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<td><strong>Author(s)</strong></td>
<td>Sasaki, Takehiko</td>
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<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 7(2) P.527-P.534</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1970</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/12688">https://doi.org/10.18910/12688</a></td>
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<tr>
<td><strong>DOI</strong></td>
<td>10.18910/12688</td>
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 Oska University
ON SOME EXTREMAL QUASICONFORMAL MAPPINGS OF DISC

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(Received June 9, 1970)

1. Introduction

A quasiconformal mapping \( w(z) \) of the unit disc \( \Delta = \{ z \mid |z| < 1 \} \) onto itself is known to have continuous boundary values, hence we may consider the class \( Q(w; \Delta, \Delta) \) of all quasiconformal mappings of \( \Delta \) onto itself that coincide with \( w(z) \) on the boundary \( \partial \Delta = \{ z \mid |z| = 1 \} \). In \( Q(w; \Delta, \Delta) \) there is at least one quasiconformal mapping whose maximal dilatation is a minimum. Such a quasiconformal mapping is called extremal in the class \( Q(w; \Delta, \Delta) \). If there exists a regular single-valued analytic function \( \varphi \) defined on \( \Delta \) and if the complex dilatation \( \mu \) of a quasiconformal mapping is written in the form

\[
\mu = k \frac{\varphi}{|\varphi|} \quad (0 < k < 1),
\]

except at zeros of \( \varphi \), then it is called a Teichmüller mapping corresponding to \( \varphi \). It was studied by K. Strebel [4] whether a quasiconformal mapping \( f(z) \) with the complex dilatation of the form (1) is extremal in the class \( Q(f; \Delta, \Delta) \) or not.

In section 2 and section 3 we prove two distortion theorems which serve to show some extremality. In section 4 some extremal quasiconformal mappings which are not Teichmüller mappings in general are considered.

2. Distortion of argument (1)

Let \( w(z) \) be a \( K \)-quasiconformal mapping which maps \( |z| < 1 \) onto \( |w| < 1 \) with \( w(0) = 0 \) and \( w(1) = 1 \) and let \( \arg w(z) = \arg w(re^{i\theta}) \) a continuous branch with \( \arg w(1) = 0 \). Then we have

**Theorem 1.** For all \( K \)-quasiconformal mappings which map \( |z| < 1 \) onto \( |w| < 1 \) with \( w(0) = 0 \) and \( w(1) = 1 \), we have

\[
\lim_{r \to 0} \frac{\arg w(r)}{\log r} \leq \frac{1}{2} \left( K - \frac{1}{K} \right).
\]
This bound is best possible.

We begin with some preliminary considerations. For \(|w(r)|\) it is well known that \(\frac{1}{4K}r^K < |w(r)| < 4r^{K-1}\). Therefore for each \(r\) we have

\[
|w(r)| = c(r)r^{\ell(r)} \text{ with } \frac{1}{4K} < c(r) < 4, \quad \frac{1}{K} \leq \ell(r) \leq K.
\] (3)

Also the next inequality is known to hold (see for example [1])

\[
\max_{0 \leq \theta < 2\pi} |w(re^{i\theta})| \leq \lambda(K) = \frac{1}{16}e^{\pi K} - \frac{1}{2} + O(e^{-\pi K}),
\]

so that we see at once

\[
\begin{align*}
\max_{0 \leq \theta < 2\pi} & |w(re^{i\theta})| \leq \lambda(K)c(r)r^{\ell(r)}, \\
\min_{0 \leq \theta < 2\pi} & |w(re^{i\theta})| \geq \lambda(K)^{-1}c(r)r^{\ell(r)}.
\end{align*}
\] (4)

In order to estimate \(\max_{0 \leq \theta < 2\pi} |\arg w(re^{i\theta}) - \arg w(r)|\), we need

**Lemma.** For \(K\)-quasiconformal mapping \(w(z)\) which maps \(|z| < 1\) onto \(|w| < 1\) with \(w(0) = 0\) we have

\[
\max_{0 \leq \theta < 2\pi} |\arg w(re^{i\theta}) - \arg w(r)| < N_1 < \infty,
\] (5)

where \(N_1\) is a constant depending only on \(K\).

