 in [5]. Then we give here a sketch of the proof. We define a mapping \( S; \mathcal{U}_A \times \mathbb{G}/\mathbb{Z} \to \mathcal{A} \), \( S(A+\alpha, g) = g(A+\alpha) \). Then \( S \) is smooth relative to the \( L^2 \) topologies and its derivative at \( \alpha = 0 \) and \( g = \) the identity is given by

\[
DS; \text{Ker} \, d_A^* \times \Omega^0(\mathfrak{g}) \to \Omega^1(\mathfrak{g}), \tag{2.1}
\]

\[
(\alpha, \phi) \mapsto \alpha + d_A \phi,
\]

which is an isomorphism since \( \text{Ker} \, d_A = 0 \) and \( \Omega^1(\mathfrak{g}) = \text{Im} \, d_A \oplus \text{Ker} \, d_A^* \). Then \( S \) gives a local diffeomorphism. Thus for a sufficiently small \( \varepsilon \) there is a neighborhood \( Q \) of \( A \) in \( \mathcal{A} \) which is written as \( S(U_{A,A} \times \mathcal{W}) \), where \( \mathcal{W} \) is a neighborhood in \( \mathbb{G} \). Namely, each \( A_1 \) in \( Q \) has a unique form \( A_1 = g(A+\beta), \beta \in U_{A,A}, g \in W \). By the aid of the semi-continuity of \( \dim \text{Ker} \, d_A \) we can assume here that each connection of \( Q \) is irreducible. The proof is completed if we use the argument given at p. 448, 449 of [3].

Let \( \mathcal{K} \) be the subset of \( \mathcal{A} \) given by \( \{[A] \in \mathcal{B}; A \) is reducible\}. Since \( F(A + \alpha) = F(A) + d_A \alpha - \alpha \wedge \alpha \), a slice neighborhood \( \mathcal{U}_{(A)} \) of \( [A] \in \mathcal{M} \setminus \mathcal{K} \) in \( \mathcal{M} \) can be given by an \( \varepsilon \)-neighborhood of a slice

\[
\{A + \alpha; |\alpha| < \varepsilon, d_A^* \alpha = 0, d_A \alpha = \alpha \wedge \alpha\}, \tag{2.2}
\]

where \( d_A^* = p_\alpha \circ d_A \) and \( \#; \Omega^1(\mathfrak{g}) \times \Omega^1(\mathfrak{g}) \to \Omega^2(\mathfrak{g}) = \Gamma(M, \Lambda^2 \otimes \mathfrak{g}) \) is defined by \( \alpha \# \beta = (1/2)p_\alpha(\alpha \wedge \beta + \beta \wedge \alpha) \).

To analyze more exactly the structure of neighborhoods of the moduli space \( \mathcal{M} \) we need notion of an elliptic complex and also the integrating method due to Kuranishi ([11]).

For any anti-self-dual \( SU(n) \)-connection \( A \) the following sequence presents an elliptic complex ([3, p. 444], [7, Proposition 2.4])

\[
0 \to \Omega^0(\mathfrak{g}) \overset{d_A}{\to} \Omega^1(\mathfrak{g}) \overset{d_A^*}{\to} \Omega^2(\mathfrak{g}) \to 0. \tag{2.3}
\]

If the connection \( A \) is irreducible, then 0-th cohomology group \( H^0_A \) vanishes. With respect to the second cohomology group \( H^2_A \) we have the following two propositions.

**Proposition 2.2.** Let \( A \) be an anti-self-dual connection. Then for each \( \Phi = \Phi^{2,0} + \Phi^{0,2} + \Phi^{0,0} \otimes \omega \in \Omega^2(\mathfrak{g}) \)

\[
|d_A^* \Phi|_{\mathfrak{g}}^2 = (1/2) \left\{ |\nabla_A \Phi^{2,0}|^2_{\mathfrak{g}} + |\nabla_A \Phi^{0,2}|^2_{\mathfrak{g}} + |d_A \Phi^0|^2_{\mathfrak{g}} \right\} + (1/4) \int_M \text{Scal}(g) \left\{ |\Phi^{2,0}|^2 + |\Phi^{0,2}|^2 \right\} dv. \tag{2.4}
\]

Here \( \nabla_A \) denotes the covariant derivative with respect to \( A \) together with the
Levi-Civita connection of the metric $g$ and $\text{Scal}(g)$ is the scalar curvature of $g$. Notice that since each $\Phi$ in $\Omega^2(\mathbb{R})$ takes values in $\mathfrak{su}(n)$, $\Phi$ satisfies the reality condition, that is, $\Phi^0 \in \Omega^0(\mathbb{R})$ and $\Phi^{0,2} = -(\Phi^{2,0})$.

**Proposition 2.3.** If an $SU(n)$-connection $A$ is anti-self-dual, then the second cohomology $H^2_A$ is $\mathbb{R}$-isomorphic to $H^2_A \oplus H$, Where $H$ denote the space of global holomorphic sections $H^0(M; \mathcal{O}(\mathbb{C} \otimes K_M))$ with respect to the holomorphic structure $g^c$ on canonically induced from the $A$.

Proof of Proposition 2.2. It suffices to show the following Bochner-Weitzenböck formula with respect to a general connection $A$:

\[
|d_A^* \Phi|^2 = (1/2) \{ |\nabla_A \Phi|_g^2 + |\nabla_A \Phi|^{0,2}_g + |d_A \Phi|^2 \} + \frac{1}{4} \int_M \text{Scal}(g) \{ |\Phi^{2,0}|^2 + |\Phi^{0,2}|^2 \} \, dv \\
+ 4 \int_M \text{Re} \langle [\Phi^0, \sqrt{-1} F^{2,0}], \Phi^{2,0} \rangle \, dv \\
- 2 \int_M \text{Re} \langle [\Phi^{0,2}, \sqrt{-1} F^0], \Phi^{0,2} \rangle \, dv
\]  
(2.5)

for $\Phi \in \Omega^2(\mathbb{R})$ and $F_+(A) = p_+ F(A) = F^{2,0} + F^{0,2} + F^0 \otimes \omega$.

