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# EXTENSIONS OF HOLOMORPHIC MOTIONS AND HOLOMORPHIC FAMILIES OF MÖBIUS GROUPS

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**Abstract**

A normalized holomorphic motion of a closed set in the Riemann sphere, defined over a simply connected complex Banach manifold, can be extended to a normalized quasiconformal motion of the sphere, in the sense of Sullivan and Thurston. In this paper, we show that if the given holomorphic motion, defined over a simply connected complex Banach manifold, has a group equivariance property, then the extended (normalized) quasiconformal motion will have the same property. We then deduce a generalization of a theorem of Bers on holomorphic families of isomorphisms of Möbius groups. We also obtain some new results on extensions of holomorphic motions. The intimate relationship between holomorphic motions and Teichmüller spaces is exploited throughout the paper.

## 1. Definitions and statements of the main theorems

In their study of the dynamics of rational maps, Mañé, Sad, and Sullivan introduced the idea of holomorphic motions (see [20]). Since then, holomorphic motions have found several interesting applications in Teichmüller theory, complex dynamics, and Kleinian groups. A central topic in the study of holomorphic motions is the question of extensions. In this paper, we obtain some new extension theorems. We also prove a generalization of a theorem of Bers on holomorphic families of isomorphisms of Möbius groups.

### 1.1. Holomorphic motions.

DEFINITION 1.1. Let $V$ be a connected complex manifold, and let $E$ be a subset of $\hat{\mathbb{C}}$. A *holomorphic family of injections of $E$ over $V$* is a family of maps $\{\phi_x\}_{x\in V}$ that has the following properties:
(i) for each $x$ in $V$, the map $\phi_x\colon E \to \hat{\mathbb{C}}$ is an injection, and,
(ii) for each $z$ in $E$, the map $x \mapsto \phi_x(z)$ is holomorphic.

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It is convenient to define $\phi\colon V \times E \to \hat{\mathbb{C}}$ as the map $\phi(x,z) := \phi_x(z)$ for all $(x,z) \in V \times E$.

If $V$ is a connected complex manifold with a basepoint $x_0$, then *a holomorphic motion of $E$ over $V$* is a holomorphic family of injections such that $\phi(x_0, z) = z$ for all $z$ in $E$.

A holomorphic motion $\phi\colon V \times E \to \hat{\mathbb{C}}$ is called *trivial* if $\phi(x,z) = z$ for all $(x,z) \in V \times E$.

We say that $V$ is the *parameter space* of the holomorphic motion $\phi$.

Unless otherwise stated, we will always assume that $\phi$ is a *normalized* holomorphic motion; i.e. 0, 1, and $\infty$ belong to $E$ and are fixed points of the map $\phi_x(\cdot)$ for every $x$ in $V$.

DEFINITION 1.2. Let $V$ and $W$ be connected complex manifolds with basepoints, and $f$ be a basepoint preserving holomorphic map of $W$ into $V$. If $\phi\colon V \times E \to \hat{\mathbb{C}}$ is a holomorphic motion, its *pullback* by $f$ is the holomorphic motion

$$f^*(\phi)(x, z) = \phi(f(x), z) \quad \text{for all} \quad (x, z) \in W \times E$$

of $E$ over $W$.

If $E$ is a proper subset of $\tilde{E}$ and $\phi\colon V \times E \to \hat{\mathbb{C}}$ and $\tilde{\phi}\colon V \times \tilde{E} \to \hat{\mathbb{C}}$ are two maps, we say that $\tilde{\phi}$ *extends* $\phi$ if $\tilde{\phi}(x, z) = \phi(x, z)$ for all $(x, z)$ in $V \times E$.

If $\phi\colon V \times E \to \hat{\mathbb{C}}$ is a holomorphic motion, it is natural to ask whether there exists a holomorphic motion $\tilde{\phi}\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\phi}$ extends $\phi$. For holomorphic motions over the open unit disk, the papers [5], [12], [20], [26], and [28] contain important results. Extensions of holomorphic motions over more general parameter spaces have been studied in the papers [13], [21], [22], and [23].

**1.2. Quasiconformal motions.** In their paper [28], Sullivan and Thurston introduced the idea of quasiconformal motions. In what follows, $\rho$ denotes the Poincaré metric on $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$.

Let $V$ be a connected Hausdorff space with a basepoint $x_0$. For any map $\phi\colon V \times E \to \hat{\mathbb{C}}$, $x$ in $V$, and any quadruplet $a, b, c, d$ of points in $E$, let $\phi_x(a, b, c, d)$ denote the cross-ratio of the values $\phi(x, a)$, $\phi(x, b)$, $\phi(x, c)$, and $\phi(x, d)$. We will write $\phi(x, z)$ as $\phi_x(z)$ for $x$ in $V$ and $z$ in $E$. So we have:

$$\phi_x(a, b, c, d) = \frac{(\phi_x(a) - \phi_x(c))(\phi_x(b) - \phi_x(d))}{(\phi_x(a) - \phi_x(d))(\phi_x(b) - \phi_x(c))} \tag{1.1}$$

for each $x$ in $V$.

DEFINITION 1.3. A *quasiconformal motion* is a map $\phi\colon V \times E \to \hat{\mathbb{C}}$ of $E$ over $V$ such that
(i) $\phi(x_0, z) = z$ for all $z$ in $E$, and
(ii) given any $x$ in $V$ and any $\epsilon > 0$, there exists a neighborhood $U_x$ of $x$ such that for any quadruplet of distinct points $a, b, c, d$ in $E$, we have

$$\rho(\phi_y(a, b, c, d), \phi_{y'}(a, b, c, d)) < \epsilon \quad \text{for all} \quad y \text{ and } y' \quad \text{in} \quad U_x.$$

We will always assume that $\phi$ is a normalized quasiconformal motion; i.e. 0, 1, and $\infty$ belong to $E$ and are fixed points of the map $\phi_x(\,\cdot\,)$ for every $x$ in $V$.

REMARK 1.4. If $\phi\colon V \times E \to \hat{\mathbb{C}}$ is a quasiconformal motion, $\phi_x(a, b, c, d)$ is a well-defined point in $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$, and then it is obvious that for each $x$ in $V$, the map $\phi_x\colon E \to \hat{\mathbb{C}}$ is injective.

We will need the following property of quasiconformal motions of the sphere. See [23] for a complete proof.

**Proposition 1.5.** *Let $\phi\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a map such that $\phi(x_0, z) = z$ for all $z$ in $\hat{\mathbb{C}}$, and for each $x$ in $V$, $\phi_x$ fixes the points* 0, 1, *and* $\infty$. *Then, $\phi$ is a quasiconformal motion of $\hat{\mathbb{C}}$ if and only if it satisfies*:
(i) *the map $\phi_x\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is quasiconformal for each $x$ in $V$, and*
(ii) *the map that sends $x$ in $V$ to the Beltrami coefficient of $\phi_x$, for each $x$ in $V$, is continuous.*

### 1.3. Some other definitions.

DEFINITION 1.6. Let $V$ be a path-connected Hausdorff space with a basepoint $x_0$. As usual, $E$ is a subset of $\hat{\mathbb{C}}$ that contains the points 0, 1, and $\infty$. A *normalized continuous motion* of $E$ over $V$ is a continuous map $\phi\colon V \times E \to \hat{\mathbb{C}}$ such that:
(i) $\phi(x_0, z) = z$ for all $z$ in $E$, and
(ii) for each $x$ in $V$, the map $\phi(x,\,\cdot\,)$ is a homeomorphism of $E$ onto its image, that fixes the points 0, 1, and $\infty$.

As usual, we will write $\phi(x,\,\cdot\,)$ as $\phi_x(\,\cdot\,)$ and we will always assume that the continuous motion $\phi$ is normalized.

We note the following fact that was proved in [23].

**Proposition 1.7.** *A quasiconformal motion $\phi\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a continuous motion.*

DEFINITION 1.8. Let $\Delta$ denote the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. A compact subset $K$ of $\Delta$ is called *AB-removable* if every bounded holomorphic function on $\Delta - K$ can be extended to a holomorphic function on $\Delta$.

For example, a compact subset $K$ of $\Delta$ with zero 1-dimensional Hausdorff measure, is $AB$-removable.

**1.4. Statements of the main theorems.** We will always assume that $E$ is a closed subset of $\hat{\mathbb{C}}$, such that 0, 1, and $\infty$ belong to $E$, and the holomorphic motions are normalized.

