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## COBORDISM OF REGULAR TORUS ACTIONS

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### 1. Introduction and definitions

Let  $S^1$  be the unit sphere in the field of complex numbers  $\mathbf{C}$ . Let  $T^k$  be the  $k$ -dimensional torus, that is,  $T^k = S^1 \times \cdots \times S^1$  ( $k$  copies). We denote elements of  $T^k$  by coordinates  $(\lambda_1, \cdots, \lambda_k)$ ,  $\lambda_i \in S^1$ . For any sequence  $(i_1, \cdots, i_n)$  of integers with  $0 \leq i_1 < i_2 < \cdots < i_n \leq k$ , we define a subgroup  $T^k_{(i_1, \dots, i_n)}$  of  $T^k$  by

$$T^k_{(i_1, \dots, i_n)} = \{(\lambda_1, \cdots, \lambda_k) \in T^k \mid \lambda_j = 1 \text{ for } j \notin \{i_1, \dots, i_n\}\}.$$

In particular,  $T^k_0 = 1 \times \cdots \times 1$ ,  $T^k_i = 1 \times \cdots \times 1 \times \underbrace{S^1}_{i} \times 1 \times \cdots \times 1$  for  $0 < i \leq k$  and

$$T^k_{(1, 2, \dots, k)} = T^k.$$

Let  $\varphi: T^k \times M \rightarrow M$  be a differentiable action of  $T^k$  on a compact oriented differentiable manifold  $M$ . We denote such an action by a pair  $(M, \varphi)$ .  $(M, \varphi)$  is called *regular*, when for any point  $x$  in  $M$  the isotropy group  $I(x) = \{\lambda \in T^k \mid \varphi(\lambda, x) = x\}$  is of the form  $T^k_{(i_1, \dots, i_n)}$  for some sequence  $(i_1, \cdots, i_n)$ .

In particular, a regular  $T^1$ -action is called a *semi-free  $S^1$ -action*. If  $(M, \varphi)$  is a regular  $T^k$ -action, then the  $S^1$ -action  $(M, \varphi|_{T^k_i \times M})$  is a semi-free  $S^1$ -action for  $1 \leq i \leq k$ . But the reverse, in general, is not true. For example, given a free  $S^1$ -action  $(M, \varphi)$ , the  $T^2$ -action  $(M, \Phi)$  defined by  $\Phi((\lambda_1, \lambda_2), x) = \varphi(\lambda_1 \lambda_2, x)$  is not regular nevertheless the  $S^1$ -actions  $(M, \Phi|_{T^2_i \times M})$  ( $i=1, 2$ ) are free.

In this paper, we study regular  $T^k$ -actions by the method of Stong [3]. In section 3, we obtain the result which asserts that a stationary point free regular  $T^k$ -action on a closed oriented differentiable manifold bounds as a regular  $T^k$ -action on a compact oriented differentiable manifold (Corollary 3-2). And in section 4, we also obtain the result which asserts that a cobordism class of a regular  $T^k$ -action  $(M, \varphi)$  on a closed oriented differentiable manifold  $M$  is determined by the normal bundle of the stationary point set in  $M$  (Theorem 4-1).

As a corollary to Theorem 4-1, we obtain that the cobordism group of regular  $T^k$ -actions on closed oriented manifolds is isomorphic to the tensor product of  $k$  copies of the cobordism group of semi-free  $S^1$ -actions on closed oriented manifolds (Corollary 4-3).

In the case of  $k=1$ , Corollary 3-2 and Theorem 4-1 have been obtained by Uchida [4].

**2. Preliminaries**

Let  $G$  be a compact Lie group. We consider a family  $\mathfrak{F}$  of subgroups of  $G$  satisfying that if  $H \in \mathfrak{F}$  then  $gHg^{-1} \in \mathfrak{F}$  for all  $g \in G$ . All families considered will be assumed to satisfy this condition.

Given families  $\mathfrak{F} \supset \mathfrak{F}'$  of subgroups of  $G$ , an  $(\mathfrak{F}, \mathfrak{F}')$ -free  $G$ -action is a pair  $(M, \varphi)$  consisting of a compact oriented differentiable manifold  $M$  and a differentiable action  $\varphi: G \times M \rightarrow M$  such that

- (1) if  $x \in M$ , then the isotropy group  $I(x) \in \mathfrak{F}$ , and
- (2) if  $x \in \partial M$ , then  $I(x) \in \mathfrak{F}'$ .

