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LOCALIZATION OF BP-MODULE SPECTRA WITH RESPECT TO BP-RELATED HOMOLOGIES

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

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1. Introduction

BP is the Brown-Peterson spectrum for a fixed prime p. It is an associative and commutative ring spectrum whose homotopy is $BP_*=Z_{(p)}[v_1,\cdots,v_n,\cdots]$. Following Ravenel [9] we denote by L_n the localization with respect to $v_n^{-1}BP_*$-homology and by L_∞ that with respect to $\bigoplus_n v_n^{-1}BP_*$-homology. Then there is a tower

$$X\to L_\infty X\cdots\to L_nX\to L_{n-1}X\to\cdots\to L_0X$$

for each CW-spectrum X. A CW-spectrum X is said to be *harmonic* if $X=L_\infty X$, and *s-harmonic* if $X=\hat{L}_\infty X$ where we put $\hat{L}_\infty X=\varprojlim_n L_nX$. X is harmonic whenever it is s-harmonic. In this paper we study some properties of s-harmonic spectra. Especially we discuss $\hat{L}_\infty E$ when E is an associative BP-module spectrum which satisfies one or two of the following conditions:

I) *E_* is v_m-torsion for any $m<n$,*
II) *E_* is v_m-torsion for any $m>n$,*
III) *$BP_*/I_m\otimes_{BP_*}E_*$ is v_m-torsion free for any $m\leqq n$,*
IV) *$\mathrm{Tor}_m^{BP_*}(BP_*/I_m, E_*)$ is v_m-divisible for any $m<n$, and*
V) *$\mathrm{hom\,dim}_{BP_*}E_*\leqq n$.*

As such associative BP-module spectra we have $P(n)$, $k(n)$, $BP\langle n\rangle$, N_nBP and so on.

We show that an associative BP-module spectrum E is s-harmonic if $\mathrm{hom\,dim}_{BP_*}E_*$ is finite (Theorem 4.8). This implies Ravenel's result ([9, Theorem 4.4] or [6, Theorem 1.3]) that a p-local connective CW-spectrum X is harmonic if $\mathrm{hom\,dim}_{BP_*}BP_*X$ is finite (Corollary 4.9). However the finiteness assumption is not necessarily essential because $L_\infty BP\langle n\rangle$ is s-harmonic although $\mathrm{hom\,dim}_{BP_*}L_\infty BP\langle n\rangle_*$ is infinite for $n\geqq 1$ (Proposition 4.12).

We intend to describe elementary properties of s-harmonic spectra corresponding to those of harmonic spectra. The product of harmonic spectra is always harmonic. But its property is not valid for s-harmonic spectra. By computing $\varprojlim_m^1 N_{m+1}(\prod_{n\geq m} E_n)_*$ where $E_n = N_{n+1}BP$ or $N_{n+1}BP\langle n\rangle$, we finally show that neither $\prod_n N_{n+1}BP$ nor $\prod_n L_\infty BP\langle n\rangle$ is s-harmonic (Theorems 6.3 and 6.4). This says that $\hat{L}_\infty$ is never a localization functor, and hence $L_\infty X \neq \hat{L}_\infty X$ in general.

2. Associative BP-module spectra N_nE and M_nE

Let us denote by L_n the localization functor with respect to the $(v_n^{-1}BP)_*$-homology, and by L_∞ and L_ω those with respect to the $(\bigvee_n v_n^{-1}BP)_*$- and $(\prod v_n^{-1}BP)_*$-homologies respectively. Then there is a tower

$$L_S = \mathrm{id} \to L_{BP} = L_\omega \to L_\infty \to \cdots \to L_n \to \cdots \to L_0 = L_{SQ}$$

consisting of localization functors.

Define cofibrations

$$N_nX \to M_nX \to N_{n+1}X \tag{2.1}$$

inductively by setting $N_0X = X$ and $M_nX = L_nN_nX$. Then there is a commutative diagram

$$\begin{array}{ccccc} & & \Sigma^{-n}M_nX & = & \Sigma^{-n}M_nX \\ & & \downarrow & & \downarrow \\ X & \longrightarrow & L_nX & \longrightarrow & \Sigma^{-n}N_{n+1}X \\ \| & & \downarrow & & \downarrow \\ X & \to & L_{n-1}X & \longrightarrow & \Sigma^{-n+1}N_nX \end{array} \tag{2.2}$$

involving four cofibrations [9, Theorem 5.10].

Lemma 2.1. i) *If E is an (associative) BP-module spectrum, then L_nE, N_nE and M_nE are all so.*
ii) *If f: $E \to F$ is a BP-module map of BP-module spectra, then L_nf, N_nf and M_nf are all so.*

Proof. i) Consider the following diagram

$$\begin{array}{ccccccc} BP_\wedge E & \to & BP_\wedge L_nE & \to & BP_\wedge\Sigma^{-n}N_{n+1}E & \to & BP_\wedge\Sigma^1E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E & \longrightarrow & L_nE & \longrightarrow & \Sigma^{-n}N_{n+1}E & \longrightarrow & \Sigma^1E \end{array}$$

with cofibering rows. There is a unique map $BP_\wedge L_nE \to L_nE$ making the left square commutative since $BP_\wedge N_{n+1}E$ is $v_n^{-1}BP_*$-acyclic. Thus L_nE inherits a BP-module structure from that of E. The associativity of L_nE is assured by the uniqueness of induced maps. Moreover there is a unique map $BP_\wedge N_{n+1}E \to N_{n+1}E$ making the other squares commutative. This also gives a BP-module structure on $N_{n+1}E$.

ii) It is easy to show ii) along the above line.

Let E be an associative BP-module spectrum such that

$(\mathrm{I})_n$ *E_* is v_m-torsion for each $m<n$.*

Notice that $BP_*E \cong BP_*BP \underset{BP_*}{\otimes} E_*$ is also v_m-torsion for each $m<n$. As is easily seen, the multiplications

$$1\otimes v_n\colon\ v_n^{-1}BP_*BP \underset{BP_*}{\otimes} E_* \to v_n^{-1}BP_*BP \underset{BP_*}{\otimes} E_*$$
$$v_n\otimes 1\colon\ BP_*BP \underset{BP_*}{\otimes} v_n^{-1}E_* \to BP_*BP \underset{BP_*}{\otimes} v_n^{-1}E_*$$

are isomorphisms. This means that both of the maps

$$1_\wedge v_n\colon\ v_n^{-1}BP_\wedge E \to v_n^{-1}BP_\wedge E \quad \text{and} \quad v_{n\wedge}1\colon\ BP_\wedge v_n^{-1}E \to BP_\wedge v_n^{-1}E$$

are homotopy equivalences. Hence the canonical maps

$$v_n^{-1}BP_\wedge E \to v_n^{-1}BP_\wedge v_n^{-1}E \leftarrow BP_\wedge v_n^{-1}E \tag{2.3}$$

are homotopy equivalences, too.

Proposition 2.2. *Let E be an associative BP-module spectrum whose homotopy E_* is v_m-torsion for any $m<n$. Then $L_mE=pt$ for any $m<n$, and $L_nE=v_n^{-1}E$.*

Proof. The canonical map $E\to v_n^{-1}E$ is a $v_n^{-1}BP_*$-equivalence. On the other hand, we consider the commutative diagram

$$\begin{array}{ccccc}
W & \xrightarrow{f} & v_n^{-1}E & & \\
\downarrow & & \downarrow & \searrow\!\!= & \\
BP_\wedge W & \longrightarrow & BP_\wedge v_n^{-1}E & \longrightarrow & v_n^{-1}E \\
\downarrow & & \| & & \\
v_n^{-1}BP_\wedge W & \longrightarrow & v_n^{-1}BP_\wedge v_n^{-1}E & &
\end{array}$$

for any map $f\colon W\to v_n^{-1}E$. The map f is trivial whenever W is $v_n^{-1}BP_*$-acyclic. This says that $v_n^{-1}E$ is $v_n^{-1}BP_*$-local. Therefore $L_nE=v_n^{-1}E$, and hence $L_mE=v_m^{-1}E=pt$ for any $m<n$.

