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On Algebras of Left Cyclic Representation Type

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§1. Let A be an associative algebra (of finite dimension) with a unit, N its radical and let $\sum_{i=1}^{n}\sum_{j=1}^{f(i)} Ae_{ij}$ be the direct decomposition of A into directly indecomposable components, where $Ae_{ij} \cong Ae_{i1} = Ae_i$. If every indecomposable A-left module is homomorphic to one of Ae_i, then we define such an algebra A to be of left cyclic representation type.

Now it is well-known that, if every indecomposable A-left module is homomorphic to one of Ae_λ and every indecomposable A-right module is homomorphic to one of $e_\mu A$, A is generalized uniserial[1].

In this paper we shall study the structure of an algebra of left cyclic representation type. The main result is as follows:

An algebra A is of left cyclic representation type if and only if the following conditions are satisfied:

(1) *Each e_iA has only one composition series.*

(2) *Each Ne_j is the direct sum of at most two cyclic left ideals, homomorphic to Ae_ν, each of which has only one composition series.*

§2. In this section we suppose that $N^2=0$ and show some lemmas which are necessary for the proof of our main theorem.

Lemma 1. *If there exists at least one e such that $eN=v_1A\oplus v_2A$, A is not of left cyclic representation type.*

The proof of this lemma is obtained from the well-known result.

Next suppose that $e'Ne=\bar{e}'\bar{A}\bar{e}'u_1\oplus\bar{e}'\bar{A}\bar{e}'u_2\oplus\bar{e}'\bar{A}\bar{e}'u_3=u_1\bar{e}\bar{A}\bar{e}$. Then it is easily shown that $e'Ne=u_2\bar{e}\bar{A}\bar{e}=u_3\bar{e}\bar{A}\bar{e}$ and there exist $\xi_1, \xi_2\in\bar{e}\bar{A}\bar{e}$ such that $u_1\xi_1=u_2$, $u_1\xi_2=u_3$. Moreover we put $S_{ij}=[\eta\,|\,u_i\eta=\eta' u_j,\ \eta\in\bar{e}\bar{A}\bar{e}$ and $\eta'\in\bar{e}'\bar{A}\bar{e}']$. Then each S_{ij} is a module and we have

Lemma 2. *Suppose that $e'Ne$ has the above structure. Then* $\bar{e}\bar{A}\bar{e}=S_{11}+S_{12}+S_{13}$, $S_{11}=S_{22}^{(1)}+S_{23}^{(2)}$, $S_{12}=S_{23}^{(1)}+S_{21}^{(2)}$, $S_{13}=S_{21}^{(1)}+S_{22}^{(2)}$, $S_{22}^{(1)}=S_{33}^{(1)}$, $S_{21}^{(1)}=S_{32}^{(1)}$, $S_{23}^{(1)}=S_{31}^{(1)}$, $S_{21}^{(2)}=S_{33}^{(2)}$, $S_{23}^{(2)}=S_{32}^{(2)}$ *and* $S_{22}^{(2)}=S_{31}^{(2)}$ *where* $S_{ij}^{(\kappa)}$ $(\kappa=1,2)$ *are submodules of* S_{ij} *such that* $S_{ij}=S_{ij}^{(1)}+S_{ij}^{(2)}$.

1) cf. T. Nakayama I.

