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<th><strong>Title</strong></th>
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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF CERTAIN NON-AUTONOMOUS DIFFERENTIAL EQUATIONS

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1. Introduction

In this paper conditions are obtained under which all solutions of certain real non-autonomous nonlinear differential equations tend to zero as \( t \to \infty \).

Theorem 1 is concerned with the system of differential equations;

\[
\dot{x} = A(t)x + f(t, x)
\]

where \( x, f \) are \( n \)-dimensional vectors, \( A(t) \) is a bounded continuously differentiable \( n \times n \) matrix for \( t \geq 0 \), and \( f(t, x) \) is continuous in \((t, x)\) for \( t \geq 0, \|x\| < \infty \), here \( \| \| \) denotes the Euclidean norm.

Theorem 2 is concerned with the differential equation of the third order;

\[
\dddot{x} + a(t)f(x, \dot{x}, \ddot{x}) + b(t)g(x, \dot{x}) + c(t)h(x) = p(t, x, \dot{x}, \ddot{x})
\]

where \( a(t), b(t), c(t) \) are positive continuously differentiable and \( f, g, h, p \) are continuous real-valued functions depending only on the arguments shown, and the dots indicate the differentiation with respect to \( t \).

The asymptotic property of solutions of third order differential equations has received a considerable amount of attention during the past two decades, particularly when (1.2) is autonomous. Many of these results are summarized in [11].

A few authors have studied non-autonomous third order differential equations. K. E. Swick [13] considered the following equations

\[
\dddot{x} + p(t)\dddot{x} + q(t)g(\dot{x}) + r(t)h(x) = 0,
\]

\[
\dddot{x} + f(x, \dot{x}, t)\dddot{x} + q(t)g(\dot{x}) + r(t)h(x) = 0,
\]

with the assumption that \( q(t), r(t) \) are positive, bounded and monotone decreasing.

In [6], the author studied the asymptotic behavior of the solutions of the equation

Hara, T.
Osaka J. Math.
12 (1975), 267–282
\[ \ddot{\mathbf{x}} + a(t)f(x, \dot{x}) + \dot{b}(t)g(x, \dot{x}) + c(t)h(x) = e(t) \]

under the assumptions that \(|a'(t)|, |b'(t)|, |c'(t)|\) and \(|e(t)|\) are integrable and suitable conditions on \(f, g, h\). Here we assume the condition

\[ \limsup_{(t, x) \rightarrow (\omega, \omega)} \int_{t}^{t+\nu} \left\{|a'(s)| + b'(s) + c'(s)|\right\} ds < \gamma \]

where \(\gamma\) is a sufficiently small positive constant.

Conditions on \(p(t, x, \dot{x}, \ddot{x})\) are also relaxed, for example \(p(t, x, y, z)\) may be unbounded for \(x^2 + y^2 + z^2\).

Recently H. O. Tejumola [15] established conditions under which all solutions of equations of the form

\[ \ddot{\mathbf{x}} + f(x, \dot{x}, x) + g(x, \dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}) \]

tend to zero as \(t \rightarrow \infty\). Theorem 2 develops Tejumola's result [15; Theorem 1] to the non-autonomous equations of the form (1.2).

The main tools used in this work are Theorem A and Liapunov functions. Theorem A would be especially convenient to study the non-autonomous differential equations.

2. Auxiliary theorem

Consider a system of differential equations

\[ x = F(t, x) \]

where \(x\) and \(F\) are ra-dimensional vectors.

**Theorem A.** Suppose that \(F(t, x)\) of (2.1) is continuous in \(I \times \mathbb{R}^n(= [0, \infty))\) and that there exists a Liapunov function \(V(t, x)\), continuously differentiable in \(I \times \mathbb{R}^n\), satisfying the following conditions;

(i) \(a(||x||) \leq V(t, x) \leq b(||x||)\), where \(a(r) \in \text{CIP} (\text{the family of continuous and increasing positive definite functions})\), \(a(r) \rightarrow \infty\) as \(r \rightarrow \infty\) and \(b(r) \in \text{CIP}\),

(ii) \(\dot{V}_{(a,b)}(t, x) = \limsup_{k \rightarrow +0} \frac{1}{L} \{V(t + h, x + hF(t, x)) - V(t, x)\} \leq -cV(t, x) + \lambda_1(t) V(t, x) + \lambda_2(t)(1 + V(t, x))\),

where \(c > 0\) is a constant and \(\lambda_i(t) \geq 0\) \((i = 1, 2)\) are continuous functions satisfying

\[ \limsup_{(t, t) \rightarrow (\omega, \omega)} \int_{t}^{t+\nu} \lambda_1(s) ds < c, \]

\[ \limsup_{(t, t) \rightarrow (\omega, \omega)} \int_{t}^{t+\nu} \lambda_2(s) ds < c, \]

\[ \limsup_{(t, t) \rightarrow (\omega, \omega)} \int_{t}^{t+\nu} \lambda_3(s) ds < c, \]
ASYMPTOTIC BEHAVIOR OF DIFFERENTIAL EQUATIONS

(2.3) \[ \int_{t}^{t+1} \lambda_2(s) ds \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty. \]

Then, any solution \( x(t) \) of (2.1) is uniform-bounded and satisfies \( x(t) \rightarrow 0 \) as \( t \rightarrow \infty. \)

The following is an immediate consequence of Theorem A.

