Chen, M.
Osaka J. Math.
38 (2001), 451–468

ON PLURICANONICAL MAPS FOR THREEFOLDS OF GENERAL TYPE, II

MENG CHEN

(Received May 31, 1999)

1. Introduction

This paper is a continuation of [4, 9, 13]. To classify algebraic varieties is one of the goals in algebraic geometry. One way to study a given variety is to understand the behavior of its pluricanonical maps. The objects concerned here are complex projective 3-folds of general type over $\mathbb{C}$. Let X be such an object and denote by ϕ_m the m-th pluricanonical map of X, which is the rational map associated with the m-canonical system $|mK_X|$. The very natural question is when $|mK_X|$ gives a birational map, a generically finite map, $\cdots$, etc. According to [2, 4, 9, 12, 13], one has the following

Theorem 0. *Let X be a complex projective 3-fold of general type with the canonical index r. Then*
(i) *when $r = 1$, ϕ_m is a birational morphism onto its image for $m \geq 6$;*
(ii) *when $r \geq 2$, ϕ_m is a birational map onto its image for $m \geq 4r + 5$.*

In this paper, we give our results on the generic finiteness of ϕ_m. By a delicate use of the Kawamata-Viehweg vanishing theorem, we reduce the problem to a parallel one for adjoint systems on some smooth surface. Reider's results as well as other theorems on surfaces make it possible for us to go on a detailed argument.

Theorem 1. *Let X be a projective 3-fold of general type with the canonical index $r \geq 2$. Then ϕ_m is generically finite for $m \geq m(r)$, where $m(r)$ is a function as follows*:
$m(2) = 11$;
$m(r) = 2r + 8$, *for* $3 \leq r \leq 5$;
$m(r) = 2r + 6$, *for* $r \geq 6$.

Theorem 2. *Let X be a projective minimal Gorenstein 3-fold of general type. Then*

The author was partially supported by the National Natural Science Foundation of China.

(1) *ϕ_5 is birational except for some 3-folds with $K_X^3 = 2$ and $p_g(X) \le 2$; ϕ_5 is generically finite of degree ≤ 8.*
(2) *ϕ_4 is birational if $K_X^3 > 2$ and $\dim \phi_1(X) = 3$; ϕ_4 is generically finite except for some 3-folds with $K_X^3 = 2$, $p_g(X) \le 1$ and $\chi(\mathcal{O}_X) = -1$.*
(3) *ϕ_3 is generically finite if $p_g(X) \ge 39$.*

For a nonsingular projective minimal 3-fold X of general type, Benveniste ([2]) proved that $\dim \phi_m(X) \ge 2$ for $m \ge 4$, i.e. $|4K_X|$ can not be composed of a pencil. Recently, it has been proved ([5]) that $|3K_X|$ also can not be composed of a pencil. (Actually, the method is also effective for Gorenstein 3-folds of general type.) Thus it is interesting whether $|2K_X|$ can be composed of a pencil and like what a bicanonical pencil behaves. So in Section 4, we study the bicanonical pencil of a Gorenstein 3-fold of general type. According to the 3-dimensional MMP, we can suppose that X is a minimal locally factorial Gorenstein 3-fold of general type. Take a birational modification $\pi : X' \longrightarrow X$ such that X' is smooth, $|\pi^*(2K_X)|$ gives a morphism and $\pi^*(2K_X)$ has supports with only normal crossings. This is possible because of Hironaka's big theorem. Let $W := \overline{\phi_2(X)}$ and take the Stein factorization

$$\phi_2 \circ \pi : X' \xrightarrow{f} C \xrightarrow{s} W.$$

Then f is a fibration onto the nonsingular curve C, we call f a *derived fibration* of ϕ_2. Denote by F a general fibre of f. Then F is a nonsingular surface of general type by virtue of the Bertini theorem. Also set $b := g(C)$, the geometric genus of C. From [7], we know that $0 \le b \le 1$. We shall prove the following

Theorem 3. *Let X be a projective minimal Gorenstein 3-fold of general type and suppose that $|2K_X|$ is composed of a pencil. Let f be the derived fibration of ϕ_2 and F be a general fibre of f. Then we have $p_g(F) = 1$ and $K_{F_0}^2 \le 3$, where F_0 is the minimal model of F.*

As an application of our method, we shall present a corollary on surfaces of general type which somewhat simplifies Xiao's theorem for the bicanonical finiteness.

2. Proof of Theorem 1

Throughout our argument, the Kawamata-Viehweg vanishing theorem is always employed as a much more effective tool. We use it in the following form.

K-V Vanishing Theorem ([10] or [17]). *Let X be a nonsingular complete variety, $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$. Assume the following two conditions*:
(1) *D is nef and big*;
(2) *the fractional part of D has the support with only normal crossings.*

Then $H^i(X, \mathcal{O}_X(\ulcorner D \urcorner + K_X)) = 0$ *for* $i > 0$, *where* $\ulcorner D \urcorner$ *is the round-up of* D, *i.e. the minimum integral divisor with* $\ulcorner D \urcorner - D \geq 0$.

Lemma 2.1 (Corollary 2 of [16]). *Let* S *be a nonsingular algebraic surface,* L *be a nef divisor on* S, $L^2 \geq 10$ *and let* ϕ *be a map defined by* $|L + K_S|$. *If* ϕ *is not birational, then* S *contains a base point free pencil* E' *with* $L \cdot E' = 1$ *or* $L \cdot E' = 2$.

Lemma 2.2. *Let* X *be a nonsingular variety of dimension* n, $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$ *be a* $\mathbb{Q}$*-divisor on* X. *Then we have the following*:
(i) *if* S *is a smooth irreducible divisor on* X, *then* $\ulcorner D \urcorner|_S \geq \ulcorner D|_S \urcorner$;
(ii) *if* $\pi : X' \longrightarrow X$ *is a birational morphism, then* $\pi^*(\ulcorner D \urcorner) \geq \ulcorner \pi^*(D) \urcorner$.

Proof. We can write D as $G + \sum_{i=1}^{t} a_i E_i$, where G is a divisor, the E_i are effective divisors for each i and $0 < a_i < 1$, $\forall i$. So we only have to prove the lemma for effective $\mathbb{Q}$-divisors. That is easy to check. $\square$

Lemma 2.3 (Lemma 2.3 of [9]). *Let* X *be a minimal threefold of general type with canonical index* r. *Then we have the plurigenus formula*

$$\begin{aligned} &h^0(X, \omega_X^{[mr+s]}) \\ &= \frac{1}{12}(mr+s)(mr+s-1)(2mr+2s-1)(K_X^3) + am + c_s \end{aligned}$$

for $0 \leq s < r$, $mr + s \geq 2$, *where* a *is a constant and* c_s *is a constant only relating to* s.

DEFINITION 2.4. Let X be a nonsingular projective variety of dimension ≥ 2. Suppose $|M|$ is a base-point-free system on X, *a general irreducible element* S *of* $|M|$ means the following:
(i) if $\dim \Phi_{|M|}(X) \geq 2$, then S is just a general member of $|M|$;
(ii) if $\dim \Phi_{|M|}(X) = 1$, taking the Stein factorization of $\Phi_{|M|}$, then we obtain a fibration $f : X \longrightarrow C$ onto a curve C. We mean S a general fibre of f.

