Title	Gröbner bases associated with positive roots and Catalan numbers
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Citation	Osaka Journal of Mathematics. 2005, 42(2), p. 421-433
Version Type	VoR
URL	https://doi.org/10.18910/12702
rights	
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Kitamura, T.
Osaka J. Math.
42 (2005), 421–433

GRÖBNER BASES ASSOCIATED WITH POSITIVE ROOTS AND CATALAN NUMBERS

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(Received July 8, 2003)

Abstract

Let $\mathbf{A}_{n-1}^{+} \subset \mathbb{Z}^n$ denote the set of positive roots of the root system $\mathbf{A}_{n-1}$ and $I_{\mathbf{A}_{n-1}^{+}}$ its toric ideal. The purpose of the present paper is to study combinatorics and algebra on $\mathbf{A}_{n-1}^{+}$ and $I_{\mathbf{A}_{n-1}^{+}}$. First, it will be proved that $I_{\mathbf{A}_{n-1}^{+}}$ induces an initial ideal $in_{<}\left(I_{\mathbf{A}_{n-1}^{+}}\right)$ which is generated by quadratic squarefree monomials together with cubic squarefree monomials. Second, we will associate each maximal face σ of the unimodular triangulation Δ arising from $in_{<}\left(I_{\mathbf{A}_{n-1}^{+}}\right)$ with a certain subgraph G_σ on $[n] = \{1, \ldots, n\}$. Third, noting that the number of maximal faces of Δ is equal to that of anti-standard trees T on $[n]$ with $T \neq \{(1,2),(1,3),\ldots,(1,n)\}$, an explicit bijection between the set $\{G_\sigma : \sigma$ is a maximal face of $\Delta\}$ and that of anti-standard trees T on $[n]$ with $T \neq \{(1,2),(1,3),\ldots,(1,n)\}$ will be constructed. In particular, a new combinatorial expression of Catalan numbers arises.

Introduction

In their study of hypergeometric functions associated with root systems, Gel'fand, Graev and Postnikov [5] studied combinatorics on the convex hull $\operatorname{conv}\left(\widetilde{\mathbf{A}}_{n-1}^{+}\right)$ of the configuration $\widetilde{\mathbf{A}}_{n-1}^{+} = \mathbf{A}_{n-1}^{+} \cup \{\mathbf{0}\} \subset \mathbb{Z}^n$, where $\mathbf{A}_{n-1}^{+}$ is the set of positive roots of the root system $\mathbf{A}_{n-1}$ and $\mathbf{0}$ is the origin of $\mathbb{R}^n$. It turned out that $\operatorname{conv}\left(\widetilde{\mathbf{A}}_{n-1}^{+}\right)$ possesses a unimodular triangulation, i.e., a triangulation such that the normalized volume of each of its maximal faces is equal to 1, and that there is an explicit bijection between the set of maximal faces of the unimodular triangulation and that of so-called "anti-standard trees" on the vertex set $[n] = \{1, \ldots, n\}$. Since the number of anti-standard trees on $[n]$ is the famous Catalan number $(1/n)\binom{2(n-1)}{n-1}$, it follows that the normalized volume of $\operatorname{conv}\left(\widetilde{\mathbf{A}}_{n-1}^{+}\right)$ is equal to $(1/n)\binom{2(n-1)}{n-1}$.

On the other hand, from the viewpoint of toric ideals, much more important results essentially appear in Gel'fand, Graev and Postnikov [5]. For example, it is proved that the toric ideal arising from the configuration $\widetilde{\mathbf{A}}_{n-1}^{+}$ induces a squarefree quadratic initial ideal. (A monomial ideal is said to be squarefree (resp. quadratic) if it is generated by squarefree (resp. quadratic) monomials.) In general, it is known in [11] that if the toric ideal arising from a configuration induces a squarefree initial ideal, then the convex hull of the configuration possesses a unimodular triangulation. Moreover, if the toric ideal arising from a configuration induces a quadratic initial ideal, then the toric ring of the configuration is a Koszul algebra (e.g., [1] and [2]).

In the recent paper [9], the existence of a squarefree quadratic initial ideal for each of the configurations $\widetilde{\mathbf{B}}_n^+ = \mathbf{B}_n^+ \cup \{\mathbf{0}\}$, $\widetilde{\mathbf{C}}_n^+ = \mathbf{C}_n^+ \cup \{\mathbf{0}\}$ and $\widetilde{\mathbf{D}}_n^+ = \mathbf{D}_n^+ \cup \{\mathbf{0}\}$ was proved, where $\mathbf{B}_n^+$ (resp. $\mathbf{C}_n^+$, $\mathbf{D}_n^+$) is the set of positive roots of the root system $\mathbf{B}_n$ (resp. $\mathbf{C}_n$, $\mathbf{D}_n$). Moreover, in [8], the problem on the existence of a squarefree initial ideal of the toric ideal of the configuration $\mathcal{A} \cup \{\mathbf{0}\}$, where $\mathcal{A}$ is a subset of $\mathbf{BC}_n^+ = \mathbf{B}_n^+ \cup \mathbf{C}_n^+$, was mainly discussed.

Stanley [10, Exercise 6.31 (b)] computed the Ehrhart polynomial of the convex hull of $\widetilde{\mathbf{A}}_{n-1}^+$. In her dissertation [4], Fong constructed an explicit triangulation of the convex hull of each of the configurations $\widetilde{\mathbf{B}}_n^+$, $\widetilde{\mathbf{C}}_n^+$ and $\widetilde{\mathbf{D}}_n^+$, and computed the normalized volume together with the Ehrhart polynomial of each of these convex hulls.

In the papers [4], [5], [8] and [9], the play of the origin is essential. For example, if $n \geq 6$, then the toric ideal of each of the configurations $\mathbf{A}_{n-1}^+$, $\mathbf{B}_n^+$, $\mathbf{C}_n^+$ and $\mathbf{D}_n^+$ can induce *no* quadratic initial ideal. However, it is reasonable to ask if the toric ideal of each of the configurations $\mathbf{A}_{n-1}^+$, $\mathbf{B}_n^+$, $\mathbf{C}_n^+$ and $\mathbf{D}_n^+$ induces a squarefree initial ideal.

