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EINSTEIN METRICS ON BOGGINO-DAMEK-RICCI TYPE SOLVABLE LIE GROUPS

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1. Introduction

A Riemannian manifold (M, g) is called an Einstein manifold if its Ricci tensor ric_g satisfies $\mathrm{ric}_g = cg$ for some constant c. This paper deals with noncompact homogeneous Einstein manifolds. All known examples of nonflat noncompact homogeneous Einstein manifolds are isometric to solvable Lie groups endowed with left invariant Einstein metrics. It has been conjectured by D.V. Alekseevskii that every noncompact homogeneous Einstein manifold has maximal compact isotropy subgroups. This conjecture implies that the classification of noncompact homogeneous Einstein manifolds is reduced to the investigation of solvable Lie groups with left invariant Einstein metrics. The conjecture is still an open problem.

The purpose of this paper is to construct a class of noncompact homogeneous Einstein manifolds, which we call Boggino-Damek-Ricci type Einstein spaces (abbreviated to BDR-type Einstein spaces). Each element of this class is represented as a simply connected solvable Lie group with a left invariant metric. In 1985 J. Boggino constructed a class of Einstein manifolds with nonpositive sectional curvature which includes rank one symmetric spaces of noncompact type ([3]). These spaces are now called Damek-Ricci Einstein spaces ([2]). The class of BDR-type Einstein spaces is constructed as a 1-dimensional solvable extension of a 2-step nilpotent Lie algebra and contains Damek-Ricci Einstein spaces. Note that Damek-Ricci Einstein space has negative sectional curvature if and only if it is symmetric space ([3], [9]). In this paper we prove that there exist nonsymmetric BDR-type Einstein spaces with negative sectional curvature.

In Section 2 we define BDR-type spaces and investigate curvature property and Einstein condition of the BDR-type spaces. Using the Kaplan's J ([7]), we give formulas for curvature and Ricci transformation of BDR-type spaces (Lemma 2.1, 2.2). From Lemma 2.2 we see that the Einstein condition is reduced to the condition of the nilpotent part of BDR-type spaces (Proposition 2.3). We also give a sufficient condition that BDR-type space has nonpositive sectional curvature in Proposition 2.5. A Damek-Ricci Einstein space satisfies the condition of Proposition 2.5 and thus this gives another proof of the fact that it has nonpositive sectional curvature.

In Section 3 we construct BDR-type Einstein spaces which are not Damek-Ricci

Einstein spaces. We define a class of BDR-type Lie algebras whose nilpotent parts are obtained by a decomposition of isometry groups of Kähler C-spaces. By computing the Ricci transformations of 2-step nilpotent algebras of the above type we construct BDR-type Einstein spaces. Note that BDR-type Einstein spaces may not always have nonpositive sectional curvature. In Theorem 3.2 we prove that there exist nonsymmetric BDR-type Einstein spaces with negative sectional curvature.

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2. Boggino-Damek-Ricci type spaces

2.1. BDR-type spaces Let $\mathfrak{n}$ denote a finite dimensional Lie algebra over $\mathbb{R}$. For each integer $i \geq 1$ we define $\mathfrak{n}_{(i)}$, an ideal of $\mathfrak{n}$, by $\mathfrak{n}_{(i)} = [\mathfrak{n}, \mathfrak{n}_{(i-1)}]$, where $\mathfrak{n}_{(0)} = \mathfrak{n}$. The Lie algebra $\mathfrak{n}$ is *n-step nilpotent* if $\mathfrak{n}_{(n)} = \{0\}$ and $\mathfrak{n}_{(n-1)} \neq \{0\}$.

Let $(\mathfrak{n}, \langle\ ,\ \rangle_{\mathfrak{n}})$ be a 2-step nilpotent Lie algebra with a positive definite inner product, $\mathfrak{a}$ a one-dimensional real vector space and A a non-zero vector in $\mathfrak{a}$. Let $\mathfrak{z}$ denote the center of $\mathfrak{n}$ and let $\mathfrak{v}$ denote the orthogonal compliment of $\mathfrak{z}$ in $\mathfrak{n}$.

We define the $\mathbb{R}$-linear map $f : \mathfrak{a} \to \mathrm{End}(\mathfrak{n})$ by

$$f(A)X = \frac{k}{2}X, \quad f(A)Z = kZ \quad \text{for} \quad X \in \mathfrak{v}, Z \in \mathfrak{z}$$

where k is a positive constant. Since the endomorphism $f(A)$ is derivation of $\mathfrak{n}$, the semi-product $\mathfrak{s} = \mathfrak{n} \times_f \mathfrak{a}$ becomes a solvable Lie algebra whose derived subalgebra is $\mathfrak{n}$.

We define an inner product $\langle\ ,\ \rangle_{\mathfrak{a}}$ on $\mathfrak{a}$ and $\langle\ ,\ \rangle$ on $\mathfrak{s}$ by

$$\langle A, A\rangle_{\mathfrak{a}} = 1, \quad \langle\ ,\ \rangle = \langle\ ,\ \rangle_{\mathfrak{n}} \oplus \langle\ ,\ \rangle_{\mathfrak{a}}.$$

In this way, $(\mathfrak{s}, \langle , \rangle)$ becomes a Lie algebra with a positive definite inner product and is called *Boggino-Damek-Ricci type Lie algebra* (abbreviated to BDR-type Lie algebra).

DEFINITION 2.1. For a BDR-type Lie algebra $(\mathfrak{s}, \langle , \rangle)$, the corresponding connected and simply connected Lie group S with the induced left-invariant Riemannian metric g is called a Boggino-Damek-Ricci type space.

We compute the Levi-Civita connection ∇, the sectional curvature K and the Ricci transformation Ric of the Boggino-Damek-Ricci type space (S, g) in terms of the metric Lie algebra $(\mathfrak{s}, \langle\ ,\ \rangle)$.

Let (G, g) be a simply connected Lie group with a left-invariant metric g and $(\mathfrak{g}, \langle\ ,\ \rangle)$ be the associated metric Lie algebra. Regarding X, Y and Z in $\mathfrak{g}$ as left invariant vector fields on G we obtain the following formulas:

$$2\langle\nabla_X Y, Z\rangle = \langle[X, Y], Z\rangle - \langle[Y, Z], X\rangle - \langle[X, Z], Y\rangle \tag{2.1}$$

$$K(X \wedge Y) = |\nabla_X Y|^2 - \langle\nabla_X X, \nabla_Y Y\rangle - \langle {ad_Y}^2 X, X\rangle - |[X, Y]|^2, \tag{2.2}$$

where $K(X \wedge Y) = \langle R(X,Y)Y, X\rangle = \langle(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Y, X\rangle$.

Ricci tensor is defined by $\mathrm{ric}(X,Y) = \mathrm{trace}(Z \to R(Z,X)Y)$ for all X, Y in $\mathfrak{g}$. The Ricci transformation $\mathrm{Ric} : \mathfrak{g} \to \mathfrak{g}$ is symmetric (1, 1)-tensor induced by the Ricci tensor. Let $(E_i)_{i\in I}$ is an orthonormal basis of $\mathfrak{g}$ with respect to $\langle\ ,\ \rangle$, then Ric is given by

$$\mathrm{Ric} = \frac{1}{4}\sum_{i\in I} ad_{E_i} \circ ad_{E_i}{}^* - \frac{1}{2}\sum_{i\in I} ad_{E_i}{}^* \circ ad_{E_i} - \frac{1}{2}B - ad_\sigma{}^s, \tag{2.3}$$

where $(\ )^*$ is the adjoint operator with respect to $\langle\ ,\ \rangle$, σ is the vector dual to the 1-form $X \to \mathrm{tr}(ad_X)$, $(\ )^s$ is symmetrizer, i.e. $ad_\sigma{}^s = (ad_\sigma + ad_\sigma{}^*)/2$, and B is the Cartan-Killing operator, i.e. $\langle B(X), Y\rangle = \mathrm{tr}(ad_X \circ ad_Y)$ (cf. [1]).

