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Abstract

We discuss the simple constituents of Specht module $S^\lambda$ for the symmetric group $S_n$ defined over the field of $p$ elements. We firstly give an easier proof to the result in [6] which asserts that there exists a simple constituent of $S^\lambda$ with the shape of “a branch” of $\lambda$ (Theorem 3.3), and secondly give a sufficient condition for $\lambda$ to have a particular type branch as a constituent (Proposition 3.4).

1. Introduction

Let $n$ be a natural number and $p$ a prime. Let $S_n$ be the symmetric group on $n$ letters and $L$ a field of characteristic $p$. Given a partition $\lambda$ of $n$, we have an $L S_n$-module $S^\lambda$ called the Specht module corresponding to $\lambda$, which is not simple in general. However if the partition $\lambda$ is $p$-regular, the head of $S^\lambda$, denoted by $D^\lambda$, is simple and they cover all the non-isomorphic simple modules as $\lambda$ runs through the $p$-regular partitions of $n$.

One of the main concerns about the Specht modules is to have informations about the simple constituents of them. Especially, using information only on $\lambda$, we would like to describe a $p$-regular partition $\mu$ for which $D^\mu$ appears as a constituent of $S^\lambda$. For this purpose, it is useful to consider the operations on the partitions $\lambda$ introduced by James and Murphy [5], each of which is roughly interpreted as a rim hook removal followed by addition on the Young diagram corresponding to $\lambda$. We shall call each of the resulting partitions a branch of $\lambda$. The Jantzen-Schaper theorem tells that if $D^\mu$ is a constituent of $S^\lambda$, it follows that $\lambda = \mu$ or $\mu$ is obtained by making branches successively beginning with $\lambda$ (cf. [6, Corollary 1]). One of the authors showed that if $\lambda$ is $p$-regular, there is a $p$-regular branch $\mu$ of $\lambda$ such that $D^\mu$ is a constituent of $S^\lambda$ (cf. [6, Theorem 2]). And he gave some applications of the result in [7]. However the proof of the result cited above is rather long and complicated. In this paper we shall show a short proof to it and a result on simple constituents of the Specht modules as a byproduct of the proof.

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2. Preliminary results

A partition of the integer $n$ is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of non-negative integers whose sum is $n$. The Young diagram $[\lambda]$ associated with $\lambda$ is the set of the ordered pairs $(i, j)$ of integers, called the nodes of $[\lambda]$, with $1 \leq i \leq h$ and $1 \leq j \leq \lambda_i$, where $h$ denotes the largest number such that $\lambda_h \neq 0$. They are illustrated as arrays of squares. We denote by $\lambda'$ the partition conjugate of $\lambda$, so $[\lambda']$ is the transposed diagram of $[\lambda]$.

Let $c$ be a column number of $[\lambda]$ and $r$ a positive integer. Then $\lambda$ is said to be $r$-singular on column $c$ if there is an integer $i \geq 0$ such that $\lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_{i+r} = c$, and is $r$-regular on column $c$ if otherwise. We also say that $\lambda$ is $r$-singular if it is $r$-singular on some column, and is $r$-regular if otherwise. For the convenience of later arguments, we understand that every partition is $r$-regular on column 0. We denote by $P(n)$ and $P(n)_0$ the sets of the partitions and $p$-regular partitions of $n$ respectively. The dominance order $\preceq$ on $P(n)$ is defined as follows: given $\lambda, \mu \in P(n)$, $\lambda \preceq \mu$ if and only if $\sum_{1 \leq i \leq j} \lambda_i \leq \sum_{1 \leq i \leq j} \mu_i$ for all $j \geq 1$.

The $(i, j)$-hook of the Young diagram $[\lambda]$ consists of the $(i, j)$-node along with the $\lambda_i - j$ nodes to the right of it (called the arm of the hook) and the $\lambda'_j - i$ nodes below it. The length of the $(i, j)$-hook of $\lambda$ is $h_{ij}(\lambda) := \lambda_i + \lambda'_j + 1 - i - j$. An $(i, j)$-rim hook is a connected part of the rim of $[\lambda]$ of length $h_{ij}(\lambda)$ beginning at the node $(\lambda'_j, j)$. We also call the integer $\lambda_i - j$ the arm length of the node $(i, j)$. Moreover, a hook of $[\lambda]$ is called a pillar if its arm length is zero.