Since

\[
d_A^* (\Phi^{1,0} + \Phi^{0,1}) = \partial_A \Phi^{1,0} + \partial_A \Phi^{0,1} \\
+ (1/2) \langle \partial_A \Phi^{1,0} + \partial_A \Phi^{0,1}, \omega \rangle \otimes \omega
\]  
(2.6)

and we have

\[
d_A^* (\Phi^{2,0} + \Phi^{0,2}) = \partial_A \Phi^{2,0} + \partial_A \Phi^{0,2},
\]  
(2.7)

and

\[
d_A^* (\Phi^0 \otimes \omega) = \sqrt{-1} (\partial_A \Phi^0 - \partial_A \Phi^0),
\]  
(2.8)

we obtain the following

\[
d_A^* d_A^* (\Phi^{2,0} + \Phi^{0,2}) = \partial_A \partial_A \Phi^{2,0} + \partial_A \partial_A \Phi^{0,2} \\
+ (1/2) \langle \partial_A \partial_A \Phi^{2,0} + \partial_A \partial_A \Phi^{0,2}, \omega \rangle \otimes \omega
\]  
(2.9)

and

\[
d_A^* d_A^* (\Phi^0 \otimes \omega) = \sqrt{-1} \{ \partial_A \partial_A \Phi^0 - \partial_A \partial_A \Phi^0 \\
+ (1/2) \langle \partial_A \partial_A \Phi^0 - \partial_A \partial_A \Phi^0, \omega \rangle \otimes \omega \}
\]  
(10.10)

Since $d_A d_A \Phi^0 = [\Phi^0, F(A)]$, (10.10) reduces to

\[
d_A^* d_A^* (\Phi^0 \otimes \omega) = \sqrt{-1} \{ [\Phi^0, F^{2,0}] - [\Phi^0, F^{0,2}] \} \\
+ (1/2) \langle \square_A \Phi^0 \rangle \otimes \omega.
\]  
(11.11)
Here we denote by $\Box_A$ the rough Laplacian $-\sum g^{\sigma\tau} \nabla_\sigma \nabla_\tau$. Hence the inner product \( \langle d_A^\dagger d_A^\ast (\Phi^0 \otimes \omega), \Phi \rangle_M \) is given by
\[
\langle d_A^\dagger d_A^\ast (\Phi^0 \otimes \omega), \Phi \rangle_M = \int_M 2 \text{Re} \langle [\Phi^0, \sqrt{-1} F^{2,0}], \Phi^{2,0} \rangle dv + \langle \Box_A \Phi^0, \Phi \rangle_M .
\]
(2.12)

On the other hand we have by an argument similar to [7, Lemma 3.3]
\[
\partial_A \partial_A^\dagger \Phi^{2,0} = (1/2) \Box_A \Phi^{2,0} + (1/4) \frac{\text{Scal}(g)}{2} \Phi^{2,0}
\]
\[-(1/2) [\Phi^{2,0}, 2\sqrt{-1} F^0] .
\]
(2.13)

By using the Ricci formula we obtain further
\[
\langle \partial_A \partial_A^\dagger \Phi^{2,0}, \omega \rangle = \sqrt{-1} \sum g^{uv} (\partial_A \partial_A^\dagger \Phi^{2,0})_{uv}
\]
\[+ (\sqrt{-1}/2) \sum g^{uv} g^{wz} [\Phi_{wz}, F_{uv}] .
\]
(2.14)

Therefore (2.5) is derived from these formulas.

Proof of Proposition 2.3. Since the curvature form \( F(A) \) is of type (1,1), the connection \( A \) induces a holomorphic structure on the complex adjoint bundle \( g^C \). Namely a smooth section \( \Phi \) of \( g^C \) satisfies \( d_A \Phi = 0 \) if and only if \( \Phi \) is holomorphic relative to the holomorphic structure. Then the space \( \{ \Phi \in \Omega^0(\mathcal{O}(C) ; g^C) \mid \partial_A \Phi = 0 \} \) is isomorphic with the second cohomology \( H^2(M; \mathcal{O}(g^C)) \) from Theorem 4.1, ch. 3 in [10].

Moreover it is isomorphic with the space \( H \) by the aid of Serre’s duality theorem and the self-duality of \( g^C \) as a vector bundle. In the course of the proof of Proposition 2.2 we can also verify that
\[
|3_A^\dagger 3_A \Phi^{0,2} |^2_M = (1/2) |\nabla_A \Phi^{0,2} |^2_M + (1/4) \int_M \text{Scal}(g) |\Phi^{0,2} |^2 dv
\]
(2.15)
for \( \Phi^{0,2} \in \Omega^0(\mathcal{O}(g^C)) \). Thus we have
\[
|d_A^\dagger \Phi |^2_M = |3_A^\dagger 3_A \Phi^{0,2} |^2_M + |3_A^\dagger \Phi^{0,2} |^2_M + |d_A \Phi^{0,2} |^2_M
\]
(2.16)
from which the proposition follows easily.

Remark 2.1. If the canonical line bundle \( K_M \) is trivial, then \( H \) is \( C \)-isomorphic to \( (H^2)_C \). On the other hand, if the metric \( g \) is of positive total scalar curvature, i.e., \( \int_M \text{Scal}(g) dv > 0 \), then \( H \) vanishes.

By applying the Atiyah-Singer index theorem to complex (2.4), we have
\[(\gamma) \gamma^0 - h^1 + h^2 = -2c_2(g^C) + 2\dim SU(n) \cdot p_a(M), \]
where \( p_a(M) \) denotes the arithmetic genus of \( M \) and \( h^i = \dim H^i_M, i = 0,1,2. \) If both \( H^0 \) and \( H^2 \) vanish, then \( H^1 \) has even dimension.

Proposition 2.4. The first cohomology group \( H^1_M \) is \( R \)-isomorphic to the com-
plex vector space $\mathcal{J}^l = \{\alpha^{(0,1)} \in \Omega^{(0,1)}(g^c), \overline{\partial}_A \alpha^{(0,1)} = 0, \overline{\partial}_A^* \alpha^{(0,1)} = 0\}$.

Proof. Each $g$-valued 1-form $\alpha$ splits into

$$\alpha = \alpha^{(1,0)} + \alpha^{(0,1)}, \quad \alpha^{(1,0)} = \sum_{\mu} \alpha_{\mu} d^{\mu} \in \Omega^{(1,0)}(g^c),$$

$$\alpha^{(0,1)} = \sum_{\mu} \alpha_{\mu} d^{\bar{\mu}} \in \Omega^{(0,1)}(g^c) \quad \text{with} \quad i(\alpha^{(1,0)}) = -\alpha^{(0,1)}.$$  

We define a mapping $h: \Omega^l(g) \to \Omega^{(0,1)}(g^c)$ by assigning $\alpha^{(0,1)}$ to $\alpha$. We show that $h_{|H^l}$ gives an isomorphism of $H^l$ to $\mathcal{J}^l$. By an argument given in [7] we see that $d_A^* \alpha = 0$ if and only if

$$\sum g^{\mu \bar{\nu}} \nabla_{\mu} \alpha_{\nu} + \sum g^{\mu \bar{\nu}} \nabla_{\mu} \alpha_{\nu} = 0 \quad (2.17)$$

and that $d_A \alpha = 0$ if and only if

$$\begin{cases}
\partial_A \alpha^{(0,0)} = 0, \\
\overline{\partial}_A \alpha^{(0,1)} = 0, \\
\sum g^{\mu \bar{\nu}} (\nabla_{\mu} \alpha_{\nu} - \nabla_{\mu} \alpha_{\nu}) = 0.
\end{cases} \quad (2.18)$$

Hence, if $\alpha$ is in $H^l$, then $\overline{\partial}_A \alpha^{(0,1)} = 0$ and $\overline{\partial}_A^* \alpha^{(0,1)} = -\sum g^{\mu \bar{\nu}} \nabla_{\mu} \alpha_{\nu} = 0$. Since $i(\alpha^{(1,0)}) = -\alpha^{(0,1)}$, the inverse implication is easily derived.

