ON PLURICANONICAL MAPS FOR THREEFOLDS
OF GENERAL TYPE, II

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1. Introduction

This paper is a continuation of [4, 9, 13]. To classify algebraic varieties is one of the goals in algebraic geometry. One way to study a given variety is to understand the behavior of its pluricanonical maps. The objects concerned here are complex projective 3-folds of general type over \( \mathbb{C} \). Let \( X \) be such an object and denote by \( \phi_m \) the \( m \)-th pluricanonical map of \( X \), which is the rational map associated with the \( m \)-canonical system \( |mK_X| \). The very natural question is when \( |mK_X| \) gives a birational map, a generically finite map, \( \cdots \), etc. According to [2, 4, 9, 12, 13], one has the following

**Theorem 0.** Let \( X \) be a complex projective 3-fold of general type with the canonical index \( r \). Then

(i) when \( r = 1 \), \( \phi_m \) is a birational morphism onto its image for \( m \geq 6 \);

(ii) when \( r \geq 2 \), \( \phi_m \) is a birational map onto its image for \( m \geq 4r + 5 \).

In this paper, we give our results on the generic finiteness of \( \phi_m \). By a delicate use of the Kawamata-Viehweg vanishing theorem, we reduce the problem to a parallel one for adjoint systems on some smooth surface. Reider’s results as well as other theorems on surfaces make it possible for us to go on a detailed argument.

**Theorem 1.** Let \( X \) be a projective 3-fold of general type with the canonical index \( r \geq 2 \). Then \( \phi_m \) is generically finite for \( m \geq m(r) \), where \( m(r) \) is a function as follows:

- \( m(2) = 11 \);
- \( m(r) = 2r + 8 \), for \( 3 \leq r \leq 5 \);
- \( m(r) = 2r + 6 \), for \( r \geq 6 \).

**Theorem 2.** Let \( X \) be a projective minimal Gorenstein 3-fold of general type. Then

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\( (1) \) \( \phi_5 \) is birational except for some 3-folds with \( K_X^3 = 2 \) and \( p_g(X) \leq 2 \); \( \phi_5 \) is generically finite of degree \( \leq 8 \).

\( (2) \) \( \phi_4 \) is birational if \( K_X^3 > 2 \) and \( \dim \phi_1(X) = 3 \); \( \phi_4 \) is generically finite except for some 3-folds with \( K_X^3 = 2 \), \( p_g(X) \leq 1 \) and \( \chi(O_X) = -1 \).

\( (3) \) \( \phi_3 \) is generically finite if \( p_g(X) \geq 39 \).

For a nonsingular projective minimal 3-fold \( X \) of general type, Benveniste ([2]) proved that \( \dim \phi_m(X) \geq 2 \) for \( m \geq 4 \), i.e. \( |4K_X| \) can not be composed of a pencil. Recently, it has been proved ([5]) that \(|3K_X|\) also can not be composed of a pencil. (Actually, the method is also effective for Gorenstein 3-folds of general type.) Thus it is interesting whether \(|2K_X|\) can be composed of a pencil and like what a bicanonical pencil behaves. So in Section 4, we study the bicanonical pencil of a Gorenstein 3-fold of general type. According to the 3-dimensional MMP, we can suppose that \( X \) is a minimal locally factorial Gorenstein 3-fold of general type. Take a birational modification \( \pi : X' \to X \) such that \( X' \) is smooth, \(|\pi^*(2K_X)|\) gives a morphism and \( \pi^*(2K_X) \) has supports with only normal crossings. This is possible because of Hironaka’s big theorem. Let \( W := \phi_2(X) \) and take the Stein factorization

\[
\phi_2 \circ \pi : X' \to C \to W.
\]

Then \( f \) is a fibration onto the nonsingular curve \( C \), we call \( f \) a derived fibration of \( \phi_2 \). Denote by \( F \) a general fibre of \( f \). Then \( F \) is a nonsingular surface of general type by virtue of the Bertini theorem. Also set \( b := g(C) \), the geometric genus of \( C \). From [7], we know that \( 0 \leq b \leq 1 \). We shall prove the following

**Theorem 3.** Let \( X \) be a projective minimal Gorenstein 3-fold of general type and suppose that \(|2K_X|\) is composed of a pencil. Let \( f \) be the derived fibration of \( \phi_2 \) and \( F \) be a general fibre of \( f \). Then we have \( p_g(F) = 1 \) and \( K_F^2 \leq 3 \), where \( F_0 \) is the minimal model of \( F \).

As an application of our method, we shall present a corollary on surfaces of general type which somewhat simplifies Xiao’s theorem for the bicanonical finiteness.

2. Proof of Theorem 1

Throughout our argument, the Kawamata-Viehweg vanishing theorem is always employed as a much more effective tool. We use it in the following form.

**K-V Vanishing Theorem** ([10] or [17]). Let \( X \) be a nonsingular complete variety, \( D \in \text{Div}(X) \otimes \mathbb{Q} \). Assume the following two conditions:

\( (1) \) \( D \) is nef and big;

\( (2) \) the fractional part of \( D \) has the support with only normal crossings.
Then $H^i(X, O_X(\lceil D^\cap + K_X)) = 0$ for $i > 0$, where $\lceil D^\cap$ is the round-up of $D$, i.e. the minimum integral divisor with $\lceil D^\cap - D \geq 0$.

**Lemma 2.1** (Corollary 2 of [16]). Let $S$ be a nonsingular algebraic surface, $L$ be a nef divisor on $S$, $L^2 \geq 10$ and let $\phi$ be a map defined by $|L + K_S|$. If $\phi$ is not birational, then $S$ contains a base point free pencil $E'$ with $L \cdot E' = 1$ or $L \cdot E' = 2$.

**Lemma 2.2.** Let $X$ be a nonsingular variety of dimension $n$, $D \in \text{Div}(X) \otimes \mathbb{Q}$ be a $\mathbb{Q}$-divisor on $X$. Then we have the following:
(i) if $S$ is a smooth irreducible divisor on $X$, then $\lceil D^\cap|_S \geq \lceil D|_S^\cap$;
(ii) if $\pi : X' \to X$ is a birational morphism, then $\pi^*(\lceil D^\cap) \geq \lceil \pi^*(D^\cap)$.

Proof. We can write $D$ as $G + \sum_{i=1}^t a_i E_i$, where $G$ is a divisor, the $E_i$ are effective divisors for each $i$ and $0 < a_i < 1$, $\forall i$. So we only have to prove the lemma for effective $\mathbb{Q}$-divisors. That is easy to check. \square

**Lemma 2.3** (Lemma 2.3 of [9]). Let $X$ be a minimal threefold of general type with canonical index $r$. Then we have the plurigenus formula

$$h^0(X, \omega_X^{\lceil mr+s}) = \frac{1}{12} (mr+s)(mr+s-1)(2mr+2s-1)(K_X^3) + \alpha m + c_s$$

for $0 \leq s < r$, $mr+s \geq 2$, where $\alpha$ is a constant and $c_s$ is a constant only relating to $s$.