Theorem 2.3. *Let E be an associative BP-module spectrum. Then the CW-spectra N_nE and M_nE are associative BP-module spectra, and moreover $M_nE=v_n^{-1}N_nE$.* (Cf., [9, Theorem 6.1]).

Proof. By induction on n we will show that N_nE is an associative BP-module spectrum whose homotopy N_nE_* is v_m-torsion for any $m<n$. By using

Proposition 2.2 the induction hypothesis implies that $M_nE=v_n^{-1}N_nE$. Hence $N_{n+1}E_*$ is clearly v_m-torsion for any $m\leqq n$. From Lemma 2.1 it follows that M_nE and $N_{n+1}E$ are associative BP-module spectra. Therefore $N_{n+1}E$ has the desired property.

Corollary 2.4. *Let E be an associative BP-module spectrum. Then* $L_nE_\wedge X=L_n(E_\wedge X)$ *and* $N_nE_\wedge X=N_n(E_\wedge X)$.

Proof. Assume that the BP-module map $N_nE_\wedge X\to N_n(E_\wedge X)$ is a homotopy equivalence. Then it follows from Theorem 2.3 that the BP-module map $M_nE_\wedge X\to M_n(E_\wedge X)$ is so, and hence the BP-module map $N_{n+1}E_\wedge X\to N_{n+1}(E_\wedge X)$ is so, too. Moreover the BP-module map $L_nE_\wedge X\to L_n(E_\wedge X)$ is also a homotopy equivalence.

Similarly we obtain

Corollary 2.5. *Let* E_λ, $\lambda\in\Lambda$, *be associative BP-module spectra. Then* $\bigvee_\lambda L_nE_\lambda=L_n(\bigvee_\lambda E_\lambda)$ *and* $\bigvee_\lambda N_nE_\lambda=N_n(\bigvee_\lambda E_\lambda)$.

Let E be an associative BP-module spectrum such that

$(\mathrm{II})_n$ *E_* is v_m-torsion for each $m>n$.*

Then $N_{n+1}E_*$ is v_m-torsion for every $m\geqq 0$. So we have

Proposition 2.6. *Let E be an associative BP-module spectrum whose homotopy E_* is v_m-torsion for any $m>n$. Then $L_\infty E=L_nE$.*

Putting Propositions 2.2 and 2.6 together we obtain

Corollary 2.7. *Let E be an associative BP-module spectrum whose homotopy E_* is v_m-torsion except for $m=n$. Then $L_\infty E=v_n^{-1}E$.*

The associative BP-module spectra $P(n)$ and $k(n)$ satisfy the condition $(\mathrm{I})_n$, and both $BP\langle n\rangle$ and $k(n)$ satisfy the condition $(\mathrm{II})_n$. So we have

$$(2.4)\qquad \begin{aligned}&L_nP(n)=v_n^{-1}P(n)=B(n)\,,\quad L_\infty k(n)=v_n^{-1}k(n)=K(n)\quad \textit{and}\\ &L_\infty BP\langle n\rangle=L_nBP\langle n\rangle\,.\end{aligned}$$

3. v_m-torsion free and v_m-divisible

Let $A=(a_0, a_1, \cdots, a_i, \cdots)$ be an infinite sequence of positive integers. Denote by BPJ_nA the associative BP-module spectrum with $BPJ_nA_*\cong BP_*/J_nA$ where $J_nA=(p^{a_0}, v_1^{a_1}, \cdots, v_{n-1}^{a_{n-1}})$. There is a cofibering

$$\Sigma^{2(p^n-1)a_n}BPJ_nA\to BPJ_nA\to BPJ_{n+1}A$$

which induces the short exact sequence $0\to BP_*/J_nA\xrightarrow{v_n^{a_n}}BP_*/J_nA\to BP_*/J_{n+1}A\to 0$

of BP_*-modules. The composite $BPJ_nA \to \Sigma^{2(p^{n-1}-1)a_{n-1}+1}BPJ_{n-1}A \to \cdots \to \Sigma^{|J_nA|+n}BP$ yields a BP-module map

$$\eta_A\colon BPJ_nA \to \Sigma^{|J_nA|}N_nBP$$

where $|J_nA| = \Sigma_{1\leq i<n}2(p^i-1)a_i$. The induced homomorphism $\eta_{A*}\colon BP_*/J_nA \to N_nBP_*$ carries 1 to $p^{-a_0}v_1^{-a_1}\cdots v_{n-1}^{-a_{n-1}}$.

For any two sequences $A=(a_0, a_1, \cdots, a_i, \cdots)$ and $A'=(a'_0, a'_1, \cdots, a'_i, \cdots)$ with $1\leqq a_i \leqq a'_i$, we write $A\leqq A'$. For such a pair $A\leqq A'$ the triangle

$$\begin{array}{ccc} BP_*/J_nA & \xrightarrow{\ \eta_{A*}\ } & \\ \big\downarrow & & N_nBP_* \\ BP_*/J_nA' & \xrightarrow[\ \eta_{A'*}\ ]{} & \end{array}$$

is commutative where the left vertical arrow is just the multiplication by $p^{b_0}v_1^{b_1}\cdots v_{n-1}^{b_{n-1}}$ with $b_i=a'_i-a_i$. So we have an isomorphism

$$\varinjlim BP_*/J_nA \to N_nBP_* \tag{3.1}$$

of BP_*-modules.

Let N be a BP_*-module. There is an exact sequence $0\to \mathrm{Tor}_n^{BP_*}(BP_*/J_nA, N) \xrightarrow{\partial_A} \mathrm{Tor}_{n-1}^{BP_*}(BP_*/J_{n-1}A, N) \xrightarrow{v_{n-1}^{a_{n-1}}} \mathrm{Tor}_{n-1}^{BP_*}(BP_*/J_{n-1}A, N)$. Hence we verify that $\mathrm{Tor}_n^{BP_*}(BP_*/J_nA, N) \cong \{x\in N;\ v_k^{a_k}x=0 \text{ for each } k<n\}$. The projection $BP_*/J_nA' \to BP_*/J_nA$ induces a homomorphism

$$\rho_{A,A'}\colon \mathrm{Tor}_n^{BP_*}(BP_*/J_nA', N) \to \mathrm{Tor}_n^{BP_*}(BP_*/J_nA, N)$$

which is just the multiplication by $p^{b_0}v_1^{b_1}\cdots v_{n-1}^{b_{n-1}}$, and the multiplication $p^{b_0}v_1^{b_1}\cdots v_{n-1}^{b_{n-1}}\colon BP_*/J_nA \to BP_*/J_nA'$ induces a homomorphism

$$\mu_{A',A}\colon \mathrm{Tor}_n^{BP_*}(BP_*/J_nA, N) \to \mathrm{Tor}_n^{BP_*}(BP_*/J_nA', N)$$

which is the inclusion. As is easily checked, we have

$$\partial_A\rho_{A,A'} = v_{n-1}^{b_{n-1}}\rho_{A,A'}\partial_{A'} \quad \text{and} \quad \partial_{A'}\mu_{A',A} = \mu_{A',A}\partial_A\,. \tag{3.2}$$

Notice that $\mathrm{Tor}_n^{BP_*}(N_nBP_*, N) \cong \{x\in N;\ x \text{ is } v_k\text{-torsion for each } k<n\}$. The BP-module map $\eta_A\colon BPJ_nA \to \Sigma^{|J_nA|}N_nBP$ yields the inclusion

$$\lambda_A\colon \mathrm{Tor}_n^{BP_*}(BP_*/J_nA, N) \to \mathrm{Tor}_n^{BP_*}(N_nBP_*, N)\,.$$

Obviously we see

$$\lambda_{A'}\mu_{A',A} = \lambda_A \quad \text{and} \quad \lambda_A\partial_A = \lambda_A\,. \tag{3.3}$$

Let E be an associative BP-module spectrum such that

$(\mathrm{III})_n$ *$BP_*/I_m \underset{BP_*}{\otimes} E_*$ is v_m-torsion free for each $m \leqq n$.*

For example, take $BP\langle n\rangle$ as E satisfying $(\mathrm{III})_n$. Given a sequence $A=(a_0, a_1, \cdots, a_i, \cdots)$ with $a_i \geqq 1$ we can show by induction on $\Sigma_{0\leqq i\leqq n} a_i \geqq n+1$ that for any $m \leqq n$,

(3.4) *$BP_*/J_m A \underset{BP_*}{\otimes} E_*$ is v_m-torsion free, and $\mathrm{Tor}_k^{BP_*}(BP_*/J_{m+1}A, E_*)=0$ for each $k \geqq 1$.*

Moreover we have an isomorphism

$$BP_*/J_{m+1}A \underset{BP_*}{\otimes} BP_*X \rightarrow BPJ_{m+1}A_*X \tag{3.5}$$

of BP_*-modules for any $m \leqq n$, when $E=BP_\wedge X$ satisfies $(\mathrm{III})_n$.