**Proof.** The lemma is a consequence of the following

**Theorem** ([2], theorem 2). Suppose that \(w\) is a \(K\)-quasiconformal mapping of the extended plane and that \(w(\infty) = \infty\). Then for each triple of distinct finite points \(z_1, z_0, z_2\)

\[
\sin \frac{1}{2} \beta \geq \varphi_K \left( \sin \frac{1}{2} \alpha \right),
\]

where \(\varphi_K(r) = \mu^{-1}(K\mu(r))\) and

\[
\alpha = \arcsin \left( \frac{|z_1 - z_2|}{|z_1 - z_0| + |z_2 - z_0|} \right), \quad \beta = \arcsin \left( \frac{|f(z_1) - f(z_2)|}{|f(z_1) - f(z_0)| + |f(z_2) - f(z_0)|} \right).
\]

Here \(\mu(r)\) is the modulus of the unit disc slit along the real axis from 0 to \(r\), and \(\mu^{-1}\) is the inverse of \(\mu\).

**Proof of lemma.** The mapping \(w(z)\) in our lemma is extended by reflexion
to a $K$-quasiconformal mapping of the extended plane with $w(\infty) = \infty$, and we apply the above theorem to the inverse $w^{-1}$.

Suppose that $w$ satisfies the inequalities:

$$n\pi \leq \max_{0 \leq \theta < 2\pi} |\arg w(re^{i\theta}) - \arg w(r)| < (n+1)\pi,$$

where $n$ is a positive integer. Then there exists at least $2n-1$ points such that $0=\theta_1 < \theta_2 < \cdots < \theta_{2n-1} = 2\pi$ and that $|\arg w(re^{i\theta_1}) - \arg w(re^{i\theta_2})| = |\arg w(re^{i\theta_{2n-1}}) - \arg w(re^{i\theta_1})| = \cdots = |\arg w(re^{i\theta_{2n-2}}) - \arg w(re^{i\theta_{2n-1}})| = \pi$. If we put $w(re^{i\theta_1})$, $w(0)$, $w(re^{i\theta_{2n-1}})$ in place of $z_1$, $z_0$, $z_2$ in (*) respectively and $z_v = re^{i\theta_v}$, $z_v = 0$, $z_{v+1} = re^{i\theta_{v+1}}$ in place of $f(z_1)$, $f(z_0)$, $f(z_2)$ in (*) respectively, ($v = 1, 2, \ldots, 2n-2$), then

$$\alpha = \arcsin \left( \frac{|w(re^{i\theta}) - w(re^{i\theta_{v+1}})|}{|w(re^{i\theta}) - w(0)| + |w(re^{i\theta_{v+1}}) - w(0)|} \right) = \arcsin 1 = \frac{\pi}{2},$$

$$\beta = \arcsin \left( \frac{|re^{i\theta_v} - re^{i\theta_{v+1}}|}{|re^{i\theta_v} - 0| + |re^{i\theta_{v+1}} - 0|} \right) = \arcsin \frac{1}{2} |1 - e^{i\theta_{v+1} - \theta_v}|$$

and

$$\sin \frac{\theta_{v+1} - \theta_v}{4} \geq \varphi_K \left( \sin \frac{\pi}{4} \right) = \varphi_K \left( \frac{1}{\sqrt{2}} \right), \quad (v = 1, 2, \ldots, 2n-2).$$

Therefore we have

$$\theta_{v+1} - \theta_v \geq 4 \arcsin \varphi_K \left( \frac{1}{\sqrt{2}} \right), \quad (v = 1, 2, \ldots, 2n-2).$$

On adding these $2n-2$ inequalities we obtain

$$2\pi \geq 4(2n-2) \arcsin \varphi_K \left( \frac{1}{\sqrt{2}} \right)$$

or

$$n+1 \leq \frac{\pi}{4 \arcsin \varphi_K \left( \frac{1}{\sqrt{2}} \right)} + 2.$$

Putting $N_1 = \left( \frac{\pi}{4 \arcsin \varphi_K \left( \frac{1}{\sqrt{2}} \right)} + 2 \right)\pi$ we have

$$\max_{0 \leq \theta < 2\pi} |\arg w(re^{i\theta}) - \arg w(r)| < N_1,$$

thus the lemma is proved.