If  $\mathfrak{F}'$  is empty, then necessarily  $\partial M = \emptyset$ .

Given  $(M, \varphi)$ , define  $-(M, \varphi) = (-M, \varphi)$ ,  $-M$  the manifold with the opposite orientation to  $M$ . Also define  $\partial(M, \varphi) = (\partial M, \varphi|_{G \times \partial M})$ ,  $\partial M$  oriented by inward normal vectors.

Two  $(\mathfrak{F}, \mathfrak{F}')$ -free  $G$ -actions  $(M, \varphi)$  and  $(M', \varphi')$  are *cobordant*, if there are an  $(\mathfrak{F}', \mathfrak{F}')$ -free  $G$ -action  $(V, \psi)$  and an  $(\mathfrak{F}, \mathfrak{F})$ -free  $G$ -action  $(W, \psi')$  such that

- (1)  $\partial(V, \psi) = \partial(M, \varphi) \cup (-\partial(M', \varphi'))$  (disjoint union)
- (2)  $\partial(W, \psi') = \partial(M, \varphi) \cup \partial(V, \psi) \cup (-\partial(M', \varphi'))$  (glueing the boundaries).

This cobordism relation is an equivalence relation. We denote by  $[M, \varphi]$  the cobordism class of  $(M, \varphi)$ .

The set of cobordism classes of  $(\mathfrak{F}, \mathfrak{F}')$ -free  $G$ -actions forms an abelian group with the operation induced by disjoint union, and this group will be denoted by  $\Omega_*(G; \mathfrak{F}, \mathfrak{F}')$ .  $\Omega_n(G; \mathfrak{F}, \mathfrak{F}')$  denotes the summand consisting of cobordism classes of  $(\mathfrak{F}, \mathfrak{F}')$ -free  $G$ -actions  $(M, \varphi)$  with  $\dim M = n$ .

By the cartesian product,  $\Omega_*(G; \mathfrak{F}, \mathfrak{F}')$  is an  $\Omega_*$ -module,  $\Omega_*$  the oriented cobordism ring.

The following two theorems are essentially the results of Conner and Floyd.

**Theorem 2-1** (see [2; (5.3)]). *Let  $G$  be a compact Lie group and  $\mathfrak{F} \supset \mathfrak{F}' \supset \mathfrak{F}''$  families of subgroups of  $G$ . Then the sequence*

$$\dots \rightarrow \Omega_n(G; \mathfrak{F}', \mathfrak{F}'') \xrightarrow{i} \Omega_n(G; \mathfrak{F}, \mathfrak{F}'') \xrightarrow{j} \Omega_n(G; \mathfrak{F}, \mathfrak{F}') \xrightarrow{\partial} \Omega_{n-1}(G; \mathfrak{F}', \mathfrak{F}'') \rightarrow \dots$$

*is exact, where  $i$  and  $j$  are induced by considering  $(\mathfrak{F}', \mathfrak{F}'')$ -free or  $(\mathfrak{F}, \mathfrak{F}'')$ -free as being  $(\mathfrak{F}, \mathfrak{F}'')$ -free or  $(\mathfrak{F}, \mathfrak{F}')$ -free respectively, and  $\partial$  is induced by sending  $[M, \varphi]$  to  $[\partial(M, \varphi)]$ .*

Turning to actions of  $T^k$ , non-trivial, non-isomorphic, and regular irreducible orthogonal representations of  $T^k$  on the 2-dimensional real vector space  $C$  are given by the complex multiplication by  $i$ -th coordinate of  $T^k$  for  $i=1, 2, \dots, k$ . So we can express a translation of [1; Theorem 38.3] into regular  $T^k$ -actions in the following fashion:

**Theorem 2-2.** *Suppose that  $\xi$  is an  $n$ -dimensional real vector bundle over a connected, locally connected, paracompact base, and that  $\varphi: T^k \times \xi \rightarrow \xi$  is a regular  $T^k$ -action which carries each fibre orthogonally onto itself such that the stationary points are only zero vectors. There are then vector subbundles  $\xi_i$  of  $\xi$ ,  $i=1, \dots, k$ , with  $\xi = \xi_1 \oplus \dots \oplus \xi_k$ , and there exists a complex vector bundle structure on each  $\xi_i$  such that  $\varphi((\lambda_1, \dots, \lambda_k), (v_1, \dots, v_k)) = (\lambda_1 \cdot v_1, \dots, \lambda_k \cdot v_k)$  for  $(\lambda_1, \dots, \lambda_k) \in T^k$  and  $v_i \in \xi_i$ ,  $i=1, \dots, k$ , where  $\cdot$  denotes the complex multiplication. In particular, each  $\xi_i$  is invariant under the action of  $T^k$ .*