Lemma 3.1. *Let E be an associative BP-module spectrum such that $BP_*/I_m \underset{BP_*}{\otimes} E_*$ is v_m-torsion free for any $m \leqq n$. Then the BP-module map $N_{m+1}BP_\wedge E \rightarrow N_{m+1}E$ induces an isomorphism $N_{m+1}BP_* \underset{BP_*}{\otimes} E_* \rightarrow N_{m+1}E_*$ of BP_*-modules for each $m \leqq n$. And the sequence $0 \rightarrow N_m E_* \rightarrow M_m E_* \rightarrow N_{m+1}E_* \rightarrow 0$ of BP_*-modules is exact for each $m \leqq n$.*

Proof. In the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Tor}_1^{BP_*}(N_{m+1}BP_*, E_*) \rightarrow & N_m BP_* \underset{BP_*}{\otimes} E_* & \rightarrow & M_m BP_* \underset{BP_*}{\otimes} E_* & \rightarrow & N_{m+1}BP_* \underset{BP_*}{\otimes} E_* & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& N_m E_* & \longrightarrow & M_m E_* & \longrightarrow & N_{m+1}E_* &
\end{array}$$

with exact rows, we observe from (3.1) and (3.4) that $\mathrm{Tor}_1^{BP_*}(N_{m+1}BP_*, E_*)=0$. Apply induction on m to obtain our result.

Corollary 3.2. *Let E be an associative BP-module spectrum as in Lemma 3.1. Then we have an isomorphism $N_{n+1}BPJ_m A_* \underset{BP_*}{\otimes} E_* \rightarrow \mathrm{Tor}_m^{BP_*}(BP_*/J_m A, N_{n+1}E_*)$ for each $m \leqq n+1$ where $A=(a_0, a_1, \cdots, a_i, \cdots)$ with $a_i \geqq 1$.*

Proof. Proceed induction on $m \geqq 0$, the $m=0$ case being immediate from Lemma 3.1.

Lemma 3.3. *Let E be an associative BP-module spectrum such that $BP_*/I_m \underset{BP_*}{\otimes} E_*$ is v_m-torsion free for any $m \leqq n$. Then $BP_*/I_{n+1} \underset{BP_*}{\otimes} E_*=0$ if and only if $N_{n+1}E=pt$.*

Proof. If $BP_*/I_{n+1} \underset{BP_*}{\otimes} E_* = 0$, then $BP_*/J_{n+1}A \underset{BP_*}{\otimes} E_* = 0$, and hence

$N_{n+1}BP_* \underset{BP_*}{\otimes} E_*=0$. By Lemma 3.1 this means that $N_{n+1}E=pt$. On the other hand, the canonical map $BP_*/I_{n+1} \underset{BP_*}{\otimes} E_* \to N_{n+1}BP_* \underset{BP_*}{\otimes} E_*$ is monic since the map $BP_*/I_{n+1} \underset{BP_*}{\otimes} E_* \to BP_*/J_{n+1}A \underset{BP_*}{\otimes} E_*$ is so. The converse is now clear.

Proposition 3.4. *Let E be an associative BP-module spectrum such that $BP_*/I_m \underset{BP_*}{\otimes} E_*$ is v_m-torsion free for any $m \leqq n$. Then $L_0E_*=E_*\otimes Q$ and the short exact sequence $0 \to E_* \to L_mE_* \to N_{m+1}E_* \to 0$ is split as a BP_*-module for each m, $1 \leqq m \leqq n$.* (Cf., [9, Theorem 6.2]).

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \to & E_* & \to & L_{m-1}E_* & \longrightarrow & N_mE_* & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M_mE_* & = & M_mE_* & & \\
 & & & & k_m \downarrow & & \downarrow i_m & & \\
0 & \to & E_* & \longrightarrow & L_mE_* & \longrightarrow & N_{m+1}E_* & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact rows and columns. Define the splitting $\phi_m: N_{m+1}E_* \to L_mE_*$ by setting $\phi_m(z)=k_m(y)$ where $z=j_m(y)$.

Corollary 3.5. *Let E be an associative BP-module spectrum as in Proposition 3.4. Then we have an exact sequence $0 \to N_{n+1}E_* \to L_nE_* \to L_mE_* \to N_{m+1}E_* \to 0$ of BP_*-modules for each $m<n$.*

Proof. Use the fact that the composition $N_{m+2}E_* \xrightarrow{\phi_{m+1}} L_{m+1}E_* \to L_mE_*$ is trivial.

Let E be an associative BP-module spectrum such that

$(\mathrm{IV})_{n+1}$ $\mathrm{Tor}_m^{BP_*}(BP_*/I_m, E_*)$ *is v_m-divisible for each $m \leqq n$.*

For example, take $N_{n+1}BP$ as E satisfying $(\mathrm{IV})_{n+1}$. As is easily shown, it follows that for any $m \leqq n$,

(3.6) $\mathrm{Tor}_m^{BP_*}(BP_*/J_mA, E_*)$ *is v_m-divisible, and* $\mathrm{Tor}_k^{BP_*}(BP_*/J_{m+1}A, E_*)=0$ *for each $k \neq m+1$,*

where $A=(a_0, a_1, \cdots, a_i, \cdots)$ with $a_i \geqq 1$. Moreover there is an isomorphism

(3.7) $$BPJ_{m+1}A_*X \to \mathrm{Tor}_{m+1}^{BP_*}(BP_*/J_{m+1}A, BP_*X)$$

of BP_*-modules for any $m \leqq n$, when $E = BP_\wedge X$ satisfies $(\mathrm{IV})_{n+1}$.

Lemma 3.6. *Let E be an associative BP-module spectrum such that $\mathrm{Tor}_m^{BP_*}(BP_*/I_m, E_*)$ is v_m-divisible for any $m \leqq n$. Then there is an isomorphism $N_{m+1}E_* \to \mathrm{Tor}_{m+1}^{BP_*}(N_{m+1}BP_*, E_*)$ of BP_*-modules for each $m \leqq n$. And the sequence $0 \to N_{m+1}E_* \to N_m E_* \to M_m E_* \to 0$ of BP_*-modules is exact for each $m \leqq n$.*

Proof. Since $\mathrm{Tor}_m^{BP_*}(N_{m+1}BP_*, E_*) = 0$ by (3.1) and (3.6), we have a commutative diagram

$$\begin{array}{ccccccc} & N_{m+1}E_* & \longrightarrow & N_m E_* & \longrightarrow & M_m E_* & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathrm{Tor}_{m+1}^{BP_*}(N_{m+1}BP_*, E_*) & \to & \mathrm{Tor}_m^{BP_*}(N_m BP_*, E_*) & \to & \mathrm{Tor}_m^{BP_*}(M_m BP_*, E_*) & \to 0 \end{array}$$

with exact rows. Apply induction on m.

Lemma 3.7. *Let E be an associative BP-module spectrum such that $\mathrm{Tor}_m^{BP_*}(BP_*/I_m, E_*)$ is v_m-divisible for any $m \leqq n$. Then $\mathrm{Tor}_{n+1}^{BP_*}(BP_*/I_{n+1}, E_*) = 0$ if and only if $N_{n+1}E = pt$.*

Proof. If $\mathrm{Tor}_{n+1}^{BP_*}(BP_*/I_{n+1}, E_*) = 0$, then we observe that $\mathrm{Tor}_{n+1}^{BP_*}(N_{n+1}BP_*, E_*) = 0$ and hence $N_{n+1}E = pt$ by Lemma 3.6. The converse is also valid since $\mathrm{Tor}_{n+1}^{BP_*}(BP_*/I_{n+1}, E_*) \to \mathrm{Tor}_{n+1}^{BP_*}(N_{n+1}BP_*, E_*)$ is monic.