Let $(b, c)$ is a node of $[\lambda]$ and suppose that $a < b$. We let $\lambda(a, b, c)$ be the partition of $n$ obtained from $\lambda$ by unwrapping the $(b, c)$-rim hook of $[\lambda]$ and wrapping the nodes back with the lowest nodes in the added rim hook lying on row $a$ (if the resulting partition fails to be a non-increasing sequence of integers, $\lambda(a, b, c)$ is not defined). We occasionally write $\lambda(a, b, c, g)$ if the highest node in the added rim hook lies in row $g$. We call here each $\lambda(a, b, c)$ a branch of $\lambda$, and set

$$\Gamma_\lambda := \{\lambda(a, b, c); v_p(h_{ac}(\lambda)) \neq v_p(h_{bc}(\lambda))\}, \quad \Gamma^0_\lambda := \Gamma_\lambda \cap P(n)_0,$$

where $v_p(m)$ denotes the largest integer $e$ such that $p^e$ divides the integer $m$.

A branch $\mu = \lambda(a, b, c)$ is called a pillar type branch if the rim hook which has been removed and the rim hook which has been added are both pillars. Suppose that $\mu = \lambda(a, b, c)$ is a pillar type branch and put $d := \lambda_a + 1$, $q := h_{bc}(\lambda)$. Then $\mu$ is obtained by unwrapping the pillar of $q$ nodes from column $c$ and wrapping it back on column $d$ (with the lowest node on row $a$). Hence we sometimes write $\mu = \lambda(c \mid d, q)$ for simplicity. For $\lambda \in P(n)$, let $SC(S^\lambda)$ be the set of simple constituents of the Specht module $S^\lambda$.

Remark. Let $\lambda \in P(n)_0$. Then if $\mu = \lambda(a, b, c)$ is a pillar type branch of $[\lambda]$, we have $h_{bc}(\lambda) \leq p - 1$. Hence $\mu$ lies in $\Gamma_\lambda$ if and only if $h_{bc}(\lambda)$ is divisible by $p$. 476 Y. HIEDA AND Y. TSUSHIMA
Now we list below some results for later use.

**Theorem 2.1** ([2], [3]). Let $\lambda \in P(n)^0$. Then $S^\lambda$ is simple if and only if $v_p(h_{ab}(\lambda)) = v_p(h_{bc}(\lambda))$ for all $a, b, c \geq 1$.

**Theorem 2.2** (Carter and Payne [1]). Suppose that $\alpha := \lambda(c \mid d, q)$ be a pillar type branch of $\lambda$ and let $a$ be the row index of $[\lambda]$ such that $d = \lambda_a + 1$. Put $e := v_p(h_{ae}(\lambda))$. If $p^e > q$, we have

$$\text{Hom}_G(S^\alpha, S^\lambda) \neq 0.$$  

In particular, it follows that $D^\alpha \in \text{SC}(S^\lambda)$ if $\alpha$ is $p$-regular.

**REMARK.** The above statement is slightly different from the corresponding theorem in [1], but can be deduced easily from it. In fact, if $\lambda$ and $\alpha$ are the same as above then with the languages in [1], $\lambda'$ is obtained from $\alpha'$ by raising $q$ nodes from row $d$ to row $c$, whence we have $\text{Hom}_G(S^\lambda', S^\alpha') \neq 0$. The rest of the proof will be done by routine arguments, using that $S^{\lambda'}$ is isomorphic to the $L$-dual of $S^\lambda \otimes S^{\lambda'}$ ([2, Theorem 8.15]).

**Theorem 2.3** ([4, Theorem 6]). Let $\lambda, \mu$ be partitions of $n$ with $\lambda$ $p$-regular. Suppose that there is a number $k$ $(1 \leq k \leq \lambda_1, \mu_1)$ such that the subdiagrams consisting of the first $k$ columns of $[\lambda]$ and $[\mu]$ are the same and that each has $m$ nodes. Let $[\lambda']$ ($[\mu']$ resp.) be the subdiagram to the right of column $k$ of $[\lambda]$ ($[\mu]$ resp.). Then the composition multiplicity of $D^\lambda$ in $S^\mu$ as $S_n$-modules equals the composition multiplicity of $D^\mu$ in $S^\lambda$ as $S_{n-m}$-modules.

**Proposition 2.4** (Jantzen-Schaper, cf. [6, Corollary 1]). Let $\lambda \in P(n)$ and let $\mu$ be a minimal element of $\Gamma_\lambda$ with respect to the dominance order. If $\mu$ is $p$-regular, $D^\mu \in \text{SC}(S^\lambda)$.