### 3. Cobordism of regular $T^k$ -actions

For  $1 \leq p \leq k+1$  let  $\mathfrak{F}_p$  denote the family of subgroups  $T^k_{(i_1, \dots, i_n)}$  of  $T^k$  such that  $\{i_1, \dots, i_n\}$  does not contain  $\{1, 2, \dots, p\}$ , and let  $\mathfrak{F}_0 = \phi$ . Note that  $\mathfrak{F}_{k+1}$  is the family of all subgroups of the type  $T^k_{(i_1, \dots, i_n)}$ . We have inclusions

$$\phi = \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \dots \subset \mathfrak{F}_k \subset \mathfrak{F}_{k+1}.$$

We note that if  $(M, \varphi)$  is a regular  $T^k$ -action, then the isotropy groups all belong to  $\mathfrak{F}_p$  if and only if  $(M, \varphi|_{T^k_{(1, \dots, p)} \times M})$  is stationary point free.

**Theorem 3-1.** *For  $0 \leq p < k$ , the sequence of Theorem 2-1 for the families  $\mathfrak{F}_{k+1} \supset \mathfrak{F}_{p+1} \supset \mathfrak{F}_p$  becomes a split exact sequence of  $\Omega_*$ -modules:*

$$0 \rightarrow \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_p) \xrightarrow{j} \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_{p+1}) \xrightarrow{\partial} \Omega_*(T^k; \mathfrak{F}_{p+1}, \mathfrak{F}_p) \rightarrow 0.$$

*Proof.* It suffices to construct an  $\Omega_*$ -module homomorphism  $\gamma: \Omega_*(T^k; \mathfrak{F}_{p+1}, \mathfrak{F}_p) \rightarrow \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_{p+1})$  satisfying  $\partial\gamma = 1$ .

Given  $[M, \varphi] \in \Omega_n(T^k; \mathfrak{F}_{p+1}, \mathfrak{F}_p)$ , let  $N$  be the stationary point set of  $T^k_{(1, \dots, p)}$ , i.e.,  $N = \{x \in M \mid \varphi(\lambda, x) = x \text{ for all } \lambda \in T^k_{(1, \dots, p)}\}$ . Then  $N$  is a  $T^k$ -invariant submanifold of  $M$  with  $\partial N = N \cap \partial M$ , and since all isotropy groups on  $\partial M$  belong to  $\mathfrak{F}_p$ ,  $N \cap \partial M = \phi$ . Let  $\nu$  be the normal bundle of  $N$  in  $M$ . The  $T^k$ -action  $\varphi$  induces a  $T^k$ -action  $\bar{\varphi}: T^k \times \nu \rightarrow \nu$  by bundle maps covering  $\varphi$ . Since  $T^k_{p+1}$  freely acts on  $N$  by  $\varphi$ ,  $T^k_{p+1}$  also freely acts on  $\nu$  by  $\bar{\varphi}$ . Let  $D$  be the unit disc in  $C$ , and let  $T^k_{p+1}$  act on  $D$  by the complex multiplication by  $(p+1)$ -th coordinate of  $T^k_{p+1}$ . Then we obtain the orbit manifold  $D(\nu) \times D/T^k_{p+1}$  by the diagonal action, where  $D(\nu)$  is the disc bundle of  $\nu$ . The submanifold  $D(\nu) \times S^1/T^k_{p+1}$  of  $D(\nu) \times D/T^k_{p+1}$  is equivariantly identified with  $D(\nu)$  by an

identification  $[v, z] \mapsto \bar{\varphi}((1, \dots, 1, z^{-1}, 1, \dots, 1), v)$  where  $z^{-1}$  lies on  $(p+1)$ -th coordinate, and  $D(\nu)$  is equivariantly identified with a tubular neighborhood  $T(N)$  of  $N$  in  $M$ . So we can form an  $(n+1)$ -dimensional manifold  $W$  from  $D(\nu) \times D/T_{p+1}^k \cup M \times [0, 1]$  by identifying  $D(\nu) \times S^1/T_{p+1}^k$  with  $T(N) \times 1$ . We may orient  $W$  such that the inclusion  $M \rightarrow M \times 0 \subset W$  is orientation-preserving.