4. Harmonic spectra and s-harmonic spectra

A CW-spectrum X is said to be *harmonic* if it is $(\bigvee_n v_n^{-1}BP)_*$-local, thus if $X = L_\infty X$. X is said to be *s-harmonic* if $X = \varprojlim_n L_n X$.

We first list elementary results on harmonic spectra [3].

(4.1) *If $X \to Y \to Z$ is a cofibering and only two of X, Y and Z are harmonic, then so is the third.*

(4.2) *A retract of a harmonic spectrum is also harmonic.*

(4.3) *The product of a set of harmonic spectra is harmonic.*

(4.4) *An s-harmonic spectrum is always harmonic.*

Lemma 4.1. *Let E be an associative BP-module spectrum which is connective. Then E is harmonic if and only if so is $BP_\wedge E$.*

Proof. Recall that $E_*BP \cong E_*[t_1, \cdots, t_n, \cdots]$. Put $t^A = t_1^{a_1} \cdots t_n^{a_n}\colon \Sigma^{|A|} \to BP_\wedge BP$ for a finite sequence $A = (a_1, \cdots, a_n, 0, \cdots)$ where $|A| = \Sigma_{1 \leqq i \leqq n} 2(p^i - 1)a_i$. All the maps t^A give rise to a BP-module map $t\colon \vee \Sigma^{|A|}E \to E_\wedge BP$, which is a homotopy equivalence. Under our assumption that E is connective, $\vee \Sigma^{|A|}E =$

$\prod \Sigma^{|A|}E$. Therefore $BP_\wedge E$ is a product of suspensions of E. So our result is evident.

Lemma 4.2. *Assume that a CW-spectrum X is connective. If $BP_\wedge X$ is harmonic, then $XZ_{(p)}$ is harmonic, too.*

Proof. Let $\overline{BP}=BP/S$ be the cofiber of the unit $S\to BP$ and put $\overline{BP}^n=\overline{BP}_\wedge\cdots_\wedge\overline{BP}$, n-times. By induction on n using Lemma 4.1 we can show that $BP_\wedge\overline{BP}^n{}_\wedge X$ is harmonic. Let K_nX be the cofiber of $\Sigma^{-n}\overline{BP}^n{}_\wedge X\to X$. Then we have a cofibering $K_{n+1}X\to K_nX\to\Sigma^{-n}BP_\wedge\overline{BP}^n{}_\wedge X$. Therefore K_nX becomes harmonic for every $n\geqq 0$. When X is connective, it follows that $XZ_{(p)}=\varprojlim_n K_nX$, and hence it is harmonic.

We next discuss elementary results on s-harmonic spectra. Put $\hat{L}_\infty X=\varprojlim_n L_nX$ and $\hat{N}_\infty X=\varprojlim_n \Sigma^{-n}N_{n+1}X$.

Lemma 4.3. *A CW-spectrum X is s-harmonic if and only if $\varprojlim_n N_{n+1}X_*=0=\varprojlim_n{}^1 N_{n+1}X_*$.*

Proof. By applying Verdier's lemma [1] we see that $X=\hat{L}_\infty X$ if and only if $\hat{N}_\infty X=pt$.

Lemma 4.4. *Let $X\to Y\to Z$ be a cofibering of CW-spectra. If any two of X, Y and Z are s-harmonic, then so is the third.*

Proof. By Verdier's lemma we obtain that $\hat{N}_\infty X=\hat{N}_\infty Y$ if and only if $\hat{N}_\infty Z=pt$.

Lemma 4.5. *Let X be a retract of a CW-spectrum Y. If Y is s-harmonic, then so is X.*

Proof. The composition $\hat{N}_\infty X\to\hat{N}_\infty Y\to\hat{N}_\infty X$ is a homotopy equivalence if the composition $X\to Y\to X$ is just the identity. Hence $\hat{N}_\infty Y=pt$ implies $\hat{N}_\infty X=pt$.

Corollary 4.6. *Let E be a BP-module spectrum. Then E is s-harmonic if so is $BP_\wedge E$.*

A CW-spectrum X is said to be *dissonant* if it is $(\bigvee_n v_n^{-1}BP)_*$-acyclic.

Lemma 4.7. *Let C be the cofiber of $X\to\hat{L}_\infty X$. Then $L_\infty X$ is s-harmonic if and only if C is dissonant.*

Proof. Note that $\hat{L}_\infty(L_\infty X)=\hat{L}_\infty X$. It is easy to show that $L_\infty X=\hat{L}_\infty X$ if and only if C is dissonant.

For a BP_*-module N we define $\mathrm{w}\dim_{\mathcal{BP}} N \leqq n$ if $\mathrm{Tor}_k^{BP_*}(N, M)=0$ for all $k>n$ and all associative BP_*BP-comodules M. Notice that $\mathrm{w}\dim_{\mathcal{BP}} v_n^{-1}N \leqq n$ for any BP_*-module N [6].

Theorem 4.8. *Let E be an associative BP-module spectrum such that $\mathrm{w}\dim_{\mathcal{BP}} E_*$ is finite. Then E is s-harmonic.*

Proof. By induction on $d=\mathrm{w}\dim_{\mathcal{BP}} E_*$. We first assume that E_* is $\mathcal{BP}$-flat. By use of Lemma 3.1 we see that the sequence $0 \to N_nE_* \to M_nE_* \to N_{n+1}E_* \to 0$ are exact for all $n \geqq 0$. This implies that $\varprojlim N_{n+1}E_*=0=\varprojlim{}^1 N_{n+1}E_*$. Therefore E is s-harmonic by Lemma 4.3. Next, take a cofibering $Y \to W \to E$ which induces a short exact sequence $0 \to BP_*Y \to BP_*W \to BP_*E \to 0$ of BP_*-modules such that BP_*W is BP_*-free. Note that $\mathrm{w}\dim_{\mathcal{BP}} BP_*E=\mathrm{w}\dim_{\mathcal{BP}} E_*$. By induction hypothesis, $BP_\wedge Y$ and $BP_\wedge W$ are both s-harmonic. Hence $BP_\wedge E$ and therefore E are s-harmonic.

Combining Theorem 4.8 with (4.4) and Lemma 4.2 we have

Corollary 4.9 [9, Theorem 4.4]. *Let X be a connective CW-spectrum such that $\mathrm{w}\dim_{\mathcal{BP}} BP_*X$ is finite. Then $XZ_{(p)}$ is harmonic.*

Remark that $\mathrm{w}\dim_{\mathcal{BP}} BP_*X$ is the same as the BP_*-projective dimension of BP_*X when X is connective.

Lemma 4.10. *Let E be an associative BP-module spectrum such that $\mathrm{w}\dim_{\mathcal{BP}} E_* \leqq n$. Then $0 \to E_* \to L_nE_* \to N_{n+1}E_* \to 0$ is a short exact sequence of BP_*-modules.*

Proof. Consider the commutative square

$$\begin{array}{ccc} E_* & \longrightarrow & L_nE_* \\ \downarrow & & \downarrow \\ v_n^{-1}BP_*E & \rightarrow & v_n^{-1}BP_*L_nE \end{array}$$

where the bottom is isomorphic. Since $\mathrm{w}\dim_{\mathcal{BP}} BP_*E \leqq n$, it follows from [8, Lemma 3.4] that BP_*E is v_n-torsion free. So the left arrow is monic, and hence the top one is monic.

By using Proposition 2.2 and Lemma 4.10 together we have

Corollary 4.11. *Let E be an associative BP-module spectrum such that E_* is v_m-torsion for any $m<n$ and $\mathrm{w}\dim_{\mathcal{BP}} E_* \leqq n$. Then E_* is v_n-torsion free.* (Cf., [8, Lemma 3.4]).