Proposition 2.5 (Lemma 3.2 of [9]). *Let* X *be a minimal threefold of general type with canonical index* $r \geq 2$. *Then* $\dim \phi_{mr+s}(X) \geq 2$ *in one of the following cases*:
(i) $r = 2$ *and* $m \geq 3$;
(ii) $r = 3$ *and* $m \geq 2$;
(iii) $r = 4, 5$, $0 \leq s \leq 2$ *and* $m \geq 2$; $r = 4, 5$, $s \geq 3$ *and* $m \geq 1$;
(iv) $r \geq 6$, $0 \leq s \leq 1$ *and* $m \geq 2$; $r \geq 6$, $s \geq 2$ *and* $m \geq 1$.

Now we modify Proposition 2.5 by virtue of Hanamura's method in order to prove our Theorem 1. The proof is due to Hamamura ([9]).

Proposition 2.6. *Let X be a minimal threefold of general type with canonical index $r \geq 2$. Then $h^0(\omega_X^{[mr+s]}) \geq 3$ in one of the following cases*:
(i) $r = 2$ *and* $m \geq 2$;
(ii) $r \geq 3$, $s = 0, 1$ *and* $m \geq 2$; $r \geq 3$, $s \geq 2$ *and* $m \geq 1$.

Proof. From Lemma 2.3, we can put

$$P(mr+s) = \frac{1}{12}(mr+s)(mr+s-1)(2mr+2s-1)(K_X^3) + am + c_s \tag{2.1}$$

where a and c_s are constants for $0 \leq s < r$. We consider the right handside of (2.1) as a polynomial in m and denote it by $P_s(m)$. Let $Q_s(m)$ be the first term of $P_s(m)$. We have

$$P_s(m) = Q_s(m) + am + c_s.$$

We see that, for $m \geq 1$ or $m = 0$ and $s \geq 2$,

$$P_s(m) \geq 0. \tag{2.2}$$

By Kollár's result ([11]) that the $\omega_X^{[mr+s]}$ are Cohen-Macaulay, using the Grothendieck duality, one can see that, for $m \leq -1$,

$$P_s(m) \leq 0. \tag{2.3}$$

Now we want to estimate both a and c_s. For any r and s, by (2.2) and (2.3), we have

$$Q_s(1) + a + c_s \geq 0 \tag{2.4}$$
$$-Q_s(-1) + a - c_s \geq 0. \tag{2.5}$$

Which induces

$$\begin{aligned} a &\geq \frac{1}{2}\left\{Q_s(-1) - Q_s(1)\right\} \\ &= -\frac{1}{12}\left\{2r^2 + (6s^2 - 6s + 1)\right\}(rK_X^3). \end{aligned} \tag{2.6}$$

When $r \geq 3$ and $s \geq 2$, we have

$$Q_s(0) + c_s \geq 0. \tag{2.7}$$

By (2.5) and (2.7), we get

$$a \geq -Q_s(0) + Q_s(-1) \tag{2.8}$$

$$= \frac{1}{12}\left\{-2r^2 + (6s-3)r - (6s^2 - 6s + 1)\right\}(rK_X^3).$$

Explicitly, we have

$$a \geq \frac{1}{12}\left\{-\frac{1}{2}r^2 + \frac{1}{2}\right\}(rK_X^3) \text{ if } r \text{ is odd} \tag{2.9}$$

$$a \geq \frac{1}{12}\left\{-\frac{1}{2}r^2 - 1\right\}(rK_X^3) \text{ if } r \text{ is even.} \tag{2.10}$$

Now we can calculate the $P(mr+s)$ case by case.

CASE 1. $r \geq 3$ and $s \geq 2$.

When r is odd, from (2.7) and (2.9), we have

$$\begin{aligned} P(mr+s) &\geq Q_s(m) - \frac{1}{12}m\left(\frac{1}{2}r^2 - \frac{1}{2}\right)(rK_X^3) - Q_s(0) \\ &= \frac{1}{12}\left\{(mr+s)(mr+s-1)(2mr+2s-1) + m\left(-\frac{1}{2}r^3 + \frac{1}{2}r\right)\right. \\ &\qquad \left. - s(s-1)(2s-1)\right\}(K_X^3). \end{aligned}$$

We get $P(mr+s) \geq 7$ for $m \geq 1$.

When r is even, from (2.7) and (2.10), we have

$$\begin{aligned} P(mr+s) &\geq Q_s(m) - \frac{1}{12}m\left(\frac{1}{2}r^2 + 1\right)(rK_X^3) - Q_s(0) \\ &= \frac{1}{12}\left\{2r^2m^3 + (6s-3)rm^2 + \left(6s^2 - 6s - \frac{1}{2}r^2\right)m\right\}(rK_X^3). \end{aligned}$$

We get $P(mr+s) \geq 5$ for $m \geq 1$.

CASE 2. $s = 1$.

From (2.4) and (2.5), we have

$$P(mr+1) \geq \frac{1}{12}r(m^2-1)(2rm+3)(rK_X^3).$$

We get $P(mr+1) \geq 6$ for $m \geq 2$.

CASE 3. $s = 0$.

By (2.4) and (2.5), we have

$$P(mr) \geq \frac{1}{12}r(m^2-1)(2rm-3)(rK_X^3).$$

We get $P(mr) \geq 3$ for $m \geq 2$. Thus we complete the proof. $\square$

In what follows we can get an improved version of Hanamura's theorem.

Theorem 2.7. *Let X be a projective threefold of general type with the canonical index $r \geq 2$. Then ϕ_m is birational onto its image for $m \geq 4r + 3$.*

Proof. We can suppose that X is a minimal 3-fold. For any $m_1 \geq r + 2$, take some blowing-ups $\pi : X' \longrightarrow X$ according to Hironaka such that X' is nonsingular and that the movable part of $|m_1 K_{X'}|$ defines a morphism. Denote by $|M|$ the moving part of $|m_1 K_{X'}|$ and by S a general irreducible element of $|M|$. Then S is a nonsingular projective surface of general type by the Bertini theorem. On X', we consider the system $|K_{X'} + 3\pi^*(rK_X) + S|$. Because $K_{X'} + 3\pi^*(rK_X)$ is effective by Proposition 2.6, so the system can distinguish general irreducible elements of $|M|$. On the other hand, the vanishing theorem gives

$$|K_{X'} + 3\pi^*(rK_X) + S||_S = |K_S + 3L|,$$

where $L := \pi^*(rK_X)|_S$ is a nef and big divisor on S and $L^2 \geq 2$. Reider's result tells that the right system gives a birational map, so does $|K_{X'} + 3\pi^*(rK_X) + S|$. Thus ϕ_m is birational for $m \geq 4r + 3$. □

Proof Theorem 1. We can suppose that X is a minimal model. If $r = 2$, then ϕ_m is birational for $m \geq 11$ according to Theorem 2.7. From now on, we assume $r \geq 3$ and define

$$m_2 = \begin{cases} r + 3, & \text{for } 3 \leq r \leq 5 \\ r + 2, & \text{for } r \geq 6. \end{cases}$$

