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ON SOME EXTENSION OF 1-SPREAD SETS

Dedicated to Professor Hirosi Nagao on his 60th birthday

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1. Introduction

A set Σ of q^2 (2,2)-matrices over $K=GF(q)$ is said to be *a 1-spread set* if it contains the zero matrix 0 and X-Y is nonsingular for any distinct $X, Y\in\Sigma$. Let Σ' be an arbitrary 1-spread set over K. Then $\Sigma'=\{\begin{pmatrix} x & y \\ g(x,y) & h(x,y)\end{pmatrix} \mid x, y \in K\}$ for suitable mappings g and h from $K\times K$ to K. Let $F=GF(q^2)\supset K$. If Char K, the characteristic of K, is odd, we can take an element $t\in F$-K with $t^2\in K$ and define a mapping f from F to itself in such a way that $f(x+yt)=g(x,y)-h(x,y)t$ for $x, y\in K$. Then f satisfies the condition

(*) $f(0)=0$ and $(x-y)(f(x)-f(y))\notin K$ for any distinct $x, y\in F$.

Furthermore the set of (2,2)-matrices

(**) $\Sigma_f=\{\begin{pmatrix} x & y \\ f(y) & x^q\end{pmatrix} \mid x, y\in F\}$

is a 1-spread set over F and the resulting translation plane of order q^4 with the kernel F, say π, has the following properties:

(A1) The linear translation complement $LC(\pi)$ has a shears group P of order at least q^2.

(A2) $LC(\pi)$ has a Baer subgroup Q of order $q+1$ with $[P, Q]\neq 1$.

In this paper we study a class of translation planes of order q^4 with the properties (A1) and (A2) as above. Let $\Omega(F)$ be the set of mappings from F to itself satisfying (*). Then the set of (2,2)-matrices Σ_f defined by (**) is a 1-spread set for any $f\in\Omega(F)$ and if Char K is odd, a 1-spread set Σ'_f over K corresponding to f is naturally defined (Proposition 2.1). Denote by $\Pi(F)$ the set of planes π_f corresponding to Σ_f with $f\in\Omega(F)$. Then $\Pi(F)$ is characterized as the set of translation planes with the kernel F having the properties (A1) and (A2).

The translation complements of these planes are solvable when $p>2$. To show this we need a result on shears groups (Theorem 3.1). Any of these planes of order q^4 is derivable and the derived plane has the kernel isomorphic to K.

Throughout the paper all sets, planes and groups are assumed to be finite. Definitions and notations are standard and taken from [7], [8] and [13].

2. Extension of 1-spread sets

Let $q=p^n$ be a power of a prime p and set $K=GF(q)$ and $F=GF(q^2)\supset K$. Denote by Sym(X) the symmetric group on a set X. Let $f\in$ Sym(F) and set $\Sigma_f=\{\begin{pmatrix} x & y \\ f(y) & \bar{x} \end{pmatrix} | x, y\in F$, where $\bar{x}=x^q$. If $p>2$, then there exists an element $t\in F-K$ with $t^2\in K$. Then f induces mappings g and h from $K\times K$ into K in such a way that $f(x+yt)=g(x, y)-h(x, y)t$ for any $x, y\in K$. Set $\Sigma'=\{\begin{pmatrix} x & y \\ g(x, y) & h(x, y) \end{pmatrix} | x, y\in K\}$. From now on 1-spread sets are called simply spread sets.

Proposition 2.1. *Let $f\in Sym(F)$ with $f(0)=0$. Then the conditions (i) and (ii) are equivalent. Furthermore, if $p>2$, then (i), (ii) and (iii) are equivalent.*

(i) $(x-y)(f(x)-f(y))\notin K$ *for any distinct* $x, y\in F$.
(ii) Σ_f *is a spread set over* $F=GF(q^2)$.
(iii) Σ'_f *is a spread set over* $K=GF(q)$.

Proof. The condition (ii) is equivalent to

$$\text{(ii)}' \quad (x-x')^{q+1}-(y-y')(f(y)-f(y'))\neq 0$$
$$\text{for any distinct} \quad (x, y), (x', y')\in F\times F\,.$$

Hence, as $\{(x-x')^{q+1} | x, x'\in K\}=K$, (i) and (ii) are equivalent.

Assume $p>2$ and set $x=a+bt$ and $y=c+dt$, where a,b,c and $d\in K$. Then (i) is equivalent to

$$\text{(i)}' \quad ((a-c)+(b-d)t)((g(a, b)-g(c, d))-(h(a, b)-h(c, d))t)\notin K$$
$$\text{for any distinct} \quad (a, b), (c\ d)\in K\times K\,.$$

As $t\notin K$ and $t^2\in K$, (i)$'$ is equivalent to

$$\text{(i)}'' \quad -(a-c)(h(a, b)-h(c, d))+(b-d)(g(a, b)-g(c, d))\neq 0\,.$$

Therefore (i) and (iii) are equivalent when $p>2$.

Denote by $\Omega(F)$ the set of all $f\in$ Sym(F) which satisfy $f(0)=0$ and the condition (i) above. Then, by the result above, Σ_f is a spread set for $f\in\Omega(F)$ and moreover Σ'_f is also a spread set when $p>2$. Denote by $\pi_f(=\pi(\Sigma_f))$ the translation plane of order q^4 which corresponds to Σ_f. Similarly, we set $\pi'_f=\pi(\Sigma'_f)$. Let $V(4, F)$ be the underlying F-vector space of π_f and set $V_{a,b}=\{(v, v$

$\begin{pmatrix} a & b \\ f(b) & a \end{pmatrix}) \mid v \in F \times F\}$ $(a, b \in F)$, $V_\infty = O \times O \times F \times F$ and $\tilde{S} = \{V_{a,b} \mid a, b \in F\} \cup \{V_\infty\}$. Let $g \in GL(4, F)$. Then $g \in LC(\pi)$ if and only if g leaves $\tilde{S}$ invariant.

The planes constructed above have the following property.

Lemma 2.2. *Set $\Pi(F) = \{\pi_f \mid f \in \Omega(F)\}$. Let $\pi \in \Pi(F)$ and let $L = LC(\pi)$ be the linear translation complement of π. Then L contains a shears group of order q^2 and a Baer subgroup of order $q+1$.*

Proof. We can easily verify that $L \geqq P = \{\begin{pmatrix} E & T \\ O & E \end{pmatrix} \mid E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ T = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix},\ x \in F\}$ and $L \geqq Q = \{\begin{pmatrix} C & O \\ O & D \end{pmatrix} \mid C = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}, D = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}, e^{q+1} = 1, e \in F\}$. Then P and Q are desired ones.

REMARK 2.3. Let $f \in \Omega(F)$. By definition, O, $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \Sigma_f$, and $O \in \Sigma'_f$, but E is not always contained in Σ'_f.