**Definition 2.4.** Let $X$ be a nonsingular projective variety of dimension $\geq 2$. Suppose $|M|$ is a base-point-free system on $X$, a general irreducible element $S$ of $|M|$ means the following:
(i) if $\dim \Phi_{|M|}(X) \geq 2$, then $S$ is just a general member of $|M|$;
(ii) if $\dim \Phi_{|M|}(X) = 1$, taking the Stein factorization of $\Phi_{|M|}$, then we obtain a fibration $f : X \to C$ onto a curve $C$. We mean $S$ a general fibre of $f$.

**Proposition 2.5** (Lemma 3.2 of [9]). Let $X$ be a minimal threefold of general type with canonical index $r \geq 2$. Then $\dim \phi_{mr+s}(X) \geq 2$ in one of the following cases:
(i) $r = 2$ and $m \geq 3$;
(ii) $r = 3$ and $m \geq 2$;
(iii) $r = 4, 5, 0 \leq s \leq 2$ and $m \geq 2$; $r = 4, 5, s \geq 3$ and $m \geq 1$;
(iv) $r \geq 6, 0 \leq s \leq 1$ and $m \geq 2$; $r \geq 6, s \geq 2$ and $m \geq 1$.

Now we modify Proposition 2.5 by virtue of Hanamura’s method in order to prove our Theorem 1. The proof is due to Hamamura ([9]).
Proposition 2.6. Let $X$ be a minimal threefold of general type with canonical index $r \geq 2$. Then $h^0(\omega_X^{|m+n|}) \geq 3$ in one of the following cases:

(i) $r = 2$ and $m \geq 2$;
(ii) $r \geq 3$, $s = 0, 1$ and $m \geq 2$; $r \geq 3$, $s \geq 2$ and $m \geq 1$.

Proof. From Lemma 2.3, we can put

\begin{equation}
P(mr + s) = \frac{1}{12}(mr + s)(mr + s - 1)(2mr + 2s - 1)(K_X^3) + am + c_s
\end{equation}

where $a$ and $c_s$ are constants for $0 \leq s < r$. We consider the right hand side of (2.1) as a polynomial in $m$ and denote it by $P_s(m)$. Let $Q_s(m)$ be the first term of $P_s(m)$. We have

\begin{equation}
P_s(m) = Q_s(m) + am + c_s.
\end{equation}

We see that, for $m \geq 1$ or $m = 0$ and $s \geq 2$,

\begin{equation}
P_s(m) \geq 0.
\end{equation}

By Kollár’s result ([11]) that the $\omega_X^{|m+n|}$ are Cohen-Macaulay, using the Grothendieck duality, one can see that, for $m \leq -1$,

\begin{equation}
P_s(m) \leq 0.
\end{equation}

Now we want to estimate both $a$ and $c_s$. For any $r$ and $s$, by (2.2) and (2.3), we have

\begin{equation}
Q_s(1) + a + c_s \geq 0
\end{equation}

\begin{equation}
- Q_s(-1) + a - c_s \geq 0.
\end{equation}

Which induces

\begin{equation}
a \geq \frac{1}{2} \left\{ Q_s(-1) - Q_s(1) \right\}
\end{equation}

\[= - \frac{1}{12} \left\{ 2r^2 + (6s^2 - 6s + 1) \right\} (rK_X^3).\]

When $r \geq 3$ and $s \geq 2$, we have

\begin{equation}
Q_s(0) + c_s \geq 0.
\end{equation}

By (2.5) and (2.7), we get

\begin{equation}
a \geq - Q_s(0) + Q_s(-1)
\end{equation}
Explicitly, we have

\begin{align*}
\frac{1}{12} \left\{ -2r^2 + (6s - 3)r - (6s^2 - 6s + 1) \right\} (rK_X^3).
\end{align*}

(2.9) \quad a \geq \frac{1}{12} \left\{ -\frac{1}{2} r^2 + \frac{1}{2} \right\} (rK_X^3) \text{ if } r \text{ is odd}

(2.10) \quad a \geq \frac{1}{12} \left\{ -\frac{1}{2} r^2 - 1 \right\} (rK_X^3) \text{ if } r \text{ is even.}

Now we can calculate the $P(mr + s)$ case by case.

**Case 1.** \( r \geq 3 \) and \( s \geq 2 \).

When \( r \) is odd, from (2.7) and (2.9), we have

\begin{align*}
P(mr + s) \geq & \quad Q_s(m) - \frac{1}{12} m \left( \frac{1}{2} r^2 - \frac{1}{2} \right) (rK_X^3) - Q_s(0) \\
= & \quad \frac{1}{12} \left\{ (mr + s)(mr + s - 1)(2mr + 2s - 1) + m \left( -\frac{1}{2} r^3 + \frac{1}{2} r \right) \\
& \quad - s(s - 1)(2s - 1) \right\} (K_X^3).
\end{align*}

We get \( P(mr + s) \geq 7 \) for \( m \geq 1 \).

When \( r \) is even, from (2.7) and (2.10), we have

\begin{align*}
P(mr + s) \geq & \quad Q_s(m) - \frac{1}{12} m \left( \frac{1}{2} r^2 + 1 \right) (rK_X^3) - Q_s(0) \\
= & \quad \frac{1}{12} \left\{ 2r^2 m^3 + (6s - 3)rm^2 + \left( 6s^2 - 6s - \frac{1}{2} r^2 \right) m \right\} (rK_X^3).
\end{align*}

We get \( P(mr + s) \geq 5 \) for \( m \geq 1 \).

**Case 2.** \( s = 1 \).

From (2.4) and (2.5), we have

\[ P(mr + 1) \geq \frac{1}{12} r(m^2 - 1)(2rm + 3)(rK_X^3). \]

We get \( P(mr + 1) \geq 6 \) for \( m \geq 2 \).

**Case 3.** \( s = 0 \).

By (2.4) and (2.5), we have

\[ P(mr) \geq \frac{1}{12} r(m^2 - 1)(2rm - 3)(rK_X^3). \]

We get \( P(mr) \geq 3 \) for \( m \geq 2 \). Thus we complete the proof. \( \square \)

In what follows we can get an improved version of Hanamura’s theorem.
Theorem 2.7. Let $X$ be a projective threefold of general type with the canonical index $r \geq 2$. Then $\phi_m$ is birational onto its image for $m \geq 4r + 3$.