Define a  $T^k$ -action  $\psi$  on  $D(\nu) \times D/T_{p+1}^k$  by  $\psi(\lambda, [v, z]) = [\bar{\varphi}(\lambda, v), z]$  for  $\lambda \in T^k, v \in D(\nu)$  and  $z \in D$ . Then  $\psi$  is compatible with  $\varphi \times 1$  on the identified part, so we obtain an  $(\mathfrak{F}_{k+1}, \mathfrak{F}_{p+1})$ -free  $T^k$ -action  $(W, \chi)$  where  $\chi$  restricts to  $\psi$  on  $D(\nu) \times D/T_{p+1}^k$  and to  $\varphi \times 1$  on  $M \times [0, 1]$ .

Performing the same construction on a cobordism shows that  $\gamma([M, \varphi]) = [W, \chi]$  defines an  $\Omega_*$ -module homomorphism. We have  $\partial\gamma([M, \varphi]) = [\partial W, \chi | T^k \times \partial W] = [M, \varphi]$ . The last equation follows by applying [2; (5.2)], in fact,  $(\partial W, \chi | T^k \times \partial W)$  and  $(M, \varphi)$  are cobordant by a cobordism  $(\partial W \times [0, 1], (\chi | T^k \times \partial W) \times 1)$ . q.e.d.

**Corollary 3-2.** *If  $(M, \varphi)$  is a stationary point free regular  $T^k$ -action on a closed oriented manifold  $M$ , then there is a regular  $T^k$ -action  $(\mathfrak{M}, \Phi)$  on a compact oriented manifold  $\mathfrak{M}$  such that  $\partial\mathfrak{M} = M$  and  $\Phi | T^k \times \partial\mathfrak{M} = \varphi$ .*

Proof. We have the exact sequence

$$\Omega_*(T^k; \mathfrak{F}_k, \mathfrak{F}_0) \xrightarrow{i} \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0) \xrightarrow{j} \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_k)$$

$\underbrace{\hspace{15em}}_{\partial}$

for the families  $\mathfrak{F}_{k+1} \supset \mathfrak{F}_k \supset \mathfrak{F}_0$ . We note that  $\Omega_*(T^k; \mathfrak{F}_k, \mathfrak{F}_0)$  is the cobordism group of stationary point free regular  $T^k$ -actions on closed oriented manifolds and  $\Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0)$  is the cobordism group of all regular  $T^k$ -actions on closed oriented manifolds. Thus it is sufficient to show that  $j$  is monic. Let  $j_p: \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_p) \rightarrow \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_{p+1})$  be the canonical homomorphism, then  $j_p$  is monic by Theorem 3-1 if  $0 \leq p < k$ . So  $j = j_{k-1} \circ \dots \circ j_1 \circ j_0$  is monic. q.e.d.

#### 4. Cobordism of bundles with regular $T^k$ -actions

In this section, we show that a cobordism class of a regular  $T^k$ -action on a closed oriented manifold is determined by the normal bundle of the stationary point set.

Given  $[M, \varphi] \in \Omega_m(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0)$ , let  $N$  be a connected component of the stationary point set of  $\varphi$ , and let  $\nu \rightarrow N$  be the normal bundle of  $N$  in  $M$ . By Theorem 2-2, we have the decomposition

$$(*) \quad \nu = \nu_1 \oplus \nu_2 \oplus \dots \oplus \nu_k$$

of complex vector bundles with the induced  $T^k$ -action  $\bar{\varphi}$  satisfying

$\bar{\varphi}((\lambda_1, \dots, \lambda_k), (v_1, \dots, v_k)) = (\lambda_1 \cdot v_1, \dots, \lambda_k \cdot v_k)$  for  $(\lambda_1, \dots, \lambda_k) \in T^k$  and  $(v_1, \dots, v_k) \in \nu$  ( $v_i \in \nu_i$ ).  $N$  may be oriented compatibly with the canonical orientation of  $\nu$  and the given orientation of  $\tau(M)$ . Let  $f_i: N \rightarrow BU(n_i)$  be the classifying map of  $\nu_i$ , then define a homomorphism

$$\alpha: \Omega_m(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0) \rightarrow \bigoplus_{m=r+2 \sum n_i} \Omega_r(BU(n_1) \times \dots \times BU(n_k))$$

by sending  $[M, \varphi]$  to  $\bigoplus [N, f_1 \times \dots \times f_k]$  where the sum is taken over all connected components of the stationary point set.