The purpose of the present paper is to study combinatorics and algebra on the configuration $\mathbf{A}_{n-1}^+$ and its toric ideal $I_{\mathbf{A}_{n-1}^+}$. First, it will be proved that $I_{\mathbf{A}_{n-1}^+}$ induces an initial ideal $in_<\left(I_{\mathbf{A}_{n-1}^+}\right)$ which is generated by quadratic squarefree monomials together with cubic squarefree monomials (Theorem 1.1). Second, we will associate each maximal face σ of the unimodular triangulation Δ arising from $in_<\left(I_{\mathbf{A}_{n-1}^+}\right)$ with a certain subgraph G_σ on $[n]$ (Theorem 2.3). On the other hand, it is easy to see that the normalized volume of $\operatorname{conv}\left(\mathbf{A}_{n-1}^+\right)$ is one less than that of $\operatorname{conv}\left(\widetilde{\mathbf{A}}_{n-1}^+\right)$. For the sake of the completeness, two proofs of this simple fact will be given in Proposition 3.3. Third, an explicit bijection between the set $\{G_\sigma : \sigma$ is a maximal face of $\Delta\}$ and that of anti-standard trees T on $[n]$ with $T \neq \{(1,2),(1,3),\ldots,(1,n)\}$ will be constructed (Theorem 3.5). In particular, a new combinatorial expression of Catalan numbers arises. Note that a list of 66 expressions of Catalan numbers is presented in [10, Exercise 6.19].

1. Squarefree initial ideals

In the present paper, we consider the configuration

$$\mathbf{A}_{n-1}^+ = \{\mathbf{e}_i - \mathbf{e}_j : 1 \leq i < j \leq n\} \subset \mathbb{Z}^n,$$

where $\mathbf{e}_i$ denotes the i-th unit coordinate vector of $\mathbb{R}^n$. The configuration $\mathbf{A}_{n-1}^+$ is the set of positive roots of the root system $\mathbf{A}_{n-1}$ (see [7]). Let $K[\mathbf{f}] = K[f_{i,j} : 1 \leq i < j \leq n]$ denote the polynomial ring over a field K, and $K[\mathbf{t}, \mathbf{t}^{-1}, s] = K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}, s]$ the Laurent polynomial ring over K. The *toric ideal* $I_{\mathbf{A}_{n-1}^+}$ of $\mathbf{A}_{n-1}^+$ is the kernel of the homomorphism $\pi : K[\mathbf{f}] \longrightarrow K[\mathbf{t}, \mathbf{t}^{-1}, s]$ defined by setting $\pi(f_{i,j}) = t_i t_j^{-1} s$ for all $1 \leq i < j \leq n$.

We recall fundamental materials on Gröbner bases from, e.g., [3]. Let M denote the set of monomials of $K[\mathbf{f}]$. In particular, the element 1 belongs to M. A *monomial order* on M is a linear (total) order $<$ on M such that (i) $1 < u$ for any $1 \neq u \in M$, and (ii) if $u, v \in M$ and $u < v$, then $uw < vw$ for any $w \in M$. Fix a monomial order $<$ on M. For $0 \neq g \in K[\mathbf{f}]$, the *initial monomial* $in_<(g)$ is the biggest monomial appearing in g with respect to $<$. The *initial ideal* of $I_{\mathbf{A}^+_{n-1}}$ with respect to $<$ is the ideal

$$in_< \left(I_{\mathbf{A}^+_{n-1}}\right) = \left\langle in_<(g) \colon 0 \neq g \in I_{\mathbf{A}^+_{n-1}} \right\rangle \subset K[\mathbf{f}].$$

For a finite subset $\mathcal{H} \subset K[\mathbf{f}]$, let $in_<(\mathcal{H}) = \langle in_<(h) \colon h \in \mathcal{H} \rangle \subset K[\mathbf{f}]$. A finite set $\mathcal{H} \subset I_{\mathbf{A}^+_{n-1}}$ is said to be a *Gröbner basis* of $I_{\mathbf{A}^+_{n-1}}$ with respect to $<$ if $in_<(\mathcal{H}) = in_< \left(I_{\mathbf{A}^+_{n-1}}\right)$. A Gröbner basis $\mathcal{H}$ of $I_{\mathbf{A}^+_{n-1}}$ with respect to $<$ is called *reduced* if it has the additional properties that, for each $h \in \mathcal{H}$, the coefficient of $in_<(h)$ is 1 and, for any two distinct elements $h, h' \in \mathcal{H}$, no term of h' is divisible by $in_<(h)$. A reduced Gröbner basis uniquely exists.

Let $<_{\text{lex}}$ be the lexicographic order induced by the ordering of variables

$$f_{n-1,n} >_{\text{lex}} f_{n-2,n-1} >_{\text{lex}} f_{n-2,n} >_{\text{lex}} \cdots >_{\text{lex}} f_{1,2} >_{\text{lex}} f_{1,3} >_{\text{lex}} \cdots >_{\text{lex}} f_{1,n},$$

and let $<_{\text{rev}}$ be the reverse lexicographic order induced by the ordering of variables

$$f_{n-1,n} >_{\text{rev}} f_{n-2,n} >_{\text{rev}} f_{n-2,n-1} >_{\text{rev}} \cdots >_{\text{rev}} f_{2,3} >_{\text{rev}} f_{1,n} >_{\text{rev}} \cdots >_{\text{rev}} f_{1,3} >_{\text{rev}} f_{1,2}.$$

First, we explicitly provide the Gröbner basis of $I_{\mathbf{A}^+_{n-1}}$ with respect to both $<_{\text{lex}}$ and $<_{\text{rev}}$ whose initial monomials are squarefree monomials of degree ≤ 3.

Theorem 1.1. *The set $\mathcal{G}$ of the binomials*

$$\begin{array}{ll} f_{i,l} f_{j,k} - f_{i,k} f_{j,l}, & i < j < k < l, \\ f_{i,j} f_{j,k} - f_{i,i+1} f_{i+1,k}, & i+1 < j < k, \\ f_{i,j} f_{k,k+1} f_{k+1,l} - f_{i,i+1} f_{i+1,j} f_{k,l}, & i+1 < j < k < l-1, \end{array}$$

is the reduced Gröbner basis of the toric ideal $I_{\mathbf{A}^+_{n-1}}$ with respect to both $<_{\text{lex}}$ and $<_{\text{rev}}$, where the initial monomial of each binomial is the first monomial.

Proof. Since $f_{i,l} <_{\text{lex}} f_{i,k} <_{\text{lex}} f_{j,l} <_{\text{lex}} f_{j,k}$ and $f_{i,k} <_{\text{rev}} f_{i,l} <_{\text{rev}} f_{j,k} <_{\text{rev}} f_{j,l}$ for $i < j < k < l$, the initial monomial of $f_{i,l} f_{j,k} - f_{i,k} f_{j,l}$ is $f_{i,l} f_{j,k}$ with respect to both $<_{\text{lex}}$ and $<_{\text{rev}}$. Similarly, it follows that the initial monomial of each binomial belonging to $\mathcal{G}$ is the first monomial with respect to both $<_{\text{lex}}$ and $<_{\text{rev}}$. Hence it is enough to show that $\mathcal{G}$ is a Gröbner basis of $I_{\mathbf{A}^+_{n-1}}$ with respect to $<_{\text{lex}}$. (Once we know that $\mathcal{G}$ is a Gröbner basis, it immediately follows that $\mathcal{G}$ is reduced.) The effective technique discussed in [9] can be also applied in the present situation.