REMARK 2.4. In Proposition 2.1 we assumed $p \geq 3$. That result is modified for any prime p as follows:

We may assume that $F = K(t)$, where $t \in F - K$ and $t^2 + t \in K$. (Note that $x^2 + x + k$ is irreducible over K for suitable $k \in K$.) Then the following hold.

(i) Let $\Sigma_{(g,h)} = \{\begin{pmatrix} x & y \\ g(x,y) & h(x,y) \end{pmatrix} \mid x, y \in K\}$ be any spread set over K. Define a mapping f from F into itself by

$$f(x+yt) = h(x,y) - g(x,y) + h(x,y)\,t \qquad \text{for} \quad x, y \in K.$$

Then $\Sigma_f = \{\begin{pmatrix} x & y \\ f(y) & x \end{pmatrix} \mid x, y \in F\}$ is a spread set over F.

(ii) Conversely, let Σ_f be any spread set over F. Define mappings g and h from $K \times K$ into K by

$$f(x+yt) = h(x,y) - g(x,y) + h(x,y)\,t \qquad \text{for} \quad x, y \in K.$$

Then $\Sigma_{(g,h)}$ is a spread set over K.

(iii) f is additive on F if and only if g and h are additive on $K \times K$. Therefore $\pi(\Sigma_f)$ is a semifield plane if and only if $\pi(\Sigma_{(g,h)})$ is a semifield plane. (Theorem 5.1.2 of [2].)

EXAMPLE 2.5. Assume $p > 2$ and let e be an element of F such that $e \notin K$ and $e^2 \in K$. Then a function f defined by $f(x) = ex^{(q^2+2q-1)/2}$ is an element of $\Omega(F)$. Moreover π_f is not a semifield plane.

Proof. Clearly $f(x) = ex^q$ or $-ex^q$ according as $x \in \tilde{F}$ or $x \notin \tilde{F}$. Here $\tilde{F}$ is the set of square elements of F.

If $xy\in\widetilde{F}$ and $x\neq y$, then $(x-y)(f(x)-f(y))=\pm e(x-y)^{q+1}\notin K$. If $xy\notin\widetilde{F}$, then $(x-y)(f(x)-f(y))=\pm(x+y)^{q+1}e(x-y)/(x+y)$. Assume $f\notin\Omega(F)$. Then $e(x-y)/(x+y)=k$ for some $k\in K$, so $(k+e)xy=(e-k)x^2=-(e+k)^q x^2$ because $e^q=-e$. Hence $xy=-(e+k)^{q-1}x^2\in\widetilde{F}$, a contradiction. Thus $f\in\Omega(F)$.

As f is not an additive function, π_f is not a semifield plane by Theorem 5.1.2 of [2].

3. Collineation groups generated by shears

The purpose of this section is to prove the following theorem, which will be required in §4 and §5.

Theorem 3.1. *Let π be a translation plane of order $q(=p^n)$ and $C(\pi)$ its translation complement. Suppose $C(\pi)$ contains an elation group P such that $|P|^2\geqq q$. Then*

(i) *π is a desarguesian plane $PG(2, q)$,*

(ii) *π is a Lüneburg plane $L(q)$ with q even or*

(iii) *the group generated by all elations in $C(\pi)$ is a p-group.*

In particular, $C(\pi)$ fixes exactly one point on the line at infinity unless $\pi\cong PG(2, q)$, $L(q)$.

The proof is divided into several steps (Lemmas 3.2–3.6).

Lemma 3.2. *Set $H=C(\pi)$ and let N be a normal subgroup of H generated by all P^x with $x\in H$. Then one of the following holds.*

(i) *N is an elementary abelian p-group.*

(ii) *$N\cong SL(2, p^m)$ for some $m\geqq n/2$.*

(iii) *$p=2$ and $N\cong Sz(2^m)$ for some $m\geqq n/2$.*

(iv) *$p=3$ and $N\cong SL(2,5)$.*

Proof. This is an immediate consequence of [6] and [14].

Lemma 3.3. *If (iv) occurs, then $\pi\cong PG(2,9)$.*

Proof. In this case we have $q\leqq 3^2$. Hence the order of π is 9. By Theorem 8.4 of [13], $\pi\cong PG(2,9)$ or π is the nearfield plane of order 9. Since, by Theorem 8.3 of [13], the nearfield plane of order 9 contains no affine elations of order 3, we have $\pi\cong PG(2,9)$. Thus the lemma holds.

Lemma 3.4. *If the case (iii) occurs, then $\pi\cong L(q)$.*

Proof. Let S be a Sylow 2-subgroup of $N(\cong Sz(2^m))$. We may assume $S\geqq P$. As P contains no elements of order 4, we may also assume $P=Z(S)$. Let A be a unique fixed point of S on l_∞. Let N_A and A^N denote the stabilizer of A in N and the N-orbit containing A, respectively. If N fixes the point A, then N is a group of perspectivities with axis OA, a contradiction. Therefore $N_A=N_N(S)$ since $N_N(S)$ is a maximal subgroup of N (cf. [1]). From this, $|A^N|=|N: N_N(S)|=2^{2m}+1=|P|^2+1\geqq 2^n+1=|l_\infty|$, whence $n=2m$ and π is a Lüneburg plane by [12].

Lemma 3.5. *Suppose* $N\cong SL(2, p^m)$ *and* $\pi\not\cong PG(2, q)$. *Then* $m<n<2m$.

Proof. We may assume P is a Sylow p-subgroup of N. Let A be a unique fixed point of P on l_∞. Since P acts semi-regularly on $l_\infty-\{A\}$, $p^n\geqq p^m$. On the other hand $p^{2m}=|P|^2\geqq p^n$ by assumption. Thus $m\leqq n\leqq 2m$. If $n=m$ or $n=2m$, then $\pi\cong PG(2, q)$ by [4] and Theorem 38.12 of [13].

Lemma 3.6. *If (ii) occurs, then* $\pi\cong PG(2, q)$.

Proof. Suppose false. Then $m<n<2m$ by Lemma 3.5. In particular $m>1$. Let P and A be as in the proof of Lemma 3.5. Let $B\in l_\infty-A^N(\neq\phi)$. Since P contains no planar elements, N_B is a p'-subgroup of N.

Since $|B^N|\leqq|l_\infty-A^N|\leqq p^{2m-1}-p^m$, $|N_B|\geqq(p^{2m}-1)/(p^{m-1}-1)\geqq p^{m+1}+3$. In particular $|N_B|\not| 2(p^m\pm 1)$, As $p\not|\,|N_B|$ and $|N_B|\geqq p^{m+1}+3\geqq 11$, applying Dickson's Theorem (Theorem 14.1 of [13]), we have a contradiction.

Proof of Theorem 3.1.