Take some blowing-ups $\pi : X' \longrightarrow X$ such that X' is nonsingular, $|m_2 K_{X'}|$ defines a morphism and the fractional part of $\pi^*(K_X)$ has supports with only normal crossings. Denote by $|M_2|$ the moving part of $|m_2 K_{X'}|$ and by S_2 a general irreducible element of $|M_2|$. For any $t \in \mathbb{Z}_{>0}$, we consider the system

$$|K_{X'} + \ulcorner (t + m_2)\pi^*(K_X) \urcorner + S_2|,$$

which is a sub-system of $|(t+2m_2+1)K_{X'}|$. Because $K_{X'} + \ulcorner (t + m_2)\pi^*(K_X) \urcorner$ is effective by Proposition 2.6, so the system can distinguish general irreducible elements of $|M_2|$. On the other hand, the K-V vanishing theorem tells that

$$\begin{aligned} & |K_{X'} + \ulcorner (t + m_2)\pi^*(K_X) \urcorner + S_2||_{S_2} \\ = \ & |G + L|, \end{aligned}$$

where $G := \{K_{X'} + \ulcorner (t + m_2)\pi^*(K_X) \urcorner\}|_{S_2}$ is effective and $L := S_2|_{S_2}$. We can see that

$$G + L \geq K_{S_2} + \ulcorner t\pi^*(K_X) \urcorner|_{S_2} + L.$$

From Proposition 2.5, we have $h^0(S_2, L) \geq 2$. Modulo blowing-ups, actually we can suppose that $|L|$ is free from base points. Let C be a general irreducible element of $|L|$. It is obvious that $|G + L|$ can distinguish gereral irreducible elements of $|L|$. On the other hand, the K-V vanishing theorem gives

$$|K_{S_2} + \ulcorner t\pi^*(K_X)|_{S_2}\urcorner + C||_C = |K_C + D|,$$

where $D := \ulcorner t\pi^*(K_X)|_{S_2}\urcorner|_C$ is a divisor of positive degree. Because C is a curve of genus ≥ 2, so $h^0(C, K_C + D) \geq 2$ and $|K_C + D|$ gives a finite map. Thus we have $\dim \Phi_{|G+L|}(C) = 1$. Therefore ϕ_m is generically finite for $m \geq 2m_2+2$, which completes the proof. $\square$

3. On Gorenstein 3-folds of general type

For a minimal threefold X of general type with canonical index 1, we can find certain birational modifications $f : X' \longrightarrow X$ according to [15] such that $c_2(X') \cdot \Delta = 0$, where Δ is the ramification divisor of f. Then we can get the same plurigenus formula as that for a nonsingular minimal threefold, i.e.

$$p(n) := h^0(X, \mathcal{O}_X(nK_X)) = (2n-1)\left[\frac{n(n-1)}{12}K_X^3 - \chi(\mathcal{O}_X)\right],$$

for $n \geq 2$. On the other hand, the Miyaoka-Yau inequality ([14]) shows that $\chi(\mathcal{O}_X) < 0$. From [4] or [12], we know that ϕ_m is birational for $m \geq 6$.

Theorem 3.1. *Let X be a projective minimal Gorenstein 3-fold of general type. Then*
(1) *ϕ_5 is birational if either $K_X^3 > 2$ (Ein-Lazarsfeld-Lee) or $p_g(X) > 2$.*
(2) *When $p_g(X) = 2$, then ϕ_5 is birational except for some 3-folds with $q(X) = h^2(\mathcal{O}_X) = 0$, and $|K_X|$ composed with a rational pencil of surfaces of general type with $(K^2, p_g) = (1, 2)$. In this situation, ϕ_5 is generically finite of degree 2.*
(3) *ϕ_5 is birational if $\dim \phi_2(X) = 1$.*

Proof. This is the main theorem in [7]. Though the objects considered there are nonsingular minimal 3-folds, the method is also effective for all Gorenstein 3-folds of general type. $\square$

DEFINITION 3.2. Let X be a projective minimal Gorenstein 3-fold of general type. Suppose $\dim \phi_i(X) \geq 2$ and set $iK_X \sim_{\text{lin}} M_i + Z_i$, where M_i is the moving part and Z_i the fixed one for any integer i. We define $\delta_i(X) := K_X^2 \cdot M_i$.

Proposition 3.3. *Let X be a projective minimal Gorenstein 3-fold of general type. Suppose $|2K_X|$ is not composed of a pencil and $K_X^3 > 2$. Then $\delta_2(X) \geq 3$.*

Proof. We have $\delta_2(X) \geq 2$ by Proposition 2.2 of [4]. Take a birational modification $f : X' \longrightarrow X$ such that $|2f^*(K_X)|$ defines a morphism. Set $2f^*(K_X) \sim_{\text{lin}} M + Z$, where M is the moving part and Z the fixed one. A general member $S \in |M|$ is an irreducible nonsingular projective surface of general type. Denote $L := f^*(K_X)|_S$. If $L^2 = f^*(K_X)^2 \cdot S = \delta_2(X) = 2$, then we have

$$4 = 2f^*(K_X)^2 \cdot S = f^*(K_X) \cdot S^2 + f^*(K_X) \cdot S \cdot Z.$$

Noting that S is nef and $S \not\approx 0$, we have $f^*(K_X) \cdot S^2 \geq 1$. Therefore four cases occur as follows:
(i) $f^*(K_X) \cdot S^2 = 4$, $f^*(K_X) \cdot S \cdot Z = 0$;
(ii) $f^*(K_X) \cdot S^2 = 3$, $f^*(K_X) \cdot S \cdot Z = 1$;
(iii) $f^*(K_X) \cdot S^2 = 2$, $f^*(K_X) \cdot S \cdot Z = 2$;
(iv) $f^*(K_X) \cdot S^2 = 1$, $f^*(K_X) \cdot S \cdot Z = 3$.

We also have

$$\begin{aligned} 2K_X^3 &= 2f^*(K_X)^3 = f^*(K_X)^2 \cdot S + f^*(K_X)^2 \cdot Z \\ &= 2 + \frac{1}{2} f^*(K_X) \cdot Z(S+Z) \\ &= 2 + \frac{1}{2} f^*(K_X) \cdot S \cdot Z + \frac{1}{2} f^*(K_X) \cdot Z^2. \end{aligned} \tag{3.1}$$

CASE (i). Noting that $f^*(K_X)$ is nef and big, we see that $mf^*(K_X)$ is linearly equivalent to a nonsingular projective surface of general type according to Kawamata for sufficiently large integer m. Then $S|_{mf^*(K_X)}$ is nef and big and, by the Hodge Index Theorem, we have $f^*(K_X) \cdot Z^2 \leq 0$. Thus (3.1) is false and this case does not occur.

CASE (ii). We have $f^*(K_X) \cdot S(S - 3Z) = 0$, then $f^*(K_X)(S - 3Z)^2 \leq 0$, which derives $f^*(K_X) \cdot Z^2 \leq 1/3$, i.e. $f^*(K_X) \cdot Z^2 \leq 0$. (3.1) is also false.