Proof. We can suppose that $X$ is a minimal 3-fold. For any $m_1 \geq r + 2$, take some blowing-ups $\pi: X' \rightarrow X$ according to Hironaka such that $X'$ is nonsingular and that the movable part of $|m_1 K_{X'}|$ defines a morphism. Denote by $|M|$ the moving part of $|m_1 K_{X'}|$ and by $S$ a general irreducible element of $|M|$. Then $S$ is a nonsingular projective surface of general type by the Bertini theorem. On $X'$, we consider the system $|K_{X'} + 3\pi^* (r K_X) + S|$. Because $K_{X'} + 3\pi^* (r K_X)$ is effective by Proposition 2.6, so the system can distinguish general irreducible elements of $|M|$. On the other hand, the vanishing theorem gives

$$|K_{X'} + 3\pi^* (r K_X) + S||_S = |K_S + 3L||,$$

where $L := \pi^* (r K_X)|_S$ is a nef and big divisor on $S$ and $L^2 \geq 2$. Reider’s result tells that the right system gives a birational map, so does $|K_{X'} + 3\pi^* (r K_X) + S|$. Thus $\phi_m$ is birational for $m \geq 4r + 3$. 

Proof Theorem 1. We can suppose that $X$ is a minimal model. If $r = 2$, then $\phi_m$ is birational for $m \geq 11$ according to Theorem 2.7. From now on, we assume $r \geq 3$ and define

$$m_2 = \begin{cases} \r + 3, & \text{for } 3 \leq r \leq 5 \\ r + 2, & \text{for } r \geq 6. \end{cases}$$

Take some blowing-ups $\pi: X' \rightarrow X$ such that $X'$ is nonsingular, $|m_2 K_{X'}|$ defines a morphism and the fractional part of $\pi^* (K_X)$ has supports with only normal crossings. Denote by $|M_2|$ the moving part of $|m_2 K_{X'}|$ and by $S_2$ a general irreducible element of $|M_2|$. For any $t \in \mathbb{Z}_{>0}$, we consider the system

$$|K_{X'} + \gamma (t + m_2) \pi^* (K_X)| + S_2|,$$

which is a sub-system of $|(t + 2m_2 + 1) K_{X'}|$. Because $K_{X'} + \gamma (t + m_2) \pi^* (K_X)|$ is effective by Proposition 2.6, so the system can distinguish general irreducible elements of $|M_2|$. On the other hand, the K-V vanishing theorem tells that

$$|K_{X'} + \gamma (t + m_2) \pi^* (K_X)| + S_2||_{S_2}$$

$$= |G + L||,$$

where $G := \{K_{X'} + \gamma (t + m_2) \pi^* (K_X)|\}|_{S_2}$ is effective and $L := S_2|_{S_2}$. We can see that

$$G + L \geq K_{S_2} + \gamma t \pi^* (K_X)|_{S_2} + L.$$
From Proposition 2.5, we have \( h^0(S_2, L) \geq 2 \). Modulo blowing-ups, actually we can suppose that \( |L| \) is free from base points. Let \( C \) be a general irreducible element of \( |L| \). It is obvious that \( |G + L| \) can distinguish general irreducible elements of \( |L| \). On the other hand, the K-V vanishing theorem gives

\[
|K_{S_2} + t \pi^*(K_X)|_{S_2} + C||_C = |K_C + D|,
\]

where \( D := t \pi^*(K_X)|_{S_2} \) is a divisor of positive degree. Because \( C \) is a curve of genus \( \geq 2 \), so \( h^0(C, K_C + D) \geq 2 \) and \( |K_C + D| \) gives a finite map. Thus we have \( \dim \Phi_{(G + L)}(C) = 1 \). Therefore \( \phi_m \) is generically finite for \( m \geq 2m_2 + 2 \), which completes the proof.

3. On Gorenstein 3-folds of general type

For a minimal threefold \( X \) of general type with canonical index 1, we can find certain birational modifications \( f : X' \to X \) according to [15] such that \( c_2(X') \cdot \Delta = 0 \), where \( \Delta \) is the ramification divisor of \( f \). Then we can get the same plurigenus formula as that for a nonsingular minimal threefold, i.e.

\[
p(n) := h^0(X, O_X(nK_X)) = (2n - 1) \left[ \frac{n(n - 1)}{12} K_X^3 - \chi(O_X) \right],
\]

for \( n \geq 2 \). On the other hand, the Miyaoka-Yau inequality ([14]) shows that \( \chi(O_X) < 0 \). From [4] or [12], we know that \( \phi_m \) is birational for \( m \geq 6 \).

**Theorem 3.1.** Let \( X \) be a projective minimal Gorenstein 3-fold of general type. Then

1. \( \phi_5 \) is birational if either \( K_X^3 > 2 \) (Ein-Lazarsfeld-Lee) or \( p_g(X) > 2 \).
2. When \( p_g(X) = 2 \), then \( \phi_5 \) is birational except for some 3-folds with \( q(X) = h^2(O_X) = 0 \), and \( |K_X| \) composed with a rational pencil of surfaces of general type with \( (K^2, p_g) = (1, 2) \). In this situation, \( \phi_5 \) is generically finite of degree 2.
3. \( \phi_5 \) is birational if \( \dim \phi_2(X) = 1 \).

Proof. This is the main theorem in [7]. Though the objects considered there are nonsingular minimal 3-folds, the method is also effective for all Gorenstein 3-folds of general type.

**Definition 3.2.** Let \( X \) be a projective minimal Gorenstein 3-fold of general type. Suppose \( \dim \phi_i(X) \geq 2 \) and set \( iK_X \sim_{\text{lin}} M_i + Z_i \), where \( M_i \) is the moving part and \( Z_i \) the fixed one for any integer \( i \). We define \( \delta_i(X) := K_X^3 \cdot M_i \).

**Proposition 3.3.** Let \( X \) be a projective minimal Gorenstein 3-fold of general type. Suppose \( |2K_X| \) is not composed of a pencil and \( K_X^3 > 2 \). Then \( \delta_2(X) \geq 3 \).
Proof. We have $\delta_2(X) \geq 2$ by Proposition 2.2 of [4]. Take a birational modification $f : X' \to X$ such that $|2f^*(K_X)|$ defines a morphism. Set $2f^*(K_X) \sim_{\text{lin}} M + Z$, where $M$ is the moving part and $Z$ the fixed one. A general member $S \in |M|$ is an irreducible nonsingular projective surface of general type. Denote $L := f^*(K_X)|_S$. If $L^2 = f^*(K_X)^2 \cdot S = \delta_2(X) = 2$, then we have

$$4 = 2f^*(K_X)^2 \cdot S = f^*(K_X) \cdot S^2 + f^*(K_X) \cdot S \cdot Z,$$

Noting that $S$ is nef and $S \not\equiv 0$, we have $f^*(K_X) \cdot S^2 \geq 1$. Therefore four cases occur as follows:

(i) $f^*(K_X) \cdot S^2 = 4$, $f^*(K_X) \cdot S \cdot Z = 0$;
(ii) $f^*(K_X) \cdot S^2 = 3$, $f^*(K_X) \cdot S \cdot Z = 1$;
(iii) $f^*(K_X) \cdot S^2 = 2$, $f^*(K_X) \cdot S \cdot Z = 2$;
(iv) $f^*(K_X) \cdot S^2 = 1$, $f^*(K_X) \cdot S \cdot Z = 3$.