Proof of theorem 1. By $Z = \log z$ and $W = \log w$ we map $|z| < 1$ and $|w| < 1$
conformally onto half strips of height $2\pi$ such that $Z(1)=\log 1=0$ and $W(1)=\log 1=0$, in the $Z$ and $W$ planes. Then a quadrilateral $q=\{z\mid r<|z|<1, 0<\arg z<2\pi\}$ in the $z$ plane will be mapped onto a quadrilateral $Q=\{Z\mid X<0, 0<Y<2\pi\}$ in the $Z=X+iY$ plane. Let $Q'$ be the quadrilateral in the $W$ plane to which $Q$ corresponds, and let $M(Q)$ and $M(Q')$ denote the moduli of $Q$ and $Q'$. Then

$$
\frac{1}{K} M(Q) \leq M(Q') \leq K M(Q), \quad M(Q) = \frac{1}{2\pi} \log \frac{1}{r}.
$$

On the other hand, applying Rengel's inequality ([3], p. 24) to $Q'$, we have

$$
\frac{(S_a(Q'))^2}{m(Q')} \leq M(Q') \leq \frac{m(Q')}{(S_a(Q'))^2},
$$

where $S_a(Q')$ and $S_b(Q')$ denote the distances between a-sides and b-sides in $Q'$ respectively and $m(Q')$ the area of $Q'$.

The lemma and (4) imply that left one of b-sides in $Q'$ must lie in $\{W=U+iV\mid U\leq \log \lambda(K)c(r)r^{f(r)}, \mid V\mid > |\arg w(r)|-N_1\}$ when $|\arg w(r)|>N_1$. Put $N_2=N_1+2\pi \sgn \arg w(r)$, when $|\arg w(r)|>N_1+2\pi$. Then it follows by use of the Pythagoras equality that

$$
(S_a(Q'))^2 \geq |\arg w(r)-N_2|^2 + \log \lambda(K)c(r)r^{f(r)} |^2.
$$

For $m(Q')$ we have by (4)

$$
m(Q') \leq 2\pi |\log \lambda(K)^{-1}c(r)r^{f(r)}|.
$$

On using these inequalities, (7) becomes

$$
\frac{|\arg w(r)-N_2|^2 + \log \lambda(K)c(r)r^{f(r)}|^2}{2\pi |\log \lambda(K)^{-1}c(r)r^{f(r)}|} \leq M(Q').
$$

If we combine above inequality with (6), it holds that

$$
|\arg w(r)-N_2|^2 \leq f(r)(K-f(r))|\log r|^2 + (2|\log \lambda(K)c(r)|f(r)
+ K|\log \lambda(K)^{-1}c(r)| |\log r| + |\log \lambda(K)c(r)|^2.
$$

Dividing both sides by $|\log r|^2$ and letting $r$ tend to 0, we have

$$
\lim_{r \to 0} \frac{\arg w(r)}{\log r} \leq \lim_{r \to 0} \sqrt{f(r)(K-f(r))} \leq \lim_{r \to 0} \frac{K}{2} = \frac{K}{2}. \quad (8)
$$