CASE (iii). $f^*(K_X) \cdot S(S - Z) = 0$ induces $f^*(K_X) \cdot Z^2 \leq 2$, then (3.1) becomes $K_X^3 \leq 2$. Thus $K_X^3 = 2$. Actually, in this case, $f^*(K_X) \cdot (S - Z) \sim_{\text{num}} 0$ (as 1-cycle).

CASE (iv). $f^*(K_X) \cdot (3S - Z)^2 \leq 0$ induces $f^*(K_X) \cdot Z^2 \leq 9$. And (3.1) becomes $K_X^3 \leq 4$. If $K_X^3 = 4$, we see that $f^*(K_X) \cdot (3S - Z) \sim_{\text{num}} 0$ as 1-cycle. Now we set $f^*(M_2) = S + E$. Then $Z = f^*(Z_2) + E$. Obviously, we have $f_*(S) = M_2$ and $f_*(Z) = Z_2$. From $f^*(M_2) \cdot f^*(K_X) \cdot (3S - Z) = 0$, we get $3K_X \cdot M_2^2 = K_X \cdot M_2 \cdot Z_2$. Then $4 = 2K_X^2 \cdot M_2 = K_X \cdot M_2^2 + K_X \cdot M_2 \cdot Z_2 = 4K_X \cdot M_2^2$, i.e. $K_X \cdot M_2^2 = 1$. Which derives a contradiction, because $K_X \cdot M_2^2$ is even. Thus $K_X^3 = 2$. □

Proposition 3.4. *Let X be a projective minimal Gorenstein 3-fold of general type. Suppose $K_X^3 > 2$ and $\dim \phi_1(X) \geq 2$. Then $\delta_1(X) \geq 3$.*

Proof. As in the proof of the previous proposition, we first take a modification $f : X' \longrightarrow X$. Set $f^*(K_X) \sim_{\text{lin}} M + Z$, where M is the moving part. A general member $S \in |M|$ is a nonsingular projective surface of general type. Also denote $L :=$

$f^*(K_X)|_S$. Then $L^2 = \delta_1(X) \geq 2$ according to Proposition 2.1 of [7]. If $L^2 = 2$, then we have

$$2 = f^*(K_X)^2 \cdot S = f^*(K_X) \cdot S^2 + f^*(K_X) \cdot S \cdot Z.$$

We also have

$$\begin{aligned} K_X^3 &= f^*(K_X)^2 \cdot S + f^*(K_X)^2 \cdot Z \\ &= 2 + f^*(K_X) \cdot S \cdot Z + f^*(K_X) \cdot Z^2. \end{aligned} \tag{3.2}$$

Similarly, $f^*(K_X) \cdot S^2 \geq 1$. If $f^*(K_X) \cdot S^2 = 2$ and $f^*(K_X) \cdot S \cdot Z = 0$, then, by the Hodge Index Theorem, $f^*(K_X) \cdot Z^2 \leq 0$. Then (3.2) becomes $K_X^3 \leq 2$, which says $K_X^3 = 2$. If $f^*(K_X) \cdot S^2 = f^*(K_X) \cdot S \cdot Z = 1$, $f^*(K_X) \cdot S \cdot (S - Z) = 0$ induces $f^*(K_X) \cdot Z^2 \leq 1$. By (3.2), we get $K_X^3 \leq 4$. If $K_X^3 = 4$, then we can see $f^*(K_X) \cdot (S - Z) \sim_{\text{num}} 0$. By the same argument as in the case (iv) of the proof of Proposition 3.3, we have $f^*(M_1) \cdot f^*(K_X) \cdot (S - Z) = 0$, i.e. $K_X \cdot M_1^2 = K_X \cdot M_1 \cdot Z_1$. We have $2 = K_X^2 \cdot M_1 = K_X \cdot M_1^2 + K_X \cdot M_1 \cdot Z_1 = 2K_X \cdot M_1^2$. Therefore $K_X \cdot M_1^2 = 1$, which is impossible. Thus $K_X^3 = 2$. □

Theorem 3.5. *Let X be a projective minimal Gorenstein 3-fold of general type. Then ϕ_5 is generically finite of degree ≤ 8. If $\deg(\phi_5) > 2$, then $K_X^3 = 2$, $\chi(\mathcal{O}_X) = -1$ and $p_g(X) = 0, 1$.*

Proof. According to Theorem 3.1, we only have to study the case when $|2K_X|$ is not composed of a pencil. Take a modification $f : X' \longrightarrow X$ according to Hironaka such that $|2f^*(K_X)|$ defines a morphism. Set $2f^*(K_X) \sim_{\text{lin}} M + Z$, where M is the moving part and Z the fixed one. A general member $S \in |M|$ is a nonsingular projective surface of general type by the Bertini Theorem. We have

$$|K_{X'} + 2f^*(K_X) + S| \subset |5K_{X'}|.$$

Because $K_{X'} + 2f^*(K_X)$ is effective, the left system can distinguish general members of $|M|$. Denote $L := f^*(K_X)|_S$, using the long exact sequence and the vanishing theorem, we have

$$|K_{X'} + 2f^*(K_X) + S||_S = |K_S + 2L|.$$

Obviously, $K_S + 2L = G + H$, where $G := (K_{X'} + 2f^*(K_X))|_S$ is effective and $H := S|_S$. Note that $h^0(S, \mathcal{O}_S(2L)) \geq h^0(S, H) \geq P(2) - 1 \geq 3$. We have two cases.

CASE 1. $|H|$ is composed of a pencil. Taking a birational modification to S if necessary, we can suppose $|H|$ is free from base points. Denote $H \sim_{\text{lin}} \sum_{i=1}^{a} C_i + E$, where E is the fixed part. In general position, $\sum_{i=1}^{a} C_i$ can be a disjoint union of nonsingular curves in a family. We have $a \geq 2$. Thus $L \sim_{\text{num}} (a/2)C + E_0$, where

$E_0 \geq (1/2)E$ is an effective $\mathbb{Q}$-divisor. If $p_g(S) = 0$, then $q(S) = 0$ and then we can see by the long exact sequence that $|K_S + H|$ can distinguish C_i's and that $|K_S + \sum_{i=1}^{a} C_i||_{C_i} = |K_{C_i}|$, which means $|K_S + 2L|$ gives at worst a generically finite map of degree 2 and so does ϕ_5. If $p_g(S) > 0$, it is obvious that $|K_S + 2L|$ can distinguish C_i's. For a general curve C which is algebraically equivalent to C_i, we consider the $\mathbb{Q}$-divisor $G := K_S + 2L - (1/2)\sum_{i=3}^{a} C_i - E_0$. We have $\ulcorner G \urcorner \leq K_S + 2L$. On the other hand, $G - C - K_S$ is nef and big, thus by the K-V vanishing we have $|\ulcorner G \urcorner||_C = |K_C + \ulcorner E_0 \urcorner|_C|$. Because $\ulcorner E_0 \urcorner|_C$ is effective, $\Phi_{|K_S+2L|}$ is at worst a generically finite map of degree 2 and so is ϕ_5 of X.