We also have

$$2K_X^3 = 2f^*(K_X)^3 = f^*(K_X)^2 \cdot S + f^*(K_X)^2 \cdot Z = 2 + \frac{1}{2} f^*(K_X) \cdot Z(S + Z) = 2 + \frac{1}{2} f^*(K_X) \cdot S \cdot Z + \frac{1}{2} f^*(K_X) \cdot Z^2.$$

**Case (i).** Noting that $f^*(K_X)$ is nef and big, we see that $mf^*(K_X)$ is linearly equivalent to a nonsingular projective surface of general type according to Kawamata for sufficiently large integer $m$. Then $S_{|mf^*(K_X)}$ is nef and big and, by the Hodge Index Theorem, we have $f^*(K_X) \cdot Z^2 \leq 0$. Thus (3.1) is false and this case does not occur.

**Case (ii).** We have $f^*(K_X) \cdot S(S - 3Z) = 0$, then $f^*(K_X)(S - 3Z)^2 \leq 0$, which derives $f^*(K_X) \cdot Z^2 \leq 1/3$, i.e. $f^*(K_X) \cdot Z^2 \leq 0$. (3.1) is also false.

**Case (iii).** $f^*(K_X) \cdot S(S - Z) = 0$ induces $f^*(K_X) \cdot Z^2 \leq 2$, then (3.1) becomes $K_X^3 \leq 2$. Thus $K_X^3 = 2$. Actually, in this case, $f^*(K_X) \cdot (S - Z) \sim_{\text{num}} 0$ (as 1-cycle).

**Case (iv).** $f^*(K_X) \cdot (3S - Z)^2 \leq 0$ induces $f^*(K_X) \cdot Z^2 \leq 9$. And (3.1) becomes $K_X^3 \leq 4$. If $K_X^3 = 4$, we see that $f^*(K_X) \cdot (3S - Z) \sim_{\text{num}} 0$ as 1-cycle. Now we set $f^*(M_2) = S + E$. Then $Z = f^*(Z_2) + E$. Obviously, we have $f_*(S) = M_2$ and $f_*(Z) = Z_2$. From $f^*(M_2) \cdot f^*(K_X) \cdot (3S - Z) = 0$, we get $3K_X \cdot M_2^2 = K_X \cdot M_2 \cdot Z_2$. Then $4 = 2K_X^3 \cdot M_2 = K_X \cdot M_2^2 + K_X \cdot M_2 \cdot Z_2 = 4K_X \cdot M_2^2$, i.e. $K_X \cdot M_2^2 = 1$. Which derives a contradiction, because $K_X \cdot M_2^2$ is even. Thus $K_X^3 = 2$.

**Proposition 3.4.** Let $X$ be a projective minimal Gorenstein 3-fold of general type. Suppose $K_X^3 > 2$ and $\dim \phi_1(X) \geq 2$. Then $\delta_1(X) \geq 3$.

Proof. As in the proof of the previous proposition, we first take a modification $f : X' \to X$. Set $f^*(K_X) \sim_{\text{lin}} M + Z$, where $M$ is the moving part. A general member $S \in |M|$ is a nonsingular projective surface of general type. Also denote $L :=$
\( f^*(K_X)|_S \). Then \( L^2 = \delta_1(X) \geq 2 \) according to Proposition 2.1 of [7]. If \( L^2 = 2 \), then we have

\[
2 = f^*(K_X)^2 \cdot S = f^*(K_X) \cdot S^2 + f^*(K_X) \cdot S \cdot Z.
\]

We also have

\[
K_X^3 = f^*(K_X)^2 \cdot S + f^*(K_X)^2 \cdot Z
\]

= \( 2 + f^*(K_X) \cdot S \cdot Z + f^*(K_X) \cdot Z^2 \).

Similarly, \( f^*(K_X) \cdot S^2 \geq 1 \). If \( f^*(K_X) \cdot S^2 = 2 \) and \( f^*(K_X) \cdot S \cdot Z = 0 \), then, by the Hodge Index Theorem, \( f^*(K_X) \cdot Z^2 \leq 0 \). Then (3.2) becomes \( K_X^3 \leq 2 \), which says \( K_X^3 = 2 \). If \( f^*(K_X) \cdot S = f^*(K_X) \cdot S \cdot Z = 1 \), \( f^*(K_X) \cdot S \cdot (S - Z) = 0 \) induces \( f^*(K_X) \cdot Z^2 \leq 1 \).

By (3.2), we get \( K_X^3 \leq 4 \). If \( K_X^3 = 4 \), then we can see \( f^*(K_X) \cdot (S - Z) \sim_{\text{num}} 0 \).

By the same argument as in the case (iv) of the proof of Proposition 3.3, we have

\[
f^*(M) \cdot f^*(K_X) \cdot (S - Z) = 0,
\]

i.e. \( K_X \cdot M_i^2 = K_X \cdot M_i \cdot Z_i \). We have \( 2 = K_X^2 \cdot M_i = K_X \cdot M_i^2 + K_X \cdot M_i \cdot Z_i = 2K_X \cdot M_i^2 \). Therefore \( K_X \cdot M_i^2 = 1 \), which is impossible. Thus \( K_X^3 = 2 \).

\[\square\]

**Theorem 3.5.** Let \( X \) be a projective minimal Gorenstein 3-fold of general type. Then \( \phi_5 \) is generically finite of degree \( \leq 8 \). If \( \deg(\phi_5) > 2 \), then \( K_X^3 = 2 \), \( \chi(O_X) = -1 \) and \( p_g(X) = 0, 1 \).

Proof. According to Theorem 3.1, we only have to study the case when \( |2K_X| \) is not composed of a pencil. Take a modification \( f : X' \to X \) according to Hironaka such that \( |2f^*(K_X)| \) defines a morphism. Set \( 2f^*(K_X) \sim_{\text{lin}} M + Z \), where \( M \) is the moving part and \( Z \) the fixed one. A general member \( S \in |M| \) is a nonsingular projective surface of general type by the Bertini Theorem. We have

\[
|K_X + 2f^*(K_X) + S| \subset |5K_X|^1.
\]

Because \( K_X + 2f^*(K_X) \) is effective, the left system can distinguish general members of \( |M| \). Denote \( L := f^*(K_X)|_S \), using the long exact sequence and the vanishing theorem, we have

\[
|K_X + 2f^*(K_X) + S||_S = |K_X + 2L||_S.
\]

Obviously, \( K_X + 2L = G + H \), where \( G := (K_X + 2f^*(K_X))|_S \) is effective and \( H := S|_S \). Note that \( h^0(S, O_S(2L)) \geq h^0(S, H) \geq P(2) - 1 \geq 3 \). We have two cases.