**Corollary.** Under the assumptions in Theorem A, if

\[(2.2') \quad \limsup_{(t, s) \rightarrow (\infty, \infty)} \int_{t}^{s} L(\tau) d\tau < 0,\]

then any solution \( x(t) \) of (2.1) is uniform bounded and satisfies \( x(t) \rightarrow 0 \) as \( t \rightarrow \infty. \)

**REMARK.** The condition (2.3) can be replaced by

\[(2.3') \quad \int_{t}^{t+1} \lambda_2(s) ds \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty\]

where \( \tau_0 > 0 \) is an arbitrary constant.

3. **Proof of Theorem A**

Proof of Theorem A. Let \( \varepsilon > 0 \) be chosen such that \( \frac{\varepsilon}{2} > \varepsilon \) and

\[(3.1) \quad \limsup_{(t, s) \rightarrow (\infty, \infty)} \int_{t}^{s} \lambda_2(\tau) d\tau \leq c - 3\varepsilon.\]

Let \( T_0 > 0 \) such that

\[ \int_{t}^{s} \lambda_2(\tau) d\tau \leq \limsup_{(t, s) \rightarrow (\infty, \infty)} \int_{t}^{s} \lambda_2(\tau) d\tau + \varepsilon \quad \text{for} \quad t \geq T_0 \]

and \( s \geq T_0 + t. \) \( T_0 \) does not depend on \( t \) and \( s. \)

Using (3.1) and above inequality, we have

\[(3.2) \quad \int_{s-t}^{s} \lambda_2(\tau) d\tau \leq c - 2\varepsilon \quad \text{for} \quad t \geq T_0 \quad \text{and} \quad s \geq T_0 + t.\]

Let \( K > 0 \) be a constant satisfying \( \exp\{c T_0 + \int_{0}^{T_0} \lambda_2(\tau) d\tau\} \leq K. \) Then it is easy to show that for all \( s \geq t \geq 0, \) we have

\[(3.3) \quad e^{-c(s-t)} + \int_{t}^{s} \lambda_2(\tau) d\tau \leq Ke^{-2\varepsilon(s-t)}.\]
Now we consider the function $U(t, x)$ defined by

$$U(t, x) = V(t, x)e^{-\int_t^\infty e^{cs} e^{-c(s-t)+\int_s^\infty \lambda_1(r) dr} ds},$$

and show that

$$e^{-c} a(||x||) \leq U(t, x) \leq \frac{K^*}{c} b(||x||) \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}^n.$$

To verify the right-hand side inequality in (3.5), we use the inequality (3.3). Then we obtain

$$e^{-cs} \leq e^{-\int_t^\infty e^{cs} e^{-c(s-t)+\int_s^\infty \lambda_1(r) dr} ds} \leq e^{-ct} e^{s/\lambda_1(\infty)}.$$

A short calculation shows the left-hand side inequality in (3.5), i.e.

$$e^{-c} V(t, x) \leq U(t, x) \leq \frac{K^*}{c} V(t, x)$$

and using the hypothesis (i) of Theorem A, we obtain (3.5).

From (3.4) it follows that

$$\dot{U}_{(x)}(t, x) = \dot{V}_{(x)}(t, x)e^{-\int_t^\infty e^{cs} e^{-c(s-t)+\int_s^\infty \lambda_1(r) dr} ds}$$

$$\quad - \epsilon V(t, x)e^{-\int_t^\infty e^{cs} e^{-c(s-t)+\int_s^\infty \lambda_1(r) dr} ds}$$

$$\quad + \{c-\lambda_1(t)\} V(t, x)e^{-\int_t^\infty e^{cs} e^{-c(s-t)+\int_s^\infty \lambda_1(r) dr} ds} - V(t, x)$$

$$\quad \leq \{c-\lambda_1(t)\} V(t, x) + \lambda_1(t)(1+V(t, x)) \} e^{-\int_t^\infty e^{cs} e^{-c(s-t)+\int_s^\infty \lambda_1(r) dr} ds}$$

$$\quad - \epsilon U(t, x) + \{c-\lambda_1(t)\} U(t, x) - V(t, x).$$

Using (3.6), we obtain
ASYMPTOTIC BEHAVIOR OF DIFFERENTIAL EQUATIONS

\[(3.9) \quad \dot{U}(t,x) \leq -\varepsilon U(t,x) + \frac{K}{\varepsilon} \lambda_\alpha(t)(1 + V(t,x)).\]

From (3.8) and (3.9), we have

\[(3.10) \quad \dot{U}_{(2.1)}(t,x) \leq \left\{ -\varepsilon + \frac{K}{\varepsilon} \right\} U(t,x) + \frac{K}{\varepsilon} \lambda_\alpha(t).\]