Proof. It is clear that $S_{11}\xi_1=S_{12}$, $S_{11}\xi_2=S_{13}$ and $\bar{e}\bar{A}\bar{e}=S_{11}+S_{12}+S_{13}$. Hence if we denote the dimension of S_{ij} by $d(S_{ij})$ we have $d(S_{11})=d(S_{12})=d(S_{13})=\frac{d(\bar{e}\bar{A}\bar{e})}{3}$. Moreover $S_{\kappa_i}\cap S_{\lambda_i}=0$ and $S_{i\kappa}\cap S_{i\lambda}=0$ if $\kappa\neq\lambda$. For if $S_{\kappa_i}\cap S_{\lambda_i}\ni\zeta\neq 0$ we have $u_\kappa\zeta=\zeta' u_i$, $u_\lambda\zeta=\zeta'' u_i$, $\zeta'^{-1}u_\kappa\zeta=\zeta''^{-1}u_\lambda\zeta$ and $\zeta'^{-1}u_\kappa=\zeta''^{-1}u_\lambda$ and this contradicts to the assumption that $\bar{e}'\bar{A}\bar{e}'u_\kappa\neq\bar{e}'\bar{A}\bar{e}'u_\lambda$. Hence $S_{22}\cap S_{11}\neq 0$ or $S_{22}\cap S_{13}\neq 0$. Now if we put $S_{22}^{(1)}=S_{22}\cap S_{11}$ and $S_{22}^{(2)}=S_{22}\cap S_{13}$. Then $S_{22}=S_{22}^{(1)}+S_{22}^{(2)}$. For $S_{22}\cap S_{12}=0$. Next $S_{21}\cap S_{11}=0$ and if we put $S_{13}\cap S_{21}=S_{21}^{(1)}$ and $S_{12}\cap S_{21}=S_{21}^{(2)}$ we have $S_{13}=S_{22}^{(2)}+S_{21}^{(1)}$ and $d(S_{21}^{(1)})=d(S_{22}^{(1)})$ and $d(S_{22}^{(2)})=d(S_{21}^{(2)})$. Similarly $S_{12}=S_{21}^{(2)}+S_{23}^{(1)}$. Moreover $S_{11}\supset S_{22}^{(1)}+S_{23}^{(2)}$. But $d(S_{23}^{(2)})=d(S_{21}^{(2)})=d(S_{22}^{(2)})$. Hence $d(S_{22}^{(1)})+d(S_{23}^{(2)})=d(S_{22}^{(1)})+d(S_{22}^{(2)})=\frac{d(\bar{e}\bar{A}\bar{e})}{3}$. Thus $S_{11}=S_{22}^{(1)}+S_{23}^{(2)}$. Next if $S_{33}\cap S_{11}\neq 0$ and $S_{33}^{(1)}=S_{33}\cap S_{11}\subsetneqq S_{22}^{(1)}$, we have $S_{33}^{(2)}=S_{12}\cap S_{33}\supsetneqq S_{21}^{(2)}$. For $d(S_{33}^{(1)})\lneqq d(S_{22}^{(1)})$ and $d(S_{33}^{(2)})\gneqq d(S_{21}^{(2)})$. Thus $S_{33}^{(2)}\cap S_{23}^{(1)}\neq 0$ but this is a contradiction and $S_{33}^{(2)}=S_{22}^{(1)}$. In the same way as above we can prove this lemma.

Next we shall show that if Ne is the direct sum of three simple components isomorphic to $\bar{A}\bar{e}'$ and if $e'N$ is a simple right ideal, then A is not of left cyclic representation type. For this purpose we shall prove

Lemma 3. *Suppose that* $e'Ne=\bar{e}'\bar{A}\bar{e}'u_1\oplus\bar{e}'\bar{A}\bar{e}'u_2\oplus\bar{e}'\bar{A}\bar{e}'u_3=u_1\bar{e}\bar{A}\bar{e}$. *Then* $\mathfrak{M}=Aem_1+Aem_2$, *where* $u_1m_1\neq 0$, $u_2m_1=0$, $u_3m_1=u_3m_2$, $n_2m_2\neq 0$ *and* $u_1m_2=0$, *is directly indecomposable.*