REMARK 2.2. Proposition 2.4 is also established for a connection which is not necessarily anti-self-dual.

Now we define for each $[A]$ in the moduli space $\mathcal{M} \setminus \mathcal{K}$ a mapping $\Phi = \Phi_A: \Omega^l(g) \to \Omega^l(\mathbf{g})$ by $\Phi(\alpha) = -d_A^* (G_A(\alpha \# \alpha))$ ([2], [4]). Here $G_A$ is the Green operator of the Laplace operator $d_A^* \circ d_A^*$. Relative to the norms $|\cdot|_k$ we have

$$|d_A \alpha|_{k+1} \leq c_k |\alpha|_k, \quad (2.19)$$

$$|G_A \Psi|_{k+2} \leq c_k |\Psi|_k \quad (2.20)$$

and

$$|\alpha \# \beta|_k \leq c_k |\alpha|_k |\beta|_k \quad (2.21)$$

for $\alpha, \beta \in L^2_2(\Omega^l(\mathbf{g}))$, $\Psi \in L^2_2(\Omega^l_2(\mathbf{g}))$, where $c_k$ is a constant depending only on the manifold $M$ (Ch. 4 of [10], [11]). Therefore the mapping $\Phi_A: L^2(\Omega^l(\mathbf{g})) \to L^2(\Omega^l(\mathbf{g}))$ is differentiable. Suppose that $H_A^\beta = 0$. Then we have on $\Omega^l_2(\mathbf{g})$ $d_A^* \circ d_A^* G_A = \text{id}$. Hence a slice neighborhood $U_{\alpha, \beta} \subset [A]$ is mapped by the $\Phi$ into $H_A^\beta$. Since the derivative of $\Phi$ at $\alpha = 0$ is identity, it has an inverse on a sufficiently small neighborhood $U_{\beta} = \{ \beta \in H_A^\beta; \ |\beta|_{M < \epsilon}\}$.

Notice that by using a prior estimates of elliptic differential operators each $\beta$ in $L^2_2(\Omega^l(\mathbf{g}))$ satisfying $(d_A d_A^* + d_A^* d_A) \beta = 0$ is a smooth section and norms $|\beta|_k$ and $|\beta|_{M}$ are equivalent.

As a consequence of these propositions we obtain
**Proposition 2.5.** Let \( M \) be a compact Kähler surface with a Kähler metric \( g \) and \( P \) a principal \( SU(n) \)-bundle with \( c_1(g^C) > 0 \). Suppose that either the canonical line bundle \( K_M \) is trivial or the metric is with positive total scalar curvature. Then, if the moduli space \( \mathcal{M} \setminus \mathcal{K} \) of irreducible anti-self-dual connections on \( P \) is not empty, it is a smooth manifold of dimension \( 2c_2(g^C) - 2(n^2 - 1) \cdot \rho_a(M) \).

**Remark 2.3.** On the subset \( \mathcal{B} \setminus \mathcal{K} = \{ [A] \in \mathcal{B}; A \text{ is irreducible} \} \) we define a metric function \( \sigma \) (see for the precise discussion p. 448 in [3]); \( \sigma([A],[A_0]) = \inf g \in \mathcal{G} | A - g(A')|_M \). Since \( \sigma \) is continuous relative to the \( L^2 \)-topology, \( \mathcal{B} \setminus \mathcal{K} \) is a Hausdorff space. Therefore the moduli space \( \mathcal{M} \setminus \mathcal{K} \), a closed subset of \( \mathcal{B} \setminus \mathcal{K} \), is also Hausdorff with respect to the relative topology.

3. \((0, 1)\)-connections and moduli space of holomorphic \((0, 1)\)-connections

We denote by \( P^C \) a smooth principal \( SL(n; C) \)-bundle given by extending the transition functions of the bundle \( P \) to \( SL(n; C) \). The complexification \( g^C \) of \( g \) clearly coincides with \( P^C \times Ad \mathfrak{sl}(n; C) \). Now we define on \( P^C \) a \((0, 1)\)-connection and a holomorphic \((0, 1)\)-connection as follows.

**Definition 3.1.** Let \( \{ W_\alpha \} \) be the open covering of \( M \) consisting of local trivializing neighborhoods of \( P \). A system \( A = \{ A_\alpha \} \), where each \( A_\alpha \) is a smooth \( \mathfrak{sl}(n; C) \)-valued \((0, 1)\)-form defined over \( W_\alpha \), is called a \((0, 1)\)-connection on \( P^C \), when it satisfies the cocycle condition

\[
A_\beta = \bar{g} g^{-1} + g \cdot A_\alpha \cdot g^{-1} \tag{3.1}
\]
on \( W_\alpha \cap W_\beta \), where \( g = g_{\alpha \beta} \) is the transition function of \( P \).

The set \( \mathcal{A}^{(0, 1)} \) of all \((0, 1)\)-connections on \( P^C \) has a structure of affine space. The group of complex gauge transformations \( \mathcal{O}^C = \Gamma(M; P^C \times Ad SL(n; C)) \) acts on \( \mathcal{A}^{(0, 1)} \) in the form

\[
g(A) = \bar{g} g^{-1} + g \cdot A \cdot g^{-1} \tag{3.2}
\]
g \( \in \mathcal{O}^C \), \( A \in \mathcal{A}^{(0, 1)} \). We denote by \( \mathcal{B}^{(0, 1)} \) the quotient space \( \mathcal{A}^{(0, 1)}/\mathcal{O}^C \).

**Remark 3.1.** By its definition, each \((0, 1)\)-connection is not a connection by itself. But we have a mapping \( h; \mathcal{A} \to \mathcal{A}^{(0, 1)}; A \mapsto A^{(0, 1)} \), where \( A^{(0, 1)} \) is the \((0, 1)\)-component of \( A \). Then \( h \) is one-to-one and onto, because for every \((0, 1)\)-connection \( A = \{ A_\alpha \} \) on \( P^C \) a system \( \tilde{A} = \{ \tilde{A}_\alpha \} \) given by \( \tilde{A}_\alpha = A_\alpha^{-1}(\tilde{A}_\alpha) \) satisfies (2.1) from (3.1) and it takes values in \( \mathfrak{su}(n) \), and hence it gives an \( SU(n) \)-connection on \( P \) and \( h(\tilde{A}) = A \).

A \((0, 1)\)-connection \( A \) is called irreducible, if \( \tilde{A}_\alpha; \Omega^0(g^C) \to \Omega^{(0, 1)}(g^C); \Psi \mapsto \tilde{\Psi} \) has trivial kernel. We call a \((0, 1)\)-connection reducible when it is not irreducible.
For each \( A \in \mathcal{A}^{(0,1)} \) the curvature form \( F(A) = \overline{\partial}A - A \wedge A \) is defined. The curvature form \( F(A) \) belongs to \( \Omega^{(0,2)}(g^C) \).