Set

\[W(t) = U(t,x(t))\]

where \(x(t), x(t_0) = x_0\) is any solution of (2.1). Then the inequality (3.10) implies

\[\frac{d}{dt} W(t) \leq \left\{ -\varepsilon + \frac{K}{\varepsilon} \right\} W(t) + \frac{K}{\varepsilon} \lambda_\alpha(t).\]

This immediately gives

\[W(t) \leq e^{\int_{t_0}^t g(\tau)d\tau} + \int_{t_0}^t e^{\int_{\tau}^t g(\sigma)d\sigma} h(\sigma) d\sigma\]

where \(g(t) = -\varepsilon + \frac{K}{\varepsilon} \lambda_\alpha(t)\) and \(h(t) = \frac{K}{\varepsilon} \lambda_\alpha(t)\)

Using the hypothesis (2.3), we can choose a constant \(T > 0\) so that

\[\frac{K \varepsilon}{8} \cdot \frac{1}{t-t_0} \int_{t_0}^t \lambda_\alpha(\tau) d\tau \leq \frac{\varepsilon}{2} \quad \text{for} \quad t \geq t_n \quad \text{and} \quad t_n \geq T.\]

Let \(K > 0\) be a constant satisfying \(\exp \left\{ \frac{\varepsilon}{2} (1 + T) + \frac{K \varepsilon}{8} \int_{t_0}^T \lambda_\alpha(\tau) d\tau \right\} \leq K\).

Then for all \(t \geq t_0 \geq 0\) we have

\[(3.11) \quad U(t,x(t)) \leq K \varepsilon \left\{ b(||x_0||) e^{-\varepsilon(t-t_0)/2} + \int_{t_0}^t e^{-\varepsilon(t-s)/2} \lambda_\alpha(s) ds \right\}.\]

Using the left-hand side of (3.5), we find that all the solutions of (2.1) are uniform-bounded.

Furthermore the condition (2.3) implies that

\[U(t,x(t)) \to 0 \quad \text{as} \quad t \to \infty.\]

Therefore by the inequality (3.5) we have

\[x(t) \to 0 \quad \text{as} \quad t \to \infty.\]

Q.E.D.

Proof of Corollary. Let \(c > 0\) be an arbitrary positive constant. By assumption (ii)', we have
It now follows from (2.2)' that
\[
\limsup_{(t,e) \to (\infty, \infty)} \frac{1}{t} \int_t^{t+\nu} \{c+L(\tau)\} d\tau < c,
\]
which establishes the assumption of Theorem A, and thus the proof is completed.

Q.E.D.

4. Theorems

Let \( A(t) \) satisfy the hypothesis (i) of the following Theorem 1 and \( P(t) \) be a solution of the matrix equation

(4.1) \[ A^T(t)P(t) + P(t)A(t) = -I. \]

Notice that \( P(t) \) is bounded for bounded \( A(t) \).

The following propositions are due to J. R. Dickerson [2].

**Proposition A.** \[ x^TP(t)x \geq C \|x\|^2, \] where \( C \) is a positive constant.

**Proposition B.** \[ \|x^T \dot{P}(t)x\| \leq 2 \|A(t)\| \|P(t)\| \|x^T P(t)x\|, \] where \( \dot{P}(t) \) and \( A(t) \) denote the time derivative of matrices \( P(t) \) and \( A(t) \) respectively.

**Theorem 1.** Suppose that the following conditions are satisfied;

(i) there exists a positive constant \( \tau_0 \) such that the real parts of all the eigenvalues of \( A(t) \) \( \leq -\tau_0 < 0 \) for all \( t \geq 0 \),

(ii) \[ \limsup_{(t,e) \to (\infty, \infty)} \frac{1}{t} \int_t^{t+\nu} \|A(s)\| ds < \frac{1}{2P_1^2} \]

where \( P_1 = \limsup_{t \to \infty} \|P(t)\| \),

(iii) \[ \|f(t, x)\| \leq \gamma(t)(1+\|x\|) \]

where \( \gamma(t) \) is a non-negative continuous function on \((0, \infty)\),

(iv) \[ \int_t^{t+1} \gamma(s) ds \to 0 \text{ as } t \to \infty \quad (i = 1, 2). \]

Then, all solutions \( x(t) \) of (1.1) are uniform-bounded and satisfy \( x(t) \to 0 \) as \( t \to \infty \).

**REMARK.** It may be shown by examples [16] that the smallness of \( \|A(t)\| \) is essential, even if the condition (i) is satisfied.

Next, we consider the equation (1.2) and assume that \( g(x, y), g_x(x, y), f(x, y, z), f_x(x, y, z) \) and \( f_z(x, y, z) \) are continuous for all \((x, y, z) \in \mathbb{R}^3\) and \( h(x) \)
is continuously differentiable for all \( x \in \mathbb{R}^3 \).