For a holomorphic motion $\phi$ of $E$ over a Riemann surface $X$, Chirka [6] announced that there exists a topological condition for the extendability of the motion $\phi$ to a holomorphic motion of $\hat{\mathbb{C}}$ over $X$. The following theorem gives an analytic condition for a complex manifold $V$ to have a non-trivial holomorphic motion of $\hat{\mathbb{C}}$ over $V$.

**Theorem 1.** (1) *Let $V$ be any connected complex Banach manifold, and let $x_0$ be any basepoint on $V$. Then there exists a non-trivial holomorphic motion of $\hat{\mathbb{C}}$ over $V$ if and only if there is a non-constant bounded holomorphic function on $V$.*
(2) *Let $V$ be a simply connected complex Banach manifold, and let $x_0$ be a basepoint on $V$. Let $E$ be a closed subset of $\hat{\mathbb{C}}$. Then there is a non-trivial holomorphic motion of $E$ over $V$ if and only if there is a non-constant bounded holomorphic function on $V$.*

The following theorem implies that an $AB$-removable set is "removable" for holomorphic motions if the motion can be extended to the whole sphere. (Here, by "removable" we mean that if the given holomorphic motion can be extended to the whole sphere, then the holomorphic motion over $\Delta - K$ can be extended to a holomorphic motion over $\Delta$.)

**Theorem 2.** *Let $K$ be a compact subset of $\Delta$. Suppose that $K$ is $AB$-removable. For a holomorphic motion $\phi : (\Delta - K) \times E \to \hat{\mathbb{C}}$, the following are equivalent*:
(1) *$\phi$ can be extended to a continuous motion $\tilde{\phi} : (\Delta - K) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.*
(2) *$\phi$ can be extended to a holomorphic motion $\hat{\phi} : (\Delta - K) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.*
(3) *$\phi$ can be extended to a holomorphic motion $\phi_0 : \Delta \times E \to \hat{\mathbb{C}}$.*
*Statement* (3) *means that $\phi_0(t, z) = \phi(t, z)$ for all $(t, z) \in (\Delta - K) \times E$.*

*If $K$ is not $AB$-removable, there exists a holomorphic motion on $(\Delta - K) \times E$ such that it cannot be extended to a holomorphic motion on $\Delta \times E$ while it can be extended to a holomorphic motion on $(\Delta - K) \times \hat{\mathbb{C}}$.*

REMARK 1.9. If $\phi$ satisfies one of the above conditions, then it can be extended to a holomorphic motion on $\Delta \times \hat{\mathbb{C}}$.

Let $V$ be a connected complex manifold. In what follows, $G$ is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$, $E$ is a closed subset of $\hat{\mathbb{C}}$ (as usual, 0, 1, and $\infty$ belong to $E$), and suppose $E$ is invariant under $G$ (which means that $g(E) = E$ for all $g$ in $G$). An isomorphism $\eta\colon G \to \mathrm{PSL}(2,\mathbb{C})$ is said to be *induced* by an injection $f : E \to \hat{\mathbb{C}}$ if

$$f(g(z)) = \eta(g)(f(z))$$

for all $g \in G$ and for all $z \in E$. An isomorphism induced by a quasiconformal self-map of $\hat{\mathbb{C}}$ is called a *quasiconformal deformation of $G$*.

DEFINITION 1.10. A *holomorphic family of isomorphisms of $G$* is a family $\{\theta_x\}_{x\in V}$ such that:

(i) for each $x \in V$, $\theta_x : G \to \mathrm{PSL}(2,\mathbb{C})$ is an isomorphism, and
(ii) for each $g \in G$, the map $x \mapsto \theta_x(g)$, for $x \in V$, is holomorphic.

DEFINITION 1.11. Let $\{\theta_x\}$ be a holomorphic family of isomorphisms of $G$. If $V$ has a basepoint, and $\tilde{\phi}\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal motion, such that

$$\tilde{\phi}_x(g(z)) = \theta_x(g)(\tilde{\phi}_x(z))$$

for all $(x, z) \in V \times \hat{\mathbb{C}}$, we say that the family $\{\theta_x\}_{x\in V}$ is *induced* by the quasiconformal motion $\tilde{\phi}$.

Let $\phi\colon V \times E \to \hat{\mathbb{C}}$ be a holomorphic motion. As above, let $G$ be a group of Möbius transformations, such that $E$ is invariant under $G$. We say that $\phi$ is *$G$-equivariant* if and only if for each $g$ in $G$, and $x$ in $V$, there exists a Möbius transformation $\theta_x(g)$ such that:

$$\phi(x, g(z)) = \theta_x(g)(\phi(x, z)) \quad \text{for all} \quad z \quad \text{in} \quad E. \tag{1.2}$$

In [12], Earle, Kra and Krushkal′ proved that if $\phi\colon \Delta \times E \to \hat{\mathbb{C}}$ is a holomorphic motion that is $G$-equivariant, there exists a holomorphic motion $\hat{\phi}\colon \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that extends $\phi$ and is also $G$-equivariant. The main idea was to use Slodkowski's theorem that every holomorphic motion of $E$ over $\Delta$ can be extended to a holomorphic motion of $\hat{\mathbb{C}}$ over $\Delta$. For proof of Slodkowski's theorem, see the papers [3], [6], [7], [26] and the book [16]. Slodkowski's theorem cannot be generalized to holomorphic motions over higher dimensional parameter spaces. The papers [13], [18] contain some examples. In the following theorem we prove a higher-dimensional analogue of the theorem of Earle, Kra, and Krushkal′.

**Theorem 3.** *Let $\phi\colon V \times E \to \hat{\mathbb{C}}$ be a holomorphic motion where $V$ is a connected complex Banach manifold, such that $\phi$ is $G$-equivariant. Suppose there exists a continuous motion $\hat{\phi}\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that extends $\phi$. Then, there exists a quasiconformal motion $\tilde{\phi}\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that*:

(1) *$\tilde{\phi}$ extends $\phi$,*
(2) *$\tilde{\phi}$ is also $G$-equivariant,*
(3) *for each $x$ in $V$, the homeomorphisms $\tilde{\phi}_x$ and $\hat{\phi}_x$ (of $\hat{\mathbb{C}}$ onto itself) are isotopic* $\mathrm{rel}\,E$.

REMARK 1.12. Note that the continuous motion $\hat{\phi}\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is *not* assumed to have the property of $G$-equivariance given in Equation (1.2).

**Corollary 1.** *If $V$ is simply connected, and $\phi\colon V \times E \to \hat{\mathbb{C}}$ is a holomorphic motion that is $G$-equivariant, then there always exists a quasiconformal motion $\tilde{\phi}\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that extends $\phi$ and has the same $G$-equivariance property.*

An immediate consequence of Theorem 3 is the following theorem on holomorphic families of isomorphisms of Möbius groups. Our result proves Proposition 1 in [4] in its fullest generality.

**Theorem 4.** *Let $V$ be a connected complex Banach manifold, and let $\{\phi_x\}_{x\in V}$ be a holomorphic family of injections of $E$ over $V$. Suppose that, for each $x$ in $V$, and for each $g$ in $G$, there exists a Möbius transformation $\theta_x(g)$ such that*

$$\phi_x(g(z)) = \theta_x(g)(\phi_x(z)) \quad \textit{for all} \quad z \in E.$$

*Then we have*:
(i) *$\{\theta_x\}_{x\in V}$ is a holomorphic family of isomorphisms of $G$, and*
(ii) *if $\theta_t$ is a quasiconformal deformation of $G$ for some $t$ in $V$, then $\theta_x$ is a quasiconformal deformation of $G$ for every $x$ in $V$.*

*Furthermore, if $V$ is simply connected, then the family $\{\theta_x\}$ is induced by a quasiconformal motion $\tilde{\phi}\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which extends $\{\phi_x\}$.*

REMARK 1.13. If the conditions of Theorem 4 are satisfied, we say that the holomorphic family $\{\phi_x\}$ of injections of $E$ *induces* the holomorphic family $\{\theta_x\}$ of isomorphisms of $G$.

The following corollary gives an infinite version of Bers' main theorem in [4].