**Definition 3.2.** A \((0,1)\)-connection \( A \) is called holomorphic if \( F(A) = 0 \).

**Remark 3.2.** Since the curvature form of a \((0,1)\)-connection \( A \) coincides with the \((0,2)\)-component of the curvature form of the \( SU(n) \)-connection \( \tilde{A} \) induced from \( A \), there exists for each holomorphic \((0,1)\)-connection \( A \) a holomorphic structure \( J = J_A \) on \( g^C \) relative to which \( A \) gives a hermitian holomorphic connection on \( g^C \) in the usual sense ([4]). Namely, there exist smooth mappings \( h_a; W_a \rightarrow SL(n; C) \) with properties that (i) \( h_{ab} = h_a \cdot g_{ap} \cdot h_p^{-1} \); \( W_a \cap W_b \rightarrow SL(n; C) \) is holomorphic for each \( \alpha \) and \( \beta \) and (ii) \( \tilde{A} \) is transformed into a \((1,0)\)-form \( h_a(\tilde{A}_a) = dh_a \cdot h_a^{-1} + h_a \cdot \tilde{A}_a \cdot h_a^{-1} \) by \( h_a \).

**Proposition 3.1.** Let \( A \) be a holomorphic connection. Then the following sequence gives an elliptic complex:

\[
0 \rightarrow \Omega^p(g^C) \overset{\overline{\partial}}{\rightarrow} \Omega^{(0,p)}(g^C) \overset{\overline{\partial}}{\rightarrow} \Omega^{(0,2)}(g^C) \rightarrow 0 \tag{3.3}
\]

**Proof.** Since \( \overline{\partial}A \overline{\partial} \Psi = [\Psi, F(A)] \) for \( \Psi \in \Omega^p(g^C) \), the above sequence gives a complex. It is easily verified that the symbol sequence of the above is exact.

On the spaces \( \Omega^{(0,p)}(g^C) \) we define inner products \( \langle \cdot, \cdot \rangle_M \) by \( \langle \Phi, \Psi \rangle_M = \int_M Tr(\Phi \wedge *'(\Psi)), \) \( p = 0,1,2. \) Notice that these products are not \( g^C \)-invariant.

We set the subspaces \( \mathcal{M}^p = \text{Ker} \Delta^p \) of \( \Omega^{(0,p)}(g^C) \) by the aid of the complex Laplacians \( \Delta^p, \) \( p = 0,1,2 \) associated to the above complex. Then by using the Atiyah-Singer index theorem we have the index of the complex (3.3) as

\[
h^0 - h^1 + h^2 = ch(g^C) \{ ch(\Lambda^{\infty}) - ch(\Lambda^{(0,1)}) + ch(\Lambda^{(0,2)}) \} \times e(TM)^{-1} \cdot \mathbb{Z}(TM^C) \ [M] \tag{3.4}
\]

where \( h^p = \dim_C \mathcal{M}^p. \) By a simple computation the index equals to \(-c_1(g^C) + (n^2 - 1) \cdot p_1(M)\).

Since the group \( \mathcal{G}^C \) leaves the set of holomorphic \((0,1)\)-connections invariant, we obtain its quotient space \( \mathcal{M}_h \), called the moduli space of holomorphic \((0,1)\)-connections.

The center of \( SL(n; C) \) which coincides with the center of \( SU(n) \) gives complex gauge transformations commuting with each \( g \) of \( \mathcal{G}^C \). In the same way as the case of \( SU(n) \) the center \( Z(\mathcal{G}^C) \) of \( \mathcal{G}^C \) is just the center \( Z \) and it acts trivially on \( \mathcal{A}^{(0,1)} \). Since \( \mathcal{G}^C \) is a subset of \( \Gamma(M; \text{End } E) = \Gamma(M; 1) \oplus \Gamma(M; g^C) \) the isotropy subgroup \( \Gamma^h \) of each irreducible \((0,1)\)-connection \( A \) reduces to \( Z \). Thus the quotient group \( \tilde{\mathcal{G}}^C = \mathcal{G}^C/Z \) acts effectively on \( \mathcal{A}^{(0,1)} \) and its action is free on the subset \( \{ A \in \mathcal{A}^{(0,1)}; A \text{ is irreducible} \}. \) Besides the inner product \( \langle \cdot, \cdot \rangle_M \)
we define on \( \Omega^{(0,0)}(g^c) \) the Sobolev's norms \(| \cdot |_k\) and let \( \mathcal{A}^{(0,1)} \) be \( \{ A_0 + \alpha; \alpha \in L^2(\Omega^{(0,1)}(g^c)) \} \) for a fixed smooth \((0,1)\)-connection \( A_0 \). In \( L^2_{k+1} \)-topology \( \mathcal{L}^c \) and hence \( \mathcal{G}^c \) acts smoothly on \( \mathcal{A}^{(0,1)} \). The quotient space \( \mathcal{B}^{(0,1)} = \mathcal{A}^{(0,1)}/\mathcal{G}^c \) gets the canonical quotient topology by the projection \( \pi'\colon \mathcal{A}^{(0,1)} \to \mathcal{B}^{(0,1)} \). We denote by \( \mathcal{K}^{(0,1)} \{ [A] \in \mathcal{B}^{(0,1)}; A \text{ is reducible} \} \), the subset of \( \mathcal{B}^{(0,1)} \).

Like an \( SU(n) \)-connection we call a subset \( V_A \) of \( \mathcal{A}^{(0,1)} \{ A + \alpha; \alpha \in L^2(\Omega^{(0,1)}(g^c)) \} \) a slice at \( A \).

**Lemma 3.2.** Let \( A \) be an irreducible \((0,1)\)-connection on \( P_c \). Then there exists for a sufficiently small \( \varepsilon > 0 \) a slice neighborhood \( V_{A,s} = \{ A + \alpha \in V_A; |\alpha|_k < \varepsilon \} \) whose image \( \pi'(V_{A,s}) \) gives a neighborhood of \([A]\) in \( \mathcal{B}^{(0,1)} \).

Proof. Define a mapping \( T\colon V_{A,s} \times \mathcal{L}^c Z \to \mathcal{A}^{(0,1)} \); \( T(A + \alpha, g) = g(A + \alpha) \). Then in a manner similar to the case of \( SU(n) \)-connections, \( T \) is smooth relative to the \( L^2 \)-topologies and its derivative at \( \alpha = 0 \) and \( g = \text{identity} \) is written by

\[
DT; \text{Ker} \mathcal{G}^c \rho(\mathcal{G}^c) \to \Omega^{(0,1)}(g^c)
\]

\[
(\alpha, \psi) \mapsto \alpha + \mathcal{G}^c \psi.
\]

Since \( \text{Ker} \mathcal{G}^c = 0 \) and \( \Omega^{(0,1)}(g^c) = \text{Im} \mathcal{G}^c \text{Ker} \mathcal{G}^c \), \( T \) is a local diffeomorphism. Therefore by using the argument which was used at the proof of Proposition 2.1 we obtain the lemma.

