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SOME REMARKS ON POLYNOMIAL RINGS

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0. Introduction. Our main result of this paper is the following

Theorem. *Let A be a noetherian unique factorization domain and let R be an A-algebra of finite type such that with $(n-1)$-indeterminates $t_1, \cdots, t_{n-1}$, $R \otimes_A A[t_1, \cdots, t_{n-1}] \cong A[x_1, \cdots, x_n] \cong$ an n-dimensional polynomial ring over A. Then R is a one-dimensional polynomial ring over A.*

When A is a field, this is a special case of

Problem of Zariski. *Let k be a field of arbitrary characteristic, let $\boldsymbol{A}^n$ and $\boldsymbol{A}^m$ be the affine spaces over k of dimensions n and m respectively and let V be an affine variety over k such that $V \times_k \boldsymbol{A}^m \cong \boldsymbol{A}^n$. Then V is isomorphic to the affine space $\boldsymbol{A}^{n-m}$ of dimension $n-m$.*

The theorem was proved independently by S. Abhyankar, P. Eakin and W. Heinzer in [1]. However the author publishes this paper because he believes that his view point is different from theirs and because he hopes that the method employed in this paper would have a contribution to further investigations of higher dimensional case.

Our method is as follows: Since $\mathrm{Spec}(R[t_1, \cdots, t_{n-1}])$ is isomorphic to $\mathrm{Spec}(R) \times_A \mathrm{Spec}(A[t_1, \cdots, t_{n-1}])$, the $(n-1)$-product $G_{a,A}^{n-1}$ of the additive group scheme $G_{a,A}$ defined over A acts canonically on $\mathrm{Spec}(R[t_1, \cdots, t_{n-1}])$, hence on the n-dimensional affine space $\boldsymbol{A}^n$ over A and the ring R is recovered as the ring of invariants with respect to the action of $G_{a,A}^{n-1}$.

The crucial results are:

1° R is a unique factorization domain with units contained in A.

2° Let K be the quotient field of A. Then $R \otimes_A K$ is a polynomial ring of dimension 1 over K.

These results 1° and 2° combined will yield a proof of the above theorem.

Moreover we shall make one remark on relationship among the problem of Zariski, the conjecture of Serre and the Jacobian conjecture. (The last two conjectures will be given later.)

Throughout this paper, all rings are noetherian commutative rings with identity element and all homomorphisms of rings send the identity element to the identity element.

1. Let A be a ring and let $G^n_{a,A}$ be the n-direct product over A of the additive group scheme $G_{a,A}$ defined over A. Let R be an A-algebra and let σ: $G^n_{a,A}\times \mathrm{Spec}(R)\to\mathrm{Spec}(R)$ be an action of $G^n_{a,A}$ on $\mathrm{Spec}(R)$. Let $\Delta: R\to R\underset{A}{\otimes}A[t_1, \cdots, t_n]$ be the coaction associated with σ. Define a set of endomorphisms of the abelian group R, $\{D_\alpha \mid \alpha=(\alpha_1, \cdots, \alpha_n),\ \alpha_i\in \boldsymbol{Z}^+=\{0, 1, \cdots\}\}$, by

$$\Delta(r) = \sum_{\alpha\geqslant 0} D_\alpha(r)t^\alpha \qquad \text{for any} \quad r\in R$$

where $\alpha=(\alpha_1, \cdots, \alpha_n)$ with $\alpha_i\geqslant 0$ and $t^\alpha=t_1^{\alpha_1}\cdots t_n^{\alpha_n}$. Then D_α's satisfy the following properties:

1) D_α is an A-linear endomorphism of R.

2) D_0=the identity endomorphism, where $0=(0, \cdots, 0)$.

3) $D_\alpha(rr')=\sum_{\beta+\gamma=\alpha} D_\beta(r)D_\gamma(r')$ for $r, r'\in R$.

4) $D_\alpha D_\beta(r)=(\alpha_1, \beta_1)\cdots(\alpha_n, \beta_n)D_{\alpha+\beta}(r)$ where $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\beta=(\beta_1, \cdots, \beta_n)$ and (α_i, β_i) is the binary coefficient $\binom{\alpha_i+\beta_i}{\alpha_i}$.