**Corollary 2.** *Let $G$ be a non-Abelian infinite group. Let $V$ be the same as in* Theorem 4 *and let $\{\theta_x\}_{x\in V}$ be a holomorphic family of isomorphisms of $G$ defined over $V$ with $\theta_t$ a quasiconformal deformation of $G$, for some $t$ in $V$. Suppose that for all $x$ in $V$,*
(i) *$\theta_x(G)$ is discrete, and*
(ii) *$\theta_x(g)$ is parabolic if and only if $g \in G$ is parabolic.*

*Then, for each $x$ in $V$, $\theta_x$ is a quasiconformal deformation of $G$. Furthermore, if $V$ is simply connected, $\{\theta_x\}_{x\in V}$ is induced by a quasiconformal motion of $\hat{\mathbb{C}}$.*

Our paper is organized as follows. In §2, we discuss some properties of the Teichmüller space of the closed set $E$, and in §3, we define the universal holomorphic motion of the closed set $E$. In §4, we prove Theorem 1, and in §5 we prove Theorem 2. In §6 we prove some propositions and then prove Theorem 3. In §7, we prove Theorem 4 and Corollary 2. In §8, we give two examples related to Theorems 1 and 2. The first example gives a non-trivial holomorphic motion of a finite set $E$ that cannot be extended to a holomorphic motion of $\hat{\mathbb{C}}$, over a suitable Riemann surface that admits no non-constant bounded holomorphic functions. The second one gives an example of a continuous motion $\phi\colon \Delta^* \times E \to \hat{\mathbb{C}}$, which can be extended to a continuous motion $\hat{\phi}\colon \Delta^* \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, but $\phi$ cannot be extended to a continuous motion $\tilde{\phi}\colon \Delta \times E \to \hat{\mathbb{C}}$; here $\Delta^* = \{z \in \mathbb{C}\colon 0 < |z| < 1\}$.

## 2. Teichmüller space of the closed set *E*

A homeomorphism of $\hat{\mathbb{C}}$ is called *normalized* if it fixes the points 0, 1, and $\infty$.

**2.1. Definition.** Two normalized quasiconformal self-mappings $f$ and $g$ of $\hat{\mathbb{C}}$ are said to be $E$-equivalent if and only if $f^{-1} \circ g$ is isotopic to the identity rel $E$. The *Teichmüller space* $T(E)$ is the set of all $E$-equivalence classes of normalized quasiconformal self-mappings of $\hat{\mathbb{C}}$.

The basepoint of $T(E)$ is the $E$-equivalence class of the identity map.

**2.2. *T(E)* as a complex manifold.** Let $M(\mathbb{C})$ be the open unit ball of the complex Banach space $L^{\infty}(\mathbb{C})$. Each $\mu$ in $M(\mathbb{C})$ is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism $w^{\mu}$ of $\hat{\mathbb{C}}$ onto itself. The basepoint of $M(\mathbb{C})$ is the zero function.

We define the quotient map

$$P_E\colon M(\mathbb{C}) \to T(E)$$

by setting $P_E(\mu)$ equal to the $E$-equivalence class of $w^{\mu}$, written as $[w^{\mu}]_E$. Clearly, $P_E$ maps the basepoint of $M(\mathbb{C})$ to the basepoint of $T(E)$.

In his doctoral dissertation ([19]), G. Lieb proved that $T(E)$ is a complex Banach manifold such that the projection map $P_E\colon M(\mathbb{C}) \to T(E)$ is a holomorphic split submersion. For more details, see §2.4.

**2.3. Two special cases.** Let $E$ be a finite set. Its complement $\Omega = \hat{\mathbb{C}} \setminus E$ is the Riemann sphere with punctures at the points of $E$. Since $T(E)$ and the classical Teichmüller space *Teich*$(\Omega)$ are quotients of $M(\mathbb{C})$ by the same equivalence relation, $T(E)$ can be naturally identified with *Teich*$(\Omega)$ (see Example 3.1 in [21]). The reader is referred to [15], [17], or [24] for standard facts on classical Teichmüller theory. This canonical identification will be useful in our paper.

When $E = \hat{\mathbb{C}}$, the space $T(\hat{\mathbb{C}})$ consists of all the normalized quasiconformal self-mappings of $\hat{\mathbb{C}}$, and the map $P_{\hat{\mathbb{C}}}$ from $M(\mathbb{C})$ to $T(\hat{\mathbb{C}})$ is bijective. We use it to identify $T(\hat{\mathbb{C}})$ biholomorphically with $M(\mathbb{C})$.

**2.4. Lieb's isomorphism theorem.** For the reader's convenience, we include a brief discussion of "Lieb's isomorphism theorem." For complete details, the reader is referred to Section 7 of [13]. In what follows, we shall assume that $E$ is infinite, and has a nonempty complement $E^c = \hat{\mathbb{C}} \setminus E$. Let $\{X_n\}$ be the connected components of $E^c$. Each $X_n$ is a hyperbolic Riemann surface; let *Teich*$(X_n)$ denote its Teichmüller space. If the number of components is finite, *Teich*$(E^c)$ is, by definition, the cartesian product of the spaces *Teich*$(X_n)$. If there are infinitely many components, then $E^c$ is the disjoint union of $X_n$'s. We define the *product Teichmüller space Teich*$(E^c)$ as follows.

For each $n \geq 1$, let $0_n$ be the basepoint of the Teichmüller space *Teich*$(X_n)$, and let $d_n$ be the Teichmüller metric on *Teich*$(X_n)$. As usual, let $M(X_n)$ denote the open unit ball of the complex Banach space $L^\infty(X_n)$, for each $n \geq 1$. By definition, the *product Teichmüller space Teich*$(E^c)$ is the set of sequences $t = \{t_n\}_{n=1}^\infty$ such that $t_n$ belongs to *Teich*$(X_n)$ for each $n \geq 1$, and

$$\sup\{d_n(0_n, t_n) \colon n \geq 1\} < \infty.$$

The basepoint of *Teich*$(E^c)$ is the sequence $0 = \{0_n\}$ whose $n$th term is the basepoint of *Teich*$(X_n)$.

Let $L^\infty(E^c)$ be the complex Banach space of sequences $\mu = \{\mu_n\}$ such that $\mu_n$ belongs to $L^\infty(X_n)$ for each $n \geq 1$ and the norm $\|\mu\|_\infty = \sup\{\|\mu_n\|_\infty \colon n \geq 1\}$ is finite. Let $M(E^c)$ be the open unit ball of $L^\infty(E^c)$. Note that if $\mu$ belongs to $M(E^c)$, then $\mu_n$ belongs to $M(X_n)$ for all $n \geq 1$ (but the converse is false).

For each $n \geq 1$, let $\Phi_n$ be the standard projection from $M(X_n)$ to *Teich*$(X_n)$ (see [15] or [17] or [24] for the basic definitions). For $\mu$ in $M(E^c)$, let $\Phi(\mu)$ be the sequence $\{\Phi_n(\mu_n)\}$. It is easy to see that $\Phi(\mu)$ belongs to *Teich*$(E^c)$, and the map $\Phi$ is surjective. We call $\Phi$ the *standard projection* of $M(E^c)$ onto *Teich*$(E^c)$. In [19] it was shown that *Teich*$(E^c)$ is a complex Banach manifold such that the map $\Phi$ is a holomorphic split submersion (see also [13] or [21]).

Let $M(E)$ be the open unit ball in $L^\infty(E)$. The product *Teich*$(E^c) \times M(E)$ is a complex Banach manifold. (If $E$ has zero area, then $M(E)$ contains only one point, and *Teich*$(E^c) \times M(E)$ is then isomorphic to *Teich*$(E^c)$.)

For $\mu$ in $L^\infty(\mathbb{C})$, let $\mu|E^c$ and $\mu|E$ be the restrictions of $\mu$ to $E^c$ and $E$ respectively. We define the projection map $\tilde{P}_E$ from $M(\mathbb{C})$ to *Teich*$(E^c) \times M(E)$ by the formula:

$$\tilde{P}_E(\mu) = (\Phi(\mu|E^c), \mu|E) \quad \text{for all} \quad \mu \in M(\mathbb{C}).$$

**Proposition 2.1** (Lieb's isomorphism theorem). *For all $\mu$ and $\nu$ in $M(\mathbb{C})$ we have $P_E(\mu) = P_E(\nu)$ if and only if $\tilde{P}_E(\mu) = \tilde{P}_E(\nu)$. Consequently, there is a well-defined* *bijection* $\theta\colon T(E) \to Teich(E^c) \times M(E)$ *such that* $\theta \circ P_E = \tilde{P}_E$, *and* $T(E)$ *has a unique complex manifold structure such that* $P_E$ *is a holomorphic split submersion.*

See Section 7.9 of [13] for a complete proof.