Let  $s: \bigoplus \Omega_r(BU(n_1) \times \dots \times BU(n_k)) \rightarrow \bigoplus' \Omega_r(BU(n_1) \times \dots \times BU(n_k))$  be the projection where the sum  $\bigoplus$  is taken over all  $(r, n_1, \dots, n_k)$  with  $m=r+2 \sum_{i=1}^k n_i$ , and  $\bigoplus'$  is taken over all  $(r, n_1, \dots, n_k)$  with  $m=r+2 \sum_{i=1}^k n_i$ , and  $n_i \neq 1$  for  $i=1, \dots, k$ .

Then we have

**Theorem 4-1.** *By the composition  $s\alpha$*

$$\Omega_m(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0) \cong \bigoplus \Omega_r(BU(n_1) \times \dots \times BU(n_k)),$$

where the sum is taken over all  $(r, n_1, \dots, n_k)$  with  $m=r+2 \sum_{i=1}^k n_i$  and  $n_i \neq 1$  for  $i=1, \dots, k$ .

As immediate corollaries we have the following Corollaries 4-2 and 4-3. (Of course Corollary 3-2 can also be obtained as a corollary to Theorem 4-1.)

**Corollary 4-2.** *Given a regular  $T^k$ -action  $(M, \varphi)$  on an  $m$ -dimensional closed oriented manifold, let  $\nu$  be the normal bundle of a connected component of the stationary point set in  $M$ . If  $\nu$  has at least one (complex) 1-dimensional summand in the decomposition (\*) for all connected components of the stationary point set, then  $[M, \varphi]=0$  in  $\Omega_m(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0)$ . In particular, if the stationary point set is 2-codimensional, then  $[M, \varphi]=0$  in  $\Omega_m(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0)$ .*

$\Omega_*(T^1; \mathfrak{F}_2, \mathfrak{F}_0)$  is the cobordism group of semi-free  $S^1$ -actions on oriented closed manifolds. We consider the tensor product  $\Omega_*(T^1; \mathfrak{F}_2, \mathfrak{F}_0) \otimes \dots \otimes \Omega_*(T^1; \mathfrak{F}_2, \mathfrak{F}_0)$ , (over  $\Omega_*$ ), of  $k$  copies of  $\Omega_*(T^1; \mathfrak{F}_2, \mathfrak{F}_0)$  and define a homomorphism  $\beta: \Omega_*(T^1; \mathfrak{F}_2, \mathfrak{F}_0) \otimes \dots \otimes \Omega_*(T^1; \mathfrak{F}_2, \mathfrak{F}_0) \rightarrow \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0)$  by sending  $[M_1, \varphi_1] \otimes \dots \otimes [M_k, \varphi_k]$  to  $[M_1 \times \dots \times M_k, \varphi_1 \times \dots \times \varphi_k]$  where  $\varphi_1 \times \dots \times \varphi_k$  is a regular  $T^k$ -action defined by  $\varphi_1 \times \dots \times \varphi_k ((\lambda_1, \dots, \lambda_k), (x_1, \dots, x_k)) = (\varphi_1(\lambda_1, x_1), \dots, \varphi_k(\lambda_k, x_k))$  for  $(\lambda_1, \dots, \lambda_k) \in T^k$  and  $x_i \in M_i$ . We then have

**Corollary 4-3.**  *$\beta$  is an isomorphism of  $\Omega_*$ -modules.*

Proof. By applying the Künneth formula to the spaces  $BU(n_1) \times \dots \times BU(n_k)$ , the Corollary follows from Theorem 4-1. q.e.d.

In order to prove Theorem 4-1, we introduce cobordism groups of complex vector bundles with regular  $T^k$ -actions.

We consider a  $(T^k, p)$ -manifold-bundle given by a collection  $((M, \varphi), (\xi_1, \varphi_1), \dots, (\xi_p, \varphi_p))$  where  $(M, \varphi)$  is a regular  $T^k$ -action on an oriented closed manifold, and  $\xi_i$  is a complex vector bundle over  $M$  with a regular  $T^k$ -action  $\varphi_i$  by complex vector bundle maps covering  $\varphi$ .

Let  $\Omega_m(T^k; n_1, \dots, n_p)$  denote the group of cobordism classes of  $(T^k, p)$ -manifold-bundles with  $\dim M = m$  and  $\dim_C \xi_i = n_i$ . Similarly, let  $\hat{\Omega}_m(T^k; n_1, \dots, n_p)$  denote the group obtained under the assumption that  $T_1^k$  freely acts on  $M$  by  $\varphi$  (i.e.  $(M, \varphi)$  is  $(\mathfrak{F}_1, \mathfrak{F}_0)$ -free).