By Lemmas 3.2–3.6, $\pi\cong PG(2, q)$, $L(q)$ or N is a p-group. If $\pi\not\cong PG(2, q)$, $L(q)$, then N fixes exactly one point on l_∞. Therefore, as $C(\pi)\trianglerighteq N$, $C(\pi)$ fixes that point.

4. A characterization of the class of planes $\Pi(F)$

In this section a characterization of the planes in $\Pi(F)$ defined in §2 is presented in terms of their collineation groups.

Theorem 4.1. *Let* π *be a translation plane of order* q^4 *having the kernel* F. *Then* π *is contained in* $\Pi(F)$ *if and only if* $LC(\pi)$ *has subgroups* P *and* Q *with the properties (A1) and (A2):*

(A1) *P is a group of elations of order at least* q^2.

(A2) *Q is a Baer subgroup of order* $q+1$ *with* $[P, Q]\neq 1$.

The "only if" part of the theorem has been proved in Lemma 2.2, so it suffices to show the "if" part of the theorem. Throughout this section π is assumed to be a translation plane of order q^4 having the kernel F and the properties (A1) and (A2). We may assume that P is a maximal elation group of $LC(\pi)$.

Lemma 4.2. *Set* $L=LC(\pi)$. *Then* $L\trianglerighteq P$ *and* L *fixes exactly one point* A *on* l_∞.

Proof. By the properties of the desarguesian plane $PG(2, q^4)$, together with Theorem 3.1, we have the lemma.

Lemma 4.3. *Let V be the underlying F-vector space of π. By choosing a suitable basis for V, Q is represented in the following form:*

$$Q = <\begin{pmatrix} Q_1 & O \\ O & Q_2 \end{pmatrix} | Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}, Q_2 = \begin{pmatrix} e' & 0 \\ 0 & 1 \end{pmatrix}> ,$$

where e and e' are some elements of $F^{\sharp}=F-\{0\}$ of order $q+1$.

Proof. Let $B\in l_\infty$ be a fixed point of Q with $B\neq A$. Let U be a L-submodule of V corresponding to the line OA and W a Q-submodule of V corresponding to the line OB. Since $p\not| q+1$, V is completely reducible as a Q-module by Maschke's theorem. Hence there exist one dimensional Q-submodules U_1, U_2, W_1 and W_2 such that $V=U_1\oplus U_2\oplus W_1\oplus W_2$, $U=U_1\oplus U_2$, $W=W_1\oplus W_2$ and $U_1\oplus W_2=\{v\in V \mid vQ=v\}$. Let $0\neq u_i\in U_i$ and $0\neq w_i\in W_i$ with $1\leqq i\leqq 2$. Then $\{u_1, u_2, w_1, w_2\}$ is a basis for V and Q is represented as a subgroup of $\{\begin{pmatrix} Q_1 & O \\ O & Q_2 \end{pmatrix} | Q_1=\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}, Q_2=\begin{pmatrix} e' & 0 \\ 0 & 1 \end{pmatrix}, e, e'\in F, e^{q+1}=(e')^{q+1}=1\}$. Each element of $Q-\{1\}$ is a Baer collineation with fixed vectors $U_1\oplus W_2$. Hence the lemma holds.

In the rest of this section we fix the basis for V as stated above and coordinatize π in such a way that $A=(\infty)$ and $B=(0)$. Let Σ be the corresponding spread set of π. We may assume that $E\in\Sigma$.

Lemma 4.4. *Set $\Psi=\{T\in\Sigma \mid T+S\in\Sigma$ for any $S\in\Sigma\}$. Then Ψ is an abelian group of order at least q^2 and $P=\{\begin{pmatrix} E & T \\ O & E \end{pmatrix} | T\in\Psi\}$.*

Proof. By Theorem 3.13 of [13], the lemma holds.

Lemma 4.5. (i) *Let e be an element of $F^{\sharp}$ of order $q+1$. Then*

$$Q = <\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}> .$$

(ii) *If $\begin{pmatrix} 0 & y \\ u & v \end{pmatrix}\in\Sigma-\{O\}$, then $v=0$.*

(iii) *If $\begin{pmatrix} x & y \\ u & v \end{pmatrix}\in\Psi$, then $\begin{pmatrix} e^m x & y \\ u & e^{-m} v \end{pmatrix}$, $\begin{pmatrix} (e-1)x & 0 \\ 0 & (e^{-1}-1)v \end{pmatrix}\in\Psi$ for any integer m.*

Proof. Since $\begin{pmatrix} Q_1 & O \\ O & Q_2 \end{pmatrix}^m$ is a collineation for any integer m, $\begin{pmatrix} x & y \\ u & v \end{pmatrix}\in\Sigma$ implies

$Q_1^{-m}\begin{pmatrix} x & y \\ u & v \end{pmatrix} Q_2^m = \begin{pmatrix} (e')^m x & y \\ (e^{-1}e')^m u & e^{-m} v \end{pmatrix} \in \Sigma$. Hence, if $x=0$ and $y \neq 0$, then $\begin{pmatrix} 0 & y \\ (e^{-1}e')^m u & e^{-m} v \end{pmatrix}$, $\begin{pmatrix} 0 & y \\ u & v \end{pmatrix} \in \Sigma$. As Σ is a spread set, $(e^{-1}e')^m u = u \neq 0$ and $e^{-m} v = v$. Therefore $e=e'$ and $v=0$ and so (i) and (ii) hold.

Let $\begin{pmatrix} x & y \\ u & v \end{pmatrix} \in \Psi$. Then, by Lemmas 4.2 and 4.4, $Q_1^{-m}\begin{pmatrix} x & y \\ u & v \end{pmatrix} Q_2^m \in \Psi$. Hence $\begin{pmatrix} e^m x & y \\ u & e^{-m} v \end{pmatrix} \in \Psi$. By definition of Ψ, $\begin{pmatrix} e^m x & y \\ u & e^{-m} v \end{pmatrix} - \begin{pmatrix} x & y \\ u & v \end{pmatrix} \in \Psi$. Thus (iii) holds.

Lemma 4.6. $\Sigma = \{\begin{pmatrix} x & y \\ \tilde{g}(y) & k\bar{x} \end{pmatrix} \mid x, y \in F\}$ *for some* $\tilde{g} \in Sym(F)$ *with* $\tilde{g}(0)=0$ *and* $k \in F^\sharp$.