**Proposition 3.3.** Each irreducible \( [A] \in \mathcal{M}_h \) has a neighborhood \( \mathcal{V}_{[A]}\) which is given by the image of \( V_{A,s} = \{ A + \alpha; \alpha \in \Omega^{(0,1)}(g^c), |\alpha|_k < \varepsilon, \mathcal{G}^c \alpha = 0, \mathcal{G}^c \alpha = \alpha \land \alpha \} \).

Proof. Since \( F(A + \alpha) = F(A) + \mathcal{G}^c \alpha - \alpha \land \alpha \), this is a direct consequence of the above lemma.

Let \( \Psi = \Psi_A \) be a mapping from \( L^2(\Omega^{(0,1)}(g^c)) \) to itself defined by \( \Psi(\alpha) = \alpha - (\mathcal{G}^c) (G^c(\alpha \land \alpha)) \). Here \( G^c \) denotes the Green operator of \( \Delta^c \). Assume now that the second cohomology group \( \mathcal{H}^2 \) vanishes. Then we see that \( \mathcal{G}^c \alpha = 0 \) and \( \mathcal{G}^c \alpha = \alpha \land \alpha \) if and only if \( \Psi(\alpha) \in \mathcal{H} \). Thus the slice neighborhood \( V_{A,s} \) is mapped through \( \Psi \) into \( \mathcal{H}^1 \). Because over \( L^2(\Omega^{(0,1)}(g^c)) \) the derivative \( D\Psi \) at \( \alpha = 0 \) is identity, \( \Psi\mid_{V_{A,s}} \) has an inverse over a small \( \varepsilon \)-neighborhood \( V_{A,s} \) of \( \mathcal{H}^1 \). We remark that \( \Psi^{-1}\mid_{V_{A,s}} \) is holomorphic as a mapping from an open subset of a Banach space to a Banach space, since \( \Psi \) is quadratic over the completed Banach space \( L^2(\Omega^{(0,1)}(g^c)) \) ([11]).

4. Canonical imbedding of \( \mathcal{H} \setminus \mathcal{K} \) into \( \mathcal{H} \setminus \mathcal{K}^{(0,1)} \)

Let \( A \) be an \( SU(n) \)-connection on the bundle \( P \). Then the \((0,1)\)-component \( A^{(0,1)} \) of \( A \) certainly defines a \((0,1)\)-connection on the complexified bundle \( P^c \) and the curvature \( F(A^{(0,1)}) \) is given by the \((0,2)\)-component of \( F(A) \). If \( A \)
is anti-self-dual, then \( F(A) \) is of type \((1,1)\), and hence \( A^{(0,1)} \) is holomorphic. Because \( \mathcal{G} \subset \mathcal{G}^c \), to each \([A] \) of \( \mathcal{M} \) we can assign \([A^{(0,1)}] \) of \( \mathcal{M}_h \). We denote this assignment by \( f \).

**Proposition 4.1.** If an anti-self-dual connection \( A \) is irreducible, then \( A^{(0,1)} \) is also irreducible.

**Proof.** Since \( A \) is anti-self-dual we have the formula \( \sum g^{\mu \nu} F_{\mu \nu}(A) = 0 \) ([7, Proposition 2.2]). Then we obtain for a nonzero \( \psi \) of \( \Omega^2(g^c) \) satisfying \( \partial_A \psi = 0 \) that

\[
\sum g^{\mu \nu} \nabla_\mu \nabla_\nu Tr(\psi \cdot \psi^*) = \sum g^{\mu \nu} Tr(\nabla_\mu \psi \cdot \psi^* \nabla_\nu \psi) = |\partial_A \psi|^2.
\]

(4.1)

We integrate this over \( M \) to get \( \partial_A \psi = 0 \), that is, \( d_A \psi = 0 \). The sections \( \phi \) and \( \phi' \) of the adjoint bundle \( g \) given by \( \phi = \psi - i \psi \) and \( \phi' = (1/\sqrt{-1})(\psi + i \psi) \), respectively, are parallel with respect to \( d_A \).

From this proposition we have \( f(\mathcal{M} \setminus \mathcal{K}) \subset \mathcal{M}_h \setminus \mathcal{K}^{(0,1)} \).

Now we show the following

**Proposition 4.2.** The mapping \( f \) restricted to \( \mathcal{M} \setminus \mathcal{K} \) is injective.

**Proof.** It suffices to verify that if there is for irreducible anti-self-dual connections \( A \) and \( A_1 \), \( g \in \mathcal{G}^c \) satisfying \( (A_1)^{(0,1)} = g(A^{(0,1)}) \), then \( g \) must lie in \( \mathcal{G} \).

By the way \( SL(n; C) \) has the following decomposition; \( SL(n; C) = H^+(n) \cdot SU(n) \), where \( H^+(n) \) means the set of all positive definite Hermitian matrices with determinant 1. This decomposition is invariant under the adjoint representation of \( SU(n) \), namely, if \( X \in SL(n; C) \) splits into \( X = X^h \cdot X^w \), \( X^h \in SU(n) \), \( X^w \in H^+(n) \), then \( Y \cdot X \cdot Y^{-1} = (Y \cdot X^h \cdot Y^{-1})(Y \cdot X^w \cdot Y^{-1}) \), \( Y \in SU(n) \) gives the decomposition of \( Y \cdot X \cdot Y^{-1} \). Therefore the complex gauge transformation \( g \) splits into \( g = g_1 \cdot g^w \), \( g^w \in \mathcal{G} \), \( g_1 \in \Gamma(M; \mathcal{P}_{SU(n)}^h(n)) \). Then we have \( (A_1)^{(0,1)} = g_1(g^w(A^{(0,1)})) \).

Moreover \( g^w(A^{(0,1)}) = (g^w A)^{(0,1)} \) and \( g^w(A) \) is anti-self-dual since \( g^w \) is unitary.

Because the exponential map \( \exp; H_0(n) \rightarrow H^+_0(n); X \mapsto \exp X \) is a diffeomorphism, here \( H_0(n) \) is the set of all Hermitian matrices of trace zero, we can lift \( \exp \) to a bundle map \( \exp; PU(\mathcal{G}) \rightarrow \mathcal{P}_{SU(n)}^h(n) \). From the fact \( H_0(n) = \sqrt{-1} \mathfrak{su}(n) \) we induce a canonical mapping from \( \mathfrak{g} \) to \( P \times \mathfrak{su}(n) H^+_0(n) \) by \( \phi \mapsto \exp \sqrt{-1} \phi \). Then there is a \( \psi \in \Omega^2(g) \) such that \( g_1 = \exp \sqrt{-1} \psi \). A one-parameter subgroup \( g_t = \exp(t \sqrt{-1} \psi) \), \( t \in \mathbf{R} \), of \( \mathcal{G}^c \) yields a one-parameter family of \((0,1)\)-connections \( \{ A_t \} \) by \( A_t = g_t((A_0)^{(0,1)}) \), where \( A_0 = g^w(A) \). Further the family \( \{ A_t \} \) defines a family of connections \( \{ A_t \} \) of \( P \) by \( A_t = A_t - t(\bar{A}_t) \). The curvature \( F_t \) of \( A_t \) is certainly of type \((1,1)\).