Suppose that

$$u = f_{i_1,j_1} \cdots f_{i_q,j_q},$$
$$u' = f_{i'_1,j'_1} \cdots f_{i'_{q'},j'_{q'}}$$

are monomials of $K[\mathbf{f}]$ with $u \notin in_{<_{\rm lex}}(\mathcal{G})$ and $u' \notin in_{<_{\rm lex}}(\mathcal{G})$, where

$$f_{i_1,j_1} \leq_{\rm lex} \cdots \leq_{\rm lex} f_{i_q,j_q},$$
$$f_{i'_1,j'_1} \leq_{\rm lex} \cdots \leq_{\rm lex} f_{i'_{q'},j'_{q'}}.$$

What we have to show is that if $\pi(u) = \pi(u')$, then $q = q'$ and $i_1 = i'_1, \ldots, i_q = i'_q$, $j_1 = j'_1, \ldots, j_q = j'_q$.

Suppose $\pi(u) = \pi(u')$. By comparing the exponent of s in $\pi(u)$ with that in $\pi(u')$, we have $q = q'$. Using the induction on q, we will show that there exists a variable appearing in both u and u'. Let m_u denote the set of all indices i such that both t_i and t_i^{-1} appear in the product $\pi(u) = \pi\left(f_{i_1,j_1}\right) \cdots \pi\left(f_{i_q,j_q}\right)$. Since $\pi(u) = \pi(u')$ and $q = q'$, it follows that $m_u = \emptyset$ if and only if $m_{u'} = \emptyset$.

CASE 1. $m_u \neq \emptyset$ and $m_{u'} \neq \emptyset$.

Let p (resp. p') be the smallest element in m_u (resp. $m_{u'}$). We may assume that $p \leq p'$. If u is devided by $f_{\mu,p} f_{p,\nu}$ for some μ and ν with $\mu + 1 < p < \nu$, then we have $u \in in_{<_{\rm lex}}(\mathcal{G})$ by the previous argument of this proof. This contradicts the assumption. Hence both $f_{p-1,p}$ and $f_{p,r}$ appear in u for some $r > p$. Similarly, both $f_{p'-1,p'}$ and $f_{p',r'}$ appear in u' for some $r' > p'$.

Suppose that $f_{p-1,p}$ does not appear in u'. Then we have $p < p'$. Since $\pi(u) = \pi(u')$ and $p - 1 \notin m_u$, there exists a variable $f_{p-1,s}$ appearing in u' with $p < s$. Since

$$\begin{array}{rcrl} p < p' < s & \Rightarrow & f_{p-1,s} f_{p'-1,p'} & \in in_{<_{\rm lex}}(\mathcal{G}), \\ p' = s & \Rightarrow & f_{p-1,s} f_{p',r'} & \in in_{<_{\rm lex}}(\mathcal{G}), \\ p' = s + 1 & \Rightarrow & f_{p-1,s} f_{p'-1,p'} & \in in_{<_{\rm lex}}(\mathcal{G}), \\ s + 1 < p' & \Rightarrow & f_{p-1,s} f_{p'-1,p'} f_{p',r'} & \in in_{<_{\rm lex}}(\mathcal{G}) \end{array}$$

hold, this contradicts $u' \notin in_{<_{\rm lex}}(\mathcal{G})$. Thus $f_{p-1,p}$ appears in both u and u'.

CASE 2. $m_u = m_{u'} = \emptyset$.

Since $i_1 \leq i_2 \leq \cdots \leq i_q$ and $i'_1 \leq i'_2 \leq \cdots \leq i'_q$, it follows that $i_q = i'_q$. If $j_q < j'_q$, then there exists h such that $1 \leq h < q$ with $j'_q = j_h$. Hence we have $i_h \leq i_q < j_q < j_h$. Since $f_{i_h,j_h} f_{i_q,j_q} \notin in_{<_{\rm lex}}(\mathcal{G})$, we have $i_h = i_q$. Thus $f_{i'_q,j'_q}$ appears in both u and u', as desired. $\square$

2. Unimodular triangulations

Let $<_{\rm lex}$ denote the lexicographic order discussed in the previous section. Let $\Delta = \Delta_{<_{\rm lex}}\left(I_{\mathbf{A}^+_{n-1}}\right)$ denote the regular triangulation ([11, Chapter 8]) of the $n-1$ dimensional convex polytope $\operatorname{conv}\left(\mathbf{A}^+_{n-1}\right)$ associated with $in_{<_{\rm lex}}\left(I_{\mathbf{A}^+_{n-1}}\right)$. Thus Δ consists of all subsets $\sigma \subset \mathbf{A}_{n-1}^{+}$ such that

$$\prod_{\mathbf{e}_i - \mathbf{e}_j \in \sigma} f_{i,j} \notin in_{<_{\text{lex}}} \left(I_{\mathbf{A}_{n-1}^{+}}\right).$$

Since $in_{<_{\text{lex}}} \left(I_{\mathbf{A}_{n-1}^{+}}\right)$ is generated by squarefree monomials by Theorem 1.1, it follows from [11, Corollary 8.9] that the triangulation Δ is unimodular, i.e., the normalized volume ([11, p.36]) of σ is 1 for every maximal face σ of Δ.

In this section, we present a graph-theoretical characterization of the maximal faces of Δ. Let $[n] = \{1, \dots, n\}$ be the vertex set and let (i, j), $1 \le i < j \le n$, be the arrow from i to j. Given a subset σ of $\mathbf{A}_{n-1}^{+}$, we write G_σ for the graph on $[n]$ having the arrows (i, j) with $\mathbf{e}_i - \mathbf{e}_j \in \sigma$.

Lemma 2.1. *Let σ be a maximal face of Δ. Then the graph G_σ associated with σ is a connected graph which has n vertices, n arrows and a cycle* $\{(q, q+1), (q, j), (q+1, j)\}$ *for some* $2 \le q+1 < j \le n$.

Proof. Theorem 1.1 guarantees that a subset σ of $\mathbf{A}_{n-1}^{+}$ is a face of Δ if and only if none of the following subgraphs appear in G_σ:
(I) $\{(i, l), (j, k)\}$ with $i < j < k < l$,
(II) $\{(i, j), (j, k)\}$ with $i + 1 < j < k$,
(III) $\{(i, j), (k, k+1), (k+1, l)\}$ with $i + 1 < j < k < l - 1$.
Let σ be a maximal face of Δ. Then σ is of dimension $n - 1$. Thus G_σ is a graph with n vertices and n arrows. Hence G_σ has at least one cycle.