**Theorem 2.** Suppose that \( a(t), b(t), c(t) \) are continuously differentiable on \([0, \infty)\) and \( g(x, 0) = h(0) = 0 \) and the following conditions are satisfied:

(i) \( A \geq a(t) \geq a_0 > 0, \quad B \geq b(t) \geq b_0 > 0, \quad C \geq c(t) \geq c_0 > 0 \)

for \( t \in I = [0, \infty) \),

(ii) \( h(x)/x \geq 0 (x \neq 0) \),

(iii) \( f \geq f(x, y, z) \geq f_0 > 0 \) for all \( (x, y, z) \) and \( g \geq g(x, y) \geq g_0 > 0 \)

for all \( y \neq 0 \) and \( x \),

(iv) \( yf_x(x, y, 0) \leq 0, \quad yf_x(x, y, z) \geq 0, \quad g_x(x, y) \leq 0 \)

for all \( (x, y, z) \in \mathbb{R}^3 \),

(v) \( \frac{a_0b_0g_0^2}{c} h_1 \geq h'(x) \),

(vi) \( \delta > \frac{\mu_0}{4c_0} \left\{ A(f - f_0) + \frac{B}{\mu_1} (g - g_0) \right\} \)

where \( \mu_1 \) and \( \mu_2 \) are arbitrarily fixed constants satisfying

\[
\frac{Ch_1}{b_0g_0} < \mu_1 < a_0f_0, \quad 0 < \mu_2 < \frac{ab_0g_0 - Ch_1}{Af_0}
\]

(vii) \( \limsup_{(t, x, y, z) \to (\infty, 0, 0, 0)} \frac{1}{\gamma} \int_{t}^{t+\gamma} \left\{ |a'(s)| + b'(s) + |c'(s)| \right\} ds < \gamma \),

where \( \gamma \) is a small positive constant whose magnitude depends only on the constants appeared in (i)~(vi), and \( b'_*(t) = \max(b'(t), 0) \),

(viii) \( |p(t, x, y, z)| \leq p(t) \{1 + (x^2 + y^2 + z^2)^{1/2} \} + \Delta(x^2 + y^2 + z^2)^{1/2} \)

where \( \Delta \) is a positive constant and \( p(t) \) is a non-negative continuous function,

(ix) \( \int_{t}^{t+1} p(s) ds \to 0 \) as \( t \to \infty \).

Then there exists a finite constant \( \varepsilon = \varepsilon(A, a_0, B, b_0, C, c_0, \delta, f_0, g, g_0, h_1) > 0 \) such that if \( \Delta \leq \varepsilon \) then every solution \( x(t) \) of (1.2) is uniform-bounded and satisfies

\( x(t) \to 0, \quad \dot{x}(t) \to 0, \quad \ddot{x}(t) \to 0 \) as \( t \to \infty \).

**Remark.** It should be pointed out that in the special case \( f \equiv 1 \) (so that the assumption (iv) is automatically satisfied) Theorem 2 reduces to the author's earlier result [7; Theorem 2]. Also in another special case in which
\( a(t)f(x, yz) = \Pi, b(t)g(x, y) = by \) and \( c(t)h(x) = cx \) in (1.2) (so that all the conditions \((ii) \sim (iv)\) and \((vi)\) are trivially fulfilled) the hypothesis \((i)\) and \((v)\) reduce to

\[ a > 0, \ b > 0, \ c > 0, \ ab - c > 0 \]

which is the Routh-Hurwitz criterion for the asymptotic stability in the large of the zero solution of the equation

\[ \ddot{x} + a\dot{x} + b\dot{x} + cx = 0. \]

5. Proof of theorems

Proof of Theorem 1. We consider the Liapunov function

\[ (5.1) \quad V(t, x) = x^T P(t) x. \]

By virtue of Proposition A and the boundedness of \( P(t) \), there exist positive constants \( C \) and \( P_2 \) such that

\[ (5.2) \quad C \|x\|^2 \leq V(t, x) \leq P_2 \|x\|^2. \]

A simple calculation shows that

\[ \dot{V}_{(G.1)}(t, x) = \dot{x}^T P(t) x + x^T P(t) \dot{x} + x^T \dot{P}(t) x \]

\[ = -x^T \dot{x} + f^T(t, x) P(t) x + x^T P(t) f(t, x) + x^T \dot{P}(t) x. \]

Applying Proposition B to the function \( x^T P(t) x \), we obtain

\[ \dot{V}_{(G.1)}(t, x) \leq -\|x\|^2 + 2\|f(t, x)\| \|P(t)\| \|x\| \]

\[ + 2\|\dot{A}(t)\| \|P(t)\| \cdot x^T P(t) x \]

Using (5.1), (5.2) and (iii) of Theorem 1, we have

\[ \dot{V}_{(G.1)}(t, x) \leq \left\{ -\frac{2}{\|P(t)\|} + 2\|P(t)\| \|\dot{A}(t)\| \right\} V(t, x) \]

\[ + 2\|P(t)\| \gamma(t) \left\{ \left( \frac{V(t, x)}{C} \right)^{1/2} + \frac{V(t, x)}{C} \right\}. \]