**Case 1.** \( |H| \) is composed of a pencil. Taking a birational modification to \( S \) if necessary, we can suppose \( |H| \) is free from base points. Denote \( H \sim_{\text{lin}} \sum_{i=1}^a C_i + E \), where \( E \) is the fixed part. In general position, \( \sum_{i=1}^a C_i \) can be a disjoint union of nonsingular curves in a family. We have \( a \geq 2 \). Thus \( L \sim_{\text{num}} (a/2)\mathcal{C} + E_0 \), where
\( E_0 \geq (1/2)E \) is an effective \( \mathbb{Q} \)-divisor. If \( p_g(S) = 0 \), then \( q(S) = 0 \) and then we can see by the long exact sequence that \( |K_S + H| \) can distinguish \( C_i \)'s and that \( |K_S + \sum_{i=1}^{a} C_i|_C = |K_G|_C \), which means \( |K_S + 2L| \) gives at worst a generically finite map of degree 2 and so does \( \phi_5 \). If \( p_g(S) > 0 \), it is obvious that \( |K_S + 2L| \) can distinguish \( C_i \)'s. For a general curve \( C \) which is algebraically equivalent to \( C_i \), we consider the \( \mathbb{Q} \)-divisor \( G := K_S + 2L - (1/2)\sum_{i=3}^{a} C_i - E_0 \). We have \( \Gamma G^2 \leq K_S + 2L \).

On the other hand, \( G - C - K_S \) is nef and big, thus by the K-V vanishing we have \( \Gamma G^2|_C = |K_C + \Gamma E_0^2|_C \). Because \( \Gamma E_0^2|_C \) is effective, \( \Phi_{|K_S + 2L|} \) is at worst a generically finite map of degree 2 and so is \( \phi_5 \) of \( X \).

**Case 2.** \( |H| \) is not composed of a pencil, so neither is \( |2L| \). Similarly, we can suppose \( |2L| \) is base point free. If \( p_g(S) = 0 \), we can use a parallel discussion to that of Case 1 to see that \( \phi_5 \) is at worst a generically finite map of degree 2. If \( p_g(S) > 0 \), then \( \Phi_{|K_S + 2L|} \) is obviously generically finite. We know that \( L^2 \geq 2 \) from Proposition 2.2 of [4]. If \( \Phi_{|K_S + 2L|} \) is not birational and \( L^2 \geq 3 \), then according to Lemma 2.1, there is a free pencil on \( S \) with a general member \( C \) such that \( C^2 = 0 \) and \( L \cdot C = 1 \). Since \( \dim \Phi_{|2L|} (C) = 1 \), then \( h^0(2L|_C) \geq 2 \) and then, by the Clifford theorem, we see that \( C \) is a curve of genus 2 and \( 2L|_C \sim_{\text{lin}} K_C \). Finally we can see that \( |2L||_C = |K_C| \).

Therefore \( \Phi_{|K_S + 2L|} \) is a generically finite map of degree 2. Therefore \( \phi_5 \) is generically finite with \( \deg(\phi_5) \leq 2 \). If \( L^2 = 2 \), then \( K_X^3 = 2 \) by the proof of Proposition 3.3. On the surface \( S \), set \( 2L \sim_{\text{lin}} C_1 + E_1 \), where \( C_1 \) is the moving part. We easily get

\[
8 = (2L)^2 \geq C_1^2 \geq d(h^0(2L) - 2) \geq d(P(2) - 3).
\]

Therefore we have

\[
d \leq \frac{8}{P(2) - 3} = \frac{8}{-3\chi(O_X) - 2}.
\]

If \( d > 2 \), then \( \chi(O_X) = -1 \). \( \square \)

For the 4-canonical map of \( X \), it is obvious that \( \phi_4 \) is not birational if \( X \) admits a pencil of surfaces of general type with \( (K^2, p_g) = (1, 2) \). Therefore it is pessimistic for us to obtain an effective sufficient condition for the birationality of \( \phi_4 \). We have a partial result as follows.

**Theorem 3.6.** Let \( X \) be a projective minimal Gorenstein 3-fold of general type. Suppose \( K_X^3 > 2 \) and \( \dim \phi_1(X) = 3 \). Then \( \phi_4 \) is a birational map onto its image.

**Proof.** Take a birational modification \( f : X' \rightarrow X \) such that the movable part of \( |f^*(K_X)| \) is base point free. Set \( f^*(K_X) \sim_{\text{lin}} S + Z \), where \( S \) is the moving part and \( Z \) the fixed one. A general member \( S \) is a nonsingular projective surface of general type. We have \( |K_{X'} + 2f^*(K_X) + S| \subset |4K_{X'}| \). Using the vanishing theorem, we have

\[
|K_{X'} + 2f^*(K_X) + S|_S = |K_S + 2L|,
\]
where $L := f^*(K_X)|_S$ is a nef and big divisor on $S$. By Proposition 3.4, we see that $L^2 \geq 3$ under the condition $K_X^3 > 2$. If $\Phi_{|K_S+2L|}$ is not birational, then, by Lemma 2.1, there is a free pencil with a general member $C$ such that $C^2 = 0$ and $L \cdot C = 1$. Because $\dim \Phi_{|L|}(S) = 2$, $h^0(C, \mathcal{O}_C(L|_C)) \geq 2$. Therefore, by the Clifford theorem, we see that $\deg(L|_C) \geq 2h^0(L|_C) - 2 \geq 2$. This is a contradiction. Therefore $\Phi_{|K_S+2L|}$ is birational and so is $\phi_4$.

Example 3.7. We give an example which shows that $\phi_4$ is not birational when $K_X^3 = 2$ and $\dim \phi_1(X) = 3$. On $\mathbb{P}^3(\mathbb{C})$, take a smooth hypersurface $S$ of degree 10, $S \sim_{\text{lin}} 10H$. Let $X$ be a double cover of $\mathbb{P}^3$ with branch locus along $S$. Then $X$ is a nonsingular canonical model, $K_X^3 = 2$ and $p_g(X) = 4$ and $\phi_1$ is a finite morphism onto $\mathbb{P}^3$ of degree 2. One can easily check that $\phi_4$ is also a finite morphism of degree 2.

Theorem 3.8. Let $X$ be a projective minimal Gorenstein 3-fold of general type. Then $\phi_4$ is generically finite when $p_g(X) \geq 2$ or when $K_X^3 > 2$ or when $\chi(\mathcal{O}_X) \neq -1$.

Proof. Part I: $p_g(X) \geq 2$.

First we make a modification $f : X' \longrightarrow X$ such that the movable part of $|f^*(K_X)|$ is free from base points and that $f^*(K_X)$ has support with only normal crossings. Set $f^*(K_X) \sim_{\text{lin}} M + Z$, where $M$ is the moving part and $Z$ the fixed one.