Proof. Set $\Psi_1 = \Psi \cap \{\begin{pmatrix} x & 0 \\ 0 & v \end{pmatrix} \mid x, v \in F\}$. Since $[P, Q] \neq 1$, $\Psi_1 \neq \{O\}$ and $q+1 \mid |\Psi_1| - 1 \neq 0$ by Lemma 4.5 (iii). As $|\Psi_1| \mid q^2$, we have $|\Psi_1| = q^2$. Hence there exists $h \in \mathrm{Sym}(F)$ such that $\Psi_1 = \{\begin{pmatrix} x & 0 \\ 0 & h(x) \end{pmatrix} \mid x \in F\}$. Let $\begin{pmatrix} x & y \\ u & v \end{pmatrix}$ be any element of Σ. Then $\begin{pmatrix} x & y \\ u & v \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & h(x) \end{pmatrix} = \begin{pmatrix} 0 & y \\ u & v-h(x) \end{pmatrix} \in \Sigma$ and so $v - h(x) = 0$, $\begin{pmatrix} 0 & y \\ u & 0 \end{pmatrix} \in \Sigma$ by Lemma 4.5 (ii). This implies that $u = \tilde{g}(y)$ for some $\tilde{g} \in \mathrm{Sym}(F)$.

Since Ψ_1 is abelian, h is an additive mapping. Set $h(x) = \sum_{i=0}^{2n-1} c_i x^{p^i}$. Moreover $h(ex) = e^{-1} h(x)$ by Lemma 4.5 (iii). Therefore $c_i e^{p^i} = e^{-1} c_i$ for each i, $0 \leqq i \leqq 2n-1$. Assume $c_i \neq 0$. Then $e^{p^i - q} = 1$ and therefore $q+1 = p^n + 1 \mid p^i - p^n$. Clearly $n \leqq i \leqq 2n-1$ and so set $i = n+r$, $0 \leqq r \leqq n-1$. As $p^{n+r} - p^n = (p^n+1)(p^r-1) - (p^r-1)$, we have $r=0$ and $h(x) = kx^{p^n}$ for some $k \in F^\sharp$.

Proof of Theorem 4.1. Since $\begin{pmatrix} 1 & 0 \\ 0 & k^{-1} \end{pmatrix} \Sigma = \{\begin{pmatrix} x & y \\ g(y) & \bar{x} \end{pmatrix} \mid x, y \in F\}$, $g(y) = k^{-1} \tilde{g}(y)$, by Lemma 4.6, π is contained in $\Pi(F)$.

REMARK 4.7. Clearly $\Psi = \{\begin{pmatrix} x & z \\ g(z) & \bar{x} \end{pmatrix} \mid x \in F, z \in U\}$, where $U = \{z \in F \mid g(y) + g(z) = g(y+z)$ for any $y \in F\}$ (See Lemma 4.4). As $g(0)=0$, $\Psi \geqq \{\begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix} \mid x \in F\}$.

REMARK 4.8. Set $P_1 = \{\begin{pmatrix} E & T_x \\ O & E \end{pmatrix} \mid T_x = \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}, x \in F\}$, $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ and $E_2 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, where $a^{q+1} = 1$. Then $\begin{pmatrix} E_1 & O \\ O & E_2 \end{pmatrix}^{-1} \begin{pmatrix} E & T_x \\ O & E \end{pmatrix} \begin{pmatrix} E_1 & O \\ O & E_2 \end{pmatrix} = \begin{pmatrix} E & T_{ax} \\ O & E \end{pmatrix}$. Hence, by the maximality of P, $P_1 \leqq P$ and $P_1 Q$ is a Frobenius group with kernel P_1.

5. Solvability of $C(\pi)$ when $p>2$

In this section we prove the solvability of $C(\pi)$ with $\pi \in \Pi(F)$. When

$p>2$. Except in Lemma 5.1 we assume that $p>2$ and $C(\pi)$ is not solvable. Let notations Σ, P, Q, L and Ψ be as in §4.

Lemma 5.1. *$L \triangleright P$ and any element of L is represented in the form $\begin{pmatrix} A & AC \\ O & AD \end{pmatrix}$, where A, C and D satisfy the following conditions.*

(i) $A \in GL(2, q^2)$, $C \in \Sigma$, $D \in \Psi - \{0\}$ *and*

(ii) $A^{-1}\Psi AD = \Psi$, $C + A^{-1}\Sigma AD = \Sigma$.

Proof. Let $g \in L$. Applying Theorems 3.1 and 4.1, $g = \begin{pmatrix} A & X \\ O & Y \end{pmatrix}$ for some $A, Y \in GL(2, q^2)$ and $X \in M(2, q^2)$ and g normalizes P. Hence $M(2, q^2)$ denotes the set of all (2, 2)-matrices over $GF(q^2)$. Since $g^{-1}\begin{pmatrix} E & T \\ O & E \end{pmatrix} g = \begin{pmatrix} E & A^{-1}TY \\ O & E \end{pmatrix}$ for any $T \in \Psi$, $A^{-1}\Psi Y = \Psi$. Set $D = A^{-1}Y \in \Psi$. Then $Y = AD$ and so $A^{-1}\Psi AD = \Psi$. On the other hand $A^{-1}(X + MY) = A^{-1}X + A^{-1}MY \in \Sigma$ for each $M \in \Sigma$. Set $C = A^{-1}X$. Then $X = AC$, $C \in \Sigma$ and $C + A^{-1}\Sigma AD = \Sigma$. Thus the lemma holds.

Let X be a noımal subgroup of L and denote by $X^{(\infty)}$ the last term of the derived series of X. By assumption, $L^{(\infty)} \neq 1$. Let r and s be homomorphisms from L to $GL(2, q^2)$ defined by $r\begin{pmatrix} A & C \\ O & B \end{pmatrix} = A$ and $s\begin{pmatrix} A & C \\ O & B \end{pmatrix} = B$, respectively. For a subgroup X of $GL(2, q^2)$, set $\bar{X} = XZ/Z$, where Z is the center of $GL(2, q^2)$.

Lemma 5.2. *If $L \triangleright X$ and $X^{(\infty)} \neq 1$, then $r(X^{(\infty)}) \neq 1$.*

Proof. Set $M = X^{(\infty)}$ and assume $r(M) = 1$ but $s(M) \neq 1$. Let $B \in s(M)$ be an element of order p. Then $g = \begin{pmatrix} E & C \\ O & B \end{pmatrix} \in M$ for some $C \in M(2, q^2)$. Hence g fixes each element of $\{(x, y) \mid x, y \in V(2, q^2), y = xC(E-B)^{-1}\}$ and some nontrivial element of $\{(0, y) \mid y \in V(2, q^2)\}$. This is a contradiction by Bruck's Theorem (cf. Theorem 3.7 of [8]). Therefore $s(M)$ is a p'-group. Applying Dickson's Theorem, $\overline{s(QM)} \simeq A_5$ or S_5 and $\overline{s(M)} \simeq A_5$. Since $p \nmid |\overline{s(M)}| = 2^2 \cdot 3 \cdot 5$, we have $q+1 \geqq 8$. However, $\overline{s(QM)} \geqq \overline{s(Q)} \simeq Z_{q+1}$, a contradiction.