**Lemma 4.4.**  $\Omega_n(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_p) \cong \bigoplus_{n=m+2\sum n_i} \Omega_m(T^{k-p}; n_1, \dots, n_p)$ .

Proof. We define homomorphisms  $\sigma: \Omega_n(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_p) \rightarrow \bigoplus_{n=m+2\sum n_i} \Omega_m(T^{k-p}; n_1, \dots, n_p)$  and  $\rho: \bigoplus_{n=m+2\sum n_i} \Omega_m(T^{k-p}; n_1, \dots, n_p) \rightarrow \Omega_n(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_p)$  satisfying  $\sigma\rho=1$  and  $\rho\sigma=1$  as follows:

Given  $[M, \varphi] \in \Omega_n(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_p)$ , let  $N$  be a connected component of the stationary point set of  $T^k_{(1, \dots, p)}$ . Then  $N$  is an oriented closed submanifold which is invariant under the action of  $T^k$ . Let  $\nu$  be the normal bundle of  $N$  in  $M$ . The  $T^k$ -action  $\varphi$  induces the  $T^k$ -action  $\bar{\varphi}: T^k \times \nu \rightarrow \nu$  by bundle maps covering  $\varphi$ . Restrict  $\bar{\varphi}$  to the action of  $T^p = T^k_{(1, \dots, p)}$ , then the restricted  $T^p$ -action on  $\nu$  is satisfied with the hypothesis of Theorem 2-2, so we have the decomposition  $\nu = \nu_1 \oplus \dots \oplus \nu_p$  of complex vector bundles with the  $T^p$ -action  $\bar{\varphi}$  satisfying  $\bar{\varphi}((\lambda_1, \dots, \lambda_p), (v_1, \dots, v_p)) = (\lambda_1 \cdot v_1, \dots, \lambda_p \cdot v_p)$  for  $(\lambda_1, \dots, \lambda_p) \in T^p$  and  $(v_1, \dots, v_p) \in \nu$ . By the commutativity of the action of  $T^k$  and the non-equivalence of  $T^p$ -representations on fibres in distinct summands of  $\nu$ , we see that the decomposition of  $\nu$  is compatible with the  $T^k$ -action  $\bar{\varphi}$ . Then we define  $\sigma$  by sending  $[M, \varphi]$  to  $\bigoplus [(N, \varphi'), (v_1, \bar{\varphi}_1), \dots, (v_p, \bar{\varphi}_p)]$  where, considering  $T^{k-p} = T^k_{(p+1, \dots, k)}$ ,  $\varphi' = \varphi|_{T^{k-p} \times N}$  and  $\bar{\varphi}_i = \bar{\varphi}|_{T^{k-p} \times \nu_i}$ , and where the sum is taken over all connected components of the stationary point set of  $T^k_{(1, \dots, p)}$ .

Next we define  $\rho$  by sending  $[(M, \varphi), (\xi_1, \varphi_1), \dots, (\xi_p, \varphi_p)] \in \bigoplus \Omega_m(T^{k-p}; n_1, \dots, n_p)$  to  $[D(\xi), \mu]$ , where  $D(\xi)$  is the disc bundle of  $\xi = \bigoplus_{i=1}^p \xi_i$  and is oriented by the complex structure and the orientation of  $M$ , and  $\mu: T^k \times D(\xi) \rightarrow D(\xi)$  is defined by  $\mu((\lambda_1, \dots, \lambda_k), (v_1, \dots, v_p)) = (\varphi_1(\lambda', \lambda_1 \cdot v_1), \varphi_2(\lambda', \lambda_2 \cdot v_2), \dots, \varphi_p(\lambda', \lambda_p \cdot v_p))$  for  $(\lambda_1, \dots, \lambda_k) \in T^k$ ,  $\lambda' = (\lambda_{p+1}, \dots, \lambda_k)$ , and  $(v_1, \dots, v_p) \in D(\xi)$ ,  $v_i \in \xi_i$ .

$\sigma\rho=1$  is clear.  $\rho\sigma([M, \varphi]) = [D(\nu), \bar{\varphi}]$  where  $D(\nu)$  is identified equivariantly with a tubular neighborhood of the stationary point set of  $T^k_{(1, \dots, p)}$  in  $M$ . From [2; (5.2)] we have  $[D(\nu), \bar{\varphi}] = [M, \varphi]$ . q.e.d.