5) For any element $r\in R$, there exists α_0 such that $D_\alpha(r)=0$ whenever $\alpha\geqslant\alpha_0$.

From these relations, we can derive easily the following: Let $D^n_i=D_{\varepsilon^n_i}$, where $\varepsilon^n_i=(0, \cdots, 0, n, 0, \cdots, 0)$ with a positive integer n placed on the i-th place and 0 elsewhere. Then $\mathcal{D}_i=\{D_0=id., D^1_i, \cdots, D^n_i, \cdots\}$ is an A-trivial iterative infinite higher derivation in R and we have $\mathcal{D}_i\mathcal{D}_j=\mathcal{D}_j\mathcal{D}_i$ for all i and j. Moreover, we have $D_{(\alpha_1, \cdots, \alpha_n)}=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$.

Conversely, suppose that we are given a set of A-trivial iterative infinite higher derivations $\{\mathcal{D}_i \mid (i=1, 2, \cdots, n)\}$ in R which satisfies the following properties:

1) $\mathcal{D}_i\mathcal{D}_j=\mathcal{D}_j\mathcal{D}_i$ for all i and j.

2) Let $D_\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ for $(\alpha_1, \cdots, \alpha_n)$. Then for any element r of R, there exists α_0 such that $D_\alpha(r)=0$ whenever $\alpha\geqslant\alpha_0$.

Then the A-linear homomorphism $\Delta: R\to R\underset{A}{\otimes}A[t_1, \cdots, t_n]$ defined by $\Delta(r)=\sum_{\alpha\geqslant 0} D_\alpha(r)t^\alpha$ for any $r\in R$ gives rise to an action of $G^n_{a,A}$ on Spec (R). For more details, we refer to [3].

With this description of an action of $G^n_{a,A}$, we have

Lemma 1. *Let A be a ring and let R be an A-algebra of finite type which is a unique factorization domain. Suppose that an action of $G^n_{a,A}$ on* Spec(R) *is given. Then the ring of invariants S in R with respect to the action of $G^n_{a,A}$ is a* *unique factorization domain too.*

Proof. Let s be an element of S. It suffices to show that if $s=r_1r_2$ with $r_1, r_2\in R$, then $r_1, r_2\in S$. Let Δ be the coaction of $G^n_{a,A}$ on R. Let $\Delta(r_i)=\sum_{\alpha\geq 0} D_\alpha(r_i)t^\alpha$, $i=1, 2$. Then $\Delta(s)=\Delta(r_1)\Delta(r_2)$. Put a lexicographic order in the set $\{\alpha=(\alpha_1, \cdots, \alpha_n) \mid \alpha_i\in \boldsymbol{Z}^+, (i=1, 2, \cdots, n)\}$ and let α_1 (or α_2) be the maximum of α such that $D_{\alpha_1}(r_1)\neq 0$ (or $D_{\alpha_2}(r_2)\neq 0$). Then $\alpha_1+\alpha_2$ is the maximum of α such that $D_\alpha(s)\neq 0$. On the other hand, $D_\alpha(s)=0$ for all $\alpha>0$. Then $D_{\alpha_1+\alpha_2}(s)\neq 0$ implies $\alpha_1=0$ and $\alpha_2=0$. Hence r_1 and $r_2\in S$. q.e.d.

Lemma 2. *Let k be an algebraically closed field of arbitrary characteristic and let $\boldsymbol{A}^n_k$ be the n-dimensional affine space over k. Assume that the $(n-1)$-product $G^{n-1}_{a,k}$ of the additive group scheme $G_{a,k}$ acts on $\boldsymbol{A}^n_k$ so that general orbits of $G^{n-1}_{a,k}$ are of dimension $n-1$. Then the ring of invariants in $k[x_1, \cdots, x_n]$ (=the affine ring of $\boldsymbol{A}^n_k$) by $G^{n-1}_{a,k}$ is a one-dimensional polynomial ring over k.*