We are primarily interested in the case that the Lie group is Boggino-Damek-Ricci type space. From now on, let (S, g) be a Boggino-Damek-Ricci type space (abbreviated to BDR-type space), $(\mathfrak{s}, \langle,\rangle)$ be the corresponding metric Lie algebra and $(\mathfrak{n}, \langle,\rangle_{\mathfrak{n}})$ be the derived subalgebra of $\mathfrak{s}$. In our case ad_A, which is defined by $f(A)$, is symmetric, hence formula (2.1) yields the following expression for the Levi-Civita connection ∇.

$$\begin{aligned} \nabla_A &= 0 \\ \nabla_X A &= -ad_A(X) \quad \text{for} \quad X \in \mathfrak{n} \\ \nabla_X Y &= \nabla^{\mathfrak{n}}{}_X Y + \langle ad_A(X), Y\rangle A \quad \text{for} \quad X, Y \in \mathfrak{n}, \end{aligned} \tag{2.4}$$

where $\nabla^{\mathfrak{n}}$ denotes the Levi-Civita connection of $(\mathfrak{n}, \langle\ ,\ \rangle_{\mathfrak{n}})$.

For 2-step nilpotent Lie algebra $\mathfrak{n}$, we define the $\mathbb{R}$-algebra homomorphism $J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ introduced by A. Kaplan in [7]. For each element $Z \in \mathfrak{z}$ we define a skew symmetric linear transformation $J(Z) : \mathfrak{v} \to \mathfrak{v}$ by

$$J(Z)X = (ad_X)^{*(\mathfrak{n})}Z \quad \text{for} \quad X \in \mathfrak{v},$$

where $(\,)^{*(\mathfrak{n})}$ denotes the adjoint operator with respect to $\langle,\rangle_{\mathfrak{n}}$. Notice that the transformations $\{J(Z) \mid Z \in \mathfrak{z}\}$ characterize the Lie algebra structure of metric 2-step nilpotent Lie algebra.

Now we can express the connection ∇ using this operator J. we obtain

$$\begin{aligned} \nabla_{V_1+Z_1}(V_2 + Z_2 + aA) = &-\frac{1}{2}J(Z_2)V_1 - \frac{1}{2}J(Z_1)V_2 + \frac{1}{2}[V_1, V_2] \\ &+ \frac{k}{2}\{\langle V_1, V_2\rangle + 2\langle Z_1, Z_2\rangle\}A - \frac{ka}{2}(V_1 + 2Z_1) \end{aligned}$$

for $V_1, V_2 \in \mathfrak{v}$, $Z_1, Z_2 \in \mathfrak{z}$, $a \in \mathbb{R}$. By a straightforward computation using (2.2) we get the following lemma.

Lemma 2.1. *For $a \in \mathbb{R}$, $X = V_1 + Z_1 + aA$ and $Y = V_2 + Z_2$,*

$$\begin{aligned}
K(X \wedge Y) &= \langle R(X, Y)Y, X\rangle \\
&= \frac{1}{4}|J(Z_1)V_2 + J(Z_2)V_1|^2 - \langle J(Z_1)V_1, J(Z_2)V_2\rangle \\
&\quad + \frac{k^2}{4}(\langle V_1, V_2\rangle + 2\langle Z_1, Z_2\rangle)^2 - \frac{k^2}{4}(|V_1|^2 + 2|Z_1|^2)(|V_2|^2 + 2|Z_2|^2) \\
&\quad - \frac{3}{4}|[V_1, V_2] + akZ_2|^2 - \frac{a^2k^2}{4}(|V_2|^2 + |Z_2|^2).
\end{aligned}$$

Next we determine the Ricci transformation of BDR-type spaces. Using the unimodularity of $\mathfrak{n}$ we obtain the following:

Lemma 2.2. *If $\{Z_1, \dots, Z_m\}$ is an orthonormal basis of $\mathfrak{z}$ and $\{V_1, \dots, V_n\}$ is an orthonormal basis of $\mathfrak{v}$, then*

(a) $$\mathrm{Ric}(A) = -k^2\left(\frac{n}{4} + m\right)A,$$

(b) $$\mathrm{Ric}(V) = \frac{1}{2}\sum_{j=1}^{m} J(Z_j)^2V - \frac{k^2}{2}\left(\frac{n}{2} + m\right)V \quad \textit{for } V \in \mathfrak{v},$$

(c) $$\mathrm{Ric}(Z) = \frac{1}{4}\sum_{i=1}^{n}[V_i, J(Z)V_i] - k^2\left(\frac{n}{2} + m\right)Z \quad \textit{for } Z \in \mathfrak{z}.$$

In particular Ric *leaves $\mathfrak{a}$, $\mathfrak{v}$ and $\mathfrak{z}$ invariant.*

Proof. Since $R(W, A)A = -{ad_A}^2W$ for $W \in \mathfrak{s}$, we obtain

$$R(V_i, A)A = -\frac{k^2}{4}V_i \quad \text{for} \quad i = 1, \dots, n, \quad R(Z_j, A)A = -k^2Z_j \quad \text{for} \quad j = 1, \dots, m,$$

and (a) follows immediately. By third equation of (2.4) we have

(2.5) $$\mathrm{Ric}(X) = \mathrm{Ric}^{\mathfrak{n}}(X) - \mathrm{tr}(ad_A)\cdot ad_AX \qquad \text{for} \quad X \in \mathfrak{n},$$

where $\mathrm{Ric}^{\mathfrak{n}}$ denotes the Ricci transformation of $(\mathfrak{n}, \langle\ ,\ \rangle_{\mathfrak{n}})$. Notice that in the case of nilpotent Lie algebra the formula (2.3) is reduced to the following form:

(2.6) $$\mathrm{Ric} = \frac{1}{4}\sum_{i\in I} ad_{E_i} \circ {ad_{E_i}}^* - \frac{1}{2}\sum_{i\in I} {ad_{E_i}}^* \circ ad_{E_i}.$$

Using (2.6) $\mathrm{Ric}^{\mathfrak{n}}$ is computed as

$$\mathrm{Ric}^{\mathfrak{n}}(V) = \frac{1}{2}\sum_{j=1}^{m} J(Z_j)^2V$$

$$\mathrm{Ric}^{\mathfrak{n}}(Z) = \frac{1}{4}\sum_{i=1}^{n}[V_i, J(Z)V_i]. \qquad \square$$

REMARK. For 2-step nilpotent metric Lie algebras, the Ricci transformations are computed in [5].