CASE 2. $|H|$ is not composed of a pencil, so neither is $|2L|$. Similarly, we can suppose $|2L|$ is base point free. If $p_g(S) = 0$, we can use a parallel discussion to that of Case 1 to see that ϕ_5 is at worst a generically finite map of degree 2. If $p_g(S) > 0$, then $\Phi_{|K_S+2L|}$ is obviously generically finite. We know that $L^2 \geq 2$ from Proposition 2.2 of [4]. If $\Phi_{|K_S+2L|}$ is not birational and $L^2 \geq 3$, then according to Lemma 2.1, there is a free pencil on S with a general member C such that $C^2 = 0$ and $L \cdot C = 1$. Since $\dim \Phi_{|2L|}(C) = 1$, then $h^0(2L|_C) \geq 2$ and then, by the Clifford theorem, we see that C is a curve of genus 2 and $2L|_C \sim_{\text{lin}} K_C$. Finally we can see that $|2L||_C = |K_C|$. Therefore $\Phi_{|K_S+2L|}$ is a generically finite map of degree 2. Therefore ϕ_5 is generically finite with $\deg(\phi_5) \leq 2$. If $L^2 = 2$, then $K_X^3 = 2$ by the proof of Proposition 3.3. On the surface S, set $2L \sim_{\text{lin}} C_1 + E_1$, where C_1 is the moving part. We easily get

$$8 = (2L)^2 \geq C_1^2 \geq d(h^0(2L) - 2) \geq d(P(2) - 3).$$

Therefore we have

$$d \leq \frac{8}{P(2) - 3} = \frac{8}{-3\chi(\mathcal{O}_X) - 2}.$$

If $d > 2$, then $\chi(\mathcal{O}_X) = -1$. □

For the 4-canonical map of X, it is obvious that ϕ_4 is not birational if X admits a pencil of surfaces of general type with $(K^2, p_g) = (1, 2)$. Therefore it is pessimistic for us to obtain an effective sufficient condition for the birationality of ϕ_4. We have a partial result as follows.

Theorem 3.6. *Let X be a projective minimal Gorenstein 3-fold of general type. Suppose $K_X^3 > 2$ and $\dim \phi_1(X) = 3$. Then ϕ_4 is a birational map onto its image.*

Proof. Take a birational modification $f : X' \longrightarrow X$ such that the movable part of $|f^*(K_X)|$ is base point free. Set $f^*(K_X) \sim_{\text{lin}} S + Z$, where S is the moving part and Z the fixed one. A general member S is a nonsingular projective surface of general type. We have $|K_{X'} + 2f^*(K_X) + S| \subset |4K_{X'}|$. Using the vanishing theorem, we have

$$|K_{X'} + 2f^*(K_X) + S||_S = |K_S + 2L|,$$

where $L := f^*(K_X)|_S$ is a nef and big divisor on S. By Proposition 3.4, we see that $L^2 \geq 3$ under the condition $K_X^3 > 2$. If $\Phi_{|K_S+2L|}$ is not birational, then, by Lemma 2.1, there is a free pencil with a general member C such that $C^2 = 0$ and $L \cdot C = 1$. Because $\dim \Phi_{|L|}(S) = 2$, $h^0(C, \mathcal{O}_C(L|_C)) \geq 2$. Therefore, by the Clifford theorem, we see that $\deg(L|_C) \geq 2h^0(L|_C) - 2 \geq 2$. This is a contradiction. Therefore $\Phi_{|K_S+2L|}$ is birational and so is ϕ_4. □

EXAMPLE 3.7. We give an example which shows that ϕ_4 is not birational when $K_X^3 = 2$ and $\dim \phi_1(X) = 3$. On $\mathbb{P}^3(\mathbb{C})$, take a smooth hypersurface S of degree 10, $S \sim_{\text{lin}} 10H$. Let X be a double cover of $\mathbb{P}^3$ with branch locus along S. Then X is a nonsingular canonical model, $K_X^3 = 2$ and $p_g(X) = 4$ and ϕ_1 is a finite morphism onto $\mathbb{P}^3$ of degree 2. One can easily check that ϕ_4 is also a finite morphism of degree 2.

Theorem 3.8. *Let X be a projective minimal Gorenstein 3-fold of general type. Then ϕ_4 is generically finite when $p_g(X) \geq 2$ or when $K_X^3 > 2$ or when $\chi(\mathcal{O}_X) \neq -1$.*

Proof. PART I: $p_g(X) \geq 2$.

First we make a modification $f : X' \longrightarrow X$ such that the movable part of $|f^*(K_X)|$ is free from base points and that $f^*(K_X)$ has support with only normal crossings. Set $f^*(K_X) \sim_{\text{lin}} M + Z$, where M is the moving part and Z the fixed one.

If $\dim \phi_1(X) = 2$, then a general member $S \in |M|$ is a nonsingular projective surface of general type. We have

$$|K_{X'} + 2f^*(K_X) + S| \subset |4K_{X'}|.$$

Using the vanishing theorem, we have $|K_{X'} + 2f^*(K_X) + S||_S = |K_S + 2L|$, where $L := f^*(K_X)|_S$ is nef and big effective divisor on S. We have $h^0(S, L) \geq 2$. Noting that $p_g(S) > 0$ in this case. And if $|L|$ is not composed of a pencil, then neither is $|K_S + 2L|$. If $|L|$ is composed of a pencil, taking a modification if possible, we can suppose that the movable part of $|L|$ is free from base points. Set $L \sim_{\text{lin}} \sum C_i + Z_0$, we can see $|K_S + L + \sum C_i||_{C_i} = |K_{C_i} + D|$, where $D := L|_{C_i}$ is effective. We easily see that $\Phi_{|K_S+2L|}$ is at worst generically finite of degree ≤ 2 and so is ϕ_4.

If $\dim \phi_1(X) = 1$, then $M \sim_{\text{num}} aF$, where F is a nonsingular projective surface of general type. $M_1 \sim_{\text{num}} aF_0$, where $F_0 = f_*(F)$ is irreducible on X. If $K_X \cdot F_0^2 = 0$, then, by Lemma 2.3 of [7], we have $\mathcal{O}_F(f^*(K_X)|_F) \cong \mathcal{O}_F(\pi^*(K_0))$, where π is the contraction map onto the minimal model and K_0 is the canonical divisor of the minimal model of F. Obviously, $|K_{X'} + 2f^*(K_X) + M|$ can distinguish general members of $|M|$. Moreover $|K_{X'} + 2f^*(K_X) + M||_F = |K_F + 2\pi^*(K_0)|$, the right system gives a generically finite map and so does ϕ_4. If $K_X \cdot F_0^2 > 0$, then

$$L^2 = f^*(K_X)^2 \cdot F = K_X^2 \cdot F_0 \geq K_X \cdot F_0^2 \geq 2.$$

It is sufficient to show that $|K_F + 2L|$ gives a generically finite map. We have $K_F + 2L \geq 3L$. If $|3L|$ is not composed of a pencil, then neither is $|K_F + 2L|$. If $|3L|$ is composed of a pencil, we claim that $h^0(F, 3L) \geq 3$. In fact, we have $|K_{X'} + f^*(K_X) + F||_F = |K_F + L|$ and $h^0(F, K_F + L) \geq 3$. Considering the natural map $H^0(X', 3K_{X'}) \xrightarrow{\alpha} H^0(F, 3K_F)$, because $K_{X'} + f^*(K_X) + F \leq 3K_{X'}$, we see that $\dim_{\mathbb{C}}(\mathrm{Im}(\alpha)) \geq h^0(K_F + L) \geq 3$. Similarly, considering another natural map $H^0(X', 3f^*(K_X)) \xrightarrow{\beta} H^0(F, 3L)$, we have

$$h^0(3L) \geq \dim_{\mathbb{C}}(\mathrm{Im}(\beta)) = \dim_{\mathbb{C}}(\mathrm{Im}(\alpha)) \geq 3.$$