Proposition 4.12. *Let $n \geqq 1$ and E be an associative BP-module spectrum such that $BP_*/I_{n+1} \otimes_{BP_*} E_* \neq 0$. Assume that $BP_*/I_m \otimes_{BP_*} E_*$ is v_m-torsion free* *for any $m \leqq n$ and E_* is v_k-torsion for any $k>n$. Then $L_\infty E$ is s-harmonic but* $\mathrm{w\,dim}_{\mathcal{BP}} L_\infty E_*$ *is infinite.*

Proof. From Proposition 2.6 it follows that $L_\infty E$ is s-harmonic and moreover that $N_{n+1}E \neq pt$ is dissonant, thus $N_{n+1}E_*$ is v_m-torsion for all $m \geqq 0$. Assume that $\mathrm{w\,dim}_{\mathcal{BP}} E_* < \infty$. Because of Lemma 3.1 it is easily checked that $\mathrm{w\,dim}_{\mathcal{BP}} N_{n+1}E_* < \infty$, which contradicts to Corollary 4.11. Therefore $\mathrm{w\,dim}_{\mathcal{BP}} E_* = \infty$, and hence also $\mathrm{w\,dim}_{\mathcal{BP}} L_\infty E_* = \infty$ by Proposition 3.4.

The $\mathcal{BP}$-weak dimensions of $P(n)_*$, $K(n)_*$ and N_nBP_* are just n, but that of $L_\infty BP\langle n\rangle_*$ is infinite when $n \geqq 1$. By Theorem 4.8 and Proposition 4.12 we obtain

(4.5) $\quad P(n),\ K(n),\ N_nBP$ *and* $L_\infty BP\langle n\rangle$ *are all s-harmonic.*

5. Cofiber of $E \to \hat{L}_\infty E = \varprojlim L_m E$

For associative BP-module spectra E_n the wedge sum $\bigvee_n E_n$ and the product $\prod_n E_n$ are both associative BP-module spectra. Denote by $\omega E_n = \prod_n E_n / \bigvee E_n$ the cofiber of the canonical map $\bigvee E_n \to \prod E_n$. This is a weak associative BP-module spectrum. We now study $\hat{L}_\infty(\bigvee E_n)$ and $\hat{L}_\infty(\prod E_n)$ for suitable BP-module spectra E_n.

Proposition 5.1. *Let E_n be associative BP-module spectra such that* $\mathrm{w\,dim}_{\mathcal{BP}} E_{n*} \leqq n$.
i) *If E_{n*} is v_m-torsion for any $m<n$, then $\hat{L}_\infty(\bigvee E_n) = \prod E_n$.*
ii) *If $\prod_{k \geqq n} E_{k*}$ is v_m-torsion for any $m<n$, then $L_\infty(\bigvee E_n) = \prod E_n$ and it is s-harmonic.*

Proof. i) Put $E = \bigvee E_m$. From Proposition 2.2 and Corollary 2.5 we observe that $L_nE = L_nE_0 \vee \cdots \vee L_nE_n$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \bigoplus_{m \leqq n} E_{m*} & \longrightarrow & L_nE_* & \longrightarrow & \bigoplus_{m \leqq n} N_{n+1}E_{m*} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bigoplus_{m \leqq n-1} E_{m*} & \to & L_{n-1}E_* & \longrightarrow & \bigoplus_{m \leqq n-1} N_nE_{m*} & \to & 0 \end{array}$$

where two rows are exact by Lemma 4.10. By induction on $n \geqq m$ we show that $\mathrm{w\,dim}_{\mathcal{BP}} N_nE_{m*} \leqq n$. Assume that $\mathrm{w\,dim}_{\mathcal{BP}} N_nE_{m*} \leqq n$, then Lemma 4.10 says that the sequence $0 \to N_nE_{m*} \to M_nE_{m*} \to N_{n+1}E_{m*} \to 0$ is exact. Since $\mathrm{w\,dim}_{\mathcal{BP}} M_nE_{m*} \leqq n$, the induction hypothesis implies that $\mathrm{w\,dim}_{\mathcal{BP}} N_{n+1}E_{m*} \leqq n+1$. Hence the right vertical arrow is trivial in the above diagram. So we obtain that $\prod E_{n*} \cong \varprojlim L_nE_*$ and $\varprojlim^1 L_nE_* = 0$. This yields that $\prod E_n = \varprojlim (E_1 \vee \cdots \vee E_n) = \varprojlim L_nE$.

ii) Note that ωE_n is clearly dissonant. Therefore $L_\infty(\vee E_n)=L_\infty(\prod E_n)=\prod E_n$, and it is s-harmonic by i) and Lemma 4.7.

Corollary 5.2. *Let E_n be associative BP-module spectra.*
i) *If E_{n*} is v_m-torsion for any $m<n$, then $\hat{L}_\infty(\vee L_nE_n)=\prod L_nE_n$.*
ii) *If $\prod_{k\geq n}E_{k*}$ is v_m-torsion for any $m<n$, then $L_\infty(\vee L_nE_n)=\prod L_nE_n$ and it is s-harmonic.*

Proof. Since $L_nE_n=v_n^{-1}E_n$ by Proposition 2.2, it satisfies the conditions stated in the above proposition.

Corollary 5.3. *Let E_n be associative BP-module spectra whose homotopy E_{n*} are v_k-torsion for any $k>n$.*
i) *If E_{n*} is v_m-torsion for any $m<n$, then $\hat{L}_\infty(\vee E_n)=\prod L_nE_n$.*
ii) *If $\prod_{k\geq n}E_{k*}$ is v_m-torsion for any $m<n$, then $L_\infty(\vee E_n)=L_\infty(\prod E_n)=\hat{L}_\infty(\prod E_n)=\prod L_nE_n$.*

Proof. i) Observe that $\hat{L}_\infty(\vee E_n)=\hat{L}_\infty(\vee L_nE_n)$ because of Proposition 2.6, then use Corollary 5.2 i).

ii) Remark that $L_\infty(\vee E_n)=L_\infty(\prod E_n)$, $L_\infty(\vee E_n)=L_\infty(\vee L_nE_n)$ and $\hat{L}_\infty(\vee E_n)=\hat{L}_\infty(\prod E_n)$. Apply Corollary 5.2 ii) and the above i) to obtain that $\hat{L}_\infty(\vee E_n)=\prod L_nE_n=L_\infty(\vee L_nE_n)$.

Applying Proposition 5.1, Corollary 5.3 and Lemma 4.7 we obtain some examples.

(5.1) *$\hat{L}_\infty(\vee N_nBP)=\prod N_nBP$ and $L_\infty(\vee N_nBP)$ is not s-harmonic.*
(5.2) *$L_\infty(\vee P(n))=\hat{L}_\infty(\vee P(n))=\prod P(n)$ and it is s-harmonic.*
(5.3) *$L_\infty(\vee K(n))=\hat{L}_\infty(\vee K(n))=\prod K(n)$ and it is s-harmonic.*
(5.4) *$L_\infty(\vee k(n))=L_\infty(\prod k(n))=\hat{L}_\infty(\vee k(n))=\hat{L}_\infty(\prod k(n))=\prod K(n)$, and it is s-harmonic.*

Proposition 5.4. *Let E_n be associative BP-module spectra such that $BP_*/I_m\underset{BP_*}{\otimes}E_{n*}$ are v_m-torsion free for any $m\leqq n$ and E_{n*} are v_k-torsion for any $k>n$. Then there is a cofibering $\vee E_n\to\hat{L}_\infty(\vee E_n)\to\prod N_{n+1}E_n$, and $\hat{L}_\infty(\prod E_n)=\prod L_nE_n$.*

Proof. Put $E=\vee E_n$. The cofibering $E\to L_mE\to N_{m+1}E$ gives us a short exact sequence $0\to E_*\to L_mE_*\to N_{m+1}E_*\to 0$. This yields that $0\to E_*\to\varprojlim L_mE_*\to\varprojlim N_{m+1}E_*\to 0$ is exact and $\varprojlim^1 L_mE_*\cong\varprojlim^1 N_{m+1}E_*$. Here we consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & N_{m+1}(\vee_{n>m}E_n)_* & \to & N_{m+1}E_* & \to & \oplus_{n\leq m}N_{n+1}E_{n*} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & N_m(\vee_{n>m-1}E_n)_* & \to & N_mE_* & \to & \oplus_{n\leq m-1}N_{n+1}E_{n*} & \to & 0 \end{array}$$

with exact rows. Since the left vertical arrow is trivial by Lemma 3.1, it is immediate that $\varprojlim N_{m+1}E_* \cong \prod N_{m+1}E_{m*}$ and $\varprojlim^1 N_{m+1}E_*=0$. Obviously the composition $E \to \hat{L}_\infty E \to \prod L_m E \to \prod N_{m+1}E \to \prod N_{m+1}E_m$ is trivial and it induces a short exact sequence $0 \to E_* \to \hat{L}_\infty E_* \to \prod N_{m+1}E_{m*} \to 0$. Hence it is easily verified that the sequence $E \to \hat{L}_\infty E \to \prod N_{m+1}E_m$ is a cofibering.