Now we apply the method of moment map developed at [6, p. 11]. Define for \( \{ A_t \} \) a function \( m; \mathbf{R} \rightarrow \mathbf{R} \) by
\[ m(t) = \int_M R_\omega(t) \wedge \omega, \]  
where \( R_\omega(t) \) is a 2-form of type (1,1) over \( M \) modulo \( \text{Im} \partial + \text{Im} \bar{\partial} \) satisfying
\[
\sqrt{-1} \partial \bar{\partial} R_\omega(t) = -Tr F_t \wedge F_t - (-Tr F_0 \wedge F_0). \tag{4.3}
\]

Then we have the following facts (Proposition 8 of [6]). Since \( A_0 \) is anti-self-dual, \( d|dt|_{t=0} m(t) = 0 \) and
\[
d^2|d^2 m(t) = |d_A \psi|^2 = 0. \tag{4.4}
\]

Because \( m(t) \) is critical at also \( t=1 \), \( d^2|d^2 m(t) = 0 \) identically, hence \( d_A \nabla = 0 \).

Using the irreducibility of \( A_0 \) we have \( \nabla = 0 \) and hence \( g_1 = \text{identity} \), that is, \( g \in \mathcal{G} \).

We define open subsets \((\mathcal{H} \backslash \mathcal{K})_0 \) and \((\mathcal{H} \backslash \mathcal{K}^{(0,1)})_0 \) of \( \mathcal{H} \backslash \mathcal{K} \) and \( \mathcal{H} \backslash \mathcal{K}^{(0,1)} \), respectively, by \((\mathcal{H} \backslash \mathcal{K})_0 = \{ [A] \in \mathcal{H} \backslash \mathcal{K}; \mathcal{H}_A = 0 \} \) and \((\mathcal{H} \backslash \mathcal{K}^{(0,1)})_0 = \{ [A] \in \mathcal{H} \backslash \mathcal{K}^{(0,1)}; \mathcal{H}_A^{(0,1)} = 0 \} \). Since from Proposition 2.3 \( \mathcal{H}^{(0,1)}_A = \mathcal{H}_A \) for the \((0,1)\)-component \( A^{(0,1)} \) of an anti-self-dual connection \( A \) we have \( f((\mathcal{H} \backslash \mathcal{K})_0) \subset (\mathcal{H} \backslash \mathcal{K}^{(0,1)})_0 \).

**Proposition 4.3.** \( f((\mathcal{H} \backslash \mathcal{K})_0; (\mathcal{H} \backslash \mathcal{K})_0 \rightarrow (\mathcal{H} \backslash \mathcal{K}^{(0,1)})_0 \) is an open mapping.

Proof. Let \( U_{LA} \) be a neighborhood of \([A] \in (\mathcal{H} \backslash \mathcal{K})_0 \) identified with a slice neighborhood \( U_{\alpha} = \{ A + \alpha; |\alpha|_k < \varepsilon, d_A^* \alpha = 0, d_A \alpha = \alpha \# \alpha \} \). We notice that if \( \alpha \) is such a one-form its (0,1)-component \( \alpha^{(0,1)} \), denoted by \( h(\alpha) \) in \$2\$, satisfies \( \partial_{A'} \alpha^{(0,1)} = \alpha^{(0,1)} \wedge \alpha^{(0,1)} \) but does not necessarily satisfy \( (\bar{\partial}_{A'} \gamma)(\alpha^{(0,1)}) = 0 \) for \( A' = A^{(0,1)} \in \mathcal{H}^{(0,1)} \). Let \( C_{LA} \) be a neighborhood of \([A'] \) in \((\mathcal{H} \backslash \mathcal{K}^{(0,1)})_0 \) written in the form of the image of a slice neighborhood \( \mathcal{V}_{A',\gamma} = \{ A' + \gamma; |\gamma|_k < \varepsilon', (\bar{\partial}_{A'} \gamma)(\alpha^{(0,1)}) = 0, \bar{\partial}_{A'} \gamma^{(0,1)} = \gamma^{(0,1)} \wedge \gamma^{(0,1)} \} \).

**Assertion.** If we choose a sufficiently small \( \varepsilon \), then for any \( A + \alpha \) in \( U_{\alpha} \), there is a unique \( g = g_{\alpha} \) in \( \mathcal{G} \) close to the identity so that \( g(A' + h(\alpha)) \) belongs to \( \mathcal{V}_{A',\gamma} \).

This assertion is shown as follows. Since \( g(A' + h(\alpha)) = (\bar{\partial}_{A'} g) \cdot g^{-1} + g \cdot h(\alpha) \cdot g^{-1} \), the \((0,1)\)-form \( \gamma \) defined by \( A' + \gamma = g(A' + h(\alpha)) \) is represented by \( \gamma = (\bar{\partial}_{A'} g) \cdot g^{-1} + g \cdot h(\alpha) \cdot g^{-1} \). The \((0,1)\)-connection \( A' + \gamma \) is indeed holomorphic and satisfies \( \bar{\partial}_{A'} \gamma - \gamma \wedge \gamma = 0 \). Then \( \gamma \) lies in \( \mathcal{V}_{A',\gamma} \) if and only if for \( \bar{\partial}_{A'} = \bar{\partial}_{A}^{(0,1)} \)
\[
(\bar{\partial}_{A'}^* (\bar{\partial}_{A'} g) \cdot g^{-1} + g \cdot h(\alpha) \cdot g^{-1} = 0 \tag{4.5}
\]
If we set \( g = \exp \psi, \psi \in \Omega^1(g \mathcal{G}) \), then we reduce \( \partial^2 \) to
\[
\bar{\partial}_{A'} \bar{\partial}_{A} \psi + \bar{\partial}_{A} h(\alpha) - \langle [\partial_{A'} \psi, h(\alpha)] \rangle + [\psi, \bar{\partial}_{A} h(\alpha)] 
+ \bar{\partial}_{A} R(\psi, h(\alpha)) = 0, \tag{4.6}
\]
here \( R(\psi, h(\alpha)) \) is the remainder term of order not less than two. We operate
the Green operator $G_{A'}$ of $\Delta^0_{\alpha}$ to (4.6) to deduce

$$
\psi + G_{A'}(\partial^*_{A'} h(\alpha)) - G_{A'}(\partial_A \psi, h(\alpha)) + G_{A'}[\psi, \partial^*_{A'} h(\alpha)] + G_A(\partial^*_{A} R) = 0. 
$$

We remark that since $\alpha = \alpha^{(0,0)} + \alpha^{(0,1)} = \sum (\alpha_\mu dx^\mu + \alpha_\beta dx^\beta)$ satisfies $d_A^* \alpha = 0$ and $d_A^* \alpha = \alpha^* \alpha$,

$$
\partial^*_{A'} h(\alpha) = -(\sqrt{-1}/2) \sum g^{\mu\nu}[\alpha_\mu, \alpha_\nu] 
$$

and hence the $|*|_{A}$-norm of $\partial^*_{A'} h(\alpha)$ is estimated by $|\alpha|_{A}$.