Lemma 5.3. *If $L \triangleright X$ and $X^{(\infty)} \neq 1$, then $s(X^{(\infty)}) \neq 1$.*

Proof. Set $M = X^{(\infty)}$ and assume $s(M) = 1$ but $r(M) \neq 1$. Let $g = \begin{pmatrix} A & C \\ O & E \end{pmatrix}$ be a p-element of M. Then g is a perspectivity with axis $x = 0$. Hence $A = E$ and so $\overline{r(M)}$ is a p'-group. By Dickson's Theorem, $\overline{r(QM)} \simeq A_5$ or S_5 and $\overline{r(M)} \simeq A_5$. By a similar argument as in the proof of Lemma 5.2, we have a contradiction.

Lemma 5.4. *Set $N = L^{(\infty)}$. Then $\overline{r(N)} = \overline{s(N)} \simeq A_5$, $PSL(2, q)$ or $PSL(2, q^2)$.*

Proof. By Dickson's Theorem, $\overline{r(N)}, \overline{s(N)} \in \{A_5, PSL(2, p^m)\}$ and it follows from Lemmas 5.2 and 5.3 that $\overline{r(N)} \cong \overline{s(N)}$. Moreover one of the followgwing occurs.

(i) $A_5 \cong \overline{r(N)} \lhd \overline{r(QN)} \leqq S_5$.

(ii) $PSL(2, p^m) \cong \overline{r(N)} \lhd \overline{r(QN)} \leqq PGL(2, p^m)$, $m \mid 2n$, $q = p^n$.

(iii) $PSL(2, p^m) \cong \overline{r(N)}$, $[\overline{r(QN)}: \overline{r(N)}] \mid 4$, $2m \mid 2n$, $q = p^n$.

We not note that $\overline{r(Q)} \cong Z_{pn+1}$.

If (ii) occurs, then $p^n+1 \mid p^m+1$ or $p^n+1 \mid p^m-1$, where $2n = mt$ for some integer t. Since $m \geqq n$, $t=1$ or 2. Therefore $\overline{r(N)} \cong PSL(2, q)$ or $PSL(2, q^2)$.

If (iii) occurs, then $p^n+1 \mid 2(p^m+1)$ or $p^n+1 \mid 2(p^m-1)$, where $n = mt$ for some integer t. Assume $t > 1$. Then $p^{2m}-1 < p^n+1 \leqq 2(p^m+1)$. Hence $3^m-1 \leqq p^m-1 < 2$, a contradiction. Thus $\overline{r(N)} \cong PSL(2, q)$.

Lemma 5.5. *Set $\Gamma = l_\infty - (\infty)$, where (∞) denotes a unique fixed point of P on l_∞. Then any nontrivial p-element of N has no fixed points on Γ.*

Proof. The lemma follows immediately from Theorem of [3].

Lemma 5.6. $\overline{r(N)} \neq A_5$.

Proof. Assume $\overline{r(N)} = A_5$. As we have seen in the proof of Lemma 5.4, $\overline{r(QN)}$ ($\leqq S_5$) must contain a cyclic subgroup isomorphic to $\overline{r(Q)} \cong Z_{q+1}$. Hence $q=3$ and $|P| \leqq 3^3$ by Lemma 5.5. Therefore any 5-element of N centralizes P. Since $N = N^{(\infty)}$, $[P, N] = 1$.

Let W be a Sylow 5-subgroup of N and Δ the set of fixed points of W on l_∞. Since $[W, P] = 1, 5 \mid |P| \mid \mid l_\infty - \Delta|$. As $|P| = 3^2$ or 3^3, we have either $l_\infty = \Delta$ or $|P| = 3^2$ and $|\Delta| = 3^2 \cdot 4 + 1$. Since $N \trianglerighteq W$, $l_\infty \neq \Delta$. Let W_1, W_2 and W_3 be a Sylow 5-subgroups of N such that $\overline{r(W_i)}$ are distinct. Since $|\Delta| - 1 > 3^3$, $|F(W_i) \cap F(W_j) \cap l_\infty| > 1$ for some distinct i and j. Since $\overline{r(\langle W_i, W_j \rangle)} \cong A_5$, $\langle W_i, W_j \rangle$ contains a Baer 3-element. This is a contradiction by Lemma 5.5.

Lemma 5.7. $r(N) = s(N) = SL(2, q)$. *In particular* $q \neq 3$.

Proof. Suppose false. Then $r(NP) = s(NP) = SL(2, q^2)$ by Lemmas 5.4 and 5.6. Therefore, by Lemma 5.5, $|NP: H| = q^4$, where H is the stabilizer of a point $B \in \Gamma$ in NP. Hence $q^4 - 1 \mid |H|$. Applying Dickson's Theorem, $r(H)Z/Z \cong PSL(2, q^2)$ and so $q^2 \mid |H|$, contrary to Lemma 5.5.

Lemma 5.8. *Set $X = r^{-1}(\langle -E \rangle) \cap N$ and $Y = s^{-1}(\langle -E \rangle) \cap N$. Then X is solvable and $X = Y$.*

Proof. By Lemmas 5.2 and 5.3, X is solvable and so $SL(2, q) \cong s(N) \rhd$

$s(X) \leqq \langle -E \rangle$. Hence $X \leqq Y$. Similarly $Y \leqq X$. Therefore $X = Y$.

Lemma 5.9. *There exists no element $g \in NQ$ which satisfies either* (i) $r(g) = E$, $s(g) = -E$ *or* (ii) $r(g) = -E$, $s(g) = E$.

Proof. Assume $r(g) = E$ and $s(g) = -E$ and set $U = r^{-1}(E) \cap N$ and $V = s^{-1}(E) \cap N$. Since $g \in r^{-1}(E) \cap NQ \leqq (r^{-1}(SL(2, q^2)) \cap Q) N = N$, g is not contained in V. Hence $N/U \cap V \rhd U/U \cap V \simeq UV/V \neq 1$. Moreover $N/V \simeq SL(2, q)$ and $U/U \cap V \simeq UV/V \simeq Z_2$. By Satz 25.7 of [9] Chapter V, $2 \mid |N/N'|$, a contradiction. Hence (i) does not occur. Similarly (ii) does not occur.

Lemma 5.10. *Eet* $I = \begin{pmatrix} E & O \\ O & E \end{pmatrix}$. *Then* $-I \in N$.

Proof. By Lemmas 5.7–5.9, there is an element $g = -\begin{pmatrix} E & C \\ O & E \end{pmatrix} \in X$. Then $g^p = -I \in X \leqq N$.

Lemma 5.11. *Set* $\Omega' = \{\Delta^x \mid x \in P\}$, *where* $\Delta = F(Q) \cap \Gamma$. *Then* Ω' *is a partition of* Γ.