Proof. Let R be the ring of invariants in $k[x_1, \cdots, x_n]$ by $G^{n-1}_{a,k}$. Then R is a unique factorization domain in virtue of Lemma 1. Let f be a non-zero element of R, which is not a constant and has a minimal total degree in $x_1, \cdots, x_n$. Then, $f-\alpha$ is an irreducible element for any $\alpha\in k$. If $R=k[f]$, our proof is done. Otherwise, take an element g in $R-k[f]$ with minimal total degree. Then, $g-\beta$ is an irreducible element in R, for any β of k. Take any k-rational point x_0 of $\boldsymbol{A}^n_k$ such that the dimension of the orbit of x_0 is $n-1$. Let $\alpha=f(x_0)$ and let $\beta=g(x_0)$. Let $V(f-\alpha)$ and $V(g-\beta)$ be the $G^{n-1}_{a,k}$-stable irreducible subvarieties in $\boldsymbol{A}^n_k$ defined by $f-\alpha$ and $g-\beta$ respectively. $V(f-\alpha)$ and $V(g-\beta)$ have non-empty intersection since they contain x_0. $V(f-\alpha)$ and $V(g-\beta)$ have dimension $n-1$. Hence, $V(f-\alpha)=V(g-\beta)=$the orbit of x_0. Then $f-\alpha$ must divide $g-\beta$. This is absurd since $g-\beta$ is irreducible. Therefore, $R=k[f]$.
q.e.d.

Lemma 3. *Let k be a field of characteristic $p>0$, let k' be a purely inseparable extension of k with $[k': k]=p$ and let R be a finitely generated k-algebra such that $R'=R\otimes_k k'=k'[x]$ is a one-dimensional polynomial ring over k', that R is a unique factorization domain and that R has a maximal ideal $\mathfrak{m}$ with $R/\mathfrak{m}\cong k$, i.e.,* Spec(R) *has a k-rational point. Then R is a one-dimensional polynomial ring over k.*

Proof. The proof consists of several steps.

(I) There exists an element a of k' such that $k'=k(a)$ and $a\notin k$. Then k' has a k-trivial derivation D such that $D(a)=1$ and $D^p=0$. k is then the ring of D-invariants in k' (we denote it by k'^D). Extend D onto $R'=R\otimes_k k'$ by $D(r\otimes\lambda)=r\otimes D(\lambda)$ for $r\in R$ and $\lambda\in k'$. Then D is an R-trivial derivation of R' such that $R'^D=R$ and $D^p=0$.

(II) Let $C(R)$ and $C(R')$ be the divisor class groups of R and R' respectively. Let L be the abelian group in R' consisting of logarithmic derivatives $D(z)/z\in R'$ for elements z in the quotient field of R' and let L_0 be the sub-abelian group of L consisting of logarithmic derivatives $D(u)/u$ for units u in R'. Then, by virtue of P. Samuel [5], we have an exact sequence of abelian groups,

$$0 \to L/L_0 \to C(R) \xrightarrow{j} C(R')$$

where j is the canonical homomorphism defined by $\cdot \otimes R'$. Since R and R' are unique factorization domains, $C(R)=C(R')=0$. Therefore $L=L_0$.
(III) Since R' is a free R-module of rank p, R' is a faithfully flat R-module. Hence $\mathfrak{m}R' \cap R=\mathfrak{m}$. Then $R/\mathfrak{m}\cong k \hookrightarrow R'/\mathfrak{m}R'$ and $k' \hookrightarrow R'/\mathfrak{m}R'$. Since $[R'/\mathfrak{m}R': R/\mathfrak{m}]=p$, we have $R'/\mathfrak{m}R'\cong k'$. Thus $\mathfrak{n}=\mathfrak{m}R'$ is a maximal ideal in R' such that $R'/\mathfrak{n}\cong k'$ and $\mathfrak{n}\cap R=\mathfrak{m}$.
(IV) Since $[R': R]=p$ and $[R': k[x^p]]=p^2$, we have $R\supsetneqq k[x^p]$. Let f be an element of $R-k[x^p]$ which is minimal in the degree with respect to x. Write $f=\alpha_0'+\alpha_1'x+\cdots+\alpha_n'x^n$ with $\alpha_i'\in k'$. We may assume that $x\equiv 0$ (modulo $\mathfrak{n}$) for the maximal ideal $\mathfrak{n}$ of R' fixed in (III). For otherwise, $x\equiv\alpha'$ (modulo $\mathfrak{n}$) for some element α' of k'. Then we have only to replace x by $x-\alpha'$. Then f (modulo $\mathfrak{m}$)$=\alpha_0'\in k$. Replacing f by $f-\alpha_0'$ we may assume that $f\equiv 0$ (modulo $\mathfrak{m}$).