Now we can write $3L \sim_{\text{lin}} \sum_{i=1}^{t} \overline{C_i} + E_0$, where E_0 is the fixed part, $t \geq 2$ and the $\overline{C_i}$ are irreducible curves. Denote by C a generic $\overline{C_i}$. Then $2L \sim_{\text{num}} (2/3)tC + (2/3)E_0$ and thus $2L - C - (1/t)E_0$ is a nef and big $\mathbb{Q}$-divisor. Setting $G := 2L - (1/t)E_0$, then we have $K_S + \ulcorner G \urcorner \leq K_S + 2L$. On the other hand, the K-V vanishing gives $|K_S + \ulcorner G \urcorner||_C = |K_C + D|$, where D is a divisor of positive degree. Noting that C is a curve of genus ≥ 2, so we see that $|K_C + D|$ gives a generically finite map. This means $|K_S + 2L|$ gives a generically finite map.

PART II: $K_X^3 > 2$ or $\chi(\mathcal{O}_X) \neq -1$.

We study ϕ_4 according to the behavior of ϕ_2. Of course, first we make a modification $f : X' \longrightarrow X$ such that the movable part of $|2f^*(K_X)|$ is free from base points and that $2f^*(K_X)$ has supports with only normal crossings. Set $2f^*(K_X) \sim_{\text{lin}} \overline{M_2} + \overline{Z_2}$, where $\overline{M_2}$ is the moving part and $\overline{Z_2}$ the fixed one.

If $\dim \phi_2(X) = 1$, then $\overline{M_2} \sim_{\text{num}} a_2 F$, where F is a nonsingular projective surface of general type. We have $\mathcal{O}_F(f^*(K_X)|_F) \cong \mathcal{O}_F(\pi^*(K_0))$ by Lemma 4.2 below in this paper. Because $K_{X'} + f^*(K_X)$ is effective, $|K_{X'} + f^*(K_X) + \overline{M_2}|$ can distinguish general F. On the other hand, we have $|K_{X'} + f^*(K_X) + \overline{M_2}||_F = |K_F + \pi^*(K_0)|$. From Theorem 3.1 of [7], we know that F is not a surface with $p_g = q = 0$. Thus $|K_F + \pi^*(K_0)|$ defines a generically finite map according to [19] and so does ϕ_4.

If $\dim \phi_2(X) \geq 2$, then a general member $S \in |\overline{M_2}|$ is a nonsingular projective surface of general type. We have $|K_{X'} + f^*(K_X) + S||_S = |K_S + L|$, where $L := f^*(K_X)|_S$. Noting that $K_S \geq L$, then we have $K_S + L \geq 2L$. Under our assumption, we have $P(2) \geq 5$. Thus $h^0(2L) \geq 4$. We may suppose that the movable part of $|2L|$ is free from base points. If $|2L|$ is not composed of a pencil, then neither is $|K_S + L|$. Otherwise we can set $2L \sim_{\text{lin}} \sum_{i=1}^{b} C_i + E_1$, where $b \geq 3$ and E_1 is the fixed part. We denote by C the general C_i. Because $L - C - (1/b)E_1$ is nef and big, therefore

$$\left| K_S + \ulcorner L - \frac{1}{b} E_1 \urcorner \right| \Big|_C = |K_C + D|,$$

where D is a divisor of positive degree. The right system obviously defines a generically finite map. Thus $|K_S + L|$ gives a generically finite map and so does ϕ_4. $\square$

Theorem 3.9. *Let X be a projective minimal Gorenstein 3-fold of general type. Then ϕ_3 is generically finite when $p_g(X) \geq 39$.*

Proof. First we make a modification $f : X' \longrightarrow X$ such that the movable part of $|f^*(K_X)|$ is free from base points and that $f^*(K_X)$ has support with only normal crossings. Set $f^*(K_X) \sim_{\text{lin}} M + Z$, where M is the moving part and Z the fixed one.

If $\dim \phi_1(X) \geq 2$, then a general member $S \in |M|$ is a nonsingular projective surface of general type. We have $|K_{X'} + f^*(K_X) + S||_S = |K_S + L|$, where $L := f^*(K_X)|_S$. When $p_g(X) \geq 4$, $h^0(S, L) \geq 3$. Noting that $p_g(S) > 0$, if $|L|$ is not composed of a pencil, then nor is $|K_S + L|$. So we may suppose that $|L|$ is composed of a pencil and the movable part of this system is free from base points. Set $L \sim_{\text{lin}} \sum_{i=1}^{a} C_i + E_0$, where we have $a \geq 2$. $|K_S + L|$ can distinguish the C_i generically. On the other hand, $L - C - (1/a)E_0$ is nef and big, we obtain by the Kawamata-Viehweg vanishing that

$$\left| K_S + \ulcorner L - \frac{1}{a} E_0 \urcorner \right| \Big|_C = \left| K_C + \ulcorner \frac{a-1}{a} L \urcorner \Big|_C \right| .$$

The right system defines a generically finite map and so does ϕ_3.

If $\dim \phi_1(X) = 1$, then $M \sim_{\text{num}} aF$, where F is a nonsingular projective surface of general type. Set $F_0 = f_*(F)$. If $K_X \cdot F_0^2 = 0$, then, by Lemma 2.3 of [7], we have $\mathcal{O}_F(f^*(K_X)|_F) \cong \mathcal{O}_F(\pi^*(K_0))$, where π is the contraction onto the minimal model and K_0 is the canonical divisor of the minimal model of F. We see that $|K_{X'} + f^*(K_X) + M||_F = |K_F + \pi^*(K_0)|$. Because $p_g(F) > 0$, the right system defines a generically finite map and so does ϕ_3. If $K_X \cdot F_0^2 > 0$, in order to prove the theorem, we have to show the generic finiteness of $\Phi_{|K_F + L|}$, where $L := f^*(K_X)|_F$ is effective. By Theorem 2 of [6], we see that $q(F) \geq 3$ when $p_g(X) \geq 39$. Then $\Phi_{|K_F|}$ is generically finite according to [18]. Therefore under the assumption of the theorem, we can obtain the generic finiteness of ϕ_3. □

4. On bicanonical systems

We suppose that X is a locally factorial Gorenstein minimal 3-fold of general type and that $|2K_X|$ be composed of a pencil. Keep the same notations as in section 1 and let $\pi : X' \longrightarrow X$ be the birational modification and $f : X' \longrightarrow C$ be the derived fibration.