We'll show that

\[ \limsup_{t \to +\infty} \frac{1}{t^v} \int_t^{t+v} \left\{ -\frac{1}{\|P(\tau)\|} + 2\|P(\tau)\| \|\dot{A}(\tau)\| \right\} d\tau < 0. \]

Let \( \rho_o \) be a positive number such that

\[ \rho_o = \frac{1}{2P_2^2} \limsup_{t \to +\infty} \int_t^{t+v} \|\dot{A}(\tau)\| d\tau. \]
Given \( \varepsilon > 0 \), there exists a positive number \( T \) such that \( \|P(\tau)\| < \frac{P_1 + \varepsilon}{P_1} \) for all \( \tau \geq T \). This implies also

\[
-\frac{1}{\|P(\tau)\|} < -\frac{1}{P_1} + \frac{\varepsilon}{P_1} < \frac{1}{\|P(\tau)\|} \quad \text{for all} \quad \tau \geq T.
\]

Using these inequalities, we have

\[
\limsup_{\tau \to \infty} \frac{1}{\tau} \left\{ \left( -\frac{1}{\|P(\tau)\|} + 2\|P(\tau)\| \|\dot{A}(\tau)\| \right) \right\} d\tau
\]

\[
= -2P_1 \rho_0 + \varepsilon \left\{ \frac{1}{P_1^2} + \limsup_{\tau \to \infty} \frac{2}{\tau} \|\dot{A}(\tau)\| d\tau \right\}.
\]

If \( \varepsilon \) is chosen so that

\[
\varepsilon \left\{ \frac{1}{P_1^2} + \limsup_{\tau \to \infty} \frac{2}{\tau} \|\dot{A}(\tau)\| d\tau \right\} < P_1 \rho_0,
\]

then we have

\[
\limsup_{\tau \to \infty} \frac{1}{\tau} \left\{ \left( -\frac{1}{\|P(\tau)\|} + 2\|P(\tau)\| \|\dot{A}(\tau)\| \right) \right\} d\tau < -P_1 \rho_0 < 0.
\]

Hence, the assumptions of Corollary hold and the proof of Theorem 1 is completed. Q.E.D.

Proof of Theorem 2. The equation (1.2) is equivalent to the system

\[
\begin{cases}
\dot{x} = y \\
\dot{y} = z \\
z = -a(t)f(x, y, z)z - b(t)g(x, y) - c(t)h(x) + p(t, x, y, z).
\end{cases}
\]

We consider the Liapunov function

\[
V(t, x, y, z) = V_1(t, x, y, z) + V_2(t, x, y, z) + V_3(t, x, y, z)
\]

where \( V_1, V_2 \) and \( V_3 \) are defined by

\[
2V_1 = 2\mu_1 c(t) \int_0^\infty h(\xi) d\xi + 2c(t) h(x)y + 2b(t)g(x, y) y
\]

\[
+ 2\mu_1 a(t) \int_0^\infty f(x, \eta, 0) \eta d\eta + 2\mu_2 x z + \mu_2 y z,
\]

\[
2V_2 = \mu_2 b(t) g(x)^2 + 2a(t) \int_0^\infty h(\xi) d\xi + \alpha(t) f_0 y^2
\]

\[
+ \mu_2 y^2 + 2b(t)g(x, \eta) \eta + x^2 + 2\mu_2 a(t) f_0 xy
\]

\[
+ 2\mu_1 x z + 2a(t) f_0 y z + 2c(t) h(x) y.
\]
and \( \mu_1 > 0, \mu_2 > 0 \) are two arbitrarily fixed constants such that

\[
\frac{Ch_1}{b_0 g_0} < \mu_1 < a_0 f_0, \quad 0 < \mu_2 < \frac{a_0 b_0 F_0 g_0 - Ch_1}{\Delta f_0}, \quad \delta > \frac{\mu_2}{\Delta f_0} \left\{ A f + B (g - g_0) \right\}.
\]

We shall prove the following two properties of \( V \):

\begin{align*}
(5.8) \quad & D_1 (x^2 + y^2 + z^2) \leq V(t, x, y, z) \leq D_2 (x^2 + y^2 + z^2) \\
(5.9) \quad & \mathcal{V}(x, y, z) \leq D_1 (x^2 + y^2 + z^2) + D_2 \left( |a'(t)| + b'_s(t) + |c'(t)| \right) (x^2 + y^2 + z^2) \\
& + D_2 \left( x^2 + y^2 + z^2 \right)^{\gamma_2} |p(t, x, y, z)|
\end{align*}

along any solution \((x(t), y(t), z(t))\) of (5.3), where \( D_1 \sim D_2 \) are certain positive constants.

At first we verify (5.8).