Proof. Suppose that $\mathfrak{M}$ is directly decomposable and $\mathfrak{M}=Aen_1\oplus Aen_2$ where $n_1=\alpha_1m_1+\alpha_2m_2$, $n_2=\beta_1m_1+\beta_2m_2$, $\alpha_i,\beta_j\in\bar{e}\bar{A}\bar{e}$ and $\bar{\alpha}_i\neq 0$, $\bar{\beta}_j\neq 0$. If $u_1n_1=0$, $u_1\alpha_1m_1+u_1\alpha_2m_2=0$. Hence we can suppose that $\alpha_1=\xi_{12}+\eta_{13}+\gamma$, $\alpha_2=\xi_{11}-\eta_{13}+\gamma'$ where $\xi_{12}\in S_{12}$, $\eta_{13}\in S_{13}$, $\xi_{11}\in S_{11}$ and $\gamma,\gamma'\in eNe$. Then we can write $\xi_{12}=\xi_{21}^{(2)}+\xi_{33}^{(1)}$, $\eta_{13}=\eta_{22}^{(2)}+\eta_{21}^{(1)}$, $\xi_{11}=\xi_{22}^{(1)}+\xi_{23}^{(2)}$ where $\xi_{ij}^{(k)}\in S_{ij}^{(k)}$. Thus $u_2n_1=u_2\alpha_1m_1+u_2\alpha_2m_2=(u_2\xi_{21}^{(2)}+u_2\xi_{23}^{(1)}+u_2\eta_{22}^{(2)}+u_2\eta_{21}^{(1)})m_1+(u_2\eta_{22}^{(2)}+u_2\eta_{21}^{(1)}+u_2\xi_{22}^{(1)}+u_2\xi_{23}^{(2)})m_2$ and if $u_2n_1=0$, we have $\xi_{23}^{(2)}=-\xi_{23}^{(1)}$, $\xi_{21}^{(2)}=-\eta_{21}^{(1)}$ and $\eta_{22}^{(2)}=-\xi_{22}^{(1)}$. But $S_{23}^{(2)}\cap S_{23}^{(1)}=0$. Hence $u_2n_1\neq 0$.

Similarly $u_3n_1\neq 0$. Moreover we can prove that $u_2n_1\neq u_3n_1$. Now suppose that $u_2n_1=u_3n_1$. First we may suppose that $u_2\rho=u_3$ where $\rho\in S_{23}^{(1)}$. For if $u_2\rho=\bar{\rho}u_3$, we can take $u_3'=\bar{\rho}u_3$ in place of u_3 and it is easily shown that $S_{\kappa\lambda}^{(i)}$ are invariant for u_1, u_2, u_3'. Then $(u_2\xi_{21}^{(2)}+u_2\xi_{23}^{(1)}+u_2\eta_{22}^{(2)}+u_2\eta_{21}^{(1)})m_1+(u_2\eta_{22}^{(2)}+u_2\eta_{21}^{(1)}+u_2\xi_{22}^{(1)}+u_2\xi_{23}^{(2)})m_2=(u_3\xi_{33}^{(2)}+u_3\xi_{31}^{(1)}+u_3\eta_{31}^{(2)}+u_3\eta_{32}^{(1)})m_1+(u_3\eta_{31}^{(2)}+u_3\eta_{32}^{(1)}+u_3\xi_{33}^{(1)}+u_3\xi_{32}^{(2)})m_2$ where we can put $\xi_{21}^{(2)}=\xi_{33}^{(2)}$, $\xi_{23}^{(1)}=\xi_{31}^{(1)}$, $\eta_{22}^{(2)}=\eta_{31}^{(2)}$, $\eta_{21}^{(1)}=\eta_{32}^{(1)}$, $\eta_{22}^{(2)}=\eta_{31}^{(2)}$, $\eta_{21}^{(1)}=\eta_{32}^{(1)}$, $\xi_{22}^{(1)}=\xi_{33}^{(1)}$ and $\xi_{23}^{(2)}=\xi_{32}^{(2)}$. Hence $u_2\xi_{21}^{(2)}m_1+u_2\xi_{23}^{(1)}m_1+u_2\eta_{21}^{(1)}m_1+u_2\eta_{22}^{(2)}m_2+u_2\xi_{22}^{(1)}m_2+u_2\xi_{23}^{(2)}m_2=u_2\rho\xi_{33}^{(2)}m_1+u_2\rho\xi_{31}^{(1)}m_1+u_2\rho\eta_{31}^{(2)}m_1+u_2\rho\eta_{32}^{(1)}m_2+u_2\rho\xi_{33}^{(1)}m_2+u_2\rho\xi_{32}^{(2)}m_2$ and from the independency of u_1m_1, u_2m_2 and $u_3m_1=u_3m_2$ we have $\xi_{21}^{(2)}+\eta_{21}^{(1)}=\rho\xi_{31}^{(1)}+\rho\eta_{31}^{(2)}$, $\xi_{23}^{(1)}+\xi_{23}^{(2)}=\rho\xi_{33}^{(2)}+\rho\xi_{33}^{(1)}$ and $\eta_{22}^{(2)}+\xi_{22}^{(1)}=\rho\eta_{32}^{(1)}+\rho\xi_{32}^{(2)}$.