**Proposition 2.5** ([6, Proposition 3]). Let $\lambda \in P(n)^0$ and let $[\mu]$ be the diagram to the right of the first column of $[\lambda]$. If $S^\mu$ is simple, $\Gamma_\lambda$ has no $p$-singular partition.

3. **Finding simple constituents of Specht modules**

We shall show a short proof to Theorem 2 of [6] and a result on simple constituents of the Specht modules. First we show

**Lemma 3.1.** Let $\lambda \in P(n)^0$. If there is a pillar type branch $\mu = \lambda(a, b, c) \in \Gamma_\lambda$ such that $\mu$ is $p$-regular on column $c - 1$, there is a pillar type branch $\tilde{\lambda}$ in $\Gamma_\lambda^0$. [Proof follows here]
Proof. We put \( r := h_{bc}(\lambda) \leq p - 1 \) and \( f := \lambda_{a} \). Note that \( h_{a}(\lambda) \) is a multiple of \( p \) since \( \mu \in \Gamma_{\lambda} \). We may assume that \( \mu \) is \( p \)-singular, so \( \mu \) is \( p \)-singular on column \( f + 1 \) by the assumption. (In the above diagram a circle in a node indicates that the hook length at the node is divisible by \( p \).

Namely \( a - \lambda'_{f+2} \geq p \), so \( a - p + 1 \geq \lambda'_{f+2} \). Put \( s_1 := a - p + 1 - \lambda'_{f+2} \geq 1 \). Then \( r - s_1 = (p - 1) - (a - r) + \lambda'_{f+2} = (p - 1) - (\lambda'_{f+1} - \lambda'_{f+2}) \geq 0 \), so \( r \geq s_1 \). Now let \( \mu(1) = \lambda(c \mid f + 2, s_1) \), which lies in \( \Gamma_{\lambda} \) since \( h_{a-r+1,c}(\lambda) \) is divisible by \( p \). Note that \( \mu(1) \) is \( p \)-regular on column \( c - 1 \). If \( \mu(1) \) is \( p \)-regular, we may take \( \mu(1) = \tilde{\lambda} \). Hence we may assume that \( \mu(1) \) is \( p \)-singular, so \( \lambda'_{f+2} \neq 0 \) and \( \mu(1) \) is \( p \)-singular on column \( f + 2 \). Namely \( (a - p + 1) - \lambda'_{f+3} \geq p \), so \( a - 2p + 2 > \lambda'_{f+3} \). Put \( s_2 := a - 2p + 2 - \lambda'_{f+3} \geq 1 \). Then \( s_1 - s_2 = (p - 1) - (\lambda'_{f+2} - \lambda'_{f+3}) \geq 0 \), so \( s_1 \geq s_2 \). Now let \( \mu(2) = \lambda(c \mid f + 3, s_2) \), which lies in \( \Gamma_{\lambda} \) since \( h_{a-2p+2,c}(\lambda) \) is divisible by \( p \). Note that \( \mu(2) \) is also \( p \)-regular on column \( c - 1 \). By repeating similar arguments we finally obtain a \( p \)-regular pillar type branch \( \mu(i) \) for some \( i \), completing the proof of the lemma.

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\begin{itemize}
  \item \textbf{Lemma 3.2.} Let \( \lambda \in P(n)^{0} \). If there is a branch \( \mu = \lambda(a, b, c) \in \Gamma_{\lambda} \) with \( c \geq 2 \) such that \( \mu \) is \( p \)-singular on column \( c - 1 \), there is a pillar type branch \( \tilde{\lambda} \) in \( \Gamma_{\lambda}^{0} \).