From the inequality \( \frac{Ch_1}{b_0 g_0} < \mu_1 < a_0 f_0 \), there exists a positive number \( \delta_0 \) such that \( \mu_0 b_0 g_0 (1 - \delta_0) > Ch_1 \) and \( a_0 f_0 (1 - \delta_0) \mu_1 \). Thus we have

\[
2V_1 = 2\mu_1 c(t) \int_0^x \left\{ 1 - \frac{h' \xi}{h_1} \right\} h(\xi) d\xi + \mu_1 c(h) \left( h(x) + \sqrt{h_1 y} \right)^2 \\
+ (\mu_2 y + z)^2 + 2 \int_0^\gamma \left\{ \mu_2 b(t) g(x, \eta) -(t) h_1 \eta \right\} d\eta \\
+ 2\mu_1 \int_0^{\eta} \left\{ a(t) f(x, \eta, 0) - \mu_1 \right\} \eta d\eta
\]

\[
= 2\mu_1 c(t) \int_0^x \left\{ 1 - \frac{h' \xi}{h_1} \right\} h(\xi) d\xi + \frac{\delta_0 \mu_1 c(t) h^2(x)}{h_1} \\
+ \mu_1 c(t) \left\{ \sqrt{1 - \frac{\delta_0}{1 - \delta_0} h(x)} + \frac{1}{\mu_1} \sqrt{\frac{h_1}{1 - \delta_0} y} \right\} y^2 \\
+ \left( \sqrt{1 - \delta_0} x + \frac{\mu_1}{\sqrt{1 - \delta_0}} y \right)^2 + 2 \int_0^\gamma \left\{ \mu_2 b(t) g(x, \eta) - c(t) h_1 \eta \right\} d\eta \\
+ 2\mu_1 \int_0^{\eta} \left\{ a(t) f(x, \eta, 0) - \frac{\mu_1}{1 - \delta_0} \right\} \eta d\eta
\]

\[
\geq \frac{\delta_0 \mu_1 c_0 \delta^2}{h_1} x^2 + \left\{ b_0 g_0 - \frac{C h_1}{\mu_1 (1 - \delta_0)} - a_0 f_0 - \frac{\mu_2^2}{1 - \delta_0} \right\} y^2 + \delta_0 z^2.
\]

Denoting \( 2D_2 = \min \left\{ \frac{\delta_0 \mu_1 c_0 \delta^2}{h_1}, b_0 g_0 - \frac{C h_1}{\mu_1 (1 - \delta_0)} + \mu_1 a_0 f_0 - \frac{\mu_2^2}{1 - \delta_0} \delta_0 \right\} \), we have
It is easy to see that there exists a positive number $D_7$ such that

$$V_1(t, x, y, z) \leq D_7 \cdot (x^2 + y^2 + z^2).$$

Hence we obtain

$$D_7 \cdot (x^2 + y^2 + z^2) \leq V_1(t, x, y, z) \leq D_7 \cdot (x^2 + y^2 + z^2).$$

As before we have

$$2V_2 = \mu_2[g \cdot b(t) - \mu_2]x^2 + \mu_2^2 x^2 + a(t) f_2 y + z^2$$

$$+ 2\mu_2 a(t) f_3 x + 2\mu_2 x + 2a(t) f_3 y z$$

$$+ 2a(t) f_4 (t) \int_0^y h(\xi) d\xi + 2b(t) \int_0^y g(x, \eta) d\eta - \mu_2 y^2 + 2c(t) h(x) y$$

$$= \mu_2[g \cdot b(t) - \mu_2]x^2 + [\mu_2 + a(t) f_2 y + z^2]$$

$$+ a(t) f_3 (t) \left\{ \int_0^x h(\xi) d\xi - \frac{1}{h_1} h'(x) \right\} + h_2 c(t) \left\{ \frac{\sqrt{a(t)} h(x) + \frac{y}{\sqrt{f_0}}}{h_1} \right\}^2$$

$$+ 2 \int_0^y \left\{ \frac{b(t) g(x, \eta)}{\eta} - \frac{h_2 c(t)}{a(t) f_0} - \mu_2 \right\} \eta d\eta.$$

We find easily that

$$[g \cdot b(t) - \mu_2] > 0,$$

$$\left\{ \int_0^x h(\xi) d\xi - \frac{1}{h_1} h'(x) \right\} = \frac{2}{h_1} \int_0^x h(\xi) \{h_1 - h'(\xi)\} d\xi \geq 0,$$

$$\left\{ \frac{b(t) g(x, \eta)}{\eta} - \frac{h_2 c(t)}{a(t) f_0} - \mu_2 \right\} > 0.$$