Now from the assumption we have $\rho \in S_{23}^{(1)} = S_{12}^{(1)} = S_{31}^{(1)}$. Hence $\rho S_{11}^{(1)} = S_{12}^{(1)}$, $\rho S_{12}^{(1)} = S_{13}^{(1)}$, $\rho S_{13}^{(1)} = S_{11}^{(1)}$, $\rho S_{12}^{(2)} = S_{13}^{(2)}$, $\rho S_{11}^{(2)} = S_{12}^{(2)}$ and $\rho S_{31}^{(2)} = S_{23}^{(2)}$. Thus we have $\eta_{21}^{(1)} = \rho\xi_{31}^{(1)}$, $\xi_{21}^{(2)} = \rho\eta_{31}^{(2)}$, $\xi_{23}^{(1)} = \rho\xi_{33}^{(1)}$, $\xi_{23}^{(2)} = \rho\xi_{33}^{(2)}$, $\eta_{22}^{(2)} = \rho\xi_{32}^{(2)}$ and $\xi_{22}^{(1)} = \rho\eta_{32}^{(1)}$. But $\xi_{22}^{(1)} = \xi_{33}^{(1)}$, $\eta_{32}^{(1)} = \eta_{21}^{(1)}$ and $\xi_{31}^{(1)} = \xi_{23}^{(1)}$. Hence we have $\rho^3 = e$ $(\rho \neq e)$. But if this is true, $e = \frac{e+\rho+\rho^2}{3} + \frac{2e-\rho-\rho^2}{3}$ is the decomposition of e into two idempotents orthogonal to each other, where we assume that the characteristic is not 2 and not 3, and this contradicts to the fact that e is a primitive idempotent. Thus we have $Au_2n_1 \neq Au_3n_1$. If the characteristic is 3, $(e-\rho)^3 = 0$ and $e-\rho \in \bar{e}\bar{A}\bar{e}$. But this is a contradiction. If the characteristic is 2, $e+\rho+\rho^2$ and $\rho+\rho^2$ are idempotents orthogonal to each other and $e = (e+\rho+\rho^2)+(\rho+\rho^2)$.

In the same way as above, if $u_1n_1 = 0$, we have $u_2n_2 \neq 0$, $u_3n_2 \neq 0$ and $Au_2n_2 \neq Au_3n_2$ and the largest completely reducible A-left submodule of $\mathfrak{M}$ is the direct sum of at least four simple components. But this contradicts to the assumption, since the largest completely reducible A-left submodule of $\mathfrak{M}$ is the direct sum of three simple components. Thus the proof of this lemma is complete.

If Ne is the direct sum of at least three simple components (not all isomorphic to each other), it is proved by the same way as above or [III] that A is not of left cyclic representation type.

Lastly we can easily prove

Lemma 4. *If $e_1 \neq e_2$ and Ne_1 and Ne_2 contain simple components isomorphic to each other, A is not of left cyclic representation type.*

Hence if A is of left cyclic representation type and Ne_1 and Ne_2 contain simple components isomorphic to each other, we have $Ae_1 \cong Ae_2$.