Proof. Assume $\Delta^x \cap \Delta^y \neq \phi$ for some $x, y \in P$. Set $z = yx^{-1}$ and let $B \in \Delta \cap \Delta^z$. Then the stabilizer $(PQ)_B \geqq \langle Q, Q^z \rangle \geqq Q$. By Lemma 5.5, $Q = Q^z$ and hence $z \in P_0$ and $x \equiv y \pmod{P_0}$, where $P_0 = C_P(Q)$. Therefore $\Delta^x = \Delta^y$. As P is abelian, P/P_0 acts regularly on Ω' and Ω' is a partition of $\bigcup_{\Delta' \in \Omega} \Delta'$. By Remark 4.8, $P_1 Q$ is a Frobenius group of order $q^2(q+1)$. Thus $|\Omega'| \geqq |P_1| = q^2$. Therefore Ω' is a partition of Γ as $|\Delta| = q^2$.

Lemma 5.12. *Set* $H = NPQ$, $W = r^{-1}(E) \cap H$ *and* $W_1 = s^{-1}(E) \cap H$. *Then* $W = W_1 = P$.

Proof. Since $W \leqq r^{-1}(SL(2, q^2)) \cap H = NP$, $W \leqq NP$. Similarly $W_1 \leqq NP$. By Lemma 5.8, W and W_1 are solvable. On the other hand, $s(W) \lhd s(NP) = s(N) \simeq SL(2, q)$ by Lemma 5.7. Therefore $s(W) \leqq \langle -E \rangle$. Applying Lemma 5.9, $s(W) = E$. This implies $W \leqq P$. Clearly $P \leqq W$ and so $W = P$. Similarly $W_1 = P$.

Lemma 5.13. *Let Z be the center of $r(H)$. Then* $Z \simeq Z_{q+1}$, $Z \cap r(Q) = 1$ *and* $r(H)/Z \simeq PGL(2, q)$.

Proof. By Lemma 5.7, $r(N) = SL(2, q)$. Hence $r(H)Z/Z \simeq PSL(2, q)$ or $PGL(2, q)$ by Dickson's Theorem. Clearly $r(H) = r(NQ) = r(N)\, r(Q)$, $r(Q) \simeq Z_{q+1}$ and $r(N) \cap r(Q) = 1$. Hence $Z = r(H) \cap \{kE \mid k \in F^*\} \simeq Z_{q+1}$ or $Z_{2(q+1)}$. Since $r(Q) \cap Z = 1$, $\overline{r(H)}$ contains a cyclic subgroup of order $q+1$. Thus $Z \simeq Z_{q+1}$ and $\overline{r(H)} = PGL(2, q)$.

Lemma 5.14. *Let* $x \in H$. *If* $0 < |F(Q) \cap F(Q^x) \cap \Gamma| < |\Delta|$. *Then* $r(Q)$

$Z=r(Q^x)Z$.

Proof. Assume $r(Q)Z \neq r(Q^x)Z$. Then $r(Q^x)Z/Z \neq r(Q)Z/Z \simeq Z_{q+1}$. Hence $<r(Q), r(Q^x)>Z/Z=r(<Q, Q^x>)\,Z/Z \simeq PGL(2, q)$ by Dickson's Theorem. In particular $p \mid |<Q, Q^x>|$. However, this contradicts Lemma 5.5.

Lemma 5.15. *Let $x \in H$. If $\Delta \cap \Delta^x \neq \phi$, then $\Delta=\Delta^x$.*

Proof. Suppose false. Then it follows from Lemma 5.14 that $r(Q)\,Z=r(Q^x)\,Z$. Let h be a natural homomorphism from H to H/W. Then $h(Q)\,h(Z)=h(Q^x)\,h(Z)$ and so $(<J>P/P)\times(<-I>P/P)=(<J^x>P/P)\times(<-I>P/P)$ by Lemma 5.12. Here $J=\begin{pmatrix} O & -J' \\ J' & O \end{pmatrix} \in Q$ with $J'=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From this $<J, -I>P=<J^x, -I>P$. Therefore $J^x=J^t$ or $-J^t$ for some $t \in P$. As $\Delta^x=\Gamma \cap F(Q^x) =\Gamma \cap F/(J^x)$ and $\Gamma \cap F(J^t)=\Gamma \cap F(-J^t)$, we have $\Delta^x=(\Gamma \cap F(J))^t=\Delta^t$. Therefore $\Delta \cap \Delta^x \neq \phi$ implies $\Delta=\Delta^x$ by Lemma 5.11.

Lemma 5.16. *$H^{\Omega'}$ is a transitive permutation group with a regular normal subgroup $P^{\Omega'}$. Moreover the global stabilizer M of Δ in H involves $PSL\ (2, q)$.*

Proof. The first part follows immediately from Lemmas 5.11 and 5.15. In Particular $H=MP$. Since $N \leqq H$ and N involves $PSL(2, q)$, M also involves $PSL(2, q)$.

We now prove the following theorem.

Theorem 5.17. *Let $\pi \in \Pi(F)$ and assume $p>2$. Then $C(\pi)$ is solvable.*

Proof. Suppose false. Then L is nonsolvable. Let R be the pointwise stabilizer of Δ in H. Then $M \triangleright R$. As $r(H)=r(MP)=r(M) \triangleright r(R)$, we have $r(M)/Z \simeq PGL(2, q) \triangleright r(R)\,Z/Z \geqq r(Q)\,Z/Z \simeq r(Q) \simeq Z_{q+1}$ by Lemma 5.13. Therefore $r(R)\,Z/Z \simeq PGL(2, q)$. In particular $p \mid |R|$, contrary to Lemma 5.5. Thus $C(\pi)$ is solvable.

6. The linear translation complements when $p>2$

In this section we determine the structure of $LC(\pi)$ with $\pi \in \Pi(F)$ and $p>2$. Let $f \in \Omega(F)$ and set $\Sigma=\Sigma_f$, $\pi=\pi_f$ and $L=LC(\pi)$. Set $M_f(a, b, c, d, e)=$

$$\begin{pmatrix} a & 0 & ac & ad \\ 0 & b & bf(d) & b\bar{c} \\ 0 & 0 & ae & 0 \\ 0 & 0 & 0 & b\bar{e} \end{pmatrix},\ g(u)=\begin{pmatrix} 0 & 1 & 0 & 0 \\ u & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & u & 0 \end{pmatrix} \text{ and } h(u)=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & 0 & w \\ 0 & 0 & u & 0 \end{pmatrix} \text{ for } u \in F^\sharp.$$

Here w is an element of $F^\sharp$ of order $2(q+1)$. The matrix $h(u)$ does exist if $p>2$.

As we have seen in the proof of Lemma 2.2, we have

Lemma 6.1. $L \geqq L_0 = \{M_f(a, b, c, 0, a^{-1}b) \mid a, b \in F^*, (a/b)^{q+1} = 1, c \in F\}$.