2.2. Einstein condition of BDR-type spaces A Riemannian metric g is called Einstein metric if its Ricci tensor satisfies $\mathrm{ric} = cg$ for some constant c. If a BDR-type space (S, g) is Einstein manifold then the Einstein constant c must be $-k^2((n/4)+m)$ from (a) of Lemma 2.2. Since $\mathrm{Ric}^{\mathfrak{n}}$ is symmetric and leaves $\mathfrak{v}$ and $\mathfrak{z}$ invariant we can choose an orthonormal basis $\{V_1, \ldots, V_n\}$ of $\mathfrak{v}$ and $\{Z_1, \ldots, Z_m\}$ of $\mathfrak{z}$ which diagonalize $\mathrm{Ric}^{\mathfrak{n}}$, that is

$$\mathrm{Ric}^{\mathfrak{n}}(V_i) = a_i V_i, \quad \mathrm{Ric}^{\mathfrak{n}}(Z_j) = b_j Z_j.$$

From (2.5) we deduce that if (S, g) is Einstein then

$$a_i = -\frac{m}{2}k^2, \quad b_j = \frac{n}{4}k^2 \quad \text{for} \quad i = 1, \ldots, n, \; j = 1, \ldots, m. \tag{2.7}$$

Conversely suppose that (2.7) holds, then it is easily checked that (S, g) is Einstein manifold with the Einstein constant $-k^2((n/4)+m)$. We therefore have the following proposition.

Proposition 2.3. *A BDR-type space* (S, g) *is Einstein manifold if and only if* $\mathrm{Ric}^{\mathfrak{n}}$ *satisfies*

$$\mathrm{Ric}^{\mathfrak{n}}|_{\mathfrak{v}} = -\frac{m}{2}k^2\mathbb{I}d_{\mathfrak{v}}, \quad \mathrm{Ric}^{\mathfrak{n}}|_{\mathfrak{z}} = \frac{n}{4}k^2\mathbb{I}d_{\mathfrak{z}},$$

where $m = \dim\mathfrak{z}$ and $n = \dim\mathfrak{v}$.

REMARK. T. Wolter [10] studied the Einstein condition of metric Lie algebra of Iwasawa type. Proposition 2.3 is regarded as the special case of Proposition 1.5 and Theorem 1.6 in [10].

EXAMPLE (Damek-Ricci space). A 2-step nilpotent Lie algebra with a positive definite inner product is of *H-type* if there exists a positive constant λ such that

$$J(Z)^2 = -\lambda^2|Z|^2\mathbb{I}d_{\mathfrak{v}}$$

for every $z \in \mathfrak{z}$. A BDR-type space (S, g) is called *Damek-Ricci space* if the Lie algebra of the nilpotent part of S equipped with the induced inner product is H-type Lie algebra. A Damek-Ricci space is called *standard* if it satisfies $\lambda = k$.

If (S, g) is DR space, equivalently $(\mathfrak{n}, \langle\ ,\ \rangle_{\mathfrak{n}})$ is of H-type, then from the definition we immediately obtain the following facts.

a) $|J(Z)V| = \lambda|Z||V| \quad \text{for} \quad Z \in \mathfrak{z}, \quad V \in \mathfrak{v}$

b) $J(Z_1) \circ J(Z_2) + J(Z_2) \circ J(Z_1) = -2\lambda^2\langle Z_1, Z_2\rangle \mathbb{I}d_{\mathfrak{v}} \quad \text{for} \quad Z_1, Z_2 \in \mathfrak{z}$

c) $\langle J(Z_1)V, J(Z_2)V\rangle = \lambda^2\langle Z_1, Z_2\rangle|V|^2 \quad \text{for} \quad Z_1, Z_2 \in \mathfrak{z}, \quad V \in \mathfrak{v}$

d) $\langle J(Z)V_1, J(Z)V_2\rangle = \lambda^2|Z|^2\langle V_1, V_2\rangle \quad \text{for} \quad Z \in \mathfrak{z}, \quad V_1, V_2 \in \mathfrak{v}$

e) $[V, J(Z)V] = \lambda^2|V|^2 Z \quad \text{for} \quad Z \in \mathfrak{z}, \quad V \in \mathfrak{v}.$

Corollary 2.4. *Let (S, g) be Damek-Ricci space, then the following* (1) *and* (2) *are equivalent.*
(1) (S, g) *is Einstein manifold.* (2) (S, g) *is standard DR space.*

Proof. From lemma (2.2), in the case of DR spaces, we have the following:

$$\mathrm{Ric}^{\mathfrak{n}}|_{\mathfrak{v}} = -\frac{m}{2}\lambda^2\mathbb{I}d_{\mathfrak{v}}, \quad \mathrm{Ric}^{\mathfrak{n}}|_{\mathfrak{z}} = \frac{n}{4}\lambda^2\mathbb{I}d_{\mathfrak{z}}. \qquad \square$$

2.3. BDR-type spaces with nonpositive curvature If the positive constant k is sufficiently large, the BDR-type space has nonpositive sectional curvature ([6]). In fact the following proposition holds:

Proposition 2.5. *Assume that there exists a positive constant C such that*

$$|J(Z)V| \leq C|Z||V| \quad \textit{for} \quad Z \in \mathfrak{z}, \quad V \in \mathfrak{v}.$$

Then if $C \leq k$ the BDR-type spaces corresponding to k have nonpositive sectional curvature.

Proof. Let S_k be a BDR-type space corresponding to k. From Lemma 2.1 we have

$$K((X + aA) \wedge Y) = K(X \wedge Y) + \frac{3}{4}|[X, Y]|^2 - \frac{3}{4}|[X, Y] + akY_{\mathfrak{z}}|^2 - \frac{a^2k^2}{4}|Y|^2,$$

where $X, Y \in \mathfrak{n}$ and $Y_{\mathfrak{z}}$ denotes the $\mathfrak{z}$-component of Y.

To prove that S_k has nonpositive sectional curvature, it is sufficient to show that if $X, Y \in \mathfrak{n}$ then

$$K(X \wedge Y) + \frac{3}{4}|[X, Y]|^2 \leq 0. \tag{2.8}$$

We divide the left-hand side of (2.8) into two parts (A) and (B) as follows:

$$(A) = \frac{1}{4}|J(Z_1)V_2|^2 + \frac{1}{4}|J(Z_2)V_1|^2 - \frac{1}{2}\langle J(Z_1)V_1, J(Z_2)V_2\rangle$$

$$
\begin{aligned}
&+\frac{k^2}{2}\{2\langle V_1, V_2\rangle\langle Z_1, Z_2\rangle - (|V_1|^2|Z_2|^2 + |V_2|^2|Z_1|^2)\},\\
(B) &= \frac{1}{2}\{\langle J(Z_2)V_1, J(Z_1)V_2\rangle - \langle J(Z_1)V_1, J(Z_2)V_2\rangle\}\\
&+\frac{k^2}{4}\{\langle V_1, V_2\rangle^2 - |V_1|^2|V_2|^2\} + k^2\{\langle Z_1, Z_2\rangle^2 - |Z_1|^2|Z_2|^2\},
\end{aligned}
$$

where $X = V_1 + Z_1,\ Y = V_2 + Z_2,\ V_1, V_2 \in \mathfrak{v},\ Z_1, Z_2 \in \mathfrak{z}$.

If X and Y are orthogonal, then $\langle V_1, V_2\rangle\langle Z_1, Z_2\rangle \leq 0$. Thus we may assume that $\langle V_1, V_2\rangle\langle Z_1, Z_2\rangle \leq 0$ without the loss of generality.