From the above lemmas we have

Theorem 1. *Suppose that $N^2 = 0$. If A is of left cyclic representation type, it satisfies the following conditions:*

(1) *Every $e_\lambda N$ is simple*

(2) *Every Ne_κ is the direct sum of at most two simple components.*

§ **3.** In this section we suppose that $N^2 \neq 0$. First of all we shall prove the following

Lemma 5. *If $Ne/N^2e = A\bar{u}_1 \oplus A\bar{u}_2$, then there exist v_1, v_2 such that $Ne = Av_1 + Av_2$ where $v_1 \equiv \bar{u}_1$ (N^2) and $v_2 \equiv \bar{u}_2 (N^2)$.*

Proof. From the assumption $Ne = Av_1 + Av_2 + N^2e$ where $v_1 \equiv \bar{u}_1(N^2)$ and $v_2 \equiv \bar{u}_2(N^2)$. Now $N^2e = Nv_1 + Nv_2 + N^3e$. Hence $Ne = Av_1 + Av_2 + N^3e$. Thus if we continue this process, we have $Ne = Av_1 + Av_2$.

Next we suppose that $Ne=Au_1+Au_2$ where $e'u_1=u_1$, $e'u_2=u_2$. Then we can put $w_1=u_1$ or $w_1=u_2$.

Thus we have

Corollary 1. *Suppose that* $Ne/N^2e=\bar{A}\bar{u}_1+\bar{A}\bar{u}_2$ *where* $\bar{A}\bar{u}_1\cong\bar{A}\bar{u}_2\cong\bar{A}\bar{e}'$, *and* $e'N/e'N^2$ *is simple. Then* $Ne=Au_1+Au_2$, $e'N=u_1A$ *and, if* $\eta,\gamma\in\bar{e}'\bar{A}\bar{e}'$, *there exist* $\eta',\gamma',\eta'',\gamma''\in\bar{e}\bar{A}\bar{e}$ *such that* $\eta u_1=u_1\eta'$, $\gamma u_2=u_1\gamma'$ *or* $\eta u_1=u_2\eta''$, $\gamma u_2=u_2\gamma''$.

From the above lemma we have also

Corollary 2. *If* $Ne_i=Au_1^{(i)}+Au_2^{(i)}$, *an arbitrary element of* N *is the sum of* $u_{\kappa_1}^{(j_1)}\cdots u_{\kappa_n}^{(j_n)}\alpha$ *where* $\alpha\in\bar{e}_{j_n}\bar{A}\bar{e}_{j_n}$.

Next suppose that $Ne=Au_1+Au_2$, $e'N=u_1A=u_2A$, $Ne'=Av_1+Av_2$ and $e''N=v_1A=v_2A$. Then $Nu_1=Ne'u_1=Av_1u_1+Av_2u_1=Av_1u_1+Av_1\alpha u_1$ $=Av_1u_1+Av_1u_1\alpha'$. Hence if $v_1u_1=0$, we have $Nu_1=0$.

Then we have

Lemma 6. *Suppose that* $Ne_1=Au_1+Au_2$ *and* $eN=u_1A=u_2A$. *If* $eN^2e_2\not\subset N^3$, *then* A *is not left cyclic representation type.*

Proof. In order to prove this lemma we have only to construct a directly indecomposable A-left module $\mathfrak{M}=Ae_1m_1+Ae_2m_2$. For this purpose we suppose that $Ne_2=Av_1$, $N^2e_1=0$ and $N^3e_2=0$. Since $eN^2e_2\not\subset N^3$, we have $e_1Ne_2\not\subset N^2$. For if $e_\xi Ne_2\not\subset N^2$ $(\xi\neq1)$, $eN^2e_2=eNe_1\cdot e_\xi Ne_2$ $\not\subset N^3$. But since $e_1e_\xi=0$, this is a contradiction.