Let $\mathcal{C}$ be a cycle of length r (≥ 3) in G_σ and let $i_0 = \min\{i : (i, j) \in \mathcal{C}\}$. Since $\mathcal{C}$ is a cycle, there exist two vertices $i_{1,1}$ and $i_{2,1}$ of G_σ such that $\{(i_0, i_{1,1}), (i_0, i_{2,1})\} \subset \mathcal{C}$ with $i_0 < i_{1,1} < i_{2,1}$. Since none of the subgraphs of type (I) appear in $\mathcal{C}$, there exists no vertex j_1 of G_σ such that $(j_1, i_{1,1}) \in \mathcal{C}$ with $i_0 < j_1 < i_{1,1}$. Thus, since $\mathcal{C}$ is a cycle, there exists a vertex $i_{1,2}$ $(> i_{1,1})$ of G_σ such that $(i_{1,1}, i_{1,2}) \in \mathcal{C}$. Note that, since none of the subgraphs of type (II) appear in $\mathcal{C}$, we have $i_{1,1} = i_0 + 1$.

Since none of the subgraphs of type (I) appear in $\mathcal{C}$, we have $i_{2,1} \le i_{1,2}$. Suppose that $i_{2,1} < i_{1,2}$. If $(i_{2,1}, i_{2,2}) \in \mathcal{C}$ with $i_{2,1} < i_{2,2}$, then we have $i_{2,1} = i_0 + 1$ since none of the subgraphs of type (II) appear in $\mathcal{C}$. This contradicts the assumption that $(i_0 + 1 =)$ $i_{1,1} < i_{2,1}$. Since $\mathcal{C}$ is a cycle, there exists a vertex j_2 of G_σ such that $(j_2, i_{2,1}) \in \mathcal{C}$ with $i_{1,1} = i_0 + 1 < j_2 < i_{2,1}$. Then the subgraph $\{(i_{1,1}, i_{1,2}), (j_2, i_{2,1})\}$ of type (I) appears in $\mathcal{C}$. Thus we have $i_{1,2} = i_{2,1}$ and $\mathcal{C} = \{(i_0, i_0 + 1), (i_0, i_{1,2}), (i_0 + 1, i_{1,2})\}$.

Suppose that two cycles $\{(p, p+1), (p, i), (p+1, i)\}$ and $\{(q, q+1), (q, j), (q+1, j)\}$ appear in G_σ. Then we have $0 \ne f_{p,p+1} f_{p+1,i} f_{q,j} - f_{p,i} f_{q,q+1} f_{q+1,j} \in I_{\mathbf{A}_{n-1}^{+}}$. Since either $f_{p,p+1} f_{p+1,i} f_{q,j}$ or $f_{p,i} f_{q,q+1} f_{q+1,j}$ belongs to $in_{<_{\text{lex}}} \left(I_{\mathbf{A}_{n-1}^{+}}\right)$, it is impossible that both $\{(p, p+1), (p+1, i), (q, j)\}$ and $\{(p, i), (q, q+1), (q+1, j)\}$ appear in G_σ. Thus G_σ has exactly one cycle.

Moreover, since G_σ has n vertices, n arrows and exactly one cycle, it follows from the following lemma that G_σ is connected. □

Lemma 2.2. *Let G be a finite graph with neither loop nor multiple edge. If G has exactly one cycle, and the number of vertices of G is equal to that of edges of G, then G is connected.*

Proof. Suppose that G is not connected. Let $\mathcal{C}_1, \dots, \mathcal{C}_s$ be connected components of G, where $s \geq 2$ and exactly one cycle of G appears in $\mathcal{C}_1$ and, for $2 \leq i \leq s$, $\mathcal{C}_i$ is a tree. Let v_i (resp. e_i) denote the number of vertices (resp. edges) of $\mathcal{C}_i$. Then we have $\sum_{j=1}^{s} e_j = \sum_{i=1}^{s} v_i$ by assumption.

For any tree T, the number of edges of T is equal to that of vertices of T minus 1 (see [6, Theorem 1.3]). Hence, since $e_1 = v_1$ and $e_k = v_k - 1$ for $2 \leq k \leq s$, it follows that $\sum_{j=1}^{s} e_j < \sum_{i=1}^{s} v_i$. This contradicts $\sum_{j=1}^{s} e_j = \sum_{i=1}^{s} v_i$. Thus G is connected. □

We now come to a graph-theoretical characterization of the maximal faces of Δ.

Theorem 2.3. *A subset σ of $\mathbf{A}_{n-1}^{+}$ is a maximal face of Δ if and only if the graph G_σ associated with σ is a connected graph with n vertices and n arrows satisfying the following condition: $G_\sigma = A \cup B \cup C$, where*

$$\begin{aligned} A &= \{(1,2),(2,3),\dots,(q-1,q),(q,q+1),(q,j),(q+1,j)\}, \\ B &= \{(q,i_1),\dots,(q,i_r)\} \end{aligned}$$

with $q+1 < i_1 < \cdots < i_r < j$ (B may be an empty set), and none of the subgraphs

(1) $\{(x,w),(y,z)\}$ *with* $x < y < z < w$,

(2) $\{(x,y),(y,z)\}$ *with* $x < y < z$

appear in C, and C is either an empty set or one of the following graphs:

CASE 1. $C = \{(q+1,s_1),\dots,(q+1,s_m),(x_{u_1},y_{u_1}),\dots,(x_{u_p},y_{u_p})\}$ *with*

$$\begin{aligned} &j < s_1 < \cdots < s_m \leq n, \\ &q+1 < x_{u_k},\ j < y_{u_k} \leq n \quad (k = 1,2,\dots,p), \\ &x_{u_k} \notin \{j, i_1, \dots, i_r\} \quad (k = 1,2,\dots,p). \end{aligned}$$

CASE 2. $C = \{(t_1,j),\dots,(t_e,j),(x_{u_1},y_{u_1}),\dots,(x_{u_p},y_{u_p})\}$ *with*

$$\begin{aligned} &q+1 < t_1 < \cdots < t_e < j, \\ &q+1 < x_{u_k},\ j < y_{u_k} \leq n \quad (k = 1,2,\dots,p), \\ &x_{u_k} \notin \{j, i_1, \dots, i_r\} \quad (k = 1,2,\dots,p). \end{aligned}$$

Proof. [**only if**] Suppose that σ is a maximal face of Δ. Then, by Lemma 2.1, the graph G_σ associated with σ is a connected graph which has n vertices, n arrows and a cycle $\{(q, q+1), (q, j), (q+1, j)\}$ of length 3. Moreover, note that none of the subgraphs of type (I), type (II) and type (III) stated in the proof of Lemma 2.1 appear in G_σ since σ is a face of Δ.