Write $f=x^r g$ where $g=\alpha_r'+\cdots+\alpha_n'x^{n-r}$ with $\alpha_r'\neq 0$ and $r\geqslant 1$. Then $x\not| g$. Now applying D to f, we have $D(f)=x^r D(g)+rx^{r-1}gD(x)=0$. Hence $xD(g)=-rgD(x)$. Therefore we have either $D(x)/x\in R'$ or $r\equiv 0$ (modulo p). In the first case, there exists a unit u in R', *i.e.*, $u\in k'$ such that $D(x)/x=D(u)/u$ (cf. (II)). Then $D(x/u)=0$. Hence $x'=x/u\in R$. Therefore $R'=k'[x]=k'[x']$ and $R=k[x']$ because the derivation D on $R'=k'[x']$ acts just on the coefficient field k' with the variable x' left invariant. In this case, we are done. In the second case, $D(g)=0$. Hence $g\in R$. Taking account of the choice of f and of the fact that $\deg(g)<\deg(f)$, we have $g\in k[x^p]$. Since $r=pr'$, we have $f=g\cdot x^r=g\cdot(x^p)^{r'}\in k[x^p]$. This is a contradiction. Therefore, only the first case takes place. Thus we have shown that $R=k[x']$. q.e.d.

Now we can prove the result 2° mentioned in the introduction.

Lemma 4. *Let k be a field of arbitrary characteristic and let R be a finitely generated k-algebra such that with $(n-1)$-indeterminates $t_1, \cdots, t_{n-1}$, $R\otimes_k k[t_1, \cdots, t_{n-1}]=k[x_1, \cdots, x_n]$ (=an n-dimensional polynomial ring over k). Then R is a one-dimensional polynomial ring over k.*

Proof. The proof consists of several steps.
(I) Let $\bar{k}$ be an algebraic closure of k and let $\bar{R}=R\otimes_k \bar{k}$. Then $\bar{R}[t_1, \cdots, t_{n-1}]=\bar{k}[x_1, \cdots, x_n]$. Let $G_{a,\bar{k}}^{n-1}$ be the $(n-1)$-product of the additive group $G_{a,\bar{k}}$ over $\bar{k}$.

Define an action of $G_{a,\bar{k}}^{n-1}$ on $\mathrm{Spec}(\bar{R}[t_1,\cdots,t_{n-1}])$ defining its coaction by $\Delta(t_i)=t_i\otimes 1+1\otimes\tau_i$ for $i=1,2,\cdots,n-1$ and $\Delta(\bar{r})=\bar{r}\otimes 1$ for $\bar{r}\in\bar{R}$, where $\tau_1,\cdots,\tau_{n-1}$ are parameters of $G_{a,\bar{k}}^{n-1}$. Thus we get an action of $G_{a,\bar{k}}^{n-1}$ on the affine n-space $A_{\bar{k}}^n$ over $\bar{k}$ by which the ring of invariants is $\bar{R}$. By virtue of Lemma 2, we know that $\bar{R}=R\otimes_k\bar{k}=\bar{k}[t]$ is a one-dimensional polynomial ring over $\bar{k}$. Once we know that $\bar{R}=\bar{k}[t]$, there exists an algebraic extension k' of k such that $R'=R\otimes_k k'=k'[t]$.