Lemma 4.1. *Let X be a projective minimal Gorenstein 3-fold of general type and suppose that $|2K_X|$ is composed of a pencil. Then $q(X) \leq 2$ and $p_g(X) \geq 1$.*

Proof. This is just a generalized version of Corollary 3.1 of [7]. Though the objects considered there are nonsingular minimal 3-folds, the method is also effective for minimal Gorenstein 3-folds. □

Lemma 4.2. *Let X be a projective minimal Gorenstein 3-fold of general type, $|2K_X|$ be composed of a pencil, $f : X' \longrightarrow C$ be the derived fibration of ϕ_2 and F be a general fibre of f. Then*

$$\mathcal{O}_F(\pi^*(K_X)|_F) \cong \mathcal{O}_F(\pi_0^*(K_{F_0})),$$

where $\pi_0 : F \longrightarrow F_0$ is the birational contraction onto the minimal model.

Proof. This is just a generalized version of Corollary 9.1 of [13]. Though the objects considered there are nonsingular minimal 3-folds, the method is also effective for minimal Gorenstein 3-folds. □

Lemma 4.3. *Under the same assumption as in* Lemma 4.2, *we have $K_{F_0}^2 \le 3$ and $1 \le p_g(F) \le 3$.*

Proof. Let $\pi^*(2K_X) \sim_{\text{lin}} g^*(H_2) + Z_2'$, where $g := \phi_2 \circ \pi$, Z_2' is the fixed part and H_2 is a general hyperplane section of the closure W of the image of X in $\mathbb{P}^{p(2)-1}$. Obviously we have $g^*(H_2) \sim_{\text{num}} a_2 F$, where $a_2 \ge p(2) - 1$. From Lemma 4.2, we have

$$K_{F_0}^2 = (\pi^*(K_X)|_F)^2 = \pi^*(K_X)^2 \cdot F.$$

Let $2K_X \sim_{\text{lin}} M_2 + Z_2$, where M_2 is the moving part and Z_2 is the fixed part. We also have $M_2 = \pi_*(g^*(H_2))$. Denote $\overline{F} := \pi_*(F)$, then $M_2 \sim_{\text{num}} a_2\overline{F}$. By the projection formula, we get

$$K_X^2 \cdot \overline{F} = \pi^*(K_X)^2 \cdot F = K_{F_0}^2.$$

Because K_X is nef and big, we have $2K_X^3 \ge a_2 K_X^2 \cdot \overline{F}$. Thus

$$K_X^2 \cdot \overline{F} \le \frac{2}{a_2} K_X^3 \le \frac{4K_X^3}{K_X^3 - 6\chi(\mathcal{O}_X) - 2} \le \frac{4K_X^3}{K_X^3 + 4} < 4,$$

which means $K_{F_0}^2 \le 3$. By Lemma 4.1, the fact that $p_g(X) \ge 1$ induces $p_g(F) > 0$. By the Noether inequality $2p_g(F_0) - 4 \le K_{F_0}^2$, we see that $p_g(F) \le 3$. □

Proof Theorem 3. In order to prove Theorem 3, we shall derive a contradiction under the assumption that $p_g(F) \ge 2$. Obviously, $|2K_{X'}|$ can distinguish general fibres of the morphism $\phi_2 \circ \pi$. We consider the system $|K_{X'} + \pi^*(K_X)|$. Write $2\pi^*(K_X) \sim_{\text{lin}} M_2' + Z_2'$, where M_2' is the moving part and Z_2' is the fixed one. Set $Z_2' = Z_v + Z_h$, where Z_v is the vertical part and Z_h is the horizontal part with respect to the fibration $f : X' \longrightarrow C$. Noting that $\pi^*(K_X)$ is effective by Lemma 4.1, Z_h should be 2-divisible, i.e. $Z_h = 2Z_0$, where Z_0 is an effective divisor. Thus we see that Z_0 is just the horizontal part of $\pi^*(K_X)$. We know that $a_2 \geq p(2) - 1 \geq 3$ and

$$\pi^*(K_X) \sim_{\text{num}} \frac{a_2}{2} F + \frac{1}{2} Z_2'.$$

Therefore $\pi^*(K_X) - F - (1/a_2)Z_2'$ is a nef and big $\mathbb{Q}$-divisor. Setting $G := \pi^*(K_X) - (1/a_2)Z_2'$, then we have $K_{X'} + \ulcorner G \urcorner \leq K_{X'} + \pi^*(K_X)$. By the Kawamata-Viehweg vanishing theorem, we see that, for a general fibre F,

$$|K_{X'} + \ulcorner G \urcorner||_F = |K_F + \ulcorner G \urcorner|_F| \supset |K_F + \ulcorner G|_F \urcorner| = \left| K_F + \ulcorner \frac{a_2 - 2}{a_2} Z_0 \Big|_F \urcorner \right|,$$

where $\ulcorner((a_2 - 2)/a_2)Z_0|_F\urcorner$ is effective on the surface F. This means that $\dim \phi_2(F) \geq 1$ under the assumption $p_g(F) \geq 2$ and then $\dim \phi_2(X) \geq 2$, a contradiction. □

The rest of this section is devoted to present an application of our method to bicanonical maps of surfaces of general type.

Theorem 4.4. *Let S be a minimal algebraic surface of general type with $p(2) \geq 4$. Then the bicanonical map of S is generically finite.*

Proof. Suppose that $|2K_S|$ is composed of a pencil, we want to derive a contradiction. Taking a birational modification $\pi : S' \longrightarrow S$ such that $|2\pi^*(K_S)|$ defines a morphism and denoting $W := \overline{\phi_2(S)}$, we obtain the following through the Stein factorization:

$$\phi_2 \circ \pi : S' \xrightarrow{f} B \longrightarrow W,$$

where B is a nonsingular curve. Denote by C a general fibre of the derived fibration f. We can write

$$\pi^*(2K_S) \sim_{\text{lin}} \sum_{i=1}^{a} C_i + Z,$$

where $a \geq p(2) - 1 \geq 3$ and Z is the fixed part. Considering the system $|K_{S'} + \pi^*(K_S)|$, we can see that the system can distinguish general fibres of ϕ_2. Setting $G := \pi^*(K_S) - (1/a)Z$, we have $K_S + \ulcorner G \urcorner \leq K_S + \pi^*(K_S)$ and $G - C \sim_{\text{num}} (a - 2/a)\pi^*(K_S)$ is nef and big. Thus, by the K-V vanishing theorem, we have

$$|K_S + \ulcorner G \urcorner||_C = |K_C + D|,$$

where $D := \ulcorner G \urcorner|_C$ is a divisor of positive degree on the curve C. Because $g(C) \geq 2$, then $h^0(C, K_C + D) \geq 2$. This means that $|K_S + \pi^*(K_S)|$ gives a generically finite map, a contradiction. □

Corollary 4.5. *Let S be a minimal algebraic surface of general type with $p_g \geq 2$. Then the bicanonical map of S is generically finite.*

Proof. If $q = 0$, then $\chi(\mathcal{O}_S) \geq 3$ and $p(2) \geq 4$. If $q > 0$, then $K_S^2 \geq 2p_g \geq 4$ by [8] and then $p(2) \geq 5$. The proof is completed by Theorem 4.4. □

Corollary 4.6. *Let S be a minimal algebraic surface of general type with $p(2) = 3$. Then $|2K_S|$ is not composed of an irrational pencil.*

Proof. This is obvious from the proof of Theorem 4.4. The critical point is that we also have $a \geq 3$ in this case. □

The remain cases are like the following:
(I) $K^2 = 1$, $p_g = 1$ and $q = 0$;
(II) $K^2 = 2$ and $p_g = q = 0$;
(III) $K^2 = 2$ and $p_g = q = 1$.