STEP 1. If an arrow $(u, q+1)$ with $u \neq q$ appears in G_σ, then the subgraph $\{(u, q+1), (q+1, j)\}$ of type (II) appears in G_σ since $q+1 > u+1$. Hence no arrow $(u, q+1)$ with $u \neq q$ appears in G_σ.

STEP 2. If an arrow $(q+1, s)$ with $s < j$ appears in G_σ, then the subgraph $\{(q, j), (q+1, s)\}$ of type (I) appears in G_σ. Hence if $(q+1, s)$ appears in G_σ, then we have $j \leq s$.

STEP 3. If an arrow (j, t) appears in G_σ, then the subgraph $\{(q, j), (j, t)\}$ of type (II) appears in G_σ. Hence no arrow (j, t) appears in G_σ.

STEP 4. If an arrow (t, j) with $t < q$ appears in G_σ, then the subgraph $\{(t, j), (q, q+1)\}$ of type (I) appears in G_σ. Hence no arrow (t, j) with $t < q$ appears in G_σ.

STEP 5. Suppose that both an arrow $(q+1, s)$ with $s \neq j$ and an arrow (t, j) with $t \neq q, q+1$ appear in G_σ. Note that $s > j$ and $t > q+1$ by STEP 2 and STEP 4. Then the subgraph $\{(q+1, s), (t, j)\}$ of type (I) appears in G_σ. Hence no subgraph $\{(q+1, s), (t, j)\}$ with $s \neq j$ and $t \neq q, q+1$ appears in G_σ.

STEP 6. If an arrow (q, i) with $i > j$ appears in G_σ, then the subgraph $\{(q, i), (q+1, j)\}$ of type (I) appears in G_σ. Hence if (q, i) appears in G_σ, then we have $i \leq j$.

STEP 7. If an arrow (k, i) with $k \neq q$ and $q+1 < i < j$ appears in G_σ, then either $k < q$ or $q+1 < k < i < j$, and either the subgraph $\{(k, i), (q, q+1)\}$ with $k < q$ of type (I) or the subgraph $\{(q+1, j), (k, i)\}$ with $q+1 < k < i < j$ of type (I) appears in G_σ. Hence no arrow (k, i) with $k \neq q$ and $q+1 < i < j$ appears in G_σ.

STEP 8. Since none of the subgraphs of type (II) appear in G_σ, no subgraph $\{(q, i), (i, l)\}$ with $q+1 < i < j$ and $i < l$ appears in G_σ.

STEP 9. Suppose that an arrow (z_1, z_2) with $z_1 < q$ appears in G_σ. If $z_2 > q+1$, then the subgraph $\{(z_1, z_2), (q, q+1)\}$ of type (I) appears in G_σ. This contradicts the assumption that $\sigma \in \Delta$. If $z_2 = q+1$, then the subgraph $\{(z_1, q+1), (q+1, j)\}$ of type (II) appears in G_σ. This contradicts the assumption that $\sigma \in \Delta$. If $z_2 = q$, then we have $z_1 = q - 1$ since none of the subgraphs of type (II) appear in G_σ. If $z_2 < q$, then we have $z_1 = z_2 - 1$ since none of the subgraphs of type (III) appear in G_σ. Thus if an arrow (z_1, z_2) with $z_1 < q$ appears in G_σ, then we have $z_2 = z_1 + 1 \leq q$. It follows from the connectedness of G_σ that $\{(1, 2), (2, 3), \ldots, (q-1, q)\}$ is a subgraph of G_σ.

Thus, from the above nine steps, we have $G_\sigma = A \cup B \cup C$, where

$$\begin{aligned} A &= \{(1, 2), (2, 3), \ldots, (q-1, q), (q, q+1), (q, j), (q+1, j)\}, \\ B &= \{(q, i_1), \ldots, (q, i_r)\} \end{aligned}$$

with $q+1 < i_1 < \cdots < i_r < j$ (B may be an empty set), and C is either an empty set or one of the following graphs:

CASE 1. $C = \{(q+1, s_1), \dots, (q+1, s_m), (x_{u_1}, y_{u_1}), \dots, (x_{u_p}, y_{u_p})\}$ with

$$\begin{aligned}&j < s_1 < \cdots < s_m \leq n,\\ &q+1 < x_{u_k},\ j < y_{u_k} \leq n \quad (k = 1, 2, \dots, p),\\ &x_{u_k} \notin \{j, i_1, \dots, i_r\} \quad (k = 1, 2, \dots, p).\end{aligned}$$

CASE 2. $C = \{(t_1, j), \dots, (t_e, j), (x_{u_1}, y_{u_1}), \dots, (x_{u_p}, y_{u_p})\}$ with

$$\begin{aligned}&q+1 < t_1 < \cdots < t_e < j,\\ &q+1 < x_{u_k},\ j < y_{u_k} \leq n \quad (k = 1, 2, \dots, p),\\ &x_{u_k} \notin \{j, i_1, \dots, i_r\} \quad (k = 1, 2, \dots, p).\end{aligned}$$

Finally, we prove that none of the subgraphs $\{(x, w), (y, z)\}$ with $x < y < z < w$ and $\{(x, y), (y, z)\}$ with $x < y < z$ appear in C. First, since none of the subgraphs of type (I) and type (II) appear in G_σ, none of the subgraphs $\{(x, w), (y, z)\}$ with $x < y < z < w$ and $\{(x, y), (y, z)\}$ with $x + 1 < y < z$ appear in C. Now, suppose that the subgraph $\{(x, x+1), (x+1, z)\}$ with $x + 1 < z$ appears in C. Note that, in both CASE 1 and CASE 2, if (z_1, z_2) appears in C, then we have $q + 1 \leq z_1 \neq j$. Hence $q + 1 \leq x$ and $x \neq j - 1, j$. If $x > j$, then the subgraph $\{(q, j), (x, x+1), (x+1, z)\}$ of type (III) appears in G_σ. This contradicts the assumption that $\sigma \in \Delta$. If $q + 1 \leq x \leq j - 2$, then the subgraph $\{(q, j), (x, x+1)\}$ of type (I) appears in G_σ. This contradicts the assumption that $\sigma \in \Delta$. Hence no subgraph $\{(x, x+1), (x+1, z)\}$ with $x + 1 < z$ appears in C.

[**if**] Let σ be a subset of $\mathbf{A}_{n-1}^{+}$ and suppose that the graph G_σ associated with σ satisfies the condition stated as above. In order to prove that σ is a maximal face of Δ, it suffices to show that σ is a face of Δ since G_σ is a connected graph with n vertices and n arrows and the dimension of maximal faces of Δ is $n - 1$. In other words, we need only prove that none of the subgraphs of type (I), type (II) and type (III) stated in the proof of Lemma 2.1 appear in G_σ.