Lemma 6.2. *Assume $q>3$ and let r and s be the homomorphisms defined in §5. Set $J=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $r(L), s(L) \leqq M = <J> \{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \mid u, v \in F^*\}$.*

Proof. By Theorem 5.17 and Lemma 5.1, $L \leqq \{\begin{pmatrix} A & C \\ O & B \end{pmatrix} \mid A, B \in GL(2, q^2), C \in M(2, q^2)\}$ and $r(L)$ and $s(L)$ are solvable. Since $r(L) \geqq r(Q) = <\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}> \cong Z_{q+1}$, $r(L) \leqq M$ or $q+1 \leqq 4$ by Dickson's Theorem. Similarly $s(L) \leqq M$. Thus the lemma holds.

Let $\Lambda(F)$ be the set of all $f \in \mathrm{Sym}(F)$ such that $f(x) = u\bar{x}$, where $u \in F-K$ and $u^{q-1} = -1$. Then it is not difficult to verify that $\Lambda(F) \subset \Omega(F)$ and π_f is a semifield plane for any $f \in \Lambda(F)$.

Proposition 6.3. *Assume $q>3$. If $f \in \Lambda(F)$ and $f(x) = u\bar{x}$ for some u with $u^{q-1} = -1$, then $L = <g(u), h(u)>H$ and $H = <M_f(a, b, c, d, e) \mid a, b, e \in F^*, c, d \in F, (b/a)^{q+1} = 1>$. In particular $|L| = 4q^4(q-1)^2(q+1)^3$ and $L \rhd H$.*

Proof. Set $g = g(u)$, $h = h(u)$ and $L_1 = <g, h, H>$. Since $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{-1} \left(\begin{pmatrix} ac & ad \\ bu\bar{d} & b\bar{c} \end{pmatrix} + \begin{pmatrix} x & y \\ u\bar{y} & \bar{x} \end{pmatrix} \begin{pmatrix} ae & 0 \\ 0 & b\bar{e} \end{pmatrix} \right) = \begin{pmatrix} c & d \\ u\bar{d} & \bar{c} \end{pmatrix} + \begin{pmatrix} ex & a^{-1}b\bar{e}y \\ ab^{-1}eu\bar{y} & \bar{e}\bar{x} \end{pmatrix}$, $M_f(a, b, c, d, e) \in L$ when $(a^{-1}b)^{q+1} = 1$. Hence $H \leqq L$. Furthermore $\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}^{-1} \begin{pmatrix} x & y \\ u\bar{y} & \bar{x} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} = \begin{pmatrix} \bar{x} & \bar{y} \\ uy & x \end{pmatrix} \in \Sigma_f$ and so we have $g \in L$. Similarly $\begin{pmatrix} -1 & 0 \\ 0 & w^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ u\bar{y} & \bar{x} \end{pmatrix} \begin{pmatrix} 0 & w \\ u & 0 \end{pmatrix} = \begin{pmatrix} -uy & -xw \\ uw^{-1}\bar{x} & u\bar{y} \end{pmatrix} = \begin{pmatrix} -uy & -xw \\ u\overline{(-xw)} & \overline{-uy} \end{pmatrix} \in \Sigma$ and so $h \in L$. Thus $L_1 \leqq L$.

Conversely, let $v \in L$. By Lemmas 6.1 and 6.2, $vv' = \begin{pmatrix} E & C \\ O & D \end{pmatrix}$, where $C = \begin{pmatrix} c & d \\ i & j \end{pmatrix}$ and $D = \begin{pmatrix} e & 0 \\ 0 & s \end{pmatrix}$ for suitable $v' \in <g, h, L_0>$. Then $C + \begin{pmatrix} x & y \\ u\bar{y} & \bar{x} \end{pmatrix} D = \begin{pmatrix} c+ex & d+sy \\ i+eu\bar{y} & j+s\bar{x} \end{pmatrix} \in \Sigma$ for any $x, y \in F$. Therefore $\bar{c} = j$, $\bar{e} = s$, $i = u\bar{d}$ and $\bar{s}u = eu$. Hence $vv' = M_f(1, 1, c, d, e) \in H$. Thus $L = <g, h>H$. Clearly $L \rhd H$ and $L/H \cong Z_2 \times Z_2$. Therefore $|L| = 4|H| = 4(q^2-1)^2(q+1)q^4 = 4q^4(q-1)^2(q+1)^3$ and the lemma holds.

Proposition 6.4. *Assume $q>3$. If $f \in \Omega(F) - \Lambda(F)$, then any element g in L is expressed in one of the following form:*

(i) *$g = M_f(a, b, c, d, e)$, where $a, b, e \in F^*$ and $c, d \in F$ satisfying $f(d + a^{-1}b\bar{e}y) = f(d) + ab^{-1}ef(y)$ for any $y \in F$.*

(ii) *$g = \begin{pmatrix} J & O \\ O & J \end{pmatrix} M_f(a, b, c, d, e)$, where $a, b, e \in F^*$ and $c, d \in F$ satisfying*

$f(d+a^{-1}b\bar{e}f(y))=f(d)+ab^{-1}ey$ for any $y\in F$.

Proof. By Lemmas 5.1 and 6.2, $g=\begin{pmatrix} A & AC \\ O & AD \end{pmatrix}$, where $A=J^m\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $C=\begin{pmatrix} c & d \\ f(d) & \bar{c} \end{pmatrix}\in\Sigma$ and $D=\begin{pmatrix} e & t \\ f(t) & \bar{e} \end{pmatrix}\in\Sigma-\{0\}$ with $a, b, c, d, e, t\in F$ and $0\leqq m\leqq 1$.

Assume $m=0$. Then $g\in L$ if and only if $Y=C+A^{-1}MAD\in\Sigma$ for any $M\in\Sigma$. Set $M=\begin{pmatrix} x & y \\ f(y) & \bar{x} \end{pmatrix}$. By calculation $Y=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, where $a'=c+ex+a^{-1}b f(t)y$, $b'=d+tx+a^{-1}b\bar{e}y$, $c'=f(d)+ab^{-1}ef(y)+f(t)\bar{x}$ and $d'=\bar{c}+ab^{-1}tf(y)+\bar{e}\bar{x}$. Hence $\overline{a^{-1}bf(t)y}=ab^{-1}tf(y)$ (*) and $f(d+tx+a^{-1}b\bar{e}y)=f(d)+ab^{-1}ef(y)+f(t)\bar{x}$. Suppose $t\neq 0$. In view of the equation (*) we have $f(y)=u\bar{y}$, where $u=(b/a)^{q+1}(f(t))^q/t$. Moreover, $u\notin K$ by Proposition 2.1. Hence $u^{1-q}=(b/a)^{q+1}\in\{\pm 1\}$ and so $u^{q-1}=-1$. This implies $f\in\Lambda(F)$, a contradiction. Thus $t=0$ and (i) follows.