**2.5. Continuous section of $P_E$.** The projection map $P_E\colon M(\mathbb{C}) \to T(E)$ has a continuous section, that will be very crucial in our paper. This was proved in [13] and also in [21]. It is an application of barycentric extensions studied in [8]. We include the discussion here, for the reader's convenience, and also to make our paper self-contained.

**Proposition 2.2.** *There is a continuous basepoint preserving map* $\hat{s}$ *from* $Teich(E^c)$ *to* $M(E^c)$ *such that* $\Phi \circ \hat{s}$ *is the identity map on* $Teich(E^c)$.

Sketch of proof. By Lemma 5 in [8], for each $n \geq 1$, there is a continuous basepoint preserving map $\hat{s}_n$ from $Teich(X_n)$ to $M(X_n)$ such that $\Phi_n \circ \hat{s}_n$ is the identity map on $Teich(X_n)$. Let

$$M_k(X_n) = \{\mu_n \in M(X_n)\colon \|\mu_n\|_\infty \leq k\}$$

for any $k$ in the open interval $(0, 1)$ and consider the map $\sigma_n = \hat{s}_n \circ \Phi_n$ from $M(X_n)$ to itself. By Propositions 3 and 7 in [8], it follows that $\sigma_n$ maps $M_k(X_n)$ into $M_{c(k)}(X_n)$, where $0 < c(k) < 1$, and $c(k)$ is independent of $n$. Furthermore, $\sigma_n$ is uniformly continuous in $M_k(X_n)$, and its modulus of continuity in $M_k(X_n)$ depends only on $k$. It can be checked that the formula $\hat{s}(t) = \{\hat{s}_n(t_n)\}$, for $t = \{t_n\}$ in $Teich(E^c)$, defines a continuous map from $Teich(E^c)$ to $M(E^c)$ with the required properties. For the details, we refer the reader to Section 7.7 in [13].

**Proposition 2.3.** *There is a continuous basepoint preserving map* $s$ *from* $T(E)$ *to* $M(\mathbb{C})$ *such that* $P_E \circ s$ *is the identity map on* $T(E)$.

Proof. By Proposition 2.2, there is a continuous basepoint preserving map $\hat{s}$ from $Teich(E^c)$ to $M(E^c)$ such that $\Phi \circ \hat{s}$ is the identity map on $Teich(E^c)$. Let $\tilde{s}$ be the map from $Teich(E^c) \times M(E)$ to $M(\mathbb{C})$ such that $\tilde{s}(\tau, \nu)$ equals $\hat{s}(\tau)$ in $E^c$ and equals $\nu$ in $E$ for each $(\tau, \nu)$ in $Teich(E^c) \times M(E)$. Clearly, $\tilde{P}_E \circ \tilde{s}$ is the identity map on $Teich(E^c) \times M(E)$. We define $s = \tilde{s} \circ \theta$, where $\theta$ is the biholomorphic map from $T(E)$ to $Teich(E^c) \times M(E)$ given in Proposition 2.1. It is clear that $s\colon T(E) \to M(\mathbb{C})$ is a continuous basepoint preserving map such that $P_E \circ s$ is the identity map on $T(E)$. □

Since $M(\mathbb{C})$ is contractible, we have the following

**Corollary 2.4.** *The Teichmüller space* $T(E)$ *is contractible.*

## 3. Universal holomorphic motion of the closed set $E$

**3.1. The general definition.** The *universal holomorphic motion* $\Psi_E$ of $E$ over $T(E)$ is defined as follows:

$$\Psi_E(P_E(\mu), z) = w^{\mu}(z) \quad \text{for} \quad \mu \in M(\mathbb{C}) \quad \text{and} \quad z \in E.$$

The definition of $P_E$ in §2.2 implies that the map $\Psi_E$ is well-defined. It is a holomorphic motion because $P_E$ is a holomorphic split submersion and $\mu \mapsto w^{\mu}(z)$ is a holomorphic map from $M(\mathbb{C})$ to $\hat{\mathbb{C}}$ for every fixed $z$ in $\hat{\mathbb{C}}$ (by Theorem 11 in [1]).

This holomorphic motion is "universal" in the following sense:

**Theorem 3.1.** *Let $\phi\colon V \times E \to \hat{\mathbb{C}}$ be a holomorphic motion. If $V$ is a simply connected complex Banach manifold with a basepoint, there is a unique basepoint preserving holomorphic map $f\colon V \to T(E)$ such that $f^*(\Psi_E) = \phi$.*

For a proof see Section 14 in [21].

Here is a special case of Theorem 3.1. Recall from §2.3, that when $E = \hat{\mathbb{C}}$, $T(\hat{\mathbb{C}})$ is canonically identified with $M(\mathbb{C})$. Therefore, the universal holomorphic motion $\Psi_{\hat{\mathbb{C}}}\colon M(\mathbb{C}) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is given by:

$$\Psi_{\hat{\mathbb{C}}}(\mu, z) = w^{\mu}(z)$$

for all $z \in \hat{\mathbb{C}}$. So, by Theorem 3.1, if $\phi\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a holomorphic motion, there exists a unique basepoint preserving holomorphic map $f\colon V \to M(\mathbb{C})$ such that $\phi(x, z) = f^*(\Psi_{\hat{\mathbb{C}}})(x, z) = \Psi_{\hat{\mathbb{C}}}(f(x), z) = w^{f(x)}(z)$ for all $(x, z)$ in $V \times \hat{\mathbb{C}}$.

We also note the following theorem that was proved in [23].

**Theorem 3.2.** *Let $\phi\colon V \times E \to \hat{\mathbb{C}}$ be a holomorphic motion where $V$ is a connected complex Banach manifold with a basepoint. Then the following are equivalent:*
(1) *There exists a continuous motion $\tilde{\phi}\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that extends $\phi$.*
(2) *There exists a quasiconformal motion $\hat{\phi}\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that extends $\phi$.*
(3) *There exists a unique basepoint preserving holomorphic map $f\colon V \to T(E)$ such that $f^*(\Psi_E) = \phi$.*

## 4. Proof of Theorem 1

(1) If there are non-constant bounded holomorphic functions on $V$, there is a non-constant holomorphic function $f$ on $V$ so that $f(x_0) = 0$ and $|f(x)| < 1$ for all $x \in V$. Take $\mu \in M(\mathbb{C})$ which does not vanish identically and put

$$\phi(x, z) = w^{f(x)\mu}(z)$$

for all $z \in \hat{\mathbb{C}}$. Then, $\phi$ is a holomorphic motion of $\hat{\mathbb{C}}$ over $V$. Since $\mu \neq 0$, the motion is non-trivial.

For the other direction, if $\phi$ is a holomorphic motion of $\hat{\mathbb{C}}$ over $V$, then, by Theorem 4 in [10] (or by Theorem 3.2 of this paper, where $E = \hat{\mathbb{C}}$ and $T(\hat{\mathbb{C}})$ is identified with $M(\mathbb{C})$), the map $F$ from $V$ to $M(\mathbb{C})$ that sends $x$ in $V$ to the Beltrami coefficient of $\phi_x$ is holomorphic. If $\phi$ is non-trivial, then $F$ is non-constant; so, $l \circ F$ is a non-constant holomorphic function on $V$ if $l$ is a suitable bounded linear functional on $L^\infty(\mathbb{C})$.

(2) If there are non-constant bounded holomorphic functions on $V$, then the same method as in (1) gives a non-trivial holomorphic motion of $\hat{\mathbb{C}}$ over $V$.