First, we show that none of the subgraphs of type (I) appear in G_σ. If a subgraph G' of type (I) appears in G_σ, then G' must be one of the following subgraphs:
(a) $\{(l_1, l_4), (l_2, l_3)\}$ with $l_1 < l_2 < l_3 < l_4$ and $(l_1, l_4), (l_2, l_3) \in A \cup B$,
(b) $\{(l_1, l_4), (l_2, l_3)\}$ with $l_1 < l_2 < l_3 < l_4$ and $(l_1, l_4) \in A \cup B$, $(l_2, l_3) \in C$,
(c) $\{(l_1, l_4), (l_2, l_3)\}$ with $l_1 < l_2 < l_3 < l_4$ and $(l_1, l_4) \in C$, $(l_2, l_3) \in A \cup B$,
(d) $\{(l_1, l_4), (l_2, l_3)\}$ with $l_1 < l_2 < l_3 < l_4$ and $(l_1, l_4), (l_2, l_3) \in C$.
However, since none of the subgraphs of type (I) appear in C, none of the subgraphs of type (d) appear in G_σ. Moreover, since

$$\begin{aligned}A \cup B = {}&\{(1, 2), (2, 3), \dots, (q-1, q), (q, q+1), (q, j), (q+1, j)\}\\ &\cup \{(q, i_1), \dots, (q, i_r)\}\end{aligned}$$

with $q + 1 < i_1 < \cdots < i_r < j$, none of the subgraphs of type (a) appear in G_σ. Note that, if (y_1, y_2) appears in $A \cup B$ and (z_1, z_2) appears in C, then we have $y_1 \leq q+1 \leq z_1$

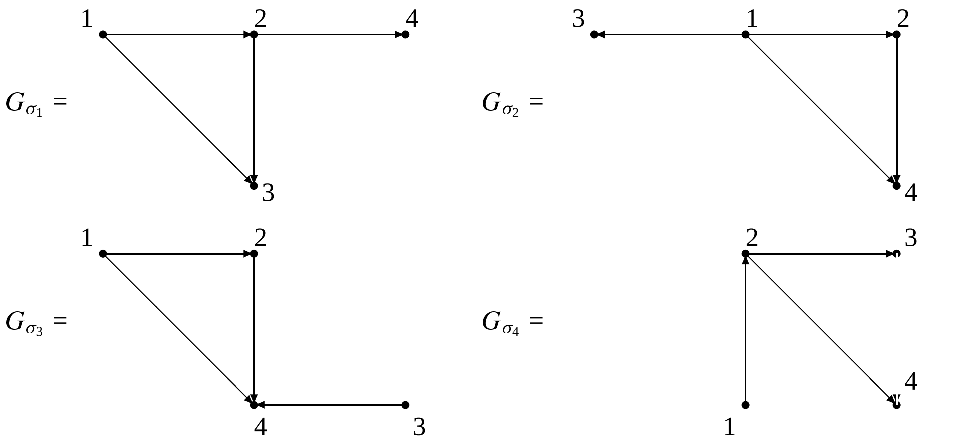

Fig. 1.

and $y_2 \le j \le z_2$. Hence none of the subgraphs of type (b) and type (c) appear in G_σ. Thus none of the subgraphs of type (I) appear in G_σ.

Second, we show that none of the subgraphs of type (II) appear in G_σ. Suppose that the subgraph $\{(l_1, l_2), (l_2, l_3)\}$ with $l_1 + 1 < l_2 < l_3$ appears in G_σ. If $(l_1, l_2) \in A \cup B$, then $l_2 \in \{j, i_1, \ldots, i_r\}$ since $l_1 + 1 < l_2$. This implies that $(l_2, l_3) \notin G_\sigma$. Hence $(l_1, l_2) \in C$. Moreover, since $q + 1 < j \le l_2 < l_3$, it follows that $(l_2, l_3) \notin A \cup B$, i.e., $(l_2, l_3) \in C$. However, none of the subgraphs of type (II) appear in C. Hence none of the subgraphs of type (II) appear in G_σ.

Finally, we show that none of the subgraphs of type (III) appear in G_σ. Suppose that the subgraph $\{(l_1, l_2), (l_3, l_3 + 1), (l_3 + 1, l_4)\}$ with $l_1 + 1 < l_2 < l_3 < l_4 - 1$ appears in G_σ. If $(l_1, l_2) \in A \cup B$, then $l_2 \in \{j, i_1, \ldots, i_r\}$ since $l_1 + 1 < l_2$. Hence, since $q + 1 < l_2 < l_3 < l_4 - 1$, it follows that $(l_3, l_3+1), (l_3+1, l_4) \notin A \cup B$, i.e., $(l_3, l_3+1), (l_3+1, l_4) \in C$. Moreover, if $(l_1, l_2) \in C$, then $(l_3, l_3+1), (l_3+1, l_4) \notin A \cup B$, i.e., $(l_3, l_3+1), (l_3+1, l_4) \in C$ since $q + 1 < j \le l_2 < l_3 < l_4 - 1$. However, no subgraph of the form $\{(v_1, v_2), (v_2, v_3)\}$ with $v_1 < v_2 < v_3$ appears in C. Hence none of the subgraphs of type (III) appear in G_σ. □

EXAMPLE 2.4. Let $n = 4$. Then the maximal faces of Δ are

$$\begin{aligned}
\sigma_1 &= \{(1, 2), (1, 3), (2, 3), (2, 4)\},\\
\sigma_2 &= \{(1, 2), (1, 3), (1, 4), (2, 4)\},\\
\sigma_3 &= \{(1, 2), (1, 4), (2, 4), (3, 4)\},\\
\sigma_4 &= \{(1, 2), (2, 3), (2, 4), (3, 4)\},
\end{aligned}$$

and, for $i = 1, 2, 3, 4$, the graph G_{σ_i} associated with σ_i is illustrated in Fig. 1.