Now we put $v_1m_2\neq0$, $u_1v_1m_2\neq0$, $u_2v_1m_2\neq0$, $u_1v_1m_2=u_1m_1$ and $u_2m_1=0$. Then we can prove that $\mathfrak{M}$ is directly indecomposable. Namely if $\mathfrak{M}$ is directly decomposable, $\mathfrak{M}=Aen_1\oplus Ae_2n_2$ where $n_2=m_2$. If $u_1n_1=0$ we have $n_1=m_1-v_1m_2$ and then $u_2n_1=u_2v_1m_2\neq0$ and $Ae_2n_2\frown Ae_1n_1\neq0$. This is a contradiction.

From this lemma we obtain

Corollary 3. *If* $Ne_1=Au_1+Au_2$ *and* $eN=u_1A=u_2A$ *we have* eN^ie' $\not\subset N^{i+1}$ *for each* i *and for every* e'.

Next suppose that A is of left cyclic representation type. Then if $Ne=Au_1+Au_2$ and $Ae_i\sim Au_i$, it is proved that $Au_1\frown Au_2=0$. Namely if $e_1\neq e_2$, we can prove this fact from Lemma 3 and Corollary 2. Next if $e_1=e_2$, then there exists α such that $u_2=u_1\alpha$ where $\alpha\in\bar{e}\bar{A}\bar{e}$. If $Au_1\frown Au_2\neq0$ then there exists $w\neq0$ such that $w=\gamma v_1\cdots v_mu_1=\beta w_1\cdots w_nu_2$ where $\gamma,\beta\in\bar{e}'\bar{A}\bar{e}'$ and we have $\gamma v_1\cdots v_mu_1=v_1\cdots v_mu_1\gamma'$ and $\beta w_1\cdots w_nu_2$ $=v_1\cdots v_mu_1\alpha\beta'$. Now since $\alpha\beta'\in S_{12}$ and $\gamma'\in S_{11}$, we have $\alpha\beta'\neq\gamma'$. Hence from $v_1\cdots u_mu_1\gamma'=v_1\cdots v_mu_1\alpha\beta'$, we have $v_1\cdots v_mu_1$ $(\gamma'-\alpha\beta')=0$ and $v_1\cdots v_mu_1=0$. But this is a contradiction.

Thus we have

Lemma 7. *If* $Ne=Au_1+Au_2$ *and* $Ae_i\sim Au_i$, *we have* $Au_1\cap Au_2=0$.

Lastly we shall prove that if $Ne=Au_1\oplus Au_2$ and A is of left cyclic representation type, each Au_i $(i=1,2)$ has only one composition series.

Now suppose that $Ne=Au_1\oplus Au_2$, where $N^ku_1=0$, $N^lu_2=0$, $N^{k-1}u_1=Av_1\oplus Av_2$ and $N^{l-1}u_2=Aw$. Then from Lemma 5 Av_1, Av_2 and Aw are simple and are not isomorphic to each other and we can construct a directly indecomposable A-left module $\mathfrak{M}=Aem_1+Aem_2$. Namely we put $v_1m_1=0$, $v_1m_2\neq 0$, $v_2m_1\neq 0$, $v_2m_2=0$ and $u_2m_1=u_2m_2$. Then we can prove that $\mathfrak{M}$ is directly indecomposable.

Moreover Lemma 6 can be obtained from the above result, Lemma 3 and Lemma 7.

Thus we have

Theorem 2. *If A is of left cyclic representation type, the following conditions are satisfied*:

(1) *Each $e_\lambda N$ has only one composition series.*

(2) *Each Ne_κ is the direct sum of at most two cyclic left ideals, homomorphic to Ae_μ, each of which has only one composition series.*

§4. In this section we shall prove that, if two conditions of Theorem 2 are satisfied, A is of left cyclic representation type.

Now from the assumption it follows that an arbitrary block of this algebra is as follows:

(1) Every Ae_i has only one composition series.