By the same way we have

**Lemma 4-5.**  $\Omega_n(T^k; \mathfrak{F}_{p+1}, \mathfrak{F}_p) \cong \bigoplus_{n=m+2\sum n_i} \hat{\Omega}_m(T^{k-p}; n_1, \dots, n_p).$

**Theorem 4-6.** *There is an exact sequence*

$$(*) \quad 0 \rightarrow \Omega_m(T^k; n_1, \dots, n_p) \xrightarrow{F} \bigoplus_{m=r+2n_{p+1}} \Omega_r(T^{k-1}; n_1, \dots, n_p, n_{p+1}) \\ \xrightarrow{S} \hat{\Omega}_{m-1}(T^k; n_1, \dots, n_p) \rightarrow 0.$$

*Proof.* First we define the homomorphism  $F$ . Given  $x = [(M, \varphi), (\xi_1, \varphi_1), \dots, (\xi_p, \varphi_p)] \in \Omega_m(T^k; n_1, \dots, n_p)$ , let  $N$  be a connected component of the stationary point set of  $T_1^k$  in  $M$ . Let  $\nu$  be the normal bundle of  $N$  in  $M$ .  $N$  may be oriented by the usual way. Then we define  $F$  by sending  $x$  to  $\bigoplus [(N, \varphi'), (\xi'_1, \varphi'_1), \dots, (\xi'_p, \varphi'_p), (\nu, \bar{\varphi}')]$ , where, considering  $T^{k-1} = T^k_{(2, \dots, k)}$ ,  $\varphi' = \varphi|_{T^{k-1} \times N}$ ,  $\xi'_i = \xi_i|_N$ ,  $\varphi'_i = \varphi_i|_{T^{k-1} \times \xi'_i}$  and  $\bar{\varphi}' = \bar{\varphi}|_{T^{k-1} \times \nu}$  with  $\bar{\varphi}$  being the  $T^k$ -action by bundle maps covering  $\varphi$ , and where the sum is taken over all connected components of the stationary point set of  $T_1^k$ .

Next we define the homomorphism  $S$ . Given  $y = [(M, \varphi), (\xi_1, \varphi_1), \dots, (\xi_{p+1}, \varphi_{p+1})] \in \Omega_r(T^{k-1}; n_1, \dots, n_{p+1})$  ( $m = r + 2n_{p+1}$ ), let  $\pi: S(\xi_{p+1}) \rightarrow M$  be the sphere bundle of  $\xi_{p+1}$ , and let  $S(\xi_{p+1})$  be oriented as the boundary of  $D(\xi_{p+1})$ ,  $D(\xi_{p+1})$  oriented by the complex structure and the given orientation of  $M$ . Let  $\pi^*\xi_i$  be the induced bundle on  $S(\xi_{p+1})$  from  $\xi_i$  by  $\pi$  for  $i = 1, \dots, p$ . Define  $\mu: T^k \times S(\xi_{p+1}) \rightarrow S(\xi_{p+1})$  by  $\mu((\lambda_1, \dots, \lambda_k), v) = \varphi_{p+1}((\lambda_2, \dots, \lambda_k), \lambda_1 \cdot v)$  for  $(\lambda_1, \dots, \lambda_k) \in T^k$  and  $v \in S(\xi_{p+1})$ , and  $\mu_i: T^k \times \pi^*\xi_i \rightarrow \pi^*\xi_i$  by  $\mu_i((\lambda_1, \dots, \lambda_k), (u, v)) = (\varphi_i((\lambda_2, \dots, \lambda_k), u), \mu((\lambda_1, \dots, \lambda_k), v))$  for  $u \in \xi_i$  and  $v \in S(\xi_{p+1})$ . Then we define  $S$  by sending  $y$  to  $[(S(\xi_{p+1}), \mu), (\pi^*\xi_1, \mu_1), \dots, (\pi^*\xi_p, \mu_p)]$ .

We take the direct sum of the sequences (\*) over all  $(m, n_1, \dots, n_p)$  with  $n = m + 2\sum_{i=1}^p n_i$ , after which this becomes precisely the sequence

$$0 \rightarrow \Omega_n(T^{k+p}; \mathfrak{F}_{k+p+1}, \mathfrak{F}_p) \xrightarrow{j} \Omega_n(T^{k+p}; \mathfrak{F}_{k+p+1}, \mathfrak{F}_{p+1}) \\ \xrightarrow{\partial} \Omega_{n-1}(T^{k+p}; \mathfrak{F}_{p+1}, \mathfrak{F}_p) \rightarrow 0$$

of Theorem 3-1, using the identifications of Lemmas 4-4 and 4-5. Thus the exactness of the sequence (\*) follows. q.e.d.