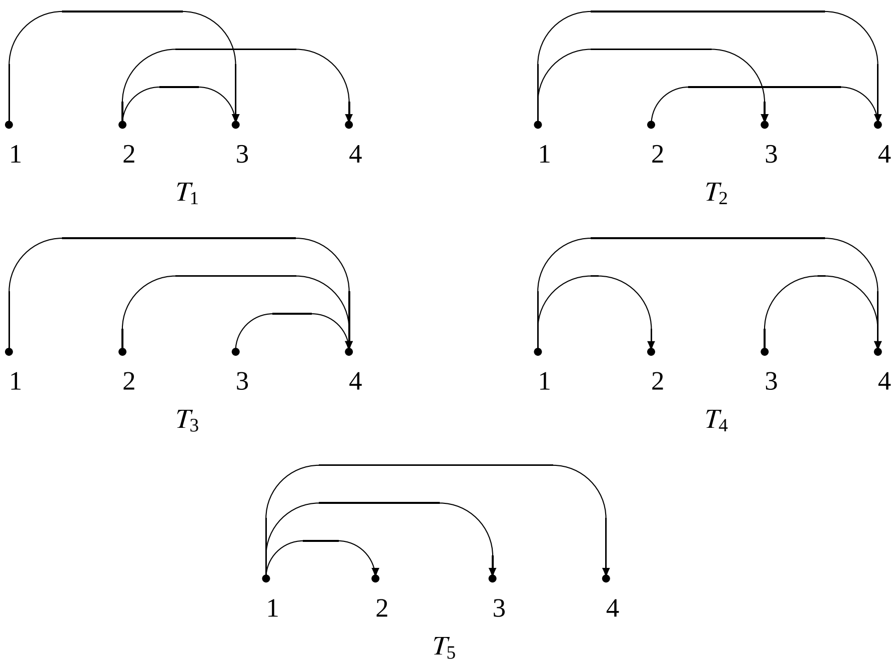

Fig. 2.

3. Catalan numbers

We now discuss the relation between the set of maximal faces of Δ and that of anti-standard trees.

A tree T on the set $[n] = \{1, \ldots, n\}$ (i.e., a connected graph T on the set $[n]$ without cycle) is called *anti-standard* if none of the following subgraphs appear in T:
(1) $\{(i,l),(j,k)\}$ with $i < j < k < l$,
(2) $\{(i,j),(j,k)\}$ with $i < j < k$.

EXAMPLE 3.1. All anti-standard trees for $n = 4$ are illustrated in Fig. 2.

Let $\mathcal{M} = \{G_\sigma : \sigma$ is a maximal face of $\Delta\}$ and let $\mathcal{T}$ denote the set of anti-standard trees on $[n]$. Recall the following result on the cadinality of $\mathcal{T}$.

Proposition 3.2 ([5, Theorem 2.3, Corollary 6.7]). (a) *The number of anti-standard trees on the set $[n]$ is equal to the Catalan number*

$$C_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}.$$

(b) *The normalized volume of* $\operatorname{conv}\left(\widetilde{\mathbf{A}}^{+}_{n-1}\right)$ *is equal to C_{n-1}.*

Even though the following result is easy to prove, we give two proofs for the sake of the completeness.

Proposition 3.3. *The normalized volume of* $\operatorname{conv}\left(\mathbf{A}_{n-1}^{+}\right)$ *is equal to* $C_{n-1}-1$.

First proof. First, we show that the set of vertices of $\operatorname{conv}\left(\widetilde{\mathbf{A}}_{n-1}^{+}\right)$ is $\widetilde{\mathbf{A}}_{n-1}^{+}$. Let $\mathbf{A}_{n-1}^{+}=\{\mathbf{v}_1,\dots,\mathbf{v}_m\}$, where $\mathbf{v}_i=\mathbf{e}_{p_i}-\mathbf{e}_{q_i}$ $(1\le p_i<q_i\le n)$ for $1\le i\le m$. Then $\widetilde{\mathbf{A}}_{n-1}^{+}=\{\mathbf{v}_1,\dots,\mathbf{v}_m\}\cup\{\mathbf{0}\}$. If $\mathbf{0}\in\operatorname{conv}(\{\mathbf{v}_1,\dots,\mathbf{v}_m\})$, then we have $\mathbf{0}=\sum_{i=1}^m a_i\mathbf{v}_i$, where $0\le a_i\in\mathbb{R}$ for $1\le i\le m$ and $\sum_{i=1}^m a_i=1$. We set $p=\min\{p_i\colon a_i\ne 0\}$. Then, the p-th component in the right-hand side is positive. However, the p-th component in the left-hand side is zero. Hence we have $\mathbf{0}\notin\operatorname{conv}(\{\mathbf{v}_1,\dots,\mathbf{v}_m\})$. Moreover, if $\mathbf{v}_1\in\operatorname{conv}(\{\mathbf{v}_2,\dots,\mathbf{v}_m\}\cup\{\mathbf{0}\})$, then we have $\mathbf{v}_1=b\mathbf{0}+\sum_{j=2}^m b_j\mathbf{v}_j$, where $0\le b\in\mathbb{R}$, $0\le b_j\in\mathbb{R}$ for $2\le j\le m$ and $b+\sum_{j=2}^m b_j=1$. Note that the first component of $\mathbf{v}_j$ is 0 or 1 for $1\le j\le m$. If $p_1>1$, then the first component in the right-hand side is $\sum_{2\le j\le m;\,p_j=1} b_j$ and the first component in the left-hand side is 0. Thus we have $b_j=0$ for every j with $p_j=1$. Hence we have $\mathbf{v}_1=b\mathbf{0}+\sum_{2\le j\le m;\,p_j\ne 1} b_j\mathbf{v}_j$. Similarly, if $p_1>2$, then we have $b_j=0$ for every j with $p_j=2$. By this argument, we may assume that $p_1=1$. Then, since the first component in the right-hand side is $\sum_{2\le j\le m;\,p_j=1} b_j$ and the first component in the left-hand side is 1, we have $b_j=0$ for every j with $p_j\ne 1$. Moreover, we have $q_j\ne q_1$ for every j with $j\ne 1$ and $p_j=1$. Thus the q_1-th component in the right-hand side is 0. However, the q_1-th component in the left-hand side is -1. Hence we have $\mathbf{v}_1\notin\operatorname{conv}(\{\mathbf{v}_2,\dots,\mathbf{v}_m\}\cup\{\mathbf{0}\})$. By the same argument, it follows that $\mathbf{v}_i\notin\operatorname{conv}(\{\mathbf{v}_1,\dots,\mathbf{v}_{i-1},\mathbf{v}_{i+1},\dots,\mathbf{v}_m\}\cup\{\mathbf{0}\})$ for $1\le i\le m$. Thus the set of vertices of $\operatorname{conv}\left(\widetilde{\mathbf{A}}_{n-1}^{+}\right)$ is $\widetilde{\mathbf{A}}_{n-1}^{+}$.