Conversely, if $\phi$ is a non-trivial holomorphic motion of some closed set $E$ ($0, 1, \infty \in E$) over $V$, then by Theorem 3.1, there exists a unique basepoint preserving holomorphic map $F : V \to T(E)$ such that $F^*(\Psi_E) = \phi$. Since $\phi$ is non-trivial, $F$ is non-constant. Lieb's isomorphism theorem (see Proposition 2.1) produces a non-constant holomorphic map $G = \theta \circ F$ from $V$ to $Teich(E^c) \times M(E)$, which is a bounded region in a complex Banach space $W$. Therefore $f = l \circ G$ is a non-constant bounded holomorphic function on $V$ if $l$ is a suitable bounded linear functional on $W$. □

REMARK 4.1. Let $V$ be a connected complex manifold with a basepoint $x_0$, and $E$ be a closed subset of $\hat{\mathbb{C}}$ (as usual, $0, 1, \infty \in E$). Let $\phi: V \times E \to \hat{\mathbb{C}}$ be a holomorphic motion. For each $\zeta \in E \setminus \{0, 1, \infty\}$, we have a holomorphic function $h_\zeta(x) := \phi(x, \zeta)$ on $V$. It is a holomorphic map from $V$ to $\mathbb{C} \setminus \{0, 1\}$. Here, we present a property of the map $h_\zeta$ which has an independent interest and may also be used to prove Theorem 1.

**Proposition 4.2.** *Suppose that $\phi: V \times E \to \hat{\mathbb{C}}$ can be extended to a continuous motion $\tilde{\phi}: V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Then, the function $h_\zeta$ can be lifted to a holomorphic function $\tilde{h}_\zeta : V \to \Delta$ (where $\Delta$ is the universal covering of $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$).*

Proof. Take any closed curve $C$ passing through $x_0$, and put $C_\phi := \phi(C, \zeta)$. Then $C_\phi$ is a closed curve in $\mathbb{C} \setminus \{0, 1\}$ passing through $\zeta$. By Theorem 3.2, there exists a quasiconformal motion $\hat{\phi}: V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that extends $\phi$. Also, by Proposition 1.5, $\hat{\phi}_x: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map, for each $x$ in $V$. Hence, there exists $\mu(x) \in M(\mathbb{C})$ for each $x \in V$ such that $h_\zeta(x) = \phi(x, \zeta) = w^{\mu(x)}(\zeta)$. Therefore,

$$C_\phi = \{w^{\mu(x)}(\zeta) \mid x \in C\}.$$

Furthermore, it follows from Proposition 1.5 that the mapping $V \ni x \mapsto \mu(x) \in M(\mathbb{C})$ is continuous on $V$. Thus, a mapping $V \ni x \mapsto w^{t\mu(x)}(\zeta) \in \mathbb{C} \setminus \{0, 1\}$ is still continuous for each $t \in [0, 1]$ and we can define a curve $C^t_\phi$ by

$$C^t_\phi = \{w^{t\mu(x)}(\zeta) \mid x \in C\} \quad (t \in [0, 1]).$$

Since $\{C_\phi^t\}_{t\in[0,1]}$ is a continuous family of curves in $\mathbb{C}\setminus\{0,1\}$ and $C_\phi^0=\{\zeta\}$, we conclude that $h_\zeta(C)=C_\phi$ is homotopic to the trivial curve in $\mathbb{C}\setminus\{0,1\}$. This implies that $h_\zeta$ can be lifted to a holomorphic function $\tilde{h}_\zeta$ from $V$ to the universal covering $\Delta$ of $\mathbb{C}\setminus\{0,1\}$, as desired. □

## 5. Proof of Theorem 2

First, we consider the case where $K$ is $AB$-removable.

(2) ⇒ (1): It is obvious.

(3) ⇒ (2): By Slodkowski's theorem, $\phi_0$ can be extended to a holomorphic motion $\hat{\phi}\colon(\Delta-K)\times\hat{\mathbb{C}}\to\hat{\mathbb{C}}$. Thus, (2) is true.

We will prove that (1) ⇒ (3). Suppose that $\phi\colon(\Delta-K)\times E\to\hat{\mathbb{C}}$ can be extended to a continuous motion $\hat{\phi}\colon(\Delta-K)\times\hat{\mathbb{C}}\to\hat{\mathbb{C}}$.

CASE 1. When $E$ is finite. Suppose $E$ contains $n$ ($\geq 4$) points. By Theorem 3.2, we have a holomorphic map $F_\phi\colon(\Delta-K)\to T(E)$ such that

$$F_\phi^*(\Psi_E)(\lambda,z)=\phi(\lambda,z)\quad\text{for all}\quad(\lambda,z)\in(\Delta-K)\times E.$$

By §2.3, $T(E)$ can be identified with the Teichmüller space of the sphere with $n$ punctures, denoted by $Teich(0,n)$. Since $Teich(0,n)$ is regarded as a bounded domain in $\mathbb{C}^{n-3}$ by Bers embedding, the holomorphic map $F_\phi$ on $\Delta-K$ can be extended to a holomorphic map $\hat{F}_\phi$ from $\Delta$ to $\overline{Teich(0,n)}$. We shall show that $\hat{F}_\phi(\lambda)\in Teich(0,n)$ for every $\lambda\in K$.

Since $K$ is $AB$-removable, the space of bounded holomorphic functions on $\Delta-K$ is the same as that on $\Delta$. Hence the Carathéodory metrics on $\Delta-K$ and on $\Delta$ are the same on $\Delta-K$. Therefore, any sequence $\{\lambda_n\}_{n=1}^\infty$ in $\Delta-K$ converging to a point $\lambda\in K$ is a Cauchy sequence with respect to the Carathéodory metric on $\Delta-K$ and $\{F_\phi(\lambda_n)\}_{n=1}^\infty$ is also a Cauchy sequence with respect to the Carathéodory metric on $Teich(0,n)$ because of the distance decreasing property of holomorphic maps. Using the completeness of the Carathéodory metric on $Teich(0,n)$ (see [9] and [25]), we conclude that $\hat{F}_\phi(\lambda)=\lim_{n\to\infty}F_\phi(\lambda_n)$ exists in $Teich(0,n)$ and the holomorphic map $\hat{F}_\phi\colon\Delta\to Teich(0,4)$ extends $F_\phi$. Therefore, $\hat{F}_\phi$ gives a holomorphic motion $\phi_0\colon\Delta\times E\to\hat{\mathbb{C}}$ defined by $\phi_0=\hat{F}_\phi^*(\Psi_E)$ and clearly, $\phi_0$ extends $\phi$.

CASE 2. When $E$ is infinite. Consider a sequence of finite subsets $\{E_n\}$ such that $\{0,1,\infty\}\subset E_n\subset E_{n+1}$ for each $n\geq 1$ and $\bigcup E_n$ is dense in $E$. Let $\phi_n=\phi|(\Delta-K)\times E_n$ for each $n\geq 1$. Consider the holomorphic motion $\phi_n\colon(\Delta-K)\times E_n\to\hat{\mathbb{C}}$; it can be extended to a continuous motion $\hat{\phi}_n\colon(\Delta-K)\times\hat{\mathbb{C}}\to\hat{\mathbb{C}}$. So, by Case 1, $\phi_n$ can be extended to a holomorphic motion $\phi_{n,0}\colon\Delta\times E_n\to\hat{\mathbb{C}}$.

Let $E_\infty=\bigcup E_n$. For $(\lambda,z)\in\Delta\times E_\infty$, let $\phi_0(\lambda,z)=\phi(\lambda,z)$ when $\lambda\notin K$. For any $z\in E_\infty$, there exists $n\in\mathbb{N}$ such that $z\in E_n$. We set $\phi_0(\lambda,z)=\phi_{n,0}(\lambda,z)$ for $\lambda\in K$. The definition of $\phi_0$ on $\Delta\times E_\infty$ is well-defined. In fact, if $z\in E_m$ for $n<m$, $\phi_m$ extends $\phi_n$ implies that $\phi_m(\lambda,z)=\phi_n(\lambda,z)$ for $\lambda\notin K$. For each $\lambda\in K$ we take a

sequence $\{\lambda_k\}_{k=1}^{\infty} \subset \Delta - K$ converging to $\lambda$ and consider the limits $\lim_{k\to\infty} \phi_{n,0}(\lambda_k, z)$ and $\lim_{k\to\infty} \phi_{m,0}(\lambda_k, z)$. Obviously, both limits coincide and do not depend on choice of the sequence. Thus, we have $\phi_{m,0}(\lambda, z) = \phi_{n,0}(\lambda, z)$ for $(\lambda, z) \in K \times E_\infty$, which shows that $\phi_0$ is well-defined.