Let  $t: \bigoplus \Omega_r(T^{k-1}; n_1, \dots, n_p, n_{p+1}) \rightarrow \bigoplus' \Omega_r(T^{k-1}; n_1, \dots, n_p, n_{p+1})$  be the projection where the sum  $\bigoplus$  is taken over all  $(r, n_{p+1})$  with  $m = r + 2n_{p+1}$ , and  $\bigoplus'$  is taken over all  $(r, n_{p+1})$  with  $m = r + 2n_{p+1}$  and  $n_{p+1} \neq 1$ .

Then we have

**Corollary 4-7.** *By the composition  $tF$ ,*

$$\Omega_m(T^k; n_1, \dots, n_p) \cong \bigoplus \Omega_r(T^{k-1}; n_1, \dots, n_p, n_{p+1})$$

where the sum is taken over all  $(r, n_{p+1})$  with  $m=r+2n_{p+1}$  and  $n_{p+1} \neq 1$ .

Proof. To prove this, it suffices to show that the homomorphism  $S$  maps the summand  $\Omega_{m-2}(T^{k-1}; n_1, \dots, n_p, 1)$  isomorphically onto  $\hat{\Omega}_{m-1}(T^k; n_1, \dots, n_p)$ . We construct the inverse  $R: \hat{\Omega}_{m-1}(T^k; n_1, \dots, n_p) \rightarrow \Omega_{m-2}(T^{k-1}; n_1, \dots, n_p, 1)$  for  $S$  on this summand as follows.

Given  $z = [(M, \varphi), (\xi_1, \varphi_1), \dots, (\xi_p, \varphi_p)] \in \hat{\Omega}_{m-1}(T^k; n_1, \dots, n_p)$ ,  $T_1^k$  freely acts on  $M$ , so we obtain the  $n_i$ -dimensional complex vector bundles  $\xi'_i \rightarrow M'$  by deviding out  $\xi_i \rightarrow M$  by the actions of  $T_1^k$  for  $i=1, \dots, p$ . Considering  $T^{k-1} = T^k_{(2, \dots, p)}$ ,  $\varphi$  and  $\varphi_i$  induce  $T^{k-1}$ -actions  $\varphi': T^{k-1} \times M' \rightarrow M'$  and  $\varphi'_i: T^{k-1} \times \xi'_i \rightarrow \xi'_i$  with  $\varphi'_i$  being actions by bundle maps covering  $\varphi'$ . Let  $L \rightarrow M'$  be the complex line bundle associated to the principal  $S^1$ -bundle  $M \rightarrow M'$ . We may give a  $T^{k-1}$ -action  $\psi$  on  $L$  by bundle maps which covers  $\varphi'$  and restricts to  $\varphi|T^{k-1} \times M$  on  $S(L) = M$ . Then we define  $R$  by sending  $z$  to  $[(M', \varphi'), (\xi'_1, \varphi'_1), \dots, (\xi'_p, \varphi'_p), (L, \psi)]$ . It is easily checked that  $R$  is the required inverse for  $S$  on the summand. q.e.d.

Proof of Theorem 4-1.  $\Omega_m(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0)$  is identified with  $\Omega_m(T^k; 0)$  the group of cobordism classes of  $(T^k, 1)$ -manifold-bundles  $((M, \varphi), (\xi, \psi))$  with  $\dim M = m$  and  $\dim_C \xi = 0$  (i.e.  $(M, \varphi) = (\xi, \psi)$ ). By repetition of Corollary 4-7,  $\Omega_m(T^k; 0)$  is isomorphic to  $\bigoplus \Omega_r(T^0; n_1, \dots, n_k)$  where the sum is taken over all  $(r, n_1, \dots, n_k)$  with  $m = r + 2 \sum_{i=1}^k n_i$  and  $n_i \neq 1$  for  $i=1, \dots, k$ . Corresponding bundles to their classifying maps,  $\Omega_r(T^0; n_1, \dots, n_k)$  is isomorphic to  $\Omega_r(BU(n_1) \times \dots \times BU(n_k))$ . Theorem 4-1 thus follows. q.e.d.

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