Proposition 4.7. *Let S be a minimal algebraic surface of type* (I). *Then the bicanonical map is generically finite.*

Proof. Suppose that $|2K_S|$ is composed of a rational pencil. We write

$$2K_S \sim_{\text{lin}} C_1 + C_2 + Z,$$

where Z is the fixed part. Denote by C a general member which is algebrally equivalent to C_i. We have $1 = K_S^2 \geq K_S \cdot C$. On the other hand, $K_S \cdot C + C^2 \geq 2$, which gives $C^2 \geq 1$. Thus $K_S \cdot C = C^2 = 1$, i.e. C is a nonsingular curve of genus two. By the index theorem, we see that $K_S \sim_{\text{num}} C$. But from [3], $\text{Pic}(S)$ is torsion free, then $K_S \sim_{\text{lin}} C$. This is impossible because $h^0(S, C) = 2$. □

Lemma 4.8 (Lemma 8 of [19]). *Let S be a surface with finite π_1. Then*

$$H^1(S, \mathcal{O}_S(\mathcal{E})) = 0$$

for any invertible torsion sheaf $\mathcal{E}$ on S.

Lemma 4.9. *Let S be a minimal surface of type* (II) *or* (III). *Suppose that $|2K_S|$ is composed of a rational pencil. Then the moving part of $|2K_S|$ is a free pencil of genus two.*

Proof. We can write $2K_S \sim_{\text{lin}} C_1 + C_2 + Z$, where Z is the fixed part. Denote by C the general member which is algebraically equivalent to C_i. If $C^2 > 0$, then

$K_S^2 \geq K_S \cdot C \geq C^2$. On the other hand, the index theorem gives $K_S^2 \times C^2 \leq (K_S \cdot C)^2$. Thus $K_S^2 = K_S \cdot C = C^2 = 2$ and then $K_S \sim_{\text{num}} C$.

If $p_g = 1$, then $Z = 0$. Let $D \in |K_S|$ be the unique effective divisor, then $2D = F_1 + F_2$, where the F_i are two fibres of ϕ_2. If $F_1 \neq F_2$, then the F_i are multiple fibres and then $D \sim_{\text{num}} 2F_0$, where F_0 is a divisor. Which implies $D^2 \geq 4$, a contradiction. If $F_1 = F_2$, then $D = F_1$ and thus $h^0(S, D) = 2$, also a contradiction.

If $p_g = 0$, because the π_1 of S is a finite group (Corary 5.8 of [1]), then $h^1(S, K_S - C) = 0$ by Lemma 4.8. Whereas we have $h^1(S, K_S - C) = h^1(S, C) = 1$ by R-R, a contradiction. Therefore we have $C^2 = 0$ and then $g(C) = 2$. □

Proposition 4.10. *Let S be a minimal surface of type* (II) *or* (III). *Then* $|2K_S|$ *can not be composed of a rational pencil of genus two.*

Proof. We refer to the proof of Proposition 3 and Theorem 3 of [19]. □

Thus we finally arrive at the following theorem of Xiao (Theorem 1 of [19]).

Theorem 4.11. *Let S be a projective surface of general type. Then* ϕ_2 *is generically finite if and only if* $h^0(S, 2K_S) > 2$.

ACKNOWLEDGEMENT. I would like to thank the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy for support during my visit. I also wish to thank Prof. L. Ein, Prof. M.S. Narasimhan and Prof. M. Reid for encouragement. Thanks are also due to the hospitality of both Prof. C. Ciliberto and Prof. E. Stagnaro who are very kind to invite me visiting their universities. During the writing of this paper, a discussion with Prof. Ciliberto was quite helpful for me to organize the last section. This paper was finally revised while I was visiting the Uinversit*ä*t G*ö*ttingen, Germany as a post-doc fellow.

References

[1] A. Beauville: *L'application canonique pour les surfaces de type général,* Invent. Math. **55** (1979), 121–140.

[2] X. Benveniste: *Sur les applications pluricanoniques des variétés de type trés gégéral en dimension* 3, Amer. J. Math. **108** (1996), 433–449.

[3] E. Bombieri: *Canonical models of surfaces of general type,* Publication I.H.E.S. **42** (1973), 171–219.

[4] M. Chen: *On pluricanonical maps for threefolds of general type,* J. Math. Soc. Japan, **50** (1998), 615–621.

[5] M. Chen: *A theorem on pluricanonical maps of nonsingular minimal threefold of general type,* Chin. Ann. Math. **19B** (1998), 415–420.

[6] M. Chen: *Complex varieties of general type whose canonical systems are composed with pen-*

cils, J. Math. Soc. Japan, **51** (1999), 331–335.
[7] M. Chen: *Kawamata-Viehweg vanishing and quint-canonical map of a complex threefold,* Comm. Algebra, **27** (1999), 5471–5486.
[8] O. Debarre: *Addendum, inégalitiés numériques pour les surfaces de type général*, Bull. Soc. Math. France, **111** (1983), 301–302.
[9] M. Hanamura: Stability of the pluricanonical maps of threefolds, Algebraic Geometry, Sendai, 1985 (Adv. Stud. in Pure Math. **10**, 185–205).
[10] Y. Kawamata: *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann. **261** (1982), 43–46.
[11] J. Kollár: *Higher direct images of dualizing sheaves* I, Ann. Math. **123** (1986), 11–42.
[12] S. Lee: *Remarks on the pluricanonical and adjoint linear series on projective threefolds,* Commun. Algebra, **27** (1999), 4459–4476.
[13] K. Matsuki: *On pluricanonical maps for* 3*-folds of general type*, J. Math. Soc. Japan, **38** (1986), 339–359.
[14] Y. Miyaoka: The Chern classes and Kodaira dimension of a minimal variety, In: Algebraic Geometry, Sendai, 1985 (Adv. Stud. in Pure Math. **10**, 1987, 449–476).
[15] M. Reid: Canonical 3-folds, in Journées de Géométrie Algébrique d'Angers (A. Beauville ed.), Sijthoff and Noordhof, Alphen ann den Rijn, 1980, 273–310.
[16] I. Reider: *Vector bundles of rank 2 and linear systems on algebraic surfaces,* Ann. Math. **127** (1988), 309–316.
[17] E. Viehweg: *Vanishing theorems*, J. reine angew. Math. **335** (1982), 1–8.
[18] G. Xiao: *L'irrégularité des surfaces de type général dont le système canonique est composé d'un pinceau*, Compositio Math. **56** (1985), 251–257.
[19] G. Xiao: *Finitude de l'application bicanonique des surfaces de type gégéral,* Bull. Soc. Math. France, **113** (1985), 23–51.

Department of Applied Mathematics
Tongji University
Shanghai 200092, China
e-mail: mchen@mail.tongji.edu.cn