Now, let $\omega=(1,2,\dots,n)\in\mathbb{Z}^n$ and $\sigma_0=\operatorname{conv}(\{\mathbf{e}_i-\mathbf{e}_{i+1}\colon 1\le i\le n-1\}\cup\{\mathbf{0}\})$. Then σ_0 is a simplex of normalized volume 1 and $\dim\sigma_0=n-1$. Since $\omega\cdot\mathbf{0}=0$ and $\omega\cdot(\mathbf{e}_i-\mathbf{e}_j)=i-j\le -1$, it follows that $\operatorname{conv}\left(\widetilde{\mathbf{A}}_{n-1}^{+}\right)$ is separated into $\operatorname{conv}\left(\mathbf{A}_{n-1}^{+}\right)$ and σ_0 by the hyperplane $\{\mathbf{x}\in\mathbb{R}^n\colon\omega\cdot\mathbf{x}=-1\}$. Hence the normalized volume of $\operatorname{conv}\left(\mathbf{A}_{n-1}^{+}\right)$ is equal to that of $\operatorname{conv}\left(\widetilde{\mathbf{A}}_{n-1}^{+}\right)$ minus 1. $\square$

Second proof. Let $<$ be a lexicographic order with the largest variable x, where x is the variable corresponding to the origin. Then, for all $1\le i<j-1\le n-1$, the binomial $xf_{i,j}-f_{i,i+1}f_{i+1,j}$, whose initial monomial is $xf_{i,j}$, belongs to $I_{\widetilde{\mathbf{A}}_{n-1}^{+}}$. Suppose that σ is a maximal face of $\Delta_<\left(I_{\widetilde{\mathbf{A}}_{n-1}^{+}}\right)$ with the origin as a vertex. If $\mathbf{e}_i-\mathbf{e}_j$ is a vertex of σ for $1\le i<j-1\le n-1$, then $x\prod_{\mathbf{e}_i-\mathbf{e}_j\in\sigma\setminus\{\mathbf{0}\}} f_{i,j}\in in_<\left(I_{\widetilde{\mathbf{A}}_{n-1}^{+}}\right)$. This contradicts the assumption that $\sigma\in\Delta_<\left(I_{\widetilde{\mathbf{A}}_{n-1}^{+}}\right)$. Thus $\mathbf{e}_i-\mathbf{e}_j$ is not a vertex of σ if $1\le i<j-1\le n-1$. Hence the vertex set of σ is $\{\mathbf{e}_i-\mathbf{e}_{i+1}\colon 1\le i\le n-1\}\cup\{\mathbf{0}\}$. Hence $\operatorname{conv}\left(\widetilde{\mathbf{A}}_{n-1}^{+}\right)$ is separated into $\operatorname{conv}\left(\mathbf{A}_{n-1}^{+}\right)$ and σ_0. Since the normalized volume of σ_0 is equal to 1, the normalized volume of $\operatorname{conv}\left(\mathbf{A}_{n-1}^{+}\right)$ is equal to that of $\operatorname{conv}\left(\widetilde{\mathbf{A}}_{n-1}^{+}\right)$ minus 1. $\square$

Since Δ is a unimodular triangulation, we have the following.

Corollary 3.4. *The number of graphs belonging to $\mathcal{M}$ is equal to $C_{n-1} - 1$.*

We set $\mathcal{T}^* = \mathcal{T} \setminus \{\{(1,2),(1,3),\dots,(1,n)\}\}$. Since the cardinality of $\mathcal{M}$ is equal to that of $\mathcal{T}^*$, it seems of interest to find an explicit bijection between $\mathcal{M}$ and $\mathcal{T}^*$.

Theorem 3.5. *The map $\varphi\colon \mathcal{M} \longrightarrow \mathcal{T}^*$ defined as follows is bijective: for each element $G_\sigma = A \cup B \cup C \in \mathcal{M}$, where*

$$A = \{(1,2),(2,3),\dots,(q-1,q),(q,q+1),(q,j),(q+1,j)\},$$
$$B = \{(q,i_1),\dots,(q,i_r)\}$$

with $q+1 < i_1 < \cdots < i_r < j$, we define $\varphi(G_\sigma) = \widetilde{A} \cup \widetilde{B} \cup C$, where

$$\widetilde{A} = \{(1,2),(1,3),\dots,(1,q),(1,j),(q+1,j)\},$$
$$\widetilde{B} = \{(1,i_1),\dots,(1,i_r)\}.$$

Proof. We take any $G_\sigma = A \cup B \cup C \in \mathcal{M}$. Since no subgraph $\{(x,w),(y,z)\}$ with $x < y < z < w$ appears in G_σ and no subgraph $\{(x,y),(y,z)\}$ with $x < y < z$ appears in C, it follows that $\varphi(G_\sigma)$ is an anti-standard tree by the definition of φ. Moreover, since the arrow $(1,q+1)$ does not appear in the graph $\varphi(G_\sigma)$, we have $\varphi(G_\sigma) \neq \{(1,2),(1,3),\dots,(1,n)\}$. Hence we have $\varphi(G_\sigma) \in \mathcal{T}^*$.

We now show that φ is injective, which implies that φ is bijective since $|\mathcal{M}| = |\mathcal{T}^*| = C_{n-1} - 1$. Suppose that $\varphi(G_\sigma) = \varphi(G_{\sigma'})$ for $G_\sigma, G_{\sigma'} \in \mathcal{M}$. We can express $G_\sigma, G_{\sigma'}$ as

$$G_\sigma = A \cup B \cup C,$$
$$G_{\sigma'} = A' \cup B' \cup C',$$

where

$$A = \{(1,2),(2,3),\dots,(q-1,q),(q,q+1),(q,j),(q+1,j)\},$$
$$B = \{(q,i_1),\dots,(q,i_r)\},$$
$$A' = \{(1,2),(2,3),\dots,(q'-1,q'),(q',q'+1),(q',j'),(q'+1,j')\},$$
$$B' = \{(q',i'_1),\dots,(q',i'_{r'})\}$$

with $q+1 < i_1 < \cdots < i_r < j$ and $q'+1 < i'_1 < \cdots < i'_{r'} < j'$. Comparing the arrows in $\varphi(G_\sigma)$ of the form $(1,k)$ with the arrows in $\varphi(G_{\sigma'})$ of the form $(1,k')$, it follows from $\varphi(G_\sigma) = \varphi(G_{\sigma'})$ that $q = q'$, $j = j'$, $r = r'$ and $i_m = i'_m$ for $1 \leq m \leq r$. Hence we have $A \cup B = A' \cup B'$. Moreover, since C and C' are invariant under the map φ, $\varphi(G_\sigma) = \varphi(G_{\sigma'})$ implies that $C = C'$. Thus $G_\sigma = G_{\sigma'}$. Hence φ is injective. $\square$

EXAMPLE 3.6. Let $n = 4$. Then $G = \{G_{\sigma_1}, G_{\sigma_2}, G_{\sigma_3}, G_{\sigma_4}\}$ and $\mathcal{T} = \{T_1, T_2, T_3, T_4, T_5\}$ are described in Example 2.4 and Example 3.1. We have $\varphi(G_{\sigma_i}) = T_i$ for $i = 1, 2, 3, 4$.

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