By using the arguments of Section 3 in Ch. 4 of [10] and also of [3], [11] we obtain for a sufficiently small $|\alpha|_{A}$ a unique smooth solution $\psi = \psi(\alpha)$ to (4.7) in a neighborhood of $0 \in \Omega^0(\gamma^0)$. We see easily that $\psi$ depends smoothly on $\alpha$ and $g_{A}(A' + h(\alpha)) \in V_{A',t'}$ for $g_{A} = \exp \psi(\alpha)$.

We remark that $\psi(0) = 0$ and from an implicit function theorem we have $(d\psi(\alpha)/d\alpha)|_{\alpha=0} = 0$ and hence $(dg_{A}(\alpha)|_{\alpha=0} = id$.

From the above assertion the mapping $f; U_{A,t} \rightarrow V_{A',t'}$ defined by $A + \alpha \mapsto g_{A}(A' + h(\alpha))$ is smooth. We show now that the composition of the following mappings

$$
U_{t}(\subset H_{A}) \xrightarrow{\Phi_{A}^{-1}} U_{A,t} \xrightarrow{f} V_{A',t'} \xrightarrow{\Phi_{A'}} V_{A'_{t'}}(\subset H_{A'})
$$

is of maximal rank at $\beta = 0$ in $H_{A}$. Since $(d\Phi_{A}/d\beta)|_{\beta=0}$ is the identity mapping of $H_{A}$ and also $(d\Psi_{A'}/d\beta')|_{\beta'=0}$ gives the identity mapping of $\mathcal{H}_{A'}$, and further $(df/d\alpha)|_{\alpha=0}(\gamma') = \lim_{t \rightarrow 0} \{g_{A}(A'h(t\gamma) - A')\}/h(\gamma)$ for each $\gamma \in H_{A}$, the derivative of the mapping at $\beta = 0$ coincides from Proposition 2.4 with $h_{A_{t}} = H_{A_{t}}$. Because $h$ is $\mathcal{R}$-isomorphic, it gives a local diffeomorphism at $\alpha = 0$ and then $f; U_{A,t} \rightarrow V_{A',t'}$ is open. Since $f$ is a lift of $f|_{U_{A,t}}$,

$$
\begin{align*}
U_{A,t} & \xrightarrow{f} V_{A',t'} & U_{t}(\subset H_{A}) & \xrightarrow{\Phi_{A}^{-1}} U_{A,t} \xrightarrow{f} V_{A',t'}(\subset H_{A'}) \xrightarrow{\Phi_{A'}} V_{A'_{t'}}(\subset H_{A'})
\end{align*}
$$

is also open from the fact that $\pi; U_{A,t} \rightarrow U_{A_{t}}$ is a homeomorphism and $\pi'; V_{A',t'} \rightarrow V_{A'_{t'}}$ is open.

**Remark 4.1.** (1) The image $f((\mathcal{M} \setminus \mathcal{K})_0)$ is an open subspace in $\mathcal{M}_h \setminus \mathcal{K}_{(0,0)}$, identified with $(\mathcal{M} \setminus \mathcal{K})_0$. (2) Although $(\mathcal{M}_h \setminus \mathcal{K}_{(0,0)})_0$ may not necessarily be Hausdorff, $f((\mathcal{M} \setminus \mathcal{K})_0)$ is surely a Hausdorff space because $(\mathcal{M} \setminus \mathcal{K})_0$ is Hausdorff from Remark 2.3. (3) Since the mapping $f; U_{A,t} \rightarrow V_{A',t'}$ provided in the above proof is locally diffeomorphic, we can choose sufficiently small $\varepsilon'$, if necessary, so that $\pi'|_{V_{A',t'}}$ gives a homeomorphism of $V_{A',t'}$ onto a neighborhood $V_{A'_{t'}}$ of
5. Complex structure of the moduli space

The aim of this section is to prove the following.

**Proposition 5.1.** The moduli space $f((\mathcal{M} \setminus \mathcal{K})_0)$ is a complex manifold of dimension $c_2(g^\mathbb{C}) - (n^2 - 1)p_3(M)$, if it is not empty.

Proof. By Propositions 4.2 and 4.3 and also from (3) of Remark 4.1 we can assume that for each $[A] \in f((\mathcal{M} \setminus \mathcal{K})_0)$ and for a sufficiently small $V_A = V_{A,\varepsilon}$ that the mapping $\Psi_A; V_A \to V_{\varepsilon} = \{\beta \in \mathcal{H}_A; |\beta|_M < \varepsilon\}$ defines a coordinate system for $f((\mathcal{M} \setminus \mathcal{K})_0)$.

Fix points $[A]$ and $[A']$ in $f((\mathcal{M} \setminus \mathcal{K})_0)$ with $\pi'(V_A) \cap \pi'(V_{A'}) = \emptyset$. We define subsets $B \subset V_A$ and $B' \subset V_{A'}$ by $B = \{A + \alpha \in V_A; \pi'(A + \alpha) \subset \pi'(V_{A'})\}$ and $B' = \{A' + \alpha' \in V_{A'}; \pi'(A' + \alpha') \subset \pi'(V_A)\}$, respectively. Then for each $A + \alpha$ in $B$ there is a $g \in \mathcal{G}^\mathbb{C}$ with $g(A + \alpha) \in B'$. Since the isotropy subgroup $\Gamma_\mathbb{C}^\mathbb{C}$ is finite, we can choose such a $g = g_\alpha$ uniquely in $\mathcal{G}^\mathbb{C}$ for $A + \alpha$.

Let $\{\beta_1, \ldots, \beta_m\}$ and $\{\beta'_1, \ldots, \beta'_m\}$ be orthonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_A'$, respectively, where $m$ is the dimension of $\mathcal{H}_A$, which is by assumption independent of $A$. Because $\Psi_A^{-1}; V_{\varepsilon} \to V_A$ is holomorphic, for $\beta(t) = \sum_{i=1}^m t_i \beta_i \in V_{\varepsilon}, t = (t_1, \ldots, t_m) \in \mathbb{C}^m \setminus \{t_i = \sqrt{\sum_i |t_i|^2} < \varepsilon\}$ $\alpha(t) = \Psi_A^{-1}(\beta(t))$ is holomorphic in $t$. Therefore, if we can show that $g_t = g_{t_0}(t)$ is holomorphic in $t$, then the composition of the mappings

$$
\Psi_A(B) (\subset V_{\varepsilon}) \xrightarrow{\Psi_A^{-1}} B(\subset V_A) \text{ the action of } g_t \xrightarrow{\Psi_{A'}} B'(\subset V_{A'}) \xrightarrow{\Psi_{A'}} \Psi_{A'}(B') (\subset V_{\varepsilon})
$$

is also holomorphic in $t$, since $\Psi_{A'}(\alpha')$ is the harmonic part of $\alpha', \sum_{i=1}^m <\alpha', \beta'_i>_M \beta_i$.