(II) Let k_s be the separable closure of k in k'. Then there exists an extension k'' of k_s and an element a of k' such that $k'=k''(a)$, $a\notin k''$ and $a^p\in k''$. On the other hand, $R''=R\otimes_k k''$ is a unique factorization domain since $R''[t_1,\cdots,t_{n-1}]=k''[x_1,\cdots,x_n]$ (cf. Lemma 1). Moreover $\mathrm{Spec}(R'')$ has a k''-rational point, since R has a maximal ideal $\mathfrak{m}=(x_1,\cdots,x_n)\cap R$ such that $R/\mathfrak{m}\cong k$. Therefore applying Lemma 3, we get $R''=k''[t]$. Proceeding by induction on $[k':k_s]$, we know that $R_s=R\otimes_k k_s=k_s[t]$.

(III) Taking a normal extension of k containing k_s, we may assume that k_s is a finite Galois extension of k with group G. Note that for any $\sigma\in G$, $R_s^\sigma=R_s$ since $R_s^\sigma[t_1,\cdots,t_{n-1}]=R_s[t_1,\cdots,t_{n-1}]$. Since any k-automorphism of $k_s[t]$ is written in the form: $t\to at+b$ with $a\in k_s^*$ and $b\in k_s$, let $\sigma(t)=a(\sigma)t+b(\sigma)$ with $a(\sigma)\in k_s^*$ and $b(\sigma)\in k_s$ for $\sigma\in G$. Then we have $a(\sigma\tau)={}^\sigma a(\tau)a(\sigma)$ and $b(\sigma\tau)={}^\sigma a(\tau)b(\sigma)+{}^\sigma b(\tau)$ for $\sigma,\tau\in G$. Hence $a(\ )$ is a 1-cocycle of G with values in k_s^*. Since $H^1(G,k_s^*)=0$ (Theorem 90 of Hilbert), there exists an element c of k_s^* such that $a(\sigma)={}^\sigma c\cdot c^{-1}$. Then $b(\ )$ satisfies $({}^{\sigma\tau}c)^{-1}b(\sigma\tau)=({}^\sigma c)^{-1}b(\sigma)+{}^\sigma(({}^\tau c)^{-1}b(\tau))$. Hence $\{({}^\sigma c)^{-1}b(\sigma)|\}\sigma\in G\}$ is a 1-cocycle of G with values in k_s. Since $H^1(G,k_s)=0$, there exists an element d of k_s such that $({}^\sigma c)^{-1}b(\sigma)={}^\sigma d-d$. Then $\sigma(t)={}^\sigma c(c^{-1}t+{}^\sigma d-d)$. Let $t'=c^{-1}t-d$. Then $\sigma(t')=t'$. Therefore $t'\in R$. This implies that $R=k[t']$. Consequently, we have proved that R is a one-dimensional polynomial ring over k. q.e.d.

We are now ready to prove the theorem. But before going to the proof of the theorem, we shall give a result which can be easily derived from Lemma 4.

Corollary 5. *Let k be a field of arbitrary characteristic p and let A_k^2 be the affine plane over k. Assume that the additive group scheme $G_{a,k}$ acts freely on A_k^2,* i.e., (σ,p_2): $G_{a,k}\times A_k^2\to A_k^2\times A_k^2$ *is a closed immersion, where σ is the action of $G_{a,k}$ on A_k^2. Then the ring of $G_{a,k}$-invariants in the affine ring of A_k^2 is a one-dimensional polynomial ring over k. Moreover, the action of $G_{a,k}$ is given as follows: There exists a pair of elements (x,y) in the affine ring of A_k^2 such that* (1) *$k[x,y]=$ the affine ring of A_k^2,* (2) *x is left fixed by $G_{a,k}$ and* (3) *${}^ty=y+ct$ (when the characteristic $p=0$) or ${}^ty=y+f_0(x)t+f_1(x)t^p+\cdots+f_n(x)t^{p^n}$ (when the characteristic $p>0$), where $t\in G_{a,k}$, $c\in k^*$, $f_0(x),\cdots,f_n(x)\in k[x]$ and where $f_0(x),\cdots,f_n(x)$ have no* *common roots in an algebraic closure of k.*