Next, put $\bar{E}=\prod E_n$. By a similar discussion to the above we can show that the sequence $\bar{E} \to \hat{L}_\infty \bar{E} \to \prod N_{m+1}E_m$ is also a cofibering, since $BP_*/I_m \underset{BP_*}{\otimes} (\prod_{k>n} E_{k*})$ is v_m-torsion free for any $m \leqq n+1$. Consider the commutative diagram

$$\begin{array}{ccccc} \bar{E} & \longrightarrow & \hat{L}_\infty \bar{E} & \longrightarrow & \prod N_{m+1}E_m \\ \downarrow & & \downarrow & & \downarrow \\ \prod \bar{E} & \longrightarrow & \prod L_m \bar{E} & \longrightarrow & \prod N_{m+1}\bar{E} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{E}=\prod E_m & \to & \prod L_m E_m & \to & \prod N_{m+1}E_m \end{array}$$

where all the rows are cofiberings. Taking the homotopy groups and using Five lemma we obtain that $\hat{L}^\infty \bar{E}=\prod L_m E_m$.

Proposition 5.5. *Let E_n be associative BP-module spectrum such that $BP_*/I_m \underset{BP_*}{\otimes} E_{n*}$ are v_m-torsion free for any $m \leqq n$. Then there is a cofibering $\vee L_n E_n \to \hat{L}_\infty(\vee L_n E_n) \to \prod N_{n+1}E_n/\vee N_{n+1}E_n$.*

Proof. Put $LE=\vee L_n E_n$ and $NE=\vee N_{n+1}E_n$. By applying Corollary 3.5 we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & N_{n+1}(\vee_{n>m}E_n)_* & \longrightarrow & LE_* & \longrightarrow & L_m LE_* & \to & N_{m+1}(\vee_{n>m}E_n)_* & \to 0 \\ & \downarrow & & \| & & \downarrow & & \downarrow & \\ 0 \to & N_{n+1}(\vee_{n>m-1}E_n)_* & \to & LE_* & \to & L_{m-1}LE_* & \to & N_m(\vee_{n>m-1}E_n)_* & \to 0 \end{array}$$

with exact rows. Then it is easily checked that the sequence $0 \to LE_* \to \varprojlim L_m LE_* \to \varprojlim^1 N_{n+1}(\vee_{n>m}E_n)_* \to 0$ is exact and $\varprojlim^1 L_m LE_*=0$, because the right arrow is trivial. Obviously the composition $LE \to \hat{L}_\infty LE \to \prod L_m LE \to \prod N_{m+1}E_m \to \omega N_{m+1}E_m$ is trivial. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & N_{n+1}(\vee_{n>m}E_n)_* & \to & LE_* & \longrightarrow & L_m LE_* & \longrightarrow & N_{m+1}(\vee_{n>m}E_n)_* & \to 0 \\ & \| & & \downarrow & & \downarrow & & & \\ 0 \to & N_{n+1}(\vee_{n>m}E_n)_* & \to & NE_* & \to & \oplus_{n \leqq m} N_{n+1}E_{n*} & \to 0 & & \end{array}$$

with exact rows. Taking the inverse limits we have the following commutative diagram

$$\begin{array}{ccccccc} 0 \to & LE_* & \longrightarrow & \hat{L}_\infty LE_* & \longrightarrow & \varprojlim^1 N_{m+1}(\vee_{m>n}E_m)_* & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ 0 \to & NE_* & \to & \prod N_{n+1}E_{n*} & \to & \varprojlim^1 N_{m+1}(\vee_{m>n}E_m)_* & \to 0 \end{array}$$

with exact rows. This means that the sequence $LE\rightarrow\hat{L}_\infty LE\rightarrow\omega N_{n+1}E_n$ induces a short exact sequence $0\rightarrow LE_*\rightarrow\hat{L}_\infty LE_*\rightarrow\prod N_{n+1}E_{n*}/\oplus N_{n+1}E_{n*}\rightarrow 0$. Therefore the sequence $LE\rightarrow\hat{L}_\infty LE\rightarrow\omega N_{n+1}E_n$ is a cofibering.

Notice that $\prod N_{n+1}BP\langle n\rangle_*$ is not v_m-torsion for every $m\geqq 0$. Combining Propositions 5.4 and 5.5 with Lemma 4.7 we have

(5.5) *$L_\infty(\vee BP\langle n\rangle)$, $L_\infty(\prod BP\langle n\rangle)$ and $L_\infty(\vee L_nBP\langle n\rangle)$ are not s-harmonic.*

We next discuss $\prod E_n/\vee E_n$ for suitable E_n.

Proposition 5.6. *Let E_n be associative BP-module spectra such that $BP_*/I_m\underset{BP_*}{\otimes}E_{n*}$ are v_m-torsion free for any $m\leqq n$. Then $\prod E_n/\vee E_n$ is s-harmonic.*

Proof. $BP_*/I_m\underset{BP_*}{\otimes}(\prod E_{n*}/\oplus E_{n*})$ is v_m-torsion free for each $m\geqq 0$, so $\prod E_{n*}/\oplus E_{n*}$ is $\mathcal{BP}$-flat. Since $BP_*\omega E_n\cong BP_*BP\underset{BP_*}{\otimes}(\prod E_{n*}/\oplus E_{n*})$ is also $\mathcal{BP}$-flat, $BP_\wedge\omega E_n$ is s-harmonic by Theorem 4.8 and hence ωE_n itself is s-harmonic by Corollary 4.6.

Combining Proposition 5.6 with Lemma 4.4 we have

Corollary 5.7. *Let E_n be associative BP-module spectra as in the above proposition. Then $L_\infty(\vee E_n)$ is s-harmonic if and only if so is $L_\infty(\prod E_n)$.*

Proposition 5.8. *Let E_n be associative BP-module spectra such that $\mathrm{Tor}_m^{BP_*}(BP_*/I_m, E_{n*})$ are v_m-divisible for any $m\leqq n$. Then $\prod E_n/\vee E_n$ is non-harmonic if $\vee N_{n+1}E_n\neq\prod N_{n+1}E_n$.*

Proof. Consider the composite map $BPI_{n+1\wedge}E_n\rightarrow\Sigma^{k_n}BP_\wedge E_n\rightarrow E_n$ where $k_n=\Sigma_{1\leqq i\leqq n}2(p^i-1)+n+1$. In the following commutative diagram

$$\begin{array}{ccccc} BPI_{n+1*}E_n & \longrightarrow & BP_*E_n & \longrightarrow & E_{n*} \\ \downarrow & & & \nearrow & \\ \mathrm{Tor}_{n+1}^{BP_*}(BP_*/I_{n+1}, BP_*E_n) & \rightarrow & \mathrm{Tor}_{n+1}^{BP_*}(BP_*/I_{n+1}, E_{n*}) & & \end{array}$$

the left vertical arrow is isomorphic by (3.7) and the bottom one is epic. Moreover the diagonal is obviously monic. So the upper composition is non-trivial if $N_{n+1}E_n\neq pt$. This shows that the induced map $\omega BPI_{n+1\wedge}E_n\rightarrow\omega E_n$ is non-trivial. Therefore ωE_n is not harmonic, because $\omega BPI_{n+1\wedge}E_n$ is dissonant.

Corollary 5.9. *Let E_n be associative BP-module spectra such that $BP_*/I_m\underset{BP_*}{\otimes}E_{n*}$ are v_m-torsion free for any $m\leqq n$. If $\prod N_{n+1}E_n/\vee N_{n+1}E_n\neq pt$, then it is non-harmonic.*

Proof. By Corollary 3.2 we observe that for each $m\leqq n$ $\mathrm{Tor}_m^{BP_*}(BP_*/I_m,$

$N_{n+1}E_{n*})\cong N_{n+1}BPI_{m*}\underset{BP_*}{\otimes}E_{n*}$ and it is v_m-divisible.