We now verify the following assertion.

**Assertion.** The complex gauge transformations $g_t$ depend holomorphically on $t$.

It suffices for this purpose to prove that for any fixed $A + \alpha(t_0) \in B$, $g_t$ is holomorphic with respect to $A + \alpha(t)$ close to $A + \alpha(t_0)$. We set $\gamma(x) = \alpha(t_0 + x) - \alpha(t_0)$ and $h_t = g_{t_0} \cdot (g_{t_0})^{-1}$. Then $\gamma(0) = 0$ and $h_0 = \text{id}$. If we define $\alpha_t$ and $\sigma(x)$ in $\Omega^{0,1}(g^\mathbb{C})$ respectively by $A' + \alpha_t = g_{t_0}(A + \alpha(t_0))$ and $\sigma(x) = g_{t_0} \cdot \gamma(x) \cdot (g_{t_0})^{-1}$, then for $t = t_0 + x$, $g_t(A + \alpha(t)) = (h_t \cdot g_{t_0})(A + \alpha(t_0) + \gamma(t))$ is written by

$$
g_t(A + \alpha(t)) = A' + \alpha'_0 + (\tilde{\beta}(\alpha'_t + \sigma_t) h_t) \cdot (h_t)^{-1} + h_t \cdot \sigma(x) \cdot (h_t)^{-1}. \tag{5.1}
$$

Since $h_t$ is close to id in $\mathcal{G}^\mathbb{C}$, there exists a unique $\varphi(x) \in \Omega^0(g^\mathbb{C})$ with $\varphi(0) = 0$.
and \( h_z = \exp \psi(z) \). Then (5.1) reduces to
\[
g_t(A + \alpha(t)) = \partial_A''\psi + A' + \sigma(z) + R(\psi, \sigma(z))
\] (5.2)
for \( A'' = A' + \alpha' \), where the remainder term \( R(\psi, \sigma) \) is given by
\[
R(\psi, \sigma) = (\partial_A'' \exp \psi) \cdot \exp(-\psi) - \partial_A'' \psi + \exp \psi \cdot \exp(-\psi) - \sigma.
\] (5.3)
Notice that the remainder term indeed including \( \partial_A'' \psi \) and \( \sigma \) as linear terms can be represented more exactly by
\[
R(\psi, \sigma) = R_1(\psi, \sigma) + \sigma, \quad \sigma = 0.
\] (5.4)
where \( R_1 \) and \( R_2 \) are written as matrix-power series of order not less than 3 with respect to \( \psi \) and \( \sigma \).

Since \( \partial_A^* \alpha'' = 0 \), we see that \( (\partial_A^* \psi)(g_t(A + \alpha(t)) - A') = 0 \), namely \( g_t(A + \alpha(t)) - A' \) belongs to the slice, if and only if from (5.2)
\[
(\partial_A^* \psi)(g_t(A + \alpha(t))) + R(\psi, \sigma) = 0.
\] (5.5)

Because \( G_A' = \Delta_A' = \text{id} \) on \( \Omega^3(g^c) \), the above reduces to
\[
\psi + G_A'(\partial_A' \psi, \alpha^\prime) + G_A'(\partial_A^* \sigma) + G_A''(\partial_A'^* \sigma) R(\psi, \sigma) = 0,
\] (5.6)
here \( \partial_A' \psi \) is the \((1,0)\)-component of \( d_A' \psi \) with respect to the \( SU(\nu) \)-connection \( A'' \) induced canonically from \( A'' \). Then by using the way quite similar to one to solve (4.7) we have a solution \( \psi = \psi(z) \) to (5.6) depending smoothly on \( z \). We operate on (5.6) \( \partial \sigma \), relative to the parameter \( z \) to obtain
\[
\partial_A \psi + G_A'(\partial_A' \psi, \alpha^\prime) + G_A'(\partial_A^* \sigma) + G_A''(\partial_A'^* \sigma) R(\psi, \sigma) = 0
\] (5.7)
since \( \partial_A \sigma(z) = 0 \) and \( \partial_A \) commutes with \( G_A'' \) and with \( d_A'' \). The term \( \partial \sigma R(\psi, \sigma) \) is obviously linear with respect to \( \partial \psi \). Define a linear operator \( L = L_{\alpha_0} \) by
\[
L(\Theta) = \Theta + G_A''(\partial_A' \Theta, \alpha^\prime), \quad \Theta \in L_{\alpha_0}^2(\Omega^3(g^c)).
\]
Then \( L \) satisfies
\[
(1 - c |\alpha_0|) |\Theta|_{k+2} \leq |L(\Theta)|_{k+2} \leq (1 + c |\alpha_0|) |\Theta|_{k+2}
\] (5.8)
for a constant \( c > 0 \), independent of \( \alpha_0 \). For each \( \alpha^\prime_0 \) in a sufficiently small slice \( V_{\alpha_0} \), \( L_{\alpha_0} \) gives a bounded linear operator from (5.8). On the other hand by the remark on \( R(\psi, \sigma) \) the norm \( |\partial \sigma R(\psi, \sigma)|_{k+1} \) is estimated by
\[
|\partial A \psi|_{k+2} \leq c_0 |\partial A \psi|_{k+1} + |\sigma|_{k+1} T_1(\psi_{k+1}) + |\psi_{k+2} - \psi_{k+1}| T_2(\psi_{k+1})
\] (5.9)
for some constant \( c_0 \), where \( T_1(s) \) and \( T_2(s) \) are power series of \( s \) with convergence radius \( \infty \).

Since \( |\sigma(z)|_{k+1} \) is sufficiently small for small \( |z| \), we can let \( |\psi(z)|_{k+2} \) be also sufficiently small from (5.5). Thus by the aid of the lower estimate of
\[
L |\partial A \psi|_{k+2} \leq c_2 |\partial A \psi|_{k+1} \leq c_2 |\partial A \psi|_{k+2}, \quad \text{when} \quad c_2 < 1 \quad \text{for sufficiently small} \quad |z|,
\]
therefore (5.7) admits only a trivial solution $\bar{\partial}_z \varphi = 0$, that is, $\varphi = \varphi(z)$ and consequently $g_t = (\exp \varphi(z)) \cdot g_t$, $t = t_0 + z$, is holomorphic.

Proposition 5.1 follows from this assertion since $\dim_c \mathcal{H}^0 = c_2(G^c) - (n^2 - 1) \cdot \mathfrak{p}_e(M)$.

The proof of Theorem 2 is now completed if we pull back to $(\mathcal{M} \setminus \mathcal{K})_0$ the complex structure of $f((\mathcal{M} \setminus \mathcal{K})_0)$ through the $f$. Theorem 1 is a direct consequence of Theorem 2 from Remark 2.1 because $H^2_\alpha \cong H^0_\alpha \oplus \mathbb{H}$ vanishes for every irreducible anti-self-dual connection $A$ over a Kähler surface $M$ which either admits a Kähler metric of positive total scalar curvature or is endowed with trivial canonical line bundle.

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References


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