Then

$$
\begin{aligned}
(A) &\leq \frac{C^2}{4}|Z_1|^2|V_2|^2 + \frac{C^2}{4}|Z_2|^2|V_1|^2 + \frac{C^2}{2}|Z_1||Z_2||V_1||V_2| - \frac{k^2}{2}(|V_1|^2|Z_2|^2 + |V_2|^2|Z_1|^2)\\
&= \frac{-C^2}{4}(|V_1||Z_2| - |V_2||Z_1|)^2 - \frac{k^2 - C^2}{2}(|V_1|^2|Z_2|^2 + |V_2|^2|Z_1|^2).
\end{aligned}
$$

Next we put

$$
\begin{aligned}
Z_2 &= \alpha Z_1 + Z_3, \langle Z_1, Z_3\rangle = 0\\
V_2 &= \beta V_1 + V_3, \langle V_1, V_3\rangle = 0,
\end{aligned}
$$

where $\alpha, \beta \in \mathbb{R}$ and $Z_3 \in \mathfrak{z},\ V_3 \in \mathfrak{v}$.

Then

$$
\begin{aligned}
(B) &= \frac{1}{2}\{\langle J(Z_3)V_1, J(Z_1)V_3\rangle - \langle J(Z_1)V_1, J(Z_3)V_3\rangle\} - \frac{k^2}{4}|V_1|^2|V_3|^2 - k^2|Z_1|^2|Z_3|^2\\
&\leq -C^2\left(\frac{1}{2}|V_1||V_3| - |Z_1||Z_3|\right)^2 - \frac{k^2 - C^2}{4}(|V_1|^2|V_3|^2 + 4|Z_1|^2|Z_3|^2). \qquad \square
\end{aligned}
$$

REMARK. In the case of DR spaces, we have $|J(Z)V| = \lambda|Z||V|$ and hence Damek-Ricci Einstein spaces have nonpositive sectional curvature.

3. BDR-type Einstein spaces

In this section, we shall construct a class of BDR-type Einstein spaces which are not DR Einstein spaces. We shall also prove that there exist nonsymmetric BDR-type Einstein spaces with negative sectional curvature. Notice that if a DR Einstein space has negative sectional curvature then it is symmetric ([3], [9]).

3.1. BDR-type Einstein spaces Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote by $\triangle$ and Π the root system and the fundamental root system relative to $\mathfrak{h}$ respectively. Take a Weyl basis $\{E_\alpha \mid \alpha \in \triangle\}$ with

$$B(E_\alpha, E_{-\alpha}) = -1 \tag{1}$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha, \beta, \alpha+\beta \in \triangle \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$N_{\alpha\beta} = N_{-\alpha-\beta} \in \mathbb{R},$$

where B is the Killing form of $\mathfrak{g}$.

Let Π_0 be a subset of Π and suppose that

$$\Pi = \{\alpha_1, \dots, \alpha_l\}, \quad \Pi_0 = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}.$$

For $(k_1, \dots, k_r) \in (\mathbb{Z}_{\geq 0})^r - \{0\}$ we put

$$\triangle(k_1, \dots, k_r) = \left\{ \sum_{j=1}^{l} m_j \alpha_j \in \triangle^+ \;\middle|\; m_{i_1} = k_1, \dots, m_{i_r} = k_r \right\}$$

where $\triangle^+$ denotes the set of all positive roots relative to Π. We define a $\mathbb{R}$-subspace of $\mathfrak{g}_{\mathbb{R}}$ associated to $\triangle(k_1, \dots, k_r)$ by

$$\mathfrak{n}(k_1, \dots, k_r) = \sum_{\alpha \in \triangle(k_1, \dots, k_r)} \mathbb{R} E_\alpha.$$

Finally we define

$$\mathfrak{n}(\Pi_0) = \sum_{k_1, \dots, k_r} \mathfrak{n}(k_1, \dots, k_r).$$

Notice that $\mathfrak{n}(\Pi_0) = \{\sum m_j \alpha_j \in \triangle^+ \mid \exists m_{i_k} > 0\}$. Then $\mathfrak{n}(\Pi_0)$ is a nilpotent Lie algebra over $\mathbb{R}$ since the structure constants $N_{\alpha\beta}$'s are $\mathbb{R}$-valued.

Let $\triangle^{\mathfrak{n}(\Pi_0)}$ be the set of all roots that define $\mathfrak{n}(\Pi_0)$, that is, $\mathfrak{n}(\Pi_0) = \sum_{\alpha \in \triangle^{\mathfrak{n}(\Pi_0)}} \mathbb{R} E_\alpha$. We define the inner product $\langle\, , \,\rangle_{\mathfrak{n}(\Pi_0)}$ on $\mathfrak{n}(\Pi_0)$ such that $\{E_\alpha \mid \alpha \in \triangle^{\mathfrak{n}(\Pi_0)}\}$ is an orthonormal basis of $\mathfrak{n}(\Pi_0)$. Then the Ricci transformation of $\mathfrak{n}(\Pi_0)$ is diagonalized with respect to $\{E_\alpha \mid \alpha \in \triangle^{\mathfrak{n}(\Pi_0)}\}$ since $\{E_\alpha \mid \alpha \in \triangle^{\mathfrak{n}(\Pi_0)}\}$ is a basis compatible with the grading of $\mathfrak{n}(\Pi_0)$.

A 2-step nilpotent Lie algebra with a positive definite inner product is said to be *nonsingular* if each $J(Z)(Z \in \mathfrak{z} - \{0\})$ is nonsingular transformation of $\mathfrak{v}$ ([5]). H-type Lie algebra is the typical example of a nonsingular 2-step nilpotent Lie algebra. On the other hand most of the 2-step nilpotent Lie algebras of the above type are singular.

Now we consider the BDR-type space whose nilpotent part is defined by a 2-step nilpotent Lie algebra of the above type. We denote by $S_k(\Pi, \Pi_0)$ and $\mathfrak{s}_k(\Pi, \Pi_0)$ the BDR-type space and its Lie algebra induced by the triple data (k, Π, Π_0) respectively.

If $\alpha,\ \beta,\ \gamma \in \triangle^{\mathfrak{n}(\Pi_0)}$ and $\alpha = \beta + \gamma$ then

$$J(E_\alpha)E_\beta = N_{\beta\gamma}E_\gamma. \tag{3.1}$$

The next lemma follows from (2.6) and (3.1).

Lemma 3.1. *Let* $(\mathfrak{n}(\Pi_0), \langle\, ,\, \rangle_{\mathfrak{n}(\Pi_0)})$ *be a* 2*-step nilpotent Lie algebra of the above type. For* $\alpha \in \triangle^{\mathfrak{n}(\Pi_0)}$ *we define*

$$\Phi(\alpha) = \{\beta \in \triangle^{\mathfrak{n}(\Pi_0)} \mid \alpha + \beta \in \triangle^{\mathfrak{n}(\Pi_0)}\}$$
$$\Psi(\alpha) = \{(\beta, \gamma) \in \triangle^{\mathfrak{n}(\Pi_0)} \times \triangle^{\mathfrak{n}(\Pi_0)} \mid \beta + \gamma = \alpha\}.$$

Then the eigenvalue ξ_α *of the Ricci transformation* $\mathrm{Ric}^{\mathfrak{n}(\Pi_0)}$ *with respect to the eigenvector* E_α *is given by*

$$\xi_\alpha = \begin{cases} -\dfrac{1}{2} \displaystyle\sum_{\beta \in \Phi(\alpha)} (N_{\alpha\beta})^2 & \text{if } E_\alpha \in \mathfrak{v} \\ \dfrac{1}{4} \displaystyle\sum_{(\beta,\gamma) \in \Psi(\alpha)} (N_{\beta\gamma})^2 & \text{if } E_\alpha \in \mathfrak{z}. \end{cases}$$

Combining this lemma with Proposition 2.3, we can decide whether a given BDR-type space $S_k(\Pi, \Pi_0)$ is Einstein manifold or not and the decision does not depend on the choice of a Weyl basis.