Next, we suppose that

$$
\lim_{r \to 0} \frac{\arg w(r)}{\log r} = \frac{1}{2} \left( K - \frac{1}{K} + \delta \right), \quad \delta \geq 0.
$$
We introduce an auxiliary \( K_{1}\)-quasiconformal mapping \( w_{1}(r e^{i\theta}) = r^{\alpha} e^{i(\gamma - \beta \log r)} \), where \( \beta^2 = -\alpha^2 + \left( K_{1} + \frac{1}{K_{1}} \right) \alpha - 1, \frac{1}{K_{1}} \leq \alpha \leq K_{1} \) and \( \text{sgn} \beta = \pm 1 \) according as

\[
\lim_{r \to 0} \frac{\arg w(r)}{\log r} = \frac{1}{2} \left( K_{1} - \frac{1}{K_{1}} + \frac{1}{K} - \frac{\delta}{K} \right) \quad \text{or} \quad \lim_{r \to 0} \frac{\arg w(r)}{\log r} = -\frac{1}{2} \left( K_{1} - \frac{1}{K_{1}} + \delta \right).
\]

The composed \( KK_{1}\)-quasiconformal mapping \( w \circ w_{1} \) maps \( |z| < 1 \) onto \( |w| < 1 \) so that \( w \circ w_{1}(0) = 0 \) and \( w \circ w_{1}(1) = 1 \). On account of (8) it satisfies

\[
\lim_{r \to 0} \frac{\arg w \circ w_{1}(r)}{\log r} \leq \frac{KK_{1}}{2}.
\]

By (5) its argument is written in the form \( \arg w \circ w_{1}(r) = \arg w(r) + \arg w_{1}(r) + A(r) \), where \( A(r) \) satisfies \( |A(r)| < N_{1} \). There is a sequence \( \{r_{i}\} \) such that

\[
\frac{KK_{1}}{2} \geq \lim_{i \to \infty} \left| \frac{\arg w(r_{i}) + \arg w_{1}(r_{i}) + A(r_{i})}{\log r_{i}} \right| = \lim_{r \to 0} \left| \frac{\arg w(r)}{\log r} \right| + |\beta| = \frac{\alpha}{2} \left( K_{1} - \frac{1}{K_{1}} + \frac{1}{K} - \frac{\delta}{K} \right) + |\beta|
\]

or

\[
\delta \leq \frac{KK_{1}}{\alpha} \frac{2|\beta|}{\alpha} - K_{1} + \frac{1}{K}.
\]

If we put \( \alpha = \frac{K_{2}^{2}}{K_{2}^{2} + 1} K_{1}\) (\( \alpha \geq \frac{1}{K_{1}} \)) then

\[
\frac{KK_{1}}{\alpha} = K_{1} + \frac{1}{K}, \quad \frac{2|\beta|}{\alpha} = \frac{2(2K^{2} + 1)}{K^{2}} \sqrt{\left( \frac{K}{K^{2} + 1} \right)^{2} + \left( \frac{K^{2}}{K^{2} + 1} - 1 \right)^{2}}.
\]

Letting \( K_{1} \to \infty \) we have

\[
\delta \leq K + \frac{1}{K} - \frac{2(K^{2} + 1)}{K^{2}} \frac{K}{K^{2} + 1} - K + \frac{1}{K} = 0
\]

Hence

\[
\lim_{r \to 0} \left| \frac{\arg w(r)}{\log r} \right| \leq \frac{1}{2} \left( K_{1} - \frac{1}{K} \right).
\]

The bound \( \frac{1}{2} \left( K_{1} - \frac{1}{K} \right) \) is attained by \( w(0) = r^{\alpha} e^{i(\gamma - \beta \log r)} \), where \( \alpha = \frac{1}{2} \left( K_{1} - \frac{1}{K} \right) \) and \( \beta = \frac{1}{2} \left( K_{1} - \frac{1}{K} \right) \).

q. e. d.
3. Distortion of argument (2)

Theorem 2. Suppose that \( w(z) \) is a \( K \)-quasiconformal mapping of \( |z| < 1 \) onto \( |w| < 1 \) with \( w(0) = 0 \) and \( w(1) = 1 \) and that

\[
|w(r)| = c(r)r^a, \quad \log c(r) = o(\log r). \tag{9}
\]

Then

\[
\lim_{r \to 0} \left| \frac{\arg w(r)}{\log r} \right| \leq \sqrt{a\left(\frac{K+1}{K}\right)-(a^2+1)}. \tag{10}
\]

For each \( a \) this bound is best possible.