Then we have a positive number $D_8$ such that

$$0 \leq V_2(t, x, y, z) \leq D_8 \cdot (x^2 + y^2 + z^2).$$

We can see also that

$$2V_3 = 2a(t) f_0 \int_0^x \{ f(x, \eta, 0) - f_0 \} \eta d\eta,$$

$$0 \leq 2V_3(t, x, y, z) \leq A^2 f_0 (f^2 - f_3) y^2.$$

Therefore there exist positive numbers $D_1$ and $D_2$ such that

$$D_1(x^2 + y^2 + z^2) \leq V(t, x, y, z) \leq D_2 \cdot (x^2 + y^2 + z^2).$$

Next we prove the inequality (5.9). Along any solution $(x(t), y(t), z(t))$ of (5.3), we have
\[ V_{1(3-3)} = - \left[ \mu \frac{b(t)g(x, y)}{y} - c(t)h'(x) \right] y^2 - [a(t)f(x, y, z) - \mu_0] z^2 \]
\[ + b(t)y \int_0^\infty g_x(x, \eta) d\eta + \mu_0a(t)y \int_0^\infty f_x(x, \eta, 0) \eta d\eta \]
\[ - \mu_0a(t)[f(x, y, z) - f(x, y, 0)] yz \]
\[ + \mu_c(t) \int_0^\infty h(\xi) d\xi + c'(t)h(x)y + b(t) \int_0^\infty g(x, \eta) d\eta \]
\[ + \mu_a'(t) \int_0^\infty f(x, \eta, 0) \eta d\eta + (\mu_0y + z)p(t, x, y, z), \]

\[ V_{2(3-3)} = - \mu_c(t)xy(x) - \mu b(t) \left[ \frac{g(x, y)}{y} - g_0 \right] xy \]
\[ - \left[ a(t)f_0b(t)g(x, y) - c(t)h'(x) - \mu_0a(t)f_0 \right] y^2 \]
\[ - a'(t)f_0[f(x, y, z) - f_0] yz + b(t) y \int_0^\infty g_x(x, \eta) d\eta \]
\[ - a(t)[f(x, y, z) - f_0] xz - \mu_0a(t)[f(x, y, z) - f_0] xz \]
\[ + \frac{1}{2} \mu_0b'(t)g_0x^2 + f_0[a'(t)c(t) + a(t)c'(t)] \int_0^\infty h(\xi) d\xi \]
\[ + a(t)a'(t) f_0 \int_0^\infty g(x, \eta) d\eta + \mu_c a'(t)f_0 xy \]
\[ + a'(t)f_0 yz + c'(t)h(x)y + [x + \mu_0x + a(t)f_0 y] p(t, x, y, z) \]

and

\[ V_{3(3-3)} = a'(t)f_0y \int_0^\infty f_x(x, \eta, 0) \eta d\eta + a'(t)f_0[f(x, y, 0) - f_0] yz \]
\[ + 2a(t)a'(t)f_0 \int_0^\infty [f(x, \eta, 0) - f_0] \eta d\eta. \]

Thus we obtain

\[ \dot{V}_{(3-3)} = - W(t, x, y, z) + 2b(t)y \int_0^\infty g_x(x, \eta) d\eta \]
\[ + a(t)[\mu_0 + a(t)f_0] y \int_0^\infty f_x(x, \eta, 0) \eta d\eta \]
\[ + \{ \mu_c(t) + f_0[a'(t)c(t) + a(t)c'(t)] \} \int_0^\infty h(\xi) d\xi \]
\[ + 2c'(t)h(x)y + 2b'(t) \int_0^\infty g(x, \eta) d\eta + \mu_c a'(t) \int_0^\infty f(x, \eta, 0) \eta d\eta \]
\[ + \frac{1}{2} \mu_0b'(t)g_0x^2 + \mu_c a'(t)f_0 xy + a'(t)f_0 yz \]
\[ + 2a(t)a'(t)f_0 \int_0^\infty f(x, \eta, 0) \eta d\eta. \]
where

\[ W = \mu_2 c(t) x t h(x) + [a(t)f(x, y, z) - \mu_1] z^2 + a(t)[f(x, y, z) - f_3] z^2 \\
+ \left\{ [\mu_1 b(t) g_6 - c(t) h'(x)] + [a(t) f(t) g_6 - c(t) h'(x) - \mu_2 a(t) f_3] \right\} y^2 \\
+ a(t) b(t) f_0 \left[ \frac{g(x, y)}{y} - g_0 \right] y^2 + a(t) b(t) \left[ \frac{g(x, y)}{y} - g_0 \right] y^2 \\
+ \mu_2 b(t) \left[ \frac{g(x, y)}{y} - g_0 \right] y^2 + \mu_2 a(t)[f(x, y, x) - f(x, y, 0)] y z \\
+ a^2(t) f_0 [f(x, y, z) - f(x, y, 0)] y z \\
+ \mu_2 a(t)[f(x, y, z) - f_3] x z \\
\geq \mu_2 c_0 \delta x^2 + [a_0 f_0 - \mu_1] z^2 \\
+ \left\{ [\mu_1 b_0 g_6 - C_h] + [a_0 b_0 f_6 g_6 - C_h - \mu_2 A f_3] \right\} y^2 \\
+ a(t) [f(x, y, z) - f_3] \{ z^2 + \mu_2 x z \} \\
+ \mu_2 b(t) \left[ \frac{g(x, y)}{y} - g_0 \right] y^2 + \mu_2 a(t) \left[ \frac{g(x, y)}{y} - g_0 \right] y^2 \\
+ a(t) \{ \mu_1 + a(t) f_3 \} [f(x, y, z) - f(x, y, 0)] y z. \]