If $\dim \phi_1(X) = 2$, then a general member $S \in |M|$ is a nonsingular projective surface of general type. We have

$$|K_{X'} + 2f^*(K_X) + S| \subset |4K_{X'}|.$$

Using the vanishing theorem, we have $|K_{X'} + 2f^*(K_X) + S||_S = |K_S + 2L|$, where $L := f^*(K_X)|_S$ is nef and big effective divisor on $S$. We have $h^0(S, L) \geq 2$. Noting that $p_g(S) > 0$ in this case. And if $|L|$ is not composed of a pencil, then neither is $|K_S + 2L|$. If $|L|$ is composed of a pencil, taking a modification if possible, we can suppose that the movable part of $|L|$ is free from base points. Set $L \sim_{\text{lin}} \sum C_i + Z_0$, we can see $|K_S + L + \sum C_i||_{C_i} = |K_{C_i} + D|$, where $D := L|_{C_i}$ is effective. We easily see that $\Phi_{|K_S+2L|}$ is at worst generically finite of degree $\leq 2$ and so is $\phi_4$.

If $\dim \phi_1(X) = 1$, then $M \sim_{\text{num}} aF$, where $F$ is a nonsingular projective surface of general type. $M_1 \sim_{\text{num}} aF_0$, where $F_0 = f_*(F)$ is irreducible on $X$. If $K_X \cdot F_0^2 = 0$, then, by Lemma 2.3 of [7], we have $\mathcal{O}_F(f^*(K_X)|_F) \cong \mathcal{O}_F(\pi^*(K_0))$, where $\pi$ is the contraction map onto the minimal model and $K_0$ is the canonical divisor of the minimal model of $F$. Obviously, $|K_{X'} + 2f^*(K_X) + M|$ can distinguish general members of $|M|$. Moreover $|K_{X'} + 2f^*(K_X) + M||_F = |K_F + 2\pi^*(K_0)|$, the right system gives a generically finite map and so does $\phi_4$. If $K_X \cdot F_0^2 > 0$, then

$$L^2 = f^*(K_X)^2 \cdot F = K_X^2 \cdot F_0 \geq K_X \cdot F_0^2 \geq 2.$$
It is sufficient to show that $|K_F + 2L|$ gives a generically finite map. We have $K_F + 2L \geq 3L$. If $|3L|$ is not composed of a pencil, then neither is $|K_F + 2L|$. If $|3L|$ is composed of a pencil, we claim that $h^0(F, 3L) \geq 3$. In fact, we have $|K_X + f^*(K_X) + F|_F = |K_F + L|$ and $h^0(F, K_F + L) \geq 3$. Considering the natural map $H^0(X', 3K_X) \to H^0(F, 3K_F)$, because $K_X + f^*(K_X) + F \leq 3K_X$, we see that $\dim_C(\text{Im}(\alpha)) \geq h^0(K_F + L) \geq 3$. Similarly, considering another natural map $H^0(X', 3f^*(K_X)) \to H^0(F, 3L)$, we have

$$h^0(3L) \geq \dim_C(\text{Im}(\beta)) = \dim_C(\text{Im}(\alpha)) \geq 3.$$ 

Now we can write $3L \sim_\text{lin} \sum_{j=1}^t C_j + E_0$, where $E_0$ is the fixed part, $t \geq 2$ and the $C_j$ are irreducible curves. Denote by $C$ a generic $C_j$. Then $2L \sim_\text{num} (2/3)C + (2/3)E_0$ and thus $2L - C - (1/t)E_0$ is a nef and big $\mathbb{Q}$-divisor. Setting $G := 2L - (1/t)E_0$, then we have $K_S + \Gamma G = K_S + 2L$. On the other hand, the K-V vanishing gives $|K_S + \Gamma G||_C = |K_C + D|$, where $D$ is a divisor of positive degree. Noting that $C$ is a curve of genus $\geq 2$, we see that $|K_C + D|$ gives a generically finite map. This means $|K_S + 2L|$ gives a generically finite map.

**PART II:** $K^2_X > 2$ or $\chi(\mathcal{O}_X) \neq -1$.

We study $\phi_2$ according to the behavior of $\phi_2$. Of course, first we make a modification $f : X' \to X$ such that the movable part of $|2f^*(K_X)|$ is free from base points and that $2f^*(K_X)$ has supports with only normal crossings. Set $2f^*(K_X) \sim_\text{lin} M_2 + \sum_i Z_i$, where $M_2$ is the moving part and $Z_i$ the fixed one.

If $\dim \phi_2(X) = 1$, then $M_2 \sim_\text{lin} a_2F$, where $F$ is a nonsingular projective surface of general type. We have $\mathcal{O}_F(f^*(K_X)|_F) \cong \mathcal{O}_F(\pi^*(K_0))$ by Lemma 4.2 below in this paper. Because $K_X + f^*(K_X)$ is effective, $|K_X + f^*(K_X) + M_2|$ can distinguish general $F$. On the other hand, we have $|K_X + f^*(K_X) + M_2|_F = |K_F + \pi^*(K_0)|$. From Theorem 3.1 of [7], we know that $F$ is not a surface with $p_g = q = 0$. Thus $|K_F + \pi^*(K_0)|$ defines a generically finite map according to [19] and so does $\phi_2$.

If $\dim \phi_2(X) \geq 2$, then a general member $S \in \sum \mathcal{M}_2$ is a nonsingular projective surface of general type. We have $|K_X + f^*(K_X) + S|_S = |K_S + L|$, where $L := f^*(K_X)|_S$. Noting that $K_S \geq L$, then we have $K_S + L \geq 2L$. Under our assumption, we have $P(2) \geq 5$. Thus $h^0(2L) \geq 4$. We may suppose that the movable part of $|2L|$ is free from base points. If $|2L|$ is not composed of a pencil, then neither is $|K_S + L|$. Otherwise we can set $2L \sim_\text{lin} \sum_{i=1}^b C_i + E_1$, where $b \geq 3$ and $E_1$ is the fixed part. We denote by $C$ the general $C_i$. Because $L - (1/b)E_1$ is nef and big, therefore

$$|K_S + \Gamma L - \frac{1}{b}E_1|_C = |K_C + D|,$$

where $D$ is a divisor of positive degree. The right system obviously defines a generically finite map. Thus $|K_S + L|$ gives a generically finite map and so does $\phi_4$. \qed
Theorem 3.9. Let $X$ be a projective minimal Gorenstein 3-fold of general type. Then $\phi_3$ is generically finite when $p_g(X) \geq 39$.

Proof. First we make a modification $f : X' \to X$ such that the movable part of $|f^*(K_X)|$ is free from base points and that $f^*(K_X)$ has support with only normal crossings. Set $f^*(K_X) \sim_{\text{lin}} M + Z$, where $M$ is the moving part and $Z$ the fixed one.