We give here the table of BDR-type Einstein spaces obtained by the construction above in the case of classical simple Lie algebras (Table 1).

REMARK. BDR-type Einstein spaces induced by type A_l algebra were also obtained in [11] in a different way from ours.

3.2. BDR-type Einstein spaces with negative curvature BDR-type Einstein spaces do not always have nonpositive sectional curvature. If $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$ then the formula of Lemma 2.1 is reduced to

$$K(V \wedge Z) = \frac{1}{4}|J(Z)V|^2 - \frac{k^2}{2}|Z|^2|V|^2.$$

By using this equation we can check that many of the BDR-type Einstein spaces obtained in the table do not have nonpositive sectional curvature. For instance let Π be the fundamental system of the type C_l algebra and Π_0 be $\{\alpha_m\}$. If $4 \le l$ and $3 \le m \le l - 1$, the BDR-type Einstein space

$$S_k(\Pi, \Pi_0), \quad k = \frac{1}{\sqrt{2m(l+1)}}$$

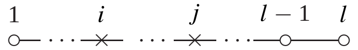
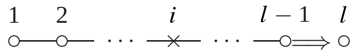
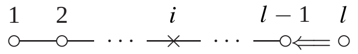
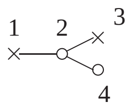
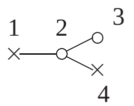
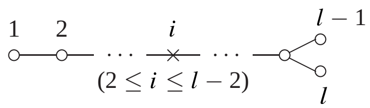
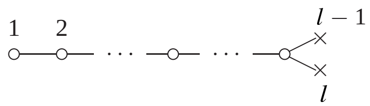
$\mathfrak{g}$	(Π, Π_0)	k	$\dim \mathfrak{z}$	$\dim \mathfrak{v}$	$S_k(\Pi, \Pi_0)$
A_l $(2 \leq l)$	 $(l+1 = i+j)$	$\frac{1}{\sqrt{2i(i+j)}}$	i^2	$2i(j-i)$	not DR $(i \neq 1)$, $\mathbb{C}H^l$ $(i = 1)$
B_l $(2 \leq l)$	 $(2 \leq i \leq l)$	$\frac{1}{\sqrt{i(2l-1)}}$	$\frac{i(i-1)}{2}$	$i(2l-2i+1)$	not DR $(i \neq 2)$, $\mathbb{C}H^{2l-2}$ $(i = 2)$
C_l $(3 \leq l)$	 $(1 \leq i \leq l-1)$	$\frac{1}{\sqrt{2i(l+1)}}$	$\frac{i(i+1)}{2}$	$2i(l-i)$	not DR $(i \neq 1)$, $\mathbb{C}H^l$ $(i = 1)$
D_4	 	$\frac{1}{3\sqrt{2}}$	3	6	not DR
D_l $(4 \leq l)$	 $(2 \leq i \leq l-2)$	$\frac{1}{\sqrt{2i(l-1)}}$	$\frac{i(i-1)}{2}$	$2i(l-i)$	not DR $(i \neq 2)$, $\mathbb{C}H^{2l-3}$ $(i = 2)$
D_l $(4 \leq l)$		$\frac{1}{\sqrt{2(l-1)}}$	$\frac{(l-1)(l-2)}{2}$	$2(l-1)$	not DR

Table 1. (The vertex $\times$ denotes the element of Π_0)

does not have nonpositive sectional curvature. To see this fact, we choose the Wyle basis E_α, $E_\beta \in \mathfrak{v}$ and $E_\gamma \in \mathfrak{z}$ such that $\alpha+\beta = \gamma$ and γ is a long root. Since $(N_{\alpha\beta})^2 = (\sqrt{2(l+1)})^{-2}$, we obtain

$$\begin{aligned} K(E_\alpha \wedge E_\gamma) &= \frac{1}{4}(N_{\alpha\beta})^2|E_\beta|^2 - \frac{1}{2}\frac{1}{2m(l+1)}|E_\alpha|^2|E_\gamma|^2 \\ &= \frac{m-2}{8m(l+1)}. \end{aligned}$$

Now we are in a position to show the following theorem.

Theorem 3.2. (i) *Let* Π *be the fundamental system of the type* B_l *algebra and* Π_0 *be* $\{\alpha_3\}$. *If* $4 \leq l$, *the BDR-type Einstein space*

$$S_k(\Pi, \Pi_0), \quad k = \frac{1}{\sqrt{3(2l-1)}}$$

has negative sectional curvature.
(ii) *Let* Π *be the fundamental system of the type* D_l *algebra and* Π_0 *be* $\{\alpha_3\}$. *If* $5 \leq l$, *the BDR-type Einstein space*

$$S_k(\Pi, \Pi_0), \quad k = \frac{1}{\sqrt{6(l-1)}}$$

has negative sectional curvature.

In particular there exist nonsymmetric BDR-type Einstein spaces with negative sectional curvature.

To prove the above theorem, we need the following lemma.

Lemma 3.3. *Let* Π *be the fundamental root system of the type* $B_l (4 \leq l)$ *and* Π_0 *be* $\{\alpha_3\}$. *If* $1/(2\sqrt{(2l-1)}) \leq k$ *then* $(A) \leq 0$ *and* $(B) \leq 0$.

Proof of Lemma 3.3. We assign the indices to the elements of $\triangle^{\mathfrak{n}(\Pi_0)}$ as follows:

$$\alpha_{i1} = \sum_{t=i}^{l} \alpha_t \quad (1 \leq i \leq 3)$$

$$\alpha_{ij} = \sum_{t=i}^{j+1} \alpha_t \quad (1 \leq i \leq 3,\ 2 \leq j \leq l-2)$$

$$\alpha_{ij} = \sum_{t=i}^{l} \alpha_t + \sum_{s=j-l+5}^{l} \alpha_s \quad (1 \leq i \leq 3,\ l-1 \leq j \leq 2l-5)$$

$$\beta_1 = \alpha_2 + \sum_{t=3}^{l} 2\alpha_t$$

$$\beta_2 = \alpha_1 + \alpha_2 + \sum_{t=3}^{l} 2\alpha_t$$

$$\beta_3 = \alpha_1 + \sum_{t=2}^{l} 2\alpha_t$$

We choose the Weyl basis such that $N_{\alpha_{i_1 j_1} \alpha_{i_2 j_2}} \geq 0$ if $i_1 < i_2$, $N_{\alpha_{i_1 j_1} \alpha_{i_2 j_2}} \leq 0$ if $i_1 > i_2$, for $1 \leq i_1, i_2 \leq 3$, $2 \leq j_1 \leq l-2$, $l-1 \leq j_2 \leq 2l-5$, where the structure constants satisfy

$$(N_{\alpha\beta})^2 = \begin{cases} \dfrac{1}{2(2l-1)} & \text{if } \alpha, \beta, \alpha+\beta \in \triangle^{\mathfrak{n}(\Pi_0)} \\ 0 & \text{otherwise.} \end{cases}$$

We put

$$V_\delta = \sum_{i,j} v_{ij}^\delta E_{\alpha_{ij}}, \quad Z_\delta = \sum_{h=1}^{3} z_h^\delta E_{\beta_h} \quad \text{for} \quad \delta = 1, 2.$$