  \begin{itemize}
    \item \textbf{Proof.} We may assume that \( \mu \) is chosen so that \( c \) is the smallest and put \( r := \lambda'_{c-1} - \lambda'_{c} \). Note that the \( (b, c) \)-rim hook of \( [\lambda] \) is a pillar if and only if \( b > \lambda'_{c+1} \).
    \begin{enumerate}
      \item \textbf{CASE I.} \( r \leq p - 2 \).
        \begin{itemize}
          \item \textbf{As \( \mu \) is \( p \)-singular on column \( c - 1 \), \( \lambda'_{c-1} - \lambda'_{c+1} \geq p - 1 \). Put \( x := (\lambda'_{c-1} - p + 2) - \lambda'_{c+1} \) and \( y := \lambda'_{c} - (\lambda'_{c-1} - p + 2) \), so \( x \geq 1 \) and \( r + y = p - 2 \).
          \begin{itemize}
            \item \textbf{SUBCASE (i)} \( x + y \leq p - 2 \).
            \begin{itemize}
              \item We have that \( x \leq r \) from \( x + y \leq p - 2 = r + y \). Now let \( \gamma = \lambda(c-1 \mid c+1, x) \in \Gamma_{\lambda} \).
              \item If \( \gamma \) is \( p \)-regular, we may take \( \gamma \) as \( \tilde{\lambda} \). Hence we may assume that \( \gamma \) is \( p \)-singular.
          \end{itemize}
          \end{itemize}
        \end{itemize}
    \end{enumerate}
  \end{itemize}
\end{itemize}
Then by the minimality of $c$, $\gamma$ must be $p$-regular on column $c - 2$ and there is a pillar type branch $\tilde{\lambda} \in \Gamma^0_{\lambda}$ by Lemma 3.1, as asserted. (In the above diagrams the boldface rim hooks will be removed to make $\mu$.)

**SUBCASE (ii)** $x + y = p - 1$.

We have $x = r + 1$. As $\mu \in \Gamma_{\lambda}$, either $h_{ic}(\lambda)$ or $h_{bc}(\lambda)$ is divisible by $p$. Let $i = a$ or $i = b$ according as $h_{ic}(\lambda)$ is divisible by $p$ or not. Let furthermore $f = \lambda_i$ and $s_i = i - \lambda'_{f+1}$. Then we see that $i \leq \lambda'_{c+1}$ since $h_{ic}(\lambda)$ is divisible by $p$. Since $s_i \leq p - 1$, we can make the pillar type branch $\gamma = \lambda(c \mid f + 1, s_i) \in \Gamma_{\lambda}$. If $s_i + r < p$, $\gamma$ is $p$-regular on column $c - 1$ and the assertion follows by Lemma 3.1. Now suppose that $s_i + r \geq p$ and put $t_i = s_i - (p - r - 1)$, so $t_i \geq 1$. Also $\lambda'_{f+1} = i - s_i < i - (p - r - 1) = i - p + r + 1$. Note that $t_i \leq r$ since $r - t_i = p - 1 - s_i \geq 0$. Hence we can make the pillar type branch $\delta = \lambda(c - 1 \mid f + 1, t_i)$, which lies in $\Gamma_{\lambda}$ since $h_{i-p+t_i+1,c-1}(\lambda) = (r + 1) + h_{ic}(\lambda) + (p - r - 1) = h_{ic}(\lambda) + p$ is divisible by $p$. By the minimality of $c$, $\delta$ is $p$-regular on column $c - 2$ and so there is a pillar type branch $\tilde{\lambda} \in \Gamma^0_{\lambda}$ by Lemma 3.1, as asserted.

**CASE II.** $r = p - 1$.

We use the same notation as in subcase (ii). Then $h_{i,c-1}(\lambda)$ is divisible by $p$, since $h_{i,c-1}(\lambda) = h_{ic}(\lambda) + p$. In the diagram below, we have $r = p - 1$, so we can make the pillar type branch $\gamma = \lambda(c - 1 \mid f + 1, s_i) \in \Gamma_{\lambda}$. By the minimality of $c$, $\gamma$ is $p$-regular on column $c - 2$ and the assertion follows by Lemma 3.1. This completes the proof of the lemma.
Now we are ready to give an alternative proof of the following theorem:

**Theorem 3.3** ([6, Theorem 2]). Let $\lambda$ be a $p$-regular partition of $n$. If $S^\lambda$ is reducible, there is a $p$-regular branch $\widetilde{\lambda} \in \Gamma_\lambda$ such that $D^{\tilde{\lambda}} \in \text{SC}(S^\lambda)$.