If $\dim \phi_3(X) \geq 2$, then a general member $S \in |M|$ is a nonsingular projective surface of general type. We have $|K_{X'} + f^*(K_X) + S|_S = |K_S + L|$, where $L : = f^*(K_X)|_S$. When $p_g(X) \geq 4$, $h^0(S, L) \geq 3$. Noting that $p_g(S) > 0$, if $|L|$ is not composed of a pencil, then nor is $|K_S + L|$. So we may suppose that $|L|$ is composed of a pencil and the movable part of this system is free from base points. Set $L \sim_{\text{lin}} \sum_{i \neq 1} C_i + E_0$, where we have $\alpha \geq 2$. $|K_S + L|$ can distinguish the $C_i$ generically. On the other hand, $L - C - (1/\alpha) E_0$ is nef and big, we obtain by the Kawamata-Viehweg vanishing that

\[
\left| K_S + \left( L - \frac{1}{\alpha} E_0 \right) \right|_C = \left| K_C + \left( \frac{a - 1}{\alpha} L \right) \right|_C.
\]

The right system defines a generically finite map and so does $\phi_3$.

If $\dim \phi_1(X) = 1$, then $M \sim_{\text{num}} aF$, where $F$ is a nonsingular projective surface of general type. Set $F_0 = f_*(F)$. If $K_X \cdot F_0^2 = 0$, then, by Lemma 2.3 of [7], we have $O_F(f^*(K_X)|_F) \cong O_F(\pi^*(K_0))$, where $\pi$ is the contraction onto the minimal model and $K_0$ is the canonical divisor of the minimal model of $F$. We see that $|K_{X'} + f^*(K_X) + M||_F = |K_F + \pi^*(K_0)|$. Because $p_g(F) > 0$, the right system defines a generically finite map and so does $\phi_3$. If $K_X \cdot F_0^2 > 0$, in order to prove the theorem, we have to show the generic finiteness of $\Phi|_{|K_F + L|}$, where $L : = f^*(K_X)|_F$ is effective. By Theorem 2 of [6], we see that $q(F) \geq 3$ when $p_g(X) \geq 39$. Then $\Phi|_{|K_F|}$ is generically finite according to [18]. Therefore under the assumption of the theorem, we can obtain the generic finiteness of $\phi_3$. \hfill $\square$

4. On bicanonical systems

We suppose that $X$ is a locally factorial Gorenstein minimal 3-fold of general type and that $|2K_X|$ be composed of a pencil. Keep the same notations as in section 1 and let $\pi : X' \to X$ be the birational modification and $f : X' \to C$ be the derived fibration.

Lemma 4.1. Let $X$ be a projective minimal Gorenstein 3-fold of general type and suppose that $|2K_X|$ is composed of a pencil. Then $q(X) \leq 2$ and $p_g(X) \geq 1$.

Proof. This is just a generalized version of Corollary 3.1 of [7]. Though the objects considered there are nonsingular minimal 3-folds, the method is also effective for minimal Gorenstein 3-folds. \hfill $\square$
Lemma 4.2. Let $X$ be a projective minimal Gorenstein 3-fold of general type, $|2K_X|$ be composed of a pencil, $f : X' \to C$ be the derived fibration of $\phi_2$ and $F$ be a general fibre of $f$. Then

$$O_F(\pi^*(K_X)|_F) \cong O_F(\pi_0^*(K_{F_0})),$$

where $\pi_0 : F \to F_0$ is the birational contraction onto the minimal model.

Proof. This is just a generalized version of Corollary 9.1 of [13]. Though the objects considered there are nonsingular minimal 3-folds, the method is also effective for minimal Gorenstein 3-folds.

Lemma 4.3. Under the same assumption as in Lemma 4.2, we have $K_{F_0}^2 \leq 3$ and $1 \leq p_g(F) \leq 3$.

Proof. Let $\pi^*(2K_X) \sim_{\text{lin}} g^*(H_2) + Z_2'$, where $g := \phi_2 \circ \pi$, $Z_2'$ is the fixed part and $H_2$ is a general hyperplane section of the closure $W$ of the image of $X$ in $\mathbb{P}^{2g-1}$. Obviously we have $g^*(H_2) \sim_{\text{num}} a_2 F$, where $a_2 \geq p(2) - 1$. From Lemma 4.2, we have

$$K_{F_0}^2 = (\pi^*(K_X)|_F)^2 = \pi^*(K_X)^2 \cdot F.$$

Let $2K_X \sim_{\text{lin}} M_2 + Z_2$, where $M_2$ is the moving part and $Z_2$ is the fixed part. We also have $M_2 = \pi_*(g^*(H_2))$. Denote $\overline{F} := \pi_*(F)$, then $M_2 \sim_{\text{num}} a_2 \overline{F}$. By the projection formula, we get

$$K_X^2 \cdot \overline{F} = \pi^*(K_X)^2 \cdot F = K_{F_0}^2.$$

Because $K_X$ is nef and big, we have $2K_X^3 \geq a_2 K_X^2 \cdot \overline{F}$. Thus

$$K_X^2 \cdot \overline{F} \leq \frac{2}{a_2} K_X^3 \leq \frac{4K_X^3}{K_X^3 - 6\chi(O_X) - 2} \leq \frac{4K_X^3}{K_X^3 + 4} < 4,$$

which means $K_{F_0}^2 \leq 3$. By Lemma 4.1, the fact that $p_g(X) \geq 1$ induces $p_g(F) > 0$. By the Noether inequality $2p_g(F_0) - 4 \leq K_{F_0}^2$, we see that $p_g(F) \leq 3$.

Proof Theorem 3. In order to prove Theorem 3, we shall derive a contradiction under the assumption that $p_g(F) \geq 2$. Obviously, $|2K_X'|$ can distinguish general fibres of the morphism $\phi_2 \circ \pi$. We consider the system $|K_{X'} + \pi^*(K_X)|$. Write $2\pi^*(K_X) \sim_{\text{lin}} M_2' + Z_2'$, where $M_2'$ is the moving part and $Z_2'$ is the fixed one. Set $Z_2' = Z_0' + Z_h'$, where $Z_0'$ is the vertical part and $Z_h'$ is the horizontal part with respect to the fibration $f : X' \to C$. Noting that $\pi^*(K_X)$ is effective by Lemma 4.1, $Z_h'$ should be 2-divisible, i.e. $Z_h' = 2Z_0$, where $Z_0$ is an effective divisor. Thus we see
that \( Z_0 \) is just the horizontal part of \( \pi^*(K_X) \). We know that \( a_2 \geq p(2) - 1 \geq 3 \) and

\[
\pi^*(K_X) \sim_{\text{num}} \frac{a_2}{2} F + \frac{1}{2} Z_2'.
\]

Therefore \( \pi^*(K_X) - F - (1/a_2)Z_2' \) is a nef and big \( \mathbb{Q} \)-divisor. Setting \( G := \pi^*(K_X) - (1/a_2)Z_2' \), then we have \( K_{X'} + \pi^* G \geq K_{X'} + \pi^*(K_X) \). By the Kawamata-Viehweg vanishing theorem, we see that, for a general fibre \( F \),

\[
|K_{X'} + \pi^* G|_F = |K_{F'} + \pi^* G|_F \geq |K_{F'} + \pi^* G|_F = \left| K_F + \frac{\pi_2 - 2}{a_2} Z_0 \right|_F,
\]

where \( \pi((a_2 - 2)/a_2)Z_0 \) is effective on the surface \( F \). This means that \( \dim \phi_2(F) \geq 1 \) under the assumption \( p_g(F) \geq 2 \) and then \( \dim \phi_2(X) \geq 2 \), a contradiction.