Then for $1 \leq \delta, \delta' \leq 2, 1 \leq j \leq 2l-5$ we get

$$\begin{aligned} &J(Z_\delta)V_{\delta'} = \\ &\frac{1}{\sqrt{2(2l-1)}} \Big[\left(z_2^\delta v_{31}^{\delta'} + z_3^\delta v_{21}^{\delta'} \right) E_{\alpha_{11}} + \left(z_1^\delta v_{31}^{\delta'} - z_3^\delta v_{11}^{\delta'} \right) E_{\alpha_{21}} - \left(z_1^\delta v_{21}^{\delta'} + z_2^\delta v_{11}^{\delta'} \right) E_{\alpha_{31}} \\ &\quad + \sum_{j=2}^{2l-5} \Big\{ \left(z_2^\delta v_{3,2l-3-j}^{\delta'} + z_3^\delta v_{2,2l-3-j}^{\delta'} \right) E_{\alpha_{1j}} + \left(z_1^\delta v_{3,2l-3-j}^{\delta'} - z_3^\delta v_{1,2l-3-j}^{\delta'} \right) E_{\alpha_{2j}} \\ &\qquad - \left(z_1^\delta v_{2,2l-3-j}^{\delta'} + z_2^\delta v_{1,2l-3-j}^{\delta'} \right) E_{\alpha_{3j}} \Big\} \Big]. \end{aligned}$$

Next we put

$$P_{\delta\delta'}^j := z_1^\delta v_{1j}^{\delta'} - z_2^\delta v_{2j}^{\delta'} + z_3^\delta v_{3j}^{\delta'} \quad \text{for} \quad 1 \leq \delta, \delta' \leq 2, 1 \leq j \leq 2l-5.$$

Then we obtain the following equation:

$$\begin{aligned} \text{(3.2)} \qquad &\frac{1}{4}\langle J(Z_{\delta_1})V_{\delta_2}, J(Z_{\delta'_1})V_{\delta'_2}\rangle - \frac{1}{8(2l-1)}\langle Z_{\delta_1}, Z_{\delta'_1}\rangle\langle V_{\delta_2}, V_{\delta'_2}\rangle \\ &\qquad = \frac{-1}{8(2l-1)} \sum_{j=1}^{2l-5} P_{\delta_1\delta'_2}^j P_{\delta'_1\delta_2}^j. \end{aligned}$$

Since $2\langle V_1, V_2\rangle\langle Z_1, Z_2\rangle - (|V_1|^2|Z_2|^2 + |V_2|^2|Z_1|^2) \leq 0$, to prove $(A) \leq 0$ it is sufficient to show that if $k = 1/(2\sqrt{2l-1})$ then $(A) \leq 0$. Using (3.2) we get

$$\begin{aligned} (A) &= \left(\frac{1}{4}|J(Z_1)V_2|^2 - \frac{1}{8(2l-1)}|V_2|^2|Z_1|^2 \right) + \left(\frac{1}{4}|J(Z_2)V_1|^2 - \frac{1}{8(2l-1)}|V_1|^2|Z_2|^2 \right) \\ &\quad - 2\left(\frac{1}{4}\langle J(Z_1)V_1, J(Z_2)V_2\rangle - \frac{1}{8(2l-1)}\langle V_1, V_2\rangle\langle Z_1, Z_2\rangle \right) \\ &= \frac{-1}{8(2l-1)} \sum_{j=1}^{2l-5} (P_{12}^j - P_{21}^j)^2 \leq 0. \end{aligned}$$

To prove $(B) \leq 0$ we will show that

$$|\langle J(Z_2)V_1, J(Z_1)V_2\rangle - \langle J(Z_1)V_1, J(Z_2)V_2\rangle| \leq \frac{1}{2(2l-1)}|Z_1||Z_2||V_1||V_2| \tag{3.3}$$

for $Z_1, Z_2 \in \mathfrak{z}$, $V_1, V_2 \in \mathfrak{v}$. If (3.3) holds, the same estimates as the proof of Proposition 2.5 accomplish the proof. In fact

$$\begin{aligned}(B) &= \frac{1}{2}\{\langle J(Z_3)V_1, J(Z_1)V_3\rangle - \langle J(Z_1)V_1, J(Z_3)V_3\rangle\}\\ &\quad -\frac{k^2}{4}|V_1|^2|V_3|^2 - k^2|Z_1|^2|Z_3|^2\\ &\leq \frac{1}{2}\{\langle J(Z_3)V_1, J(Z_1)V_3\rangle - \langle J(Z_1)V_1, J(Z_3)V_3\rangle\}\\ &\quad -k^2|V_1||V_3||Z_1||Z_3|, \end{aligned} \tag{3.4}$$

therefore if (3.3) holds we get $(B) \leq 0$.

We put

$$V_1 = \sum_{j=1}^{2l-5} V_1^{(j)}, \qquad V_2 = \sum_{j=1}^{2l-5} V_2^{(j)},$$

where $V_1^{(j)}, V_2^{(j)} \in \operatorname{Span}\{E_{\alpha_{1j}}, E_{\alpha_{2j}}, E_{\alpha_{3j}}\}$.

Using Schwarz's inequality we get

$$\begin{aligned}&|\langle J(Z_2)V_1^{(1)}, J(Z_1)V_2^{(1)}\rangle - \langle J(Z_1)V_1^{(1)}, J(Z_2)V_2^{(1)}\rangle|\\ &= \frac{1}{2(2l-1)}|(z_1^1z_2^2 - z_2^1z_1^2)(v_{21}^1v_{11}^2 - v_{11}^1v_{21}^2) + (z_1^1z_3^2 - z_3^1z_1^2)(v_{11}^1v_{31}^2 - v_{31}^1v_{11}^2)\\ &\qquad + (z_2^1z_3^2 - z_3^1z_2^2)(v_{31}^1v_{21}^2 - v_{21}^1v_{31}^2)|\\ &\leq \frac{1}{2(2l-1)}\sqrt{\sum_{1\leq i<j\leq 3}(z_i^1z_j^2 - z_j^1z_i^2)^2}\sqrt{\sum_{1\leq i<j\leq 3}(v_{i1}^1v_{j1}^2 - v_{j1}^1v_{i1}^2)^2}\\ &= \frac{1}{2(2l-1)}\sqrt{|Z_1|^2|Z_2|^2 - \langle Z_1, Z_2\rangle^2}\sqrt{|V_1^{(1)}|^2|V_2^{(1)}|^2 - \langle V_1^{(1)}, V_2^{(1)}\rangle^2}\\ &\leq \frac{1}{2(2l-1)}|Z_1||Z_2||V_1^{(1)}||V_2^{(1)}|.\end{aligned}$$

Similarly for $2 \leq j \leq l-2, l-1 \leq j' \leq 2l-5$ we get

$$\begin{aligned}&|\langle J(Z_2)V_1^{(j)}, J(Z_1)V_2^{(j)}\rangle - \langle J(Z_1)V_1^{(j)}, J(Z_2)V_2^{(j)}\rangle|\\ &\leq \frac{1}{2(2l-1)}|Z_1||Z_2||V_1^{(j+l-3)}||V_2^{(j+l-3)}|.\\ &|\langle J(Z_2)V_1^{(j')}, J(Z_1)V_2^{(j')}\rangle - \langle J(Z_1)V_1^{(j')}, J(Z_2)V_2^{(j')}\rangle|\end{aligned}$$

$$\leq \frac{1}{2(2l-1)}|Z_1||Z_2||V_1^{(j'-l+3)}||V_2^{(j'-l+3)}|.$$

Then

$$\begin{aligned}
&|\langle J(Z_2)V_1, J(Z_1)V_2\rangle - \langle J(Z_1)V_1, J(Z_2)V_2\rangle| \\
&\quad \leq \sum_{j=1}^{2l-5}|\langle J(Z_2)V_1^{(j)}, J(Z_1)V_2^{(j)}\rangle - \langle J(Z_1)V_1^{(j)}, J(Z_2)V_2^{(j)}\rangle| \\
&\quad \leq \frac{1}{2(2l-1)}|Z_1||Z_2|\left(\sum_{j=1}^{2l-5}|V_1^{(j)}||V_2^{(j)}|\right) \\
&\quad \leq \frac{1}{2(2l-1)}|Z_1||Z_2||V_1||V_2|. \qquad \square
\end{aligned}$$

Proof of Theorem 3.2. We will prove (i). (ii) can be proved in the same way as (i) and hence the proof of (ii) is omitted.