Proof. Since $S^\lambda$ is reducible, there is a column number $c$ such that $v_p(h_{a,c}(\lambda)) \neq v_p(h_{b,c}(\lambda))$ for some $a, b$ with $1 \leq a, b \leq \lambda'_c$. Let $c$ be the largest number that satisfies the condition. Let $[\delta]$ be the subdiagram of $[\lambda]$ with column $c$ as the first column, $[\gamma]$ the remaining diagram and write $\lambda = (\gamma, \delta)$. Then every branch in $\Gamma_\delta$ is $p$-regular by Proposition 2.5. Hence, if $\tilde{\delta}$ is a minimal element of $\Gamma_\delta$ with respect to the dominance order, $D^{\tilde{\delta}} \in \text{SC}(S^\delta)$ by a direct consequence of the Jantzen-Schaper theorem (see Proposition 2.4). Put $\mu := (\gamma, \tilde{\delta}) \in \Gamma_\lambda$. If $\mu$ is $p$-singular on column $c - 1$, then $c$ must be greater than 1 and there is a pillar type branch $\widetilde{\lambda} \in \Gamma^0_\lambda$ by Lemma 3.2. Thus we have $D^{\tilde{\lambda}} \in \text{SC}(S^\lambda)$ by the Carter and Payne theorem (see Theorem 2.2). So we may assume that $\mu$ is $p$-regular on column $c - 1$. Then $\mu \in \Gamma^0_\lambda$ and we have $D^{\mu} \in \text{SC}(S^\lambda)$ by Theorem 2.3. This completes the proof of the theorem.
Now a node \((b, c)\) is called a \((p, 1)\)-point of \([\lambda]\) with arm length one, if \(h_{bc}(\lambda) = p\) and \(h_{\lambda',c}(\lambda) = 1\).

**Proposition 3.4.** Suppose \(p > 2\) and that \(S^\lambda\) is not simple. Let \(\lambda\) be a \((p - 1)\)-regular partition of \(n\). Then

1. If \([\lambda]\) has no \((p, 1)\)-point with arm length one, we have \(\Gamma_\lambda^0 = \Gamma_\lambda\). Hence \(D^\mu \in \text{SC}(S^\lambda)\) for any minimal element \(\mu\) of \(\Gamma_\lambda\) with respect to the dominance order.
2. If \([\lambda]\) has a \((p, 1)\)-point with arm length one, there is a pillar type branch \(\mu = \lambda(c \mid d, q)\) such that \(D^\mu \in \text{SC}(S^\lambda)\) for some \(c, d, q\) with \(q \leq p - 2\).

**Proof.** (1) The second half follows immediately from the first half and Proposition 2.4. So we need only prove the first half. Suppose the contrary and take a \(p\)-singular branch, say \(\mu = \lambda(a, b, c, g)\), from \(\Gamma_\lambda\).

**CASE I.** \(\mu\) is \(p\)-singular on column \(c - 1\) (hence \(c \geq 2\)).

Since \(\lambda\) is \((p - 1)\)-regular, it follows that \(\lambda'_c - 1 - p + 2 \leq \lambda'_c\) and so \((\lambda'_c - 1 - p + 2, c - 1)\) is a \((p, 1)\)-point of \([\lambda]\) with arm length one, being contrary to the assumption.

**CASE II.** \(\mu\) is \(p\)-singular on column \(\lambda_g - 1\) (hence \(g \geq 2\)).

As \(\lambda\) is \((p - 1)\)-regular, we find easily that \(\lambda_{g-1} = \lambda_g + 1\). Let \(f = \lambda_{g-1}\).
Then $\lambda'_{f-1} - \lambda'_{f+1} \geq p - 1$, and the node $(\lambda'_{f-1} - p + 2, f - 1)$ is a $\langle p, 1 \rangle$-point of $[\lambda]$ with arm length one, being contrary to the assumption. This completes the proof of (1).

(2) Let $(i, j)$ be a $\langle p, 1 \rangle$-point of $[\lambda]$ with arm length one and $m := \lambda'_{j+1}$. Then $i \leq m < \lambda'_{j} = i + p - 2$ and $\lambda_{m-1} = \lambda_{m+1}$.

Now we assume that the above $(i, j)$ is chosen so that $j$ is the smallest. Let $m_1 := \lambda'_{j+2}$ and $r := h_{m+1,j}(\lambda) = i + p - 2 - m$. Then $m_1 < i$ since the node $(i, j)$ has arm length one. Let $r_1 := i - m_1$. Then $r - r_1 = (p - 2) - (m - m_1) \geq 0$, so $r_1 \leq r$. Therefore we can make the pillar type branch $\mu = \lambda(j \mid j + 2, r_1) \in \Gamma_{\lambda}$. If $\mu$ is $p$-singular on column $j - 1$, then $j$ must be greater than 1 and $(\lambda'_{j-1} - (p - 2), j - 1)$ is a $\langle p, 1 \rangle$-point of $[\lambda]$ with arm length one, contradicting the minimality of $j$. Hence $\mu$ is $p$-regular on column $j - 1$ and by Lemma 3.1, there is a pillar type branch in $\Gamma^0_{\lambda}$, whence the assertion follows by the Carter and Payne theorem.

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References


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