Now, we show that $\phi_0$ is a holomorphic motion of $\Delta \times E_\infty$. It is easily seen that $\phi_0(\,\cdot\,, z)$ is holomorphic on $\Delta$ for each fixed $z \in E_\infty$. We check injectivity. For $z$, $z'$ in $E_\infty$, where $z \neq z'$, there exists $n \in \mathbb{N}$ such that $z$, $z'$ are in $E_n$. Now, $\phi_0(\lambda, z) = \phi_{n,0}(\lambda, z) \neq \phi_{n,0}(\lambda, z') = \phi_0(\lambda, z')$. We have therefore shown that $\phi_0\colon \Delta \times E_\infty \to \hat{\mathbb{C}}$ is a holomorphic motion.

Finally, by the $\lambda$-lemma in [20], it follows that $\phi_0$ can be extended to a holomorphic motion (still called) $\phi_0\colon \Delta \times E \to \hat{\mathbb{C}}$.

Now, we consider the case where $K$ is not $AB$-removable. We may assume that $\Delta - K \ni 0$ and $E = \{0, 1, z_0, \infty\}$ for some $z_0 \neq 0, 1, \infty$. Let $\eta$ be a holomorphic quadratic differential on $X := \hat{\mathbb{C}} - E$ with $\|\eta\| = 1$, where $\|\eta\| = \sup_{z\in X} \rho(z)^{-2}|\eta(z)|$ for the hyperbolic metric $\rho$ of $X$.

Since $K$ is not $AB$-removable, there exists a bounded holomorphic function $f$ on $\Delta - K$ such that it cannot be extended to a holomorphic function on $\Delta$. We may assume that $f(0) = 0$ and $|f(\lambda)| < 1$ for each $\lambda \in \Delta - K$. Then, we define a holomorphic map $F\colon \Delta - K \to M(\mathbb{C})$ by

$$F(\lambda) = f(\lambda)\frac{\bar{\eta}}{|\eta|} \quad (\lambda \in \Delta - K)$$

and a holomorphic motion $\Psi_f\colon (\Delta - K) \times E \to \hat{\mathbb{C}}$ by

$$\Psi_f(\lambda, z) = w^{F(\lambda)}(z) \quad (z \in E).$$

Obviously, the holomorphic motion $\Psi_f$ can be extended to a holomorphic motion $\hat{\Psi}_f(\lambda, \zeta) = w^{F(\lambda)}(\zeta)$ on $(\Delta - K) \times \hat{\mathbb{C}}$.

Suppose that $\Psi_f$ can be extended to a holomorphic motion $\tilde{\Psi}_f\colon \Delta \times E \to \hat{\mathbb{C}}$. Then, we have a holomorphic map $G\colon \Delta \to T(E) = \mathit{Teich}(0, 4)$ such that

$$\tilde{\Psi}_f(\lambda, z) = \Psi_E(G(\lambda), z) \tag{5.1}$$

for every $(\lambda, z) \in \Delta \times E$. Since $\dim_{\mathbb{C}} \mathit{Teich}(0, 4) = 1$, the Teichmüller space $\mathit{Teich}(0, 4)$ is biholomorphic to the Teichmüller space of $X$; and

$$\mathit{Teich}(X) = \left\{ \lambda\frac{\bar{\eta}}{|\eta|} \;\middle|\; \lambda \in \Delta \right\}$$

by Teichmüller's theorem. Hence, the map $G$ gives a unique map $g$ from $\Delta$ to itself such that

$$G(\lambda) = P_E\left( g(\lambda)\frac{\bar{\eta}}{|\eta|} \right) \quad \text{for all} \quad \lambda \in \Delta. \tag{5.2}$$

Since $G$ is holomorphic and $P_E$ is a holomorphic split submersion, (5.2) implies that $g$ is a holomorphic function on $\Delta$.

Now, (5.1), (5.2), and the definition of $P_E$ imply that

$$\tilde{\Psi}_f(\lambda, z) = w^{g(\lambda)\bar{\eta}/|\eta|}(z)$$

for all $(\lambda, z) \in \Delta \times E$. Since the holomorphic motion $\tilde{\Psi}_f$ extends $\Psi_f$, it follows by Teichmüller's uniqueness theorem that

$$f(\lambda) = g(\lambda)$$

for $\lambda \in \Delta - K$ which implies that $g$ extends $f$. This is a contradiction. $\square$

## 6. Proof of Theorem 3

Let $G$ be a group of Möbius transformations that map $E$ onto itself. For each $g$ in $G$, there exists a biholomorphic map $\rho_g \colon T(E) \to T(E)$ (also called a "geometric isomorphism" induced by $g$) which is defined as follows: for each $\mu$ in $M(\mathbb{C})$,

$$\rho_g([w^\mu]_E) = [\hat{g} \circ w^\mu \circ g^{-1}]_E$$

where $\hat{g}$ is the unique Möbius transformation such that $\hat{g} \circ w^\mu \circ g^{-1}$ fixes the points 0, 1, and $\infty$. See Remark 3.4 in [11] for a discussion on "geometric isomorphisms" of $T(E)$.

It follows from the definition that, for each $g$ in $G$, $\rho_g$ is basepoint preserving.

We need the following

**Lemma 6.1.** *Let $B$ be a path-connected topological space and $f$, $g$ be continuous maps from $B$ to $T(E)$ satisfying*:

(i) $\Psi_E(f(t), e) = \Psi_E(g(t), e)$ *for all $e$ in $E$, and*

(ii) $f(t_0) = g(t_0)$ *for some $t_0$ in $B$,*

*then $f(t) = g(t)$ for all $t$ in $B$.*

For a proof see Lemma 12.2 in [21].

In the next proposition, let $V$ be a simply connected complex Banach manifold with a basepoint $x_0$. If $\phi\colon V \times E \to \hat{\mathbb{C}}$ is a holomorphic motion, by Theorem 3.1, there exists a unique basepoint preserving holomorphic map $f\colon V \to T(E)$ such that $f^*(\Psi_E) = \phi$.

Let $G$ be a group of Möbius transformations that map $E$ onto itself. Recall the definition of $G$-equivariance in Equation (1.2).

**Proposition 6.2.** *The holomorphic motion $\phi\colon V \times E \to \hat{\mathbb{C}}$ is $G$-equivariant if and only if $f$ maps $V$ into the set of points in $T(E)$ that are fixed by $\rho_g$ for each $g$ in $G$.*

*Proof.* Suppose $f$ maps $V$ into the set of points in $T(E)$ that are fixed by $\rho_g$ for all $g$ in $G$. Let $g \in G$, $x \in V$, and $f(x) = P_E(\mu)$. So, $\phi(x, z) = \Psi_E(f(x), z) = w^\mu(z)$ for all $z$ in $E$.

Now, $\rho_g(f(x)) = f(x)$ implies that

$$[w^\mu]_E = [\theta_x(g) \circ w^\mu \circ g^{-1}]_E$$

where $\theta_x(g)$ is the unique Möbius transformation such that $\theta_x(g) \circ w^\mu \circ g^{-1}$ fixes 0, 1, and $\infty$. This means that $\theta_x(g) \circ w^\mu \circ g^{-1} = w^\mu$ on $E$. Therefore, we have

$$\theta_x(g)(w^\mu(z)) = w^\mu(g(z)) \quad \text{for all} \quad z \in E.$$

We conclude that $\phi(x, g(z)) = \theta_x(g)(\phi(x, z))$ for all $z$ in $E$, and so, $\phi$ satisfies Equation 1.2.

Next, suppose the holomorphic motion $\phi$ satisfies Equation 1.2. Let $x \in V$ and $f(x) = [w^\mu]_E$. For $x \in V$, and $g \in G$, there exists a Möbius transformation $\theta_x(g)$ such that

$$\phi(x, g(z)) = \theta_x(g)(\phi(x, z)) \quad \text{for all} \quad z \in E.$$

Since $f(x) = [w^\mu]_E$, we have $\phi(x, g(z)) = w^\mu(g(z))$ for all $z$ in $E$. Therefore, $w^\mu(g(z)) = \theta_x(g)(w^\mu(z))$ for all $z \in E$. We conclude that $w^\mu = \theta_x(g) \circ w^\mu \circ g^{-1}$ on $E$. Since the quasiconformal map $w^\mu$ fixes 0, 1, and $\infty$, it follows that $\theta_x(g) \circ w^\mu \circ g^{-1}$ fixes 0, 1, and $\infty$.