From Lemma 3.3, if $4 \leq l$ then $S_k(\Pi, \Pi_0)$ $(k = 1/\sqrt{3(2l-1)})$ has nonpositive sectional curvature. To prove that $S_k(\Pi, \Pi_0)$ has negative sectional curvature, it is sufficient to show that if $X, Y \in \mathfrak{n}$ and X and Y are linearly independent then

$$K(X \wedge Y) + \frac{3}{4}|[X, Y]|^2 < 0. \tag{3.5}$$

Again we divide the left-hand side of (3.5) into two parts (A) and (B) as the proof of Proposition 2.5. Next we assume that there exist $X, Y \in \mathfrak{n}$ such that

$$K(X \wedge Y) + \frac{3}{4}|[X, Y]|^2 = 0.$$

We have $(A) \leq 0, (B) \leq 0$ from Lemma 3.3. Combining this fact with the assumption, we get $(A) = (B) = 0$ when $k = 1/\sqrt{3(2l-1)}$.

From Schwarz's inequality notice that

$$\begin{aligned}
&2\langle V_1, V_2\rangle\langle Z_1, Z_2\rangle - (|V_1|^2|Z_2|^2 + |V_2|^2|Z_1|^2) \leq 0 \\
&\langle V_1, V_2\rangle^2 - |V_1|^2|V_2|^2 \leq 0 \\
&\langle Z_1, Z_2\rangle^2 - |Z_1|^2|Z_2|^2 \leq 0,
\end{aligned}$$

and hence both (A) and (B) are decreasing functions with respect to k. Since $(A) \leq 0, (B) \leq 0$ when $k = 1/(2\sqrt{(2l-1)})$, we conclude that $(A) = (B) = 0$ when $k = 1/(2\sqrt{(2l-1)})$.

Then

$$0 = (A)|_{k=1/\sqrt{3(2l-1)}} \quad - \quad (A)|_{k=1/(2\sqrt{(2l-1)})}$$

$$= \frac{1}{24(2l-1)}\{2\langle V_1, V_2\rangle\langle Z_1, Z_2\rangle - (|V_1|^2|Z_2|^2 + |V_2|^2|Z_1|^2)\}$$

Combining similar comuputations for (B), we get

$$\begin{aligned}
&2\langle V_1, V_2\rangle\langle Z_1, Z_2\rangle - (|V_1|^2|Z_2|^2 + |V_2|^2|Z_1|^2) = 0\\
&\langle V_1, V_2\rangle^2 - |V_1|^2|V_2|^2 = 0\\
&\langle Z_1, Z_2\rangle^2 - |Z_1|^2|Z_2|^2 = 0
\end{aligned}$$

and hence X and Y are not linearly independent. So (3.5) holds.

As for the symmetricity, we can check that ∇R does not vanish for the BDR-type Einstein spaces above. □

REMARK. For the type B_3 algebra, BDR-type Einstein space $S_k(\Pi, \{\alpha_3\})$ ($k = 1/\sqrt{15}$) has negative sectional curvature. For the type D_4 algebra, BDR-type Einstein spaces $S_k(\Pi, \{\alpha_3, \alpha_4\})$, $S_k(\Pi, \{\alpha_1, \alpha_3\})$, $S_k(\Pi, \{\alpha_1, \alpha_4\})$ ($k = 1/(3\sqrt{2})$) have negative sectional curvature.

3.3. BDR-type Einstein spaces with nonpositve curvature As for the type A_l and C_l algebra we have the following theorem.

Theorem 3.4. (i) *Let Π be the fundamental system of the type A_l algebra and Π_0 be $\{\alpha_2, \alpha_{l-1}\}$. If $4 \leq l$, the BDR-type Einstein space*

$$S_k(\Pi, \Pi_0), \quad k = \frac{1}{2\sqrt{l+1}}$$

has nonpositive sectional curvature.
(ii) *Let Π be the fundamental system of the type C_l algebra and Π_0 be $\{\alpha_2\}$. If $3 \leq l$, the BDR-type Einstein space*

$$S_k(\Pi, \Pi_0), \quad k = \frac{1}{2\sqrt{l+1}}$$

has nonpositive sectional curvature.

Proof. We will prove (ii). (i) can be proved in the same way as (ii) and hence the proof of (i) is omitted.

We assign the indices to the elements of $\triangle^{\mathrm{n}(\Pi_0)}$ as follows:

$$\beta_1 = \alpha_1 + \sum_{t=2}^{l-1} \alpha_t + \alpha_l$$

$$\beta_2 = \sum_{t=2}^{l-1} 2\alpha_t + \alpha_l$$

$$\beta_3 = \sum_{t=1}^{l-1} 2\alpha_t + \alpha_l$$

$$\alpha_{ij} = \sum_{t=i}^{j+1} \alpha_t \quad (1 \leq i \leq 2,\ 1 \leq j \leq l-2)$$

$$\alpha_{ij} = \beta_{6-i} - \sum_{t=i-2}^{j+1} \alpha_t \quad (3 \leq i \leq 4,\ 1 \leq j \leq l-2)$$

We choose the Weyl basis such that $N_{\alpha_{i_1 j_1}\alpha_{i_2 j_2}} \geq 0$ if $i_1 < i_2$, $N_{\alpha_{i_1 j_1}\alpha_{i_2 j_2}} \leq 0$ if $i_1 > i_2$, for $1 \leq i_1,\ i_2 \leq 4$, $1 \leq j_1 \leq l-2$, where the structure constants satisfy

$$(N_{\alpha\beta})^2 = \begin{cases} \dfrac{1}{2(l+1)} & \text{if } \alpha, \beta \in \triangle^{\mathfrak{n}(\Pi_0)} \text{ and } \alpha+\beta \in \{\beta_2, \beta_3\} \\ \dfrac{1}{4(l+1)} & \text{if } \alpha, \beta \in \triangle^{\mathfrak{n}(\Pi_{0q})} \text{and } \alpha+\beta = \beta_1 \\ 0 & \text{otherwise.} \end{cases}$$

We put

$$V_\delta = \sum_{i,j} v_{ij}^\delta E_{\alpha_{ij}}, \quad Z_\delta = \sum_{h=1}^{3} z_h^\delta E_{\beta_h} \quad \text{for} \quad \delta = 1, 2.$$