Proof. Let Q be the $(fpqc)$-quotient sheaf of A_k^2 by the action σ of $G_{a,k}$. Let $\bar{k}$ be an algebraic closure of k. Then $\bar{Q}=Q\otimes_k \bar{k}$ is the $(fpqc)$-quotient sheaf of $A_{\bar{k}}^2$ by the action $\bar{\sigma}=\sigma\otimes_k \bar{k}$ of $G_{a,\bar{k}}$. By virtue of [4], Example 4.24, $\bar{Q}$ is representable by an affine $\bar{k}$-scheme and $(\bar{\sigma}, p_2)$: $G_{a,\bar{k}}\times_{\bar{k}} A_{\bar{k}}^2 \to A_{\bar{k}}^2 \times_{\bar{Q}} A_{\bar{k}}^2$ is an isomorphism. Then the $(fpqc)$-descent theory for affine schemes shows that Q is representable by an affine k-scheme and that (σ, p_2): $G_{a,k}\times_k A_k^2 \to A_k^2 \times_Q A_k^2$ is an isomorphism. This implies that A_k^2 is a $G_{a,k}$-principal homogeneous space over Q. Hence $A_k^2=G_{a,k}\times_k Q$. Let R be the affine ring of Q and let y be a parameter of $G_{a,k}$. Then $R[y]$ is a two-dimensional polynomial ring over k. Applying Lemma 4, R is a one-dimensional polynomial ring over k, $k[x]$, and the affine ring of A_k^2 is isomorphic to $k[x, y]$. The remaining of Corollary 5 is now easy to prove (cf. [3], [4]). q.e.d.

2. **The proof of the theorem.** First of all, note that R is a unique factorization domiain (cf. lemma 2). Let K be the quotient field of A and let $R_K=R\otimes_A K$. Then we have $R_K[t_1, \cdots, t_{n-1}]=K[x_1, \cdots, x_n]$. By virtue of Lemma 4, $R_K=K[t]$ for some element t of R which is algebraically independent over K. We may assume that t is not divisible by any irreducible element of A. Moreover, since $R[t_1, \cdots, t_{n-1}]=A[x_1, \cdots, x_n]$, Spec$(R)$ has an A-rational point. Namely R has a prime ideal $\mathfrak{p}$ such that $R/\mathfrak{p}\cong A$. In fact, $\mathfrak{p}=R\cap(x_1, \cdots, x_n)A[x_1, \cdots, x_n]$. Then $t\equiv a$ (modulo $\mathfrak{p}$) for some element a of A. Replacing t by $t-$a, we may assume that $t\equiv 0$ (modulo $\mathfrak{p}$). With this situation, we shall prove that $R=A[t]$. Assume that $R\supsetneqq A[t]$. Note here that since $R_K=K[t]$, for any element u of R, there exists an element v of A such that $vu\in A[t]$. For any element u of R, we define: $\deg_t u=\deg_t(vu)$, where v is an element of A such that $vu\in A[t]$. Among elements u of $R-A[t]$ with minimal degree in t, take an element u_0 satisfying the following property: There exists an element v of A such that $vu_0\in A[t]$ and that in a decomposition $v=cv_1^{\alpha_1}\cdots v_r^{\alpha_r}(\alpha_1, \cdots, \alpha_r\geqslant 1)$ with a unit c of A and with mutually distinct irreducible elements $v_1, \cdots, v_r$, the number r is minimal.

Let $vu_0=a_0+a_1t+\cdots+a_lt^l$ with $a_0, a_1 \cdots, a_l\in A$. Consider this relation modulo $v_1^{\alpha_1}A[x_1, \cdots, x_n]+(x_1, \cdots, x_n)A[x_1, \cdots, x_n]$. Since vu_0 and t belong to $v_1^{\alpha_1}A[x_1, \cdots, x_n]+(x_1, \cdots, x_n)A[x_1, \cdots, x_n]$, we have $a_0\in A\cap(v_1^{\alpha_1}A[x_1, \cdots, x_n]+(x_1, \cdots, x_n)A[x_1, \cdots, x_n])=v_1^{\alpha_1}A$. Thus $a_0=v_1^{\alpha_1}a_0'$ with $a_0'\in A$. Let $u'=-a_0'+cv_2^{\alpha_2}\cdots v_r^{\alpha_r}u_0$. Then we have $v_1^{\alpha_1}u'=t(a_1+a_2t+\cdots+a_lt^{l-1})$. Since t and v_1 have no common divisors in R other than units, t must divide u'. Let $u'=u''t$ with $u''\in R$. Then we have $v_1^{\alpha_1}u''=a_1+a_2t+\cdots+a_lt^{l-1}$. From the choice of u_0, $u''\in A[t]$. Hence we have $(cv_2^{\alpha_2}\cdots v_r^{\alpha_r})u_0\in A[t]$. Again from the choice of v, $u_0\in$

$A[t]$. This is a contradiction. Therefore we have shown that $R=A[t]$. q.e.d.