By definition of $\rho_g$, we have

$$\rho_g([w^\mu]_E) = [\hat{g} \circ w^\mu \circ g^{-1}]_E$$

where $\hat{g}$ is the unique Möbius transformation such that $\hat{g} \circ w^\mu \circ g^1$ fixes 0, 1, and $\infty$. It follows that $\hat{g} = \theta_x(g)$. Therefore, we have

$$f(x) = [w^\mu]_E$$

and

$$\rho_g(f(x)) = [\theta_x(g) \circ w^\mu \circ g^{-1}]_E.$$

Since $f$ and $\rho_g$ are both basepoint preserving, we have $f(x_0) = \rho_g(f(x_0))$. And since $w^\mu = \theta_x(g) \circ w^\mu \circ g^{-1}$ on $E$, we have $\Psi_E(f(x), z) = \Psi_E(\rho_g(f(x)), z)$ for all $z$ in $E$. It follows by Lemma 6.1 that $f(x) = \rho_g(f(x))$ for any $x$ in $V$. This means, that $f$ maps $V$ into the set of points in $T(E)$ that are fixed by $\rho_g$ for each $g$ in $G$. □

**Proposition 6.3.** *If $\tau$ is in $T(E)$ such that $\rho_g(\tau) = \tau$ for every $g$ in $G$, then $s(\tau) = \mu$ satisfies*

$$(\mu \circ g)\frac{\overline{g'}}{g'} = \mu \quad \textit{for each} \quad g \in G. \tag{6.1}$$

The proof follows easily from the construction of the map $s\colon T(E) \to M(\mathbb{C})$ in Proposition 2.3.

We need the following simple lemma. Let $B$ be a path-connected topological space and $\mathcal{H}(\hat{\mathbb{C}})$ be the group of homeomorphisms of $\hat{\mathbb{C}}$ onto itself, with the topology of uniform convergence in the spherical metric.

**Lemma 6.4.** *Let $h\colon B \to \mathcal{H}(\hat{\mathbb{C}})$ be a continuous map such that $h(t)(e) = e$ for all $t$ in $B$ and for all $e$ in $E$. If $h(t_0)$ is isotopic to the identity* rel $E$ *for some fixed $t_0$ in $B$, then $h(t)$ is isotopic to the identity* rel $E$ *for all $t$ in $B$.*

For a proof see Lemma 12.1 in [21].

We are now ready to prove Theorem 3.

Proof of Theorem 3. By Theorem 3.2, there exists a unique basepoint preserving holomorphic map $f\colon V \to T(E)$ such that $f^*(\Psi_E) = \phi$. Since $\phi$ is $G$-equivariant, it follows by Proposition 6.2 , that $f$ maps $V$ into the set of points in $T(E)$ that are fixed by $\rho_g$ for each $g$ in $G$. If $f(x) = \tau$, then by Proposition 6.3, it follows that $s(\tau) = \mu$ where $\mu$ satisfies Equation (6.1).

Define $\tilde{f} = s \circ f$ and let $\tilde{\phi}(x, z) = w^{\tilde{f}(x)}(z)$ for all $(x, z) \in V \times \hat{\mathbb{C}}$. Since $\tilde{f}\colon V \to M(\mathbb{C})$ is a continuous map, it follows by Proposition 1.5 that $\tilde{\phi}$ is a quasiconformal motion.

Also, $\tilde{\phi}$ extends $\phi$, because for all $(x, z) \in V \times E$, we have

$$\tilde{\phi}(x, z) = w^{\tilde{f}(x)}(z) = \Psi_E(P_E(s(f(x))), z) = \Psi_E(f(x), z) = \phi(x, z).$$

This proves (1).

Since $s(f(x)) = \mu$ satisfies Equation (6.1), it follows that for each $g$ in $G$, $w^\mu \circ g \circ (w^\mu)^{-1}$ is a Möbius transformation that depends on $g$ and on $\mu$ (and therefore on $x$ in $V$). So, we write this Möbius transformation as $\theta_x(g)$. We therefore have, $w^\mu(g(z)) = \theta_x(g)(w^\mu(z))$ for all $z$ in $\hat{\mathbb{C}}$. Hence, we conclude that $\tilde{\phi}(x, g(z)) = \theta_x(g)(\tilde{\phi}(x, z))$ for all $(x, z)$ in $V \times \hat{\mathbb{C}}$ i.e. $\tilde{\phi}$ is $G$-equivariant. This proves (2).

Finally, define maps $f$ and $g$ from $\mathcal{H}(\hat{\mathbb{C}})$ by $f(x)(z) = \tilde{\phi}(x, z)$ and $g(x)(z) = \hat{\phi}(x, z)$ for $x$ in $V$ and $z$ in $\hat{\mathbb{C}}$. Since $\tilde{\phi}$ is a quasiconformal motion, by Proposition 1.7, $\tilde{\phi}$ is also a continuous motion. So, both $\tilde{\phi}$ and $\hat{\phi}$ are continuous maps. Hence, by Theorem 5 in [2], the maps $f$ and $g$ are continuous. Therefore, the map $h\colon V \to \mathcal{H}(\hat{\mathbb{C}})$ defined by $h(x) = g(x)^{-1} \circ f(x)$ for $x$ in $V$, is continuous. Clearly, $h(x_0)$ is the identity map on $\hat{\mathbb{C}}$. Since both $\tilde{\phi}$ and $\hat{\phi}$ extend $\phi$, $h(x)$ fixes $E$ pointwise, for every $x$ in $V$. Hence, by Lemma 6.4, it follows that $h(x)$ is isotopic to the identity rel $E$ for each $x$ in $V$. This proves (3). □

Proof of Corollary 1. If $V$ is simply connected, by Theorem 3.1, there must always exist a basepoint preserving holomorphic map $f\colon V \to T(E)$ such that $f^*(\Psi_E) =$

$\phi$. Hence, if $\phi\colon V \times E \to \hat{\mathbb{C}}$ is a holomorphic motion satisfying Equation (1.2), there will always be a quasiconformal motion $\tilde{\phi}\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\phi}$ extends $\phi$ and also satisfies Equation (1.2). □

## 7. Proof of Theorem 4

The proof of (i) is easy; we follow exactly the first part of the arguments in the proof of Theorem 1 of [12].

For (ii), it clearly suffices to prove the theorem when $V$ is simply connected. Also, by considering $\theta_x \circ \theta_t^{-1}$, we may assume that $\theta_t = id$. Then, $\phi$ is a holomorphic motion of $E$ over $V$ with basepoint $t$. Hence, by Corollary 1, there exists a quasiconformal motion $\tilde{\phi}\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that:

(i) $\tilde{\phi}$ extends $\phi$, and

(ii) $\tilde{\phi}_x(g(z)) = \theta_x(g)(\tilde{\phi}_x(z))$ for all $z$ in $\hat{\mathbb{C}}$.

Also, by Proposition 1.5, for each $x \in V$, $\tilde{\phi}_x\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map. This means $\theta_x$ is a quasiconformal deformation of $G$ for each $x$ in $V$. □

Proof of Corollary 2. We may assume that $\theta_t = id$ and $V$ is simply connected. Let $E$ be the set of fixed points of loxodromic elements of $G$. For each $z \in E$, there exists a primitive loxodromic element $g \in G$ such that $z$ is the attracting fixed point of $g$. Let us denote the attracting fixed point of a loxodromic element $g \in \mathrm{PSL}(2, \mathbb{C})$ by $\alpha[g]$. Then, for each $x \in V$, we define

$$\phi(x, \alpha[g]) = \alpha[\theta_x(g)]$$

for each $z = \alpha[g] \in E$. Since $\theta_x(G)$ is discrete, for distinct primitive loxodromic elements $g, g' \in G$, we have $\alpha[g] \neq \alpha[g']$ and $\alpha[\theta_x(g)] \neq \alpha[\theta_x(g')]$. Therefore, $\phi$ is a holomorphic motion of $E$ over $V$.

Furthermore, $\phi_x(z)$ induces $\theta_x$. Indeed, for $g \in G$ and for $\alpha[h] \in E$ $(h \in G)$,

$$\begin{aligned}\phi_x(g(\alpha[h])) &= \phi_x(\alpha[g \circ h \circ g^{-1}]) \\ &= \alpha[\theta_x(g \circ h \circ g^{-1})] = \alpha[\theta_x(g) \circ \theta_x(h) \circ \theta_x(g)^{-1}] \\ &= \theta_x(g)(\alpha[\theta_x(h)]) = \theta_x(g)(\phi_x(\alpha[h]).\end{aligned}$$

Therefore the conclusion follows from Theorem 4. □

The following proposition generalizes Proposition 2 in [4], and also Theorem 3 in [27].