Proof. We remark first that \( a \) lies between \( \frac{1}{K} \) and \( K \). We use the same reasoning as in section 2 to obtain

\[
|w(r)| = c(r)r^a, \quad \log c(r) = o(\log r) \tag{3'}
\]

\[
\max_{0 \leq \theta < 2\pi} |w(re^{i\theta})| \leq \lambda(K)c(r)r^a, \quad \log c(r) = o(\log r) \tag{4'}
\]

\[
\min_{0 \leq \theta < 2\pi} |w(re^{i\theta})| \geq \lambda(K)^{-1}c(r)r^a, \quad \log c(r) = o(\log r). \tag{8'}
\]

Putting \( f(r) = a \) in (8) we have

\[
\lim_{r \to 0} \left| \frac{\arg w(r)}{\log r} \right| \leq \sqrt{a(K-a)}. \tag{8'}
\]

We suppose that

\[
\lim_{r \to 0} \left| \frac{\arg w(r)}{\log r} \right| = \sqrt{a\left(\frac{K+1}{K}\right)-(a^2+1)+\delta}, \quad \delta \geq 0.
\]

Using an auxiliary \( K_1 \)-quasiconformal mapping \( \phi(r) = r^\beta e^{i(\alpha \log r)} \), we have in the same manner as in section 2

\[
\alpha \left(\sqrt{a\left(\frac{K+1}{K}\right)-(a^2+1)+\delta}\right) + \sqrt{\alpha \left(\frac{K_1+1}{K_1}\right)-(\alpha^2+1)} \leq \sqrt{a\alpha(KK_1-a\alpha)},
\]

because the composed \( KK_1 \)-quasiconformal mapping satisfies (9) with \( a\alpha \) in place of \( a \). Therefore we have

\[
\delta \leq \sqrt{a\left(\frac{K}{\alpha}K_1-a\right)} - \sqrt{\frac{K_1+1}{\alpha K} - \left(1+\frac{1}{\alpha^2}\right)} - \sqrt{a\left(\frac{K+1}{K}\right)-(a^2+1)} \tag{11'}
\]

For \( a = \frac{1}{K} \) we put \( \alpha = K^{-1} \), and let \( K_1 \to \infty \), so that we have

\[
\delta \leq \sqrt{\left(\frac{aK(K-a)}{aK-1}\right)} - \sqrt{\frac{1-aK^{-1}}{aK-1}} - \sqrt{a\left(\frac{K+1}{K}\right)-(a^2+1)} = 0.
\]
Hence we have the desired inequality (10). For \( a = \frac{1}{K} \) (11) becomes
\[
\delta \leq \sqrt{\frac{K_1}{\alpha} - 1} - \sqrt{\frac{K_1}{\alpha} + \frac{1}{\alpha K_1} - \left(1 + \frac{1}{\alpha^2}\right)}.
\]
(11')

On putting \( \alpha = \sqrt{K_1} \) and letting \( K_1 \to \infty \), we have also \( \delta = 0 \). For each \( a \) the bound is attained by \( w(re^{i\theta}) = r^a e^{i(a - b \log r)}, b^2 = a \left( K + \frac{1}{K} \right) - (a^2 + 1). \) q. e. d.