3. In this section, we shall show

Proposition. *Let k be a field of characteristic zero, let V be an affine variety over k such that $V\times \boldsymbol{A}^m \cong \boldsymbol{A}^n$ and let $R=k[x_1, \cdots, x_n]$ be an n-dimensional polynomial ring over k which is the affine algebra of $\boldsymbol{A}^n$. Then V is k-isomorphic to $\boldsymbol{A}^{n-m}$ if one assumes the following two conjectures:*

Conjecture of Serre: Let P be a finitely generated projective module over R with rank $n-m$. Then P is R-free.

Jacobian conjecture: Let $(f_1, \cdots, f_n)$ be a set of elements of R such that the Jacobian $J(f_1, \cdots, f_n/x_1, \cdots, x_n)$ is a non-zero constant. Then $R=k[f_1, \cdots, f_n]$.

Proof. Denote by $R^{(a)}$ a free R-module of rank a. With the affine algebra B of V, we have an exact sequence of R-modules,

$$0 \to \Omega^1_{B/k} \otimes_B R \to R^{(n)} \to R^{(m)} \to 0$$

which follows from the well known exact sequence of modules of Kähler 1-differentials applied to the projection q: $V \times_k \boldsymbol{A}^m \to V$,

$$0 \to q^*\Omega^1_{V/k} \to \Omega^1_{A^n/k} \to \Omega^1_{A^n/V} \to 0 .$$

Then $(\Omega^1_{B/k} \otimes_B R) \oplus R^{(m)} \cong R^{(n)}$. Since $\Omega^1_{B/k} \otimes_B R$ is a finitely generated projective R-module of rank $n-m$, the assumed conjecture of Serre implies that $\Omega^1_{B/k} \otimes_B R$ is a free R-module. On the other hand, since R is an m-dimensional polynomial ring $B[t_1, \cdots, t_m]$ over B, we can conclude easily that $\Omega^1_{B/k}$ is a free B-module of rank $n-m$. {In fact, let $e_1, \cdots, e_{n-m}$ be a R-free basis of $\Omega^1_{B/k} \otimes_B R$ and write $e_i = \sum_j a_{ij} f_j$ with $a_{ij} \in B[t_1, \cdots, t_m]$ and $f_j \in \Omega^1_{B/k}$. Let α_{ij} be the constant term of a_{ij}. Then $e_i' = \sum_j \alpha_{ij} f_j$ $(i=1, \cdots, n-m)$ form a B-free basis of $\Omega^1_{B/k}$}.

Let $db_1, \cdots, db_{n-m}$ be a B-free basis of $\Omega^1_{B/k}$. Then $db_1, \cdots, db_{n-m}, dt_1, \cdots, dt_m$ form a R-free basis of $R^{(n)} = \Omega^1_{A^n/k}$. Hence $b_1, \cdots, b_{n-m}$ and $t_1, \cdots, t_m$ are polynomials in $x_1, \cdots, x_n$ with coefficients in k such that the Jacobian $J(b_1, \cdots, b_{n-m}, t_1, \cdots, t_m/x_1, \cdots, x_n)$ is a non-zero constant. Apply here the assumed Jacobian conjecture which asserts $k[b_1, \cdots, b_{n-m}, t_1, \cdots, t_m] = k[x_1, \cdots, x_n]$. On the other hand, $B[t_1, \cdots, t_m] = k[x_1, \cdots, x_n]$. Thence we conclude easily that $B = k[b_1, \cdots, b_{n-m}]$. This shows that $V \cong \boldsymbol{A}^{n-m}$. q.e.d.

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