Assume $m=1$. By a similar argument as above $Y=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\in\Sigma$, where $a'=c+a^{-1}bf(t)f(y)+e\bar{x}$, $b'=d+a^{-1}b\bar{e}f(y)+t\bar{x}$, $c'=f(d)+f(t)x+ab^{-1}ey$ and $d'=\bar{c}+\bar{e}x+ab^{-1}ty$ for any $x, y\in F$. From this, $\overline{a^{-1}bf(t)f(y)}=ab^{-1}ty$ (**) and $f(d+a^{-1}b\bar{e}f(y)+t\bar{x})=f(d)+f(t)x+ab^{-1}ey$. If $t\neq 0$, then $f(y)=u\bar{y}$ by (**), where $u=(a/b)^{q+1}(t^q/f(t))=(a/b)^{q+1}/u$. Hence $u^2=(a/b)^{q+1}$ and so $u^{q-1}\in\pm\{1\}$. By Proposition 2.1, $u\notin K$ and therefore $u^{q-1}=-1$, which implies $f\in\Lambda(F)$. This is a contradiction. Thus $t=0$ and (ii) follows.

REMARK 6.5. As we have shown in Lemma 6.1, many collineations of the form (i) actually exist. However, collineations of the form (ii) do not necessarily exist and the existence depends on the property of the function $f\in\Omega(F)$.

7. Derivations

In this section we show that any plane in the class $\Pi(F)$ is derivable. The content of this section was suggested by V. Jha and N.L. Johnson [11].

We consider an arbitrary fixed element f of $\Omega(F)$. Set $\Sigma=\Sigma_f$ and $\pi=\pi_f$. We denote the elements of Σ by $M(x, y)=\begin{pmatrix} x & y \\ f(y) & \bar{x} \end{pmatrix}$ for all $x, y\in F$. Let $\tilde{S}$ consist of the following 2-dimensional F-subspaces of $V(=F^4)$.:

$$V_\infty=0\times 0\times F\times F\,,\quad V_{a,b}=\{(v, vM(a, b))\,|\,v\in F\times F\}\quad (a, b\in F)\,.$$

Then, $\tilde{S}$ is the spread of V concerned with Σ. Set $\tilde{R}=\{V_\infty, V_{a,0}\,|\,a\in F\}$.

Lemma 7.1. *Let g be an element of $LC(\pi)$ which leaves the set $\tilde{R}$ invariant. Then $g=\begin{pmatrix} J & O \\ O & J \end{pmatrix}^i\begin{pmatrix} A & AC \\ O & AD \end{pmatrix}$, where $A=\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $C=\begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix}$, $D=\begin{pmatrix} d & 0 \\ 0 & \bar{d} \end{pmatrix}$, $J=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $a, b, c, d\in F^*$ and $i\in\{0, 1\}$.*

Proof. By Lemma 5.1, $g=\begin{pmatrix} A & AC \\ O & AD \end{pmatrix}$ for some $A\in GL(2, q^2)$ and $C, D\in\Sigma$. Since $A^{-1}(AC+MAD)\in\Sigma_0=\{\begin{pmatrix} h & 0 \\ 0 & \bar{h} \end{pmatrix} | h\in F\}$ for each $M\in\Sigma_0$, we have $C=\begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix}$ and $D=\begin{pmatrix} d & 0 \\ 0 & \bar{d} \end{pmatrix}$ for some $c\in F$ and $d\in F^\sharp$. Hence $A^{-1}MA\in\Sigma_0$ for any $M\in\Sigma_0$.

There exists $j\in\{0, 1\}$ such that $J^jA=\begin{pmatrix} a & s \\ t & b \end{pmatrix}\in GL(2, q^2)$ and $a\neq 0$. Since $J^{-j}\Sigma_0 J^j=\Sigma_0$, $\begin{pmatrix} a & s \\ t & b \end{pmatrix}^{-1}\begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}\begin{pmatrix} a & s \\ t & b \end{pmatrix}=\frac{1}{ab-st}\begin{pmatrix} abx-ts\bar{x} & bs(x-\bar{x}) \\ at(\bar{x}-x) & ab\bar{x}-tsx \end{pmatrix}\in\Sigma_0$ for any $x\in F$. From this $at=bs=0$. Hence $t=s=0$ and the lemma follows.

Theorem 7.2. *$\tilde{R}$ is a derivable partial spread of $\tilde{S}$. The kernel of the derived plane π' of π with respect to $\tilde{R}$ is isomorphic to K.*

Proof. Put $W_\infty=0\times F\times 0\times F$ and $W_a=\{(x, \bar{x}a, y, \bar{y}a) | x, y\in F\}$ for $a\in F$. It is easy to see that $\tilde{R}'=\{W_a | a\in F\cup\{\infty\}\}$ is the derived partial spread of $\tilde{R}$. Since the group of kern homologies of π' contains $\{k\begin{pmatrix} E & O \\ O & E \end{pmatrix} | k\in K^\sharp\}$, the kernel K' of π' is isomorphic to $GF(q)$, $GF(q^2)$ or $GF(q^4)$.

Assume $K'\cong GF(q^4)$. Then π is a Hall plane of order q^4 (cf. [13] Chapter 13). But, we obtain a contradiction by Lemma 5.1.

Assume $K'\cong GF(q^2)$ and let $K_0=\langle w\rangle$ be the group of the kern homologies of π'. Then $|K_0|=q^2-1$ and $K_0\leqq C(\pi)$ by Theorem 10.6 of [8]. Set $q=p^n$ and $g=w^{2n}$. Since $C(\pi)\leqq\Gamma L(4, F)$ by Theorem 1.10 of [13], $g\in LC(\pi)$ and therefore g can be expressed in the form described in Lemma 7.1. Let a, g, c, d and i be as in the lemma.

If $i=1$, then $(W_\infty)g=W_0$, a contradiction. If $i=0$, then $W_k=(W_k)g=W_l$, $l=ka^{-q}b$ for all $k\in F$. Thus $b=a^q$. Moreover, as $C+A^{-1}M(s, t)AD=\begin{pmatrix} c+sd & a^{q-1}td^q \\ * & * \end{pmatrix}=M(c+sd, a^{q-1}td^q)$, $V_{s,t}=(V_{s,t})g=V_{c+sd,a^{q-1}td^q}$ for all $s\in F$ and $t\in F^\sharp$. Thus $c=0$, $d=1$ and $a^{q-1}=1$. Therefore $g=a\begin{pmatrix} E & O \\ O & E \end{pmatrix}$ with $a\in K^\sharp$. It follows that $|g|\leqq p^n-1$. On the ther hand $|g|\geqq(p^{2n}-1)/2n$. Hence $2n\geqq p^n+1\geqq 2^n+1$, a contradiction. Therefore $K'=GF(q)$.

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