6. Harmonic but not s-harmonic spectra

Let E_n be associative BP-module spectra such that $\mathrm{Tor}_m^{BP_*}(BP_*/I_m, E_{n*})$ are v_m-divisible for any $m\leqq n$. For each $A=(a_0, a_1, \cdots, a_i, \cdots)$ with $a_i\geqq 1$ the BP-module map $\prod BPJ_{m+1}A_\wedge E_n\to\prod\Sigma^\alpha BP_\wedge E_n\to\prod\Sigma^\alpha E_n$ has a factorization

$$\prod BPJ_{m+1}A_\wedge E_n\to\Sigma^{-m-1+\alpha}N_{m+1}(\prod E_n)\to\Sigma^\alpha\prod E_n$$

where the product $\prod$ runs through all $n\geqq m+s$, $s\geqq 0$ and $\alpha=|J_{m+1}A|+m+1=\Sigma_{1\leqq i\leqq m}2(p^i-1)a_i+m+1$. Consider the commutative diagram

$$\begin{array}{ccccc}\prod BPJ_{m+1}A_*E_n & \longrightarrow & N_{m+1}(\prod E_n)_* & \longrightarrow & \prod E_{n*}\\ \downarrow & & \nwarrow & & \nearrow\\ \prod\mathrm{Tor}_{m+1}^{BP_*}(BP_*/J_{m+1}A, BP_*E_n) & \to & \prod\mathrm{Tor}_{m+1}^{BP_*}(BP_*/J_{m+1}A, E_{n*}) & & \end{array}$$

in which the left vertical arrow is isomorphic by (3.7), the bottom is epic and the diagonal is monic. Note that every homomorphism $\prod\mathrm{Tor}_{m+1}^{BP_*}(BP_*/J_{m+1}A, E_{n*})\to L_m(\prod E_n)_*$ is trivial because $\prod BPJ_{m+1}A_*E_n$ is v_k-torsion for any $k\leqq m$. So there exists a dotted arrow

$$\gamma_A:\ \prod_{n\geqq m+s}\mathrm{Tor}_{m+1}^{BP_*}(BP_*/J_{m+1}A, E_{n*})\to N_{m+1}(\prod_{n\geqq m+s}E_n)_* \tag{6.1}$$

making the square and the triangle commutative in the above diagram. As is easily seen, the triangle

$$\begin{array}{ccc}\prod\mathrm{Tor}_{m+1}^{BP_*}(BP_*/J_{m+1}A, E_{n*}) & \overset{\gamma_A}{\searrow} & \\ & & N_{m+1}(\prod E_n)_*\\ \prod\mathrm{Tor}_{m+1}^{BP_*}(BP_*/J_{m+1}A', E_{n*}) & \underset{\gamma_{A'}}{\nearrow} & \end{array}\tag{6.2}$$

is commutative for any pair $A\leqq A'$.

Lemma 6.1. *Let E_n be associative BP-module spectra such that* $\mathrm{Tor}_m^{BP_*}(BP_*/I_m, E_{n*})$ *are v_m-divisible for any $m\leqq n$. The homomorphisms γ_A induce an isomorphism*

$$\gamma:\ \varinjlim_A\prod_{n\geqq m+s}\mathrm{Tor}_{m+1}^{BP_*}(BP_*/J_{m+1}A, E_{n*})\to N_{m+1}(\prod_{n\geqq m+s}E_n)_*$$

for every $m\geqq 0$ where $s\geqq 0$.

Proof. There is a short exact sequence $0\to\mathrm{Tor}_{m+1}^{BP_*}(BP_*/J_{m+1}A, E_{n*})\to\mathrm{Tor}_m^{BP_*}(BP_*/J_mA, E_{n*})\xrightarrow{v_m^{a_m}}\mathrm{Tor}_m^{BP_*}(BP_*/J_mA, E_{n*})\to 0$ for any $n\geqq m$. So we consider the following commutative diagram

$$\begin{array}{ccc} 0 \to \varinjlim \prod \mathrm{Tor}_{m+1}^{BP_*}(BP_*/J_{m+1}A, E_{n*}) & \to & \varinjlim \prod \mathrm{Tor}_m^{BP_*}(BP_*/J_mA, E_{n*}) \\ \downarrow & & \downarrow \\ N_{m+1}(\prod E_n)_* & \longrightarrow & N_m(\prod E_n)_* \end{array}$$

$$\begin{array}{c} \to \varinjlim v_m^{-1} \prod \mathrm{Tor}_m^{BP_*}(BP_*/J_mA, E_{n*}) \to 0 \\ \downarrow \\ \longrightarrow M_m (\prod E_n)_* \end{array}$$

with exact rows, where the direct limit $\varinjlim$ runs through all sequences $A=(a_0, \cdots, a_i, \cdots, a_m, 0, \cdots)$ with $a_i \geqq 1$ and the product $\prod$ does through all $n \geqq m+s+1$. When the central arrow is isomorphic, the right is so and hence the left is also so. Therefore we can show our result by induction on m.

Lemma 6.2. *Let E_n be associative BP-module spectra such that* $\mathrm{Tor}_m^{BP_*}(BP_*/I_m, E_{n*})$ *are v_m-divisible for any $m \leqq n$. Then* $\varprojlim_m{}^1 N_{m+1}(\prod_{n \geqq m} E_n)_* \neq 0$ *if* $\bigvee_n N_{n+1}E_n \neq \prod N_{n+1}E_n$.

Proof. For all $n \geqq 0$ we may assume that $N_{n+1}E_n \neq pt$, thus $\mathrm{Tor}_{n+1}^{BP_*}(BP_*/I_{n+1}, E_{n*}) \neq 0$. Denote by $\rho_A\colon \mathrm{Tor}_m^{BP_*}(BP_*/J_mA, E_{n*}) \to \mathrm{Tor}_m^{BP_*}(BP_*/I_m, E_{n*})$ the induced homomorphism from the projection $BP_*/J_mA \to BP_*/I_m$. Clearly it is epic for each $m \leqq n+1$. Pick up an element $y_{m,n}$ in $\mathrm{Tor}_m^{BP_*}(BP_*/J_mA_m, E_{n*})$ for each $m \leqq n+1$ such that $\rho_{A_m}(y_{m,n}) \neq 0$ where $A_m=(m, \cdots, m, \cdots)$. These elements form an element $y_m=\{y_{m,n}\}_{A_m}$ in $\varinjlim \prod_{n \geqq m-1} \mathrm{Tor}_m^{BP_*}(BP_*/J_mA, E_{n*}) \cong N_m(\prod_{n \geqq m-1} E_n)_*$.

We here assume that $\varprojlim{}^1 N_m(\prod_{n \geqq m-1} E_n)_*=0$. Then there exist elements x_m in $N_m(\prod_{n \geqq m-1} E_n)_*$ such that $y_m=x_m-\delta(x_{m+1})$ where $\delta\colon N_{m+1}(\prod_{n \geqq m} E_n)_* \to N_mE_{m-1*} \oplus N_m(\prod_{n \geqq m} E_n)_*$. Notice that for every m, there is a certain sequence $A_X=(a_0, a_1, \cdots, a_i, \cdots)$ with $a_i \geqq 1$ and elements $x_{m,n}$ in $\mathrm{Tor}_m^{BP_*}(BP_*/J_mA_X, E_{n*})$ such that $x_m=\{x_{m,n}\}_{A_X}$ in $\varinjlim \prod_{n \geqq m-1} \mathrm{Tor}_m^{BP_*}(BP_*/J_mA, E_{n*})$. Using the inclusion $\lambda_A\colon \mathrm{Tor}_m^{BP_*}(BP_*/J_mA, E_{n*}) \to \mathrm{Tor}_m^{BP_*}(N_mBP_*, E_{n*}) \cong N_mE_{n*}$, we obtain the relation that $\lambda_{A_m}(y_{m,n})=\lambda_{A_X}(x_{m,n})-\lambda_{A_Y}(x_{m+1,n})$.