Then we get

$$J(Z_\delta)V_{\delta'} = \frac{-1}{2\sqrt{(l+1)}} \sum_{j=1}^{l-2} \{(z_1^\delta v_{4j}^{\delta'} + \sqrt{2} z_3^\delta v_{3j}^{\delta'})E_{\alpha_{1j}} + (z_1^\delta v_{3j}^{\delta'} + \sqrt{2} z_2^\delta v_{4j}^{\delta'})E_{\alpha_{2j}}$$
$$-(z_1^\delta v_{2j}^{\delta'} + \sqrt{2} z_3^\delta v_{1j}^{\delta'})E_{\alpha_{3j}} - (z_1^\delta v_{1j}^{\delta'} + \sqrt{2} z_2^\delta v_{2j}^{\delta'})E_{\alpha_{4j}}\}.$$

Hence we obtain the following equation:

$$\begin{aligned} &\frac{1}{4}\langle J(Z_{\delta_1})V_{\delta_2}, J(Z_{\delta'_1})V_{\delta'_2}\rangle - \frac{1}{8(l+1)}\langle Z_{\delta_1}, Z_{\delta'_1}\rangle\langle V_{\delta_2}, V_{\delta'_2}\rangle \\ &= \frac{-1}{16(l+1)} \sum_{j=1}^{l-2} \{(z_1^{\delta_1} v_{1j}^{\delta'_2} - \sqrt{2} z_3^{\delta_1} v_{2j}^{\delta'_2})(z_1^{\delta'_1} v_{1j}^{\delta_2} - \sqrt{2} z_3^{\delta'_1} v_{2j}^{\delta_2}) \\ &\qquad + (z_1^{\delta_1} v_{2j}^{\delta'_2} - \sqrt{2} z_2^{\delta_1} v_{1j}^{\delta'_2})(z_1^{\delta'_1} v_{2j}^{\delta_2} - \sqrt{2} z_2^{\delta'_1} v_{1j}^{\delta_2}) \\ &\qquad + (z_1^{\delta_1} v_{3j}^{\delta'_2} - \sqrt{2} z_3^{\delta_1} v_{4j}^{\delta'_2})(z_1^{\delta'_1} v_{3j}^{\delta_2} - \sqrt{2} z_3^{\delta'_1} v_{4j}^{\delta_2}) \\ &\qquad + (z_1^{\delta_1} v_{4j}^{\delta'_2} - \sqrt{2} z_2^{\delta_1} v_{3j}^{\delta'_2})(z_1^{\delta'_1} v_{4j}^{\delta_2} - \sqrt{2} z_2^{\delta'_1} v_{3j}^{\delta_2})\}. \end{aligned} \tag{3.6}$$

Using (3.6) we get

$$\begin{aligned}
(A) \quad = \quad & \left(\frac{1}{4}|J(Z_1)V_2|^2 - \frac{1}{8(l+1)}|V_2|^2|Z_1|^2\right) + \left(\frac{1}{4}|J(Z_2)V_1|^2 - \frac{1}{8(l+1)}|V_1|^2|Z_2|^2\right) \\
& -2\left(\frac{1}{4}\langle J(Z_1)V_1, J(Z_2)V_2\rangle - \frac{1}{8(l+1)}\langle V_1, V_2\rangle\langle Z_1, Z_2\rangle\right) \\
= \quad & \frac{-1}{16(l+1)}\sum_{j=1}^{l-2}\Big[\{(z_1^1 v_{1j}^2 - \sqrt{2}z_3^1 v_{2j}^2) - (z_1^2 v_{1j}^1 - \sqrt{2}z_3^2 v_{2j}^1)\}^2 \\
& + \{(z_1^1 v_{2j}^2 - \sqrt{2}z_2^1 v_{1j}^2) - (z_1^2 v_{2j}^1 - \sqrt{2}z_2^2 v_{1j}^1)\}^2 \\
& + \{(z_1^1 v_{3j}^2 - \sqrt{2}z_3^1 v_{4j}^2) - (z_1^2 v_{3j}^1 - \sqrt{2}z_3^2 v_{4j}^1)\}^2 \\
& + \{(z_1^1 v_{4j}^2 - \sqrt{2}z_2^1 v_{3j}^2) - (z_1^2 v_{4j}^1 - \sqrt{2}z_2^2 v_{3j}^1)\}^2\Big] \\
\leq 0.
\end{aligned}$$

To prove $(B) \leq 0$ it is sufficient to show the following inequality (see (3.4)):

$$|\langle J(Z_2)V_1, J(Z_1)V_2\rangle - \langle J(Z_1)V_1, J(Z_2)V_2\rangle| \leq \frac{1}{2(l+1)}|Z_1||Z_2||V_1||V_2| \tag{3.7}$$

for $Z_1, Z_2 \in \mathfrak{z}$, $V_1, V_2 \in \mathfrak{v}$. (3.7) can be proved in the same way as the proof of (3.3). In fact the left-hand side of (3.7) is given by

$$\begin{aligned}
\frac{\sqrt{2}}{4(l+1)} \Bigg| & \sum_{j=1}^{l-2}(z_1^1 z_2^2 - z_1^2 z_2^1)(v_{1j}^2 v_{2j}^1 - v_{1j}^1 v_{2j}^2) + \sum_{j=1}^{l-2}(z_1^1 z_3^2 - z_1^2 z_3^1)(v_{1j}^1 v_{2j}^2 - v_{1j}^2 v_{2j}^1) \\
& + \sum_{j=1}^{l-2}(z_1^1 z_2^2 - z_1^2 z_2^1)(v_{3j}^2 v_{4j}^1 - v_{3j}^1 v_{4j}^2) + \sum_{j=1}^{l-2}(z_1^1 z_3^2 - z_1^2 z_3^1)(v_{3j}^1 v_{4j}^2 - v_{3j}^2 v_{4j}^1)\Bigg|. \qquad \square
\end{aligned}$$

References

[1] D.V. Alekseevskii: *Homogeneous Riemannian spaces of negative curvature*, Mat. Sb. **25** (1975), 87–109; English translation: Math. USSR-Sb **96** (1975), 93–117.

[2] J. Berndt, F. Tricerri and L. Vanhecke: Generalized Heisenberg Groups and Damek-Ricci Harmonic Spaces, Springer-Verlag 1995.

[3] J. Boggino: *Generalized Heisenberg groups and solvmanifolds naturally associated*, Rend. Sem. Mat. Univ. Politec. Torino **43** (1985), 529–547.

[4] E. Damek and F. Ricci: *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. **27** (1992), 139–142.

[5] P. Eberlein: *Geometry of 2-step nilpotent groups with a left invariant metric*, Ann. Sci. École Norm. Sup. (4) **27** (1994), 611–660.

[6] E. Heintze: *On homogeneous manifolds of negative curvature*, Math. Ann. **211** (1974), 23–34.
[7] A. Kaplan: *Riemannian nilmanifolds attached to Clifford modules*, Geom. Dedicata **11** (1981), 127–136.
[8] A. Kaplan: *On the geometry of groups of Heisenberg type*, Bull. London math. Soc. **15** (1983), 35–42.
[9] M. Lanzendorf: *Einstein metrics with nonpositive sectional curvature on extensions of Lie algebras of Heisenberg type*, Geom. Dedicata **66** (1997), 187–202.
[10] T. Wolter: *Einstein metrics on solvable groups*, Math. Z. **206** (1991), 457–471.
[11] K. Yamada: *Einstein metrics on certain solvable groups*, Master thesis, Osaka university (1996), in Japanese.

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