Let $V$ be a simply connected complex Banach manifold with basepoint $x_0$. Let

$$U = \left\{x \in V\colon \rho_V(x, x_0) < \rho_\Delta\left(0, \frac{1}{3}\right)\right\}$$

where $\rho_V$ is the Kobayashi metric on $V$ and $\rho_\Delta$ is the Poincaré metric on $\Delta$.

Let $G$ be a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ and let $E$ be a closed subset of $\hat{\mathbb{C}}$ (as usual, $0, 1, \infty$ belong to $E$) that is invariant under $G$.

**Proposition 7.1.** *Suppose that the holomorphic family $\{\phi_x\}_{x \in V}$ of injections of $E$ induces the holomorphic family $\{\theta_x\}_{x \in V}$ of isomorphisms of $G$. If $\phi_{x_0} = id$, then there exists a holomorphic family $\{\tilde{\phi}_x\}$ of quasiconformal self-maps of $\hat{\mathbb{C}}$ defined over $U$ such that $\tilde{\phi}_{x_0} = id$ and $\tilde{\phi}_x$ induces $\theta_x$ for each $x \in U$.*

Proof. By Theorem B in [21], there exists a *unique* holomorphic motion $\tilde{\phi}\colon U \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\phi}(x, z) = \phi(x, z)$ for all $(x, z) \in U \times E$ with the following properties:

(i) $\tilde{\phi}_x\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map for each $x$ in $U$,

(ii) the Beltrami coefficient of $\tilde{\phi}_x$ depends holomorphically with respect to $x$ for each $x$ in $U$, and

(iii) the Beltrami coefficient of $\tilde{\phi}_x$ is harmonic in each component of $\hat{\mathbb{C}} \setminus E$ for each $x$ in $U$.

We now follow Bers' arguments in [4]. For some $g \in G$, let $\tilde{F}_x = \theta_x(g)^{-1} \circ \tilde{\phi}_x \circ g$ for each $x$ in $U$. Then, $\{\tilde{F}_x\}$ is a holomorphic family of quasiconformal self-maps of $\hat{\mathbb{C}}$, defined over $U$ and $\tilde{F}_0 = id$.

We are given that $\phi_x(g(z)) = \theta_x(g)(\phi_x(z))$ for all $z \in E$. Therefore, for all $z$ in $E$, we have $\tilde{F}_x(z) = \theta_x(g)^{-1}(\tilde{\phi}_x(g(z))) = \theta_x(g)^{-1}(\phi_x(g(z)))$ (since $\tilde{\phi}_x(z) = \phi_x(z)$ for all $z$ in $E$) which is equal to $\phi_x(z)$.

Let the Beltrami coefficient of $\tilde{F}_x$ be $\tilde{\mu}_x$. It can be easily shown that $\tilde{\mu}_x$ is harmonic on each component of $\hat{\mathbb{C}} \setminus E$. Therefore, by the uniqueness part of Theorem B in [21], it follows that $\tilde{F}_x = \tilde{\phi}_x$ for every $g \in G$ and for all $x \in U$. Therefore, $\theta_x(g) = \tilde{\phi}_x \circ g \circ \tilde{\phi}_x^{-1}$ for each $x \in U$ and for all $g \in G$. □

REMARK 7.2. If $E$ is not a closed set we can use Theorem 2 in [18] to extend $\phi$ to a holomorphic motion of $\overline{E}$ (the closure of $E$) over $V$.

REMARK 7.3. We can follow Bers' methods in [4] and use Proposition 7.1 to give another proof of Theorem 4. However, we want to emphasize that the statements of Corollary 1 and of Theorem 4 for a simply connected $V$ imply a global property like Slodkowski's theorem; that means, there exists a quasiconformal motion $\tilde{\phi}\colon V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that extends the given holomorphic motion $\phi$.

## 8. Examples

EXAMPLE 8.1. Let $X_0$ be a Riemann surface that admits no non-constant bounded holomorphic functions, and let $f$ be a non-constant meromorphic function on $X_0$. Fix a point $x_0 \in X_0$ as a basepoint. Let $E_0 = \{0, 1, \infty, a_1, \dots, a_n\}$ be any finite set. We may assume that $f(x_0) \notin E_0$. Then put $\Lambda = f^{-1}(E_0)$. The set $\Lambda$, which is possibly an empty set, is a discrete subset of $X_0$. Since $X_0$ admits no non-constant bounded holomorphic function, $X := X_0 \setminus \Lambda$ also admits no non-constant bounded holomorphic functions. For $E = E_0 \cup \{f(x_0)\}$, we define a holomorphic motion $\phi\colon X \times E \to \hat{\mathbb{C}}$ by

$$\phi(x, z) = \begin{cases} z & (z \in E), \\ f(x) & (z \notin E). \end{cases}$$

Since $f$ is non-constant, the motion is non-trivial. But Theorem 1 guarantees that $\phi$ cannot be extended to a holomorphic motion of $\hat{\mathbb{C}}$ over $X$.

EXAMPLE 8.2. In Theorem 2, we gave equivalent conditions for a holomorphic motion $\phi$: $(\Delta - K) \times E \to \hat{\mathbb{C}}$ to be extended to a holomorphic motion $\phi_0$: $\Delta \times E \to \hat{\mathbb{C}}$. In this example, we shall show that the holomorphicity of $\phi$ cannot be relaxed by giving a counter-example. We construct an example of a continuous motion $\phi\colon \Delta^* \times E \to \hat{\mathbb{C}}$, which can be extended to a continuous motion $\hat{\phi}\colon \Delta^* \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, but $\phi$ cannot be extended to a continuous motion $\tilde{\phi}\colon \Delta \times E \to \hat{\mathbb{C}}$.

Let $E = \{0, 1, \infty, 1/3\}$. We define $\phi(\lambda, 0) = 0$, $\phi(\lambda, 1) = 1$ and $\phi(\lambda, \infty) = \infty$, for $\lambda \in \Delta^*$. And for $(\lambda, 1/3) \in \Delta^* \times \{1/3\}$, $\lambda = re^{i\theta}$, $0 < r < 1$, we define $\phi(\lambda, 1/3) = re^{i\theta} 1/3$ for $0 \le \theta \le \pi$, and $\phi(\lambda, 1/3) = re^{i(2\pi - \theta)} 1/3$ for $\pi \le \theta \le 2\pi$.

It is easy to check that $\phi\colon \Delta^* \times E \to \hat{\mathbb{C}}$ is a continuous motion. Also, $\phi$ cannot be extended to a continuous motion $\tilde{\phi}\colon \Delta \times E \to \hat{\mathbb{C}}$.

We now construct a continuous motion $\hat{\phi}\colon \Delta^* \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that extends $\phi$. For $0 < |z| \le 1/3$, we define $\hat{\phi}(\lambda, z) = re^{i\theta} z$ for $0 \le \theta \le \pi$, and $\hat{\phi}(\lambda, z) = re^{i(2\pi - \theta)} z$ for $\pi \le \theta \le 2\pi$.

For all $|z| \ge 2/3$, set $\hat{\phi}(\lambda, z) = z$.

Finally, for $1/3 < |z| < 2/3$, we define $\hat{\phi}(re^{i\theta}, z)$ as follows: for $0 \le \theta \le \pi$, define

$$\hat{\phi}(re^{i\theta}, z) = r^{2-3|z|} \exp\left( i \left( \frac{-\theta}{\log 2} \left( \log |z| - \log \frac{2}{3} \right) \right) \right) z$$

and for $\pi \le \theta \le 2\pi$ define

$$\hat{\phi}(re^{i\theta}, z) = r^{2-3|z|} \exp\left( i \left( -\frac{2\pi - \theta}{\log 2} \left( \log|z| - \log \frac{2}{3} \right) \right) \right) z.$$

It can be checked that $\hat{\phi}\colon \Delta^* \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a continuous motion that extends the given continuous motion $\phi\colon \Delta^* \times E \to \hat{\mathbb{C}}$.

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