4. Some extremal mappings

Let \( w(z) \) be an extremal quasiconformal mapping in \( Q(w; \Delta, \Delta) \). If we map \( \Delta \) conformally onto a Jordan region \( D \) by \( \varphi \) and map \( \Delta \) onto another Jordan region \( D' \) by \( \psi \), then we obtain a boundary correspondence given by \( \psi \circ w \circ \varphi^{-1} \) between \( D \) and \( D' \). In \( Q(\psi \circ w \circ \varphi^{-1}; D, D') \), \( \psi \circ w \circ \varphi^{-1} \) is again extremal and if \( W \) is extremal in \( Q(\psi \circ w \circ \varphi^{-1}; D, D') \) then \( \varphi^{-1} \circ W \circ \psi \) is extremal in \( Q(w; \Delta, \Delta) \). By this reason extremal quasiconformal mappings which we are going to deal with are those which have the boundary correspondence between \( D \) and \( D' \).

**Theorem 3.** Let \( W_0(Z) = f(X) + i(Y + g(X)) \) be a \( K \)-quasiconformal mapping of \( D = \{ Z = X + iY \mid X < 0, 0 < Y < 2\pi \} \) such that \( f(0) = g(0) = 0 \) and that \( f(X) \to -\infty \) as \( X \to -\infty \). If
\[
\lim_{X \to -\infty} \frac{|g(X)|}{X} = \frac{1}{2} \left( K - \frac{1}{K} \right).
\]
then \( W_0(Z) \) is extremal in \( Q(W_0; D, D') \), where \( D' = W_0(D) \).

Proof. Let \( W(Z) \) be a \( K' \)-quasiconformal mapping in \( Q(W_0; D, D') \). We map \( D \) into \( \Delta \) by \( z = e^z \) and \( D' \) into another \( \Delta \). Then \( e^{W(\log r)} \) with \( \log 1 = 0 \) is a \( K' \)-quasiconformal mapping of \( \Delta \) onto \( \Delta \) with \( e^{W(\log 0)} = 0 \) and \( e^{W(\log 1)} = 1 \), because it is topological on \( \Delta \) and \( K' \)-quasiconformal in \( \Delta - \{ z = x + iy \mid 0 \leq x < 1, y = 0 \} \), ([3], I. Satz 8, 3). Theorem 1 asserts that
\[
\frac{1}{2} \left( K' - \frac{1}{K'} \right) \geq \lim_{r \to 0} \left| \arg e^{W(\log r)} \right| = \lim_{r \to 0} \left| \arg e^{W_0(\log r)} \right| = \lim_{X \to -\infty} \left| \frac{g(X)}{X} \right| = \frac{1}{2} \left( K - \frac{1}{K} \right).
\]
Therefore we have \( K' \geq K \) and it is shown that \( W_0(Z) \) is extremal in \( Q(W_0; D, D') \). q. e. d.

The complex dilatation of \( W_0(Z) \) is of the form as follows;
\[
\mu_{W_0}(Z) = \frac{f'(X) + ig'(X) - 1}{f'(X) + ig'(X) + 1}.
\]

If \( \mu_{W_0}(Z) = k \frac{\varphi(Z)}{|\varphi(Z)|} \), with analytic \( \varphi \) in \( D \), then \( \arg \varphi = \text{constant on each } X = c, -\infty < c < 0, \) and \( \varphi = e^{az+b} \). But it is not difficult to see that \( a = 0 \).

We conclude that \( W_0(Z) \) is not Teichmuller mapping except the case when \( W_0(Z) \) is an affine mapping.

By the same reasoning as before and by theorem 2 we obtain the following

**Theorem 4.** Let \( W_a(Z) = f(X) + i(Y + g(X)) \) be \( K \)-quasiconformal mapping of \( D = \{ Z = X + iY | X < 0, 0 < Y < 2\pi \} \) such that \( f(0) = g(0) = 0, f(X) \to \infty \) as \( X \to -\infty \) and that \( \lim_{x \to -\infty} \left| \frac{f(X)}{X} \right| = a, \frac{1}{K} \leq a \leq K \). If

\[
\lim_{x \to -\infty} \left| \frac{g(X)}{X} \right| = \sqrt{\frac{a(K+1)}{K}}(a^2+1),
\]

then \( W_a(Z) \) is extremal in \( Q(W_a; D, D') \), where \( D' = W_a(D) \).

**References**