(2) $\{Ae_1,\cdots,Ae_{r-1},Ae_r,Ae_{r+1},\cdots,Ae_n\}$, which has the following properties:

(a) Every Ne_i $(i=1,\cdots,r-1)$ has only one composition series or $Ne_i=Au_i^{(1)}\oplus Au_i^{(2)}$ $(i=1,\cdots,r-1)$, where $Ae_{\kappa_1}\sim Au_i^{(1)}$, $Ae_{\kappa_2}\sim Au_i^{(2)}$, $e_{\kappa_1}\neq e_{\kappa_2}$ and $Ae_{\kappa-1}\sim Ne_\kappa$.

(b) $Ne_r=Au_1\oplus Au_2$ where $Ae_{r-1}\sim Au_1\cong Au_2$ and Au_i has only one composition series.

(c) $N^2e_i=0$ $(i=r+1,\cdots,n)$.

(3) $\{Ae_1,\cdots,Ae_n\}$ where $Ne_i=Au_1^{(i)}\oplus Au_2^{(i)}$, $Ae_\kappa\sim Au_1^{(i)}$, $Ae_\lambda\sim Au_2^{(i)}$ and $e_\kappa\neq e_\lambda$.

In the case (1) we can prove it by the same way as [I].

Now we shall prove it in the case (2).

Let $\mathfrak{M}=\sum_\kappa\sum_{i_\kappa}Ae_\kappa m_{\kappa,i_\kappa}$ be an arbitrary A-left module. Then it is clear that $\sum_{i_{r+1}}Ae_{r+1}m_{r+1,i_{r+1}},\cdots,\sum_{i_n}Ae_nm_{n,i_n}$ are the direct components of $\mathfrak{M}$. Now if we prove that $\sum_{i_r}Ae_rm_{r,i_r}$ is the direct sum of Ae_rn_{r,i_r},

$\sum_{i_\kappa} Ae_\kappa m_{\kappa, i_\kappa}$ $(\kappa = r+1, \cdots, n)$ are also the direct sums of $Ae_\kappa n_{\kappa, i_\kappa}$.

First we state the following

Lemma 8. *If $e_\lambda w_1 = w_1$ and $e_\lambda w_2 = w_2$ where $w_1, w_2 \in Ne_r$, then there exists $\xi \in \bar{e}_r \bar{A} \bar{e}_r$ such that $w_1 = w_2 \xi$.*

The proof of this lemma is easy from Corollary 2.

Now suppose that $\mathfrak{M} = (Ae_r m_1 \oplus \cdots \oplus Ae_r m_{n-1}) + Ae_r m_n$ and $(Ae_r m_1 \oplus \cdots \oplus Ae_r m_{n-1}) \frown Ae_r m_n \neq 0$. Moreover we assume that $Ne_r m_n = Au_1 m_n + Au_2 m_n$. Then we can prove that $\mathfrak{M}$ is the direct sum of $Ae_r n_{i_r}$ in the following way:

(a) If $N^i u_1 m_n \subset (Ae_r m_1 \oplus \cdots \oplus Ae_r m_{n-1})$ we can put $vu_1 m_n = \alpha_1 vu_1 m_1 + \beta_1 vu_2 m_1 + \cdots + \alpha_{n-1} vu_1 m_{n-1} + \beta_{n-1} vu_2 m_{n-1}$, where $N^i e_{r-1} = Av$. Now if we put $m_1' = \alpha_1' m_1 + \beta_1' \alpha m_1, \cdots, m_{n-1}' = \alpha_{n-1}' m_{n-1} + \beta_{n-1}' \alpha m_{n-1}$, where $\alpha_i vu_1 = vu_1 \alpha_i'$, we have $vu_1 m_n = vu_1 m_1' + \cdots + vu_1 m_{n-1}'$. Moreover we can assume that the length of $Au_1 m_n$ is larger than any $Au_1 m_i$ $(i \leq n-1)$ and the length of $Au_2 m_n$ is larger than any $Au_2 m_\kappa$ such that the lengths of all $Au_1 m_\kappa$ $(\kappa = \kappa_1, \cdots, \kappa_s)$ are equal. Then if we put $m_n' = m_n - m_{\kappa_1}' - \cdots - m_{\kappa_s}'$, we have $vu_1 m_n' = vu_1 m_{\lambda_1}' + \cdots + vu_1 m_{\lambda_{n-s}}'$ and $\mathfrak{M} = Ae_r m_{\kappa_1}' \oplus \cdots \oplus Ae_r m_{\kappa_s}' \oplus \{(Ae_r m_{\lambda_1} \oplus \cdots \oplus Ae_r m_{\lambda_{n-s}}) + Ae_r m_n'\}$. By the same way as above, we can prove that $\mathfrak{M} = Ae_r n_1 \oplus \cdots \oplus Ae_r n_n$.