By induction on $n-m \geqq -1$ we will show that there exist elements $x'_{m,n}$ in $\mathrm{Tor}_m^{BP_*}(BP_*/J_mA_{n+1}, E_{n*})$ such that $\lambda_{A_{n+1}}(x'_{m,n})=\lambda_{A_X}(x_{m,n})$ and $\rho_{A_{n+1}}(x'_{m,n}) \neq 0$. First put $x'_{m,m-1}=y_{m,m-1}$ since $\lambda_{A_m}(y_{m,m-1})=\lambda_{A_X}(x_{m,m-1})$. We next suppose that there exists an element $x'_{m+1,n}$ in $\mathrm{Tor}_{m+1}^{BP_*}(BP_*/J_{m+1}A_{n+1}, E_{n*})$ such that $\lambda_{A_{n+1}}(x'_{m+1,n})=\lambda_{A_Y}(x_{m+1,n})$ and $\rho_{A_{n+1}}(x'_{m+1,n}) \neq 0$. Put $x'_{m,n}=\mu_{A_{n+1},A_m}(y_{m,n})+\partial_{A_{n+1}}(x'_{m+1,n})$, using the inclusions $\mu_{A_{n+1},A_m}\colon \mathrm{Tor}_m^{BP_*}(BP_*/J_mA_m, E_{n*}) \to \mathrm{Tor}_m^{BP_*}(BP_*/J_mA_{n+1}, E_{n*})$ and $\partial_{A_{n+1}}\colon \mathrm{Tor}_{m+1}^{BP_*}(BP_*/J_{m+1}A_{n+1}, E_{n*}) \to \mathrm{Tor}_m^{BP_*}(BP_*/J_mA_{n+1}, E_{n*})$. By use of (3.3) we see that $\lambda_{A_X}(x_{m,n})=\lambda_{A_m}(y_{m,n})+\lambda_{A_{n+1}}(x'_{m+1,n})=\lambda_{A_{n+1}}(x'_{m,n})$. Moreover it follows from (3.2) that $v_m^n \rho_{A_{n+1}}(x'_{m,n})=\partial_{A_1}\rho_{A_{n+1}}(x'_{m+1,n}) \neq 0$ in $\mathrm{Tor}_m^{BP_*}(BP_*/I_m, E_{n*})$, because $\rho_{A_{n+1}}\mu_{A_{n+1},A_m}=0$ for $n+1>m$. This says that $\rho_{A_{n+1}}(x'_{m,n}) \neq 0$.

We now set $n=\mathrm{Max}(a_0, a_1, \cdots, a_{m-1})$ for the above A_X. Then $\lambda_{A_X}(x_{m,n})=\lambda_{A_{n+1}}\mu_{A_{n+1},A_X}(x_{m,n})$ and $\lambda_{A_X}(x_{m,n})=\lambda_{A_{n+1}}(x'_{m,n})$, therefore $x'_{m,n}=\mu_{A_{n+1},A_X}(x_{m,n})$. This implies that $\rho_{A_{n+1}}(x'_{m,n})=\rho_{A_{n+1}}\mu_{A_{n+1},A_X}(x_{m,n})=0$, which is a contradiction.

At last we can state our main results.

Theorem 6.3. *Let E_n be associative BP-module spectra such that* $\mathrm{Tor}_m^{BP_*}(BP_*/I_m, E_{n*})$ *are v_m-divisible for any $m\leqq n$ and* $\mathrm{w}\dim_{\mathcal{BP}} E_{n*}\leqq n+1$. *If* $\bigvee_n N_{n+1}E_n \neq \prod_n N_{n+1}E_n$, *then*

i) $\bigvee_n E_n$ *is not harmonic, and*

ii) $\prod_n E_n$ *is harmonic, but not s-harmonic.*

Proof. In the cofibering $\vee E_n \to \prod E_n \to \omega E_n$, $\prod E_n$ is harmonic by Theorem 4.8 and (4.3). However ωE_n is not harmonic by Proposition 5.8. Hence $\vee E_n$ is not harmonic by (4.1).

Put $\bar{E}=\prod E_n$, then consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & N_{m+1}(\prod_{n\geqq m} E_n)_* & \to & N_{m+1}\bar{E}_* & \to & \oplus_{n<m} N_{m+1}E_{n*} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & N_m(\prod_{n\geqq m-1} E_n)_* & \longrightarrow & N_m\bar{E}_* & \longrightarrow & \oplus_{n<m-1} N_m E_{n*} & \to 0 \end{array}$$

with exact rows. From Lemma 3.6 it follows immediately that $\mathrm{w}\dim_{\mathcal{BP}} N_{k+1}E_{n*}\leqq n+1$ for each $k\leqq n$, in particular $\mathrm{w}\dim_{\mathcal{BP}} N_{n+1}E_{n*}\leqq n+1$. Making use of Lemma 4.10 we observe that $\mathrm{w}\dim_{\mathcal{BP}} N_m E_{n*}\leqq m$ and $0\to N_mE_{n*}\to M_mE_{n*}\to N_{m+1}E_{n*}\to 0$ is exact for every $m\geqq n+1$. Hence the right vertical arrow is trivial in the above diagram. So $\varprojlim N_{m+1}(\prod_{n\geqq m}E_n)=\varprojlim N_{m+1}\bar{E}$. However Lemma 6.2 shows that $\varprojlim N_{m+1}(\prod_{n\geqq m}E_n)\neq pt$, and hence $\bar{E}$ is not s-harmonic.

Theorem 6.4. *Let E_n be associative BP-module spectra such that* $BP_*/I_m \underset{BP_*}{\otimes} E_{n*}$ *are v_m-torsion free for any $m\leqq n$ and E_{n*} are v_k-torsion for any $k>n$.*

i) *If* $\bigvee_n N_{n+1}E_n=\prod_n N_{n+1}E_n$, *then* $\bigvee_n L_nE_n$ *and* $\prod_n L_nE_n$ *are both s-harmonic.*

ii) *If* $\bigvee_n N_{n+1}E_n \neq \prod_n N_{n+1}E_n$, *then* $\bigvee_n L_nE_n$ *is not harmonic, and* $\prod_n L_nE_n$ *is harmonic but not s-harmonic.*

Proof. i) From Proposition 5.5 it follows that $\vee L_nE_n$ is s-harmonic. Since $\omega E_n=\omega L_nE_n$ and it is s-harmonic by Proposition 5.6, $\prod L_nE_n$ is also s-harmonic.

ii) By Proposition 5.5 and Corollary 5.9 $\vee L_nE_n$ is not harmonic. Put $\overline{LE}=\prod L_nE_n$, then $N_{m+1}(\overline{LE})=N_{m+1}(\prod_{n\geqq m}L_nE_n)$. So we have a commutative diagram

$$\begin{array}{ccccc} N_{m+1}(\prod_{n\geqq m} E_n) & \to & N_{m+1}(\overline{LE}) & \to & N_{m+1}(\prod_{n\geqq m} N_{n+1}E_n) \\ \downarrow & & \downarrow & & \downarrow \\ N_m(\prod_{n\geqq m-1} E_n) & \longrightarrow & N_m(\overline{LE}) & \longrightarrow & N_m(\prod_{n\geqq m-1} N_{n+1}E_n) \end{array}$$

with cofibering rows. By use of Lemma 3.1 we see that $\varprojlim N_{m+1}(\overline{LE})=\varprojlim N_{m+1}(\prod_{n\geq m}N_{n+1}E_n)$. However Lemma 6.2 insists that $\varprojlim N_{m+1}(\prod_{n\geq m}N_{n+1}E_n)\neq pt$ because $\mathrm{Tor}_m^{BP_*}(BP_*/I_m, N_{n+1}E_{n*})\cong N_{n+1}BPI_{m*}\underset{BP_*}{\otimes}E_n^*$ is v_m-divisible for each $m\leqq n$. Therefore $\overline{LE}$ is not s-harmonic.

By applying Theorems 6.3 and 6.4 we have

(6.3) i) $\vee N_{n+1}BP$ *is not harmonic, and*
ii) $\prod N_{n+1}BP$ *is harmonic, but not s-harmonic.*

(6.4) i) $\vee L_nBP\langle n\rangle$ *is not harmonic, and*
ii) $\prod L_nBP\langle n\rangle$ *is harmonic, but not s-harmonic.*

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