Hence

\[ W \geq \mu_2 \left\{ c_0 \delta - [f(x, y, z) - f_3] \frac{A \mu_2}{4} - \left[ \frac{g(x, y)}{y} - g_0 \right] \frac{B \mu_2}{4 \mu_1} \right\} x^2 \\
+ \left\{ [\mu_1 b_0 g_6 - C_h] + [a_0 b_0 f_6 g_6 - C_h - \mu_2 A f_3] \right\} y^2 \\
+ [a_0 f_0 - \mu_1] z^2 + a(t) [f(x, y, z) - f_3] \left( z + \frac{1}{2} \mu_2 x \right)^2 \\
+ \mu_2 b(t) \left[ \frac{g(x, y)}{y} - g_0 \right] \left( y + \frac{1}{2} \mu_2 x \right)^2 + a(t) b(t) f_0 \left[ \frac{g(x, y)}{y} - g_0 \right] y^2 \\
+ a(t) \{ \mu_1 + a(t) f_3 \} [f(x, y, z) - f(x, y, 0)] y z. \]

By the assumptions

\[ \left\{ c_0 \delta - [f(x, y, z) - f_3] \frac{A \mu_2}{4} - \left[ \frac{g(x, y)}{y} - g_0 \right] \frac{B \mu_2}{4 \mu_1} \right\} \]

\[ \geq c_0 \delta - [f - f_3] \frac{A \mu_2}{4} - \left[ \frac{g - g_0}{4} \right] \frac{B \mu_2}{4 \mu_1} \]

\[ = c_0 \delta - \frac{\mu_2}{4} \left\{ A(f - f_3) + B \left( \frac{g - g_0}{\mu_1} \right) \right\} > 0, \]

\[ [\mu_1 b_0 g_6 - C_h] > \frac{C_h}{b_0 g_6} b_0 g_6 - C_h = 0, \]

\[ [a_0 b_0 f_6 g_6 - C_h - \mu_2 A f_3] \]
Applying the Mean Value Theorem, we have for \(0<|z|<|c|\),

\[
[f(x,y,z) - f(x,y,0)]yz = f_4(x,y,z)yz^2 \geq 0.
\]

Therefore there exists a positive number \(D_5\) such that for all \((x,y,z)\in \mathbb{R}^3\)

\[
W \geq D_5 \cdot (x^2+y^2+z^2).
\]

From (ii), (iii), (iv) and (v) it is easy to show that there exists a positive number \(D_6\) such that

\[
\dot{V}_{(5.3)} \leq -D_6(x^2+y^2+z^2)+D_5 \cdot (|a'(t)| + b'_4(t) + |c'(t)|)(x^2+y^2+z^2) + \{\mu_2 x + [\mu_1 + a(t)f_7]y + 2z\} p(t, x, y, z).
\]

Setting \(D_9 = \max \{\mu_2, \mu_1 + Af_6\}\), we have

\[
\{\mu_2 x + [\mu_1 + a(t)f_7]y + 2z\} p(x, y, z) \leq D_9 \cdot (|x| + |y| + |z|) |p(t, x, y, z)| \leq \sqrt{3} D_9 \cdot (x^2+y^2+z^2)^{1/2} |p(t, x, y, z)|.
\]

Let \(D_5 = \sqrt{3} D_9\). Then we obtain the inequality (5.9).

We are now ready for the principal subsidiary results needed for the completion of the proof of Theorem 2. Application to (5.9) with the assumption (viii) leads to

\[
\dot{V}_{(5.3)} \leq -D_5 \cdot (x^2+y^2+z^2)+D_5 \cdot (|a'(t)| + b'_4(t) + |c'(t)|)(x^2+y^2+z^2) + D_5 p(t)[(x^2+y^2+z^2)^{1/2} + (x^2+y^2+z^2)] + \Delta D_5 \cdot (x^2+y^2+z^2).
\]

Let \(\Delta\) be fixed, in what follows, to satisfy

\[
\Delta \leq \frac{D_5}{2D_9}.
\]

Using the inequalities (5.8) and (5.10), we have

\[
\dot{V}_{(5.3)} \leq -\frac{D_2}{2D_2} \cdot V + \frac{D_5}{D_1} (|a'(t)| + b'_4(t) + |c'(t)|) \cdot V + D_5 p(t) \left[ \left( \frac{V}{D_1} \right)^{1/2} + \frac{V}{D_1} \right].
\]

Assume

\[
\limsup_{(t,s) \rightarrow (c_0,b_0,0)} \int_t^{b+c} \{ |a'(s)| + b'_4(s) + |c'(s)| \} ds < \frac{D_7 D_9}{2D_2 D_4}.
\]
Now Theorem A will be used to prove the uniform boundedness of solutions of (1.2) and that for any solution $x(t)$

$$x^2(t) + x^2(t) + x^2(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$  

This completes the proof of Theorem 2. Q.E.D.

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