(b) Suppose that $N^i u_1 m_n \subset (Ae_r m_1 \oplus \cdots \oplus Ae_r m_{n-1})$ and $N^j u_2 m_n \subset (Ae_r m_1 \oplus \cdots \oplus Ae_r m_{n-1})$. Then we can put $vu_1 m_n = \alpha_1 vu_1 m_1 + \beta_1 vu_2 m_1 + \cdots + \alpha_{n-1} vu_1 m_{n-1} + \beta_{n-1} vu_2 m_{n-1}$ and $wu_2 m_n = \gamma_1 wu_1 m_1 + \xi_1 wu_2 m_1 + \cdots + \gamma_{n-1} wu_1 m_{n-1} + \xi_{n-1} wu_2 m_{n-1}$ where $N^i e_{r-1} = Av$ and $N^j e_{r-1} = Aw$. First if we take $m_n' = m_n - (\alpha_1' + \beta_1' \alpha) m_1 - \cdots - (\alpha_{n-1}' + \beta_{n-1}' \alpha) m_{n-1}$ in place of m_n, we have $vu_1 m_n' = 0$ and we can reduce this case to the case (a).

Next we shall show that $\sum_{\kappa=1}^{r} \sum_{i_\kappa} Ae_\kappa m_{\kappa, i_\kappa}$ is the direct sum of $Ae_\kappa n_{\kappa, j_\kappa}$. From the above result and from [I] each $\sum_{i_\lambda} Ae_\lambda m_{\lambda, i_\lambda}$ $(\lambda = 1, \cdots, r)$ is the direct sum of $Ae_\kappa n_{\lambda, i_\lambda}$. Hence we assume that $Ae_i m_i \frown (Ae_{i+1} m_{i+1} \oplus \cdots \oplus Ae_r m_r) \neq 0$ and $N^i e_i m_i \subset Ae_{i+1} m_{i+1} \oplus \cdots \oplus Ae_r m_r$. Here we remark that if $e' w_1 = w_1$ and $e' w_2 = w_2$ where $w \in Ne_\lambda$ and $w_2 \in Ne_{\lambda+j}$, there exists $p \in e_\lambda Ne_{\lambda+j}$ such that $w_1 p = w_1$.

Now suppose that $wm_i = \alpha_1 w_1 m_{i+1} + \cdots + \alpha_{r-i} w_{r-i} m_r$. Then from the above remark we have $w_1 = wp_1, \cdots, w_{r-i} = wp_{r-i}$ and if we take $m_i' = m_i - \alpha_1' p_1 m_{i+1} - \cdots - \alpha_{r-i}' p_{r-i} m_r$ in place of m_i, $Ae_i m_i' \frown (Ae_{i+1} m_{i+1} \oplus \cdots \oplus Ae_r m_r) = 0$.

In the case (3) we can prove by the same way as above.

Thus we have

Theorem 3. *An algebra A is of left cyclic representation type if* *and only if the following conditions are satisfied*:

(1) *Each* $e_\lambda N$ *has only one composition series.*

(2) *Each* Ne_κ *is the direct sum of at most two cyclic left ideals, homomorphic to* Ae_μ, *each of which has only one composition series.*

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