The rest of this section is devoted to present an application of our method to bicanonical maps of surfaces of general type.

**Theorem 4.4.** Let \( S \) be a minimal algebraic surface of general type with \( p(2) \geq 4 \). Then the bicanonical map of \( S \) is generically finite.

Proof. Suppose that \( |2K_S| \) is composed of a pencil, we want to derive a contradiction. Taking a birational modification \( \pi : S' \longrightarrow S \) such that \( |2\pi^*(K_S)| \) defines a morphism and denoting \( W := \overline{\phi_2(S)} \), we obtain the following through the Stein factorization:

\[
\phi_2 \circ \pi : S' \xrightarrow{\pi} B \longrightarrow W,
\]

where \( B \) is a nonsingular curve. Denote by \( C \) a general fibre of the derived fibration \( \pi \). We can write

\[
\pi^*(2K_S) \sim_{\text{lin}} \sum_{i=1}^{\alpha} C_i + Z,
\]

where \( \alpha \geq p(2) - 1 \geq 3 \) and \( Z \) is the fixed part. Considering the system \( |K_S + \pi^*(K_S)| \), we can see that the system can distinguish general fibres of \( \phi_2 \). Setting \( G := \pi^*(K_S) - (1/\alpha)Z \), we have \( K_S + \pi^* G \geq K_S + \pi^*(K_S) \) and \( G - C \sim_{\text{num}} (\alpha - 2/\alpha)\pi^*(K_S) \) is nef and big. Thus, by the K-V vanishing theorem, we have

\[
|K_S + \pi^* G|_C = |K_C + D|,
\]

where \( D := \pi^* G \) is a divisor of positive degree on the curve \( C \). Because \( g(C) \geq 2 \), then \( h^0(C, K_C + D) \geq 2 \). This means that \( |K_S + \pi^*(K_S)| \) gives a generically finite map, a contradiction. \( \square \)
Corollary 4.5. Let $S$ be a minimal algebraic surface of general type with $p_g \geq 2$. Then the bicanonical map of $S$ is generically finite.

Proof. If $q = 0$, then $\chi(O_S) \geq 3$ and $p(2) \geq 4$. If $q > 0$, then $K_S^2 \geq 2p_g \geq 4$ by [8] and then $p(2) \geq 5$. The proof is completed by Theorem 4.4. \qed

Corollary 4.6. Let $S$ be a minimal algebraic surface of general type with $p(2) = 3$. Then $|2K_S|$ is not composed of an irrational pencil.

Proof. This is obvious from the proof of Theorem 4.4. The critical point is that we also have $a \geq 3$ in this case. \qed

The remain cases are like the following:

(I) $K^2 = 1$, $p_g = 1$ and $q = 0$;

(II) $K^2 = 2$ and $p_g = q = 0$;

(III) $K^2 = 2$ and $p_g = q = 1$.

Proposition 4.7. Let $S$ be a minimal algebraic surface of type (I). Then the bicanonical map is generically finite.

Proof. Suppose that $|2K_S|$ is composed of a rational pencil. We write

$$2K_S \sim_{\text{lin}} C_1 + C_2 + Z,$$

where $Z$ is the fixed part. Denote by $C$ a general member which is algebraically equivalent to $C_i$. We have $1 = K_S^2 \geq K_S \cdot C$. On the other hand, $K_S \cdot C + C^2 \geq 2$, which gives $C^2 \geq 1$. Thus $K_S \cdot C = C^2 = 1$, i.e. $C$ is a nonsingular curve of genus two. By the index theorem, we see that $K_S \sim_{\text{sing}} C$. But from [3], Pic($S$) is torsion free, then $K_S \sim_{\text{lin}} C$. This is impossible because $h^0(S, C) = 2$. \qed

Lemma 4.8 (Lemma 8 of [19]). Let $S$ be a surface with finite $\pi_1$. Then

$$H^1(S, O_S(E)) = 0$$

for any invertible torsion sheaf $E$ on $S$.

Lemma 4.9. Let $S$ be a minimal surface of type (II) or (III). Suppose that $|2K_S|$ is composed of a rational pencil. Then the moving part of $|2K_S|$ is a free pencil of genus two.

Proof. We can write $2K_S \sim_{\text{lin}} C_1 + C_2 + Z$, where $Z$ is the fixed part. Denote by $C$ the general member which is algebraically equivalent to $C_i$. If $C^2 > 0$, then
$K^2_S \geq K_S \cdot C \geq C^2$. On the other hand, the index theorem gives $K^2_S \times C^2 \leq (K_S \cdot C)^2$. Thus $K^2_S = K_S \cdot C = C^2 = 2$ and then $K_S \sim_{\text{num}} C$.

If $p_g = 1$, then $Z = 0$. Let $D \in |K_S|$ be the unique effective divisor, then $2D = F_1 + F_2$, where the $F_i$ are two fibres of $\phi_2$. If $F_1 \neq F_2$, then the $F_i$ are multiple fibres and then $D \sim_{\text{num}} 2F_0$, where $F_0$ is a divisor. Which implies $D^2 \geq 4$, a contradiction. If $F_1 = F_2$, then $D = F_1$ and thus $h^0(S, D) = 2$, also a contradiction.

If $p_g = 0$, because the $\pi_1$ of $S$ is a finite group (Corary 5.8 of [1]), then $h^1(S, K_S - C) = 0$ by Lemma 4.8. Whereas we have $h^1(S, K_S - C) = h^1(S, C) = 1$ by R-R, a contradiction. Therefore we have $C^2 = 0$ and then $g(C) = 2$.

**Proposition 4.10.** Let $S$ be a minimal surface of type (II) or (III). Then $|2K_S|$ can not be composed of a rational pencil of genus two.

Proof. We refer to the proof of Proposition 3 and Theorem 3 of [19].

Thus we finally arrive at the following theorem of Xiao (Theorem 1 of [19]).

**Theorem 4.11.** Let $S$ be a projective surface of general type. Then $\phi_2$ is generically finite if and only if $h^0(S, 2K_S) > 2$.

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