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MINIMAL L-SPACE AND HALMOS-SAVAGE CRITERION FOR MAJORIZED EXPERIMFNTS

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1. Introduction

An experiment $\mathcal{E}$ is a triplet $(X, \mathcal{A}, \mathcal{P})$, where $\mathcal{P}$ is a non-empty set of probability measures on a σ-field $\mathcal{A}$ of subsets of a set X. $ca(\mathcal{A})$ denotes the space of all bounded signed measurs on $\mathcal{A}$. The closed vector sublattice $L_m(\mathcal{E})$ of $ca(\mathcal{A})$ generated by $\mathcal{P}$ is called the minimal L-space of the experiment $\mathcal{E}$ (Le Cam [8], p. 41). $\mathcal{E}$ is said to be majorized if there exists a measure μ on $\mathcal{A}$ such that each $P\in\mathcal{P}$ has a density with respect to μ. In this case, μ is called a majorizing measure for $\mathcal{E}$. The class of majorized experiments includes the weakly dominated experiments, where μ is localizable (see Mussmann [12]), the Σ-finite experiments (see Le Cam [8], p. 13 and p. 667), where μ is decomposable, the semi-decomposable experiments (see Luschgy and Mussmann [10]), and the discrete experiments, where μ is the counting measure on 2^X (see Basu and Ghosh [1]).

For $\nu\in ca(\mathcal{A})^+$, a set S in $\mathcal{A}$which satisfies $\nu(S^c)=0$ and $P(\cdot\cap S)\ll\nu$ for all $P\in\mathcal{P}$ is called an $\mathcal{E}$-support of ν. $\mathcal{E}$ is majorized if and only if each $P\in\mathcal{P}$ has an $\mathcal{E}$-support (cf. Diepenbrock [2], Lemma 9.3, Ramamoorthi and Yamada [15], Proposition 1, or Luschgy and Mussmann [9], Theorem 1). Throughout the present paper we assume that $\mathcal{E}$ is majorized. For a set H of measures on $\mathcal{A}$, put $N(H)=\{A\in\mathcal{A}: \nu(A)=0 \text{ for all } \nu\in H\}$. If $\{h_i, i\in I\}$ is a family of $\mathcal{A}$-measurable functions, then $\sigma(h_i, i\in I)\vee N(H)$ denotes the smallest sub-σ-field (subfield, for short) of $\mathcal{A}$ which contains $N(H)$ and for which each h_i, $i\in I$, is measurable. A subfield $\mathcal{B}$ of $\mathcal{A}$ is said to be PSS (pairwise sufficient containing supports) for $\mathcal{E}$ if $\mathcal{B}$ is pairwise sufficient for $\mathcal{E}$ and each $P\in\mathcal{P}$ has an $\mathcal{E}$-support belonging to $\mathcal{B}$. An equivalent majorizing measure μ is called pivotal measure for $\mathcal{E}$ if the following condition is satisfied: a subfield $\mathcal{B}$ of $\mathcal{A}$ is PSS for $\mathcal{E}$ if and only if each $P\in\mathcal{P}$ has a $\mathcal{B}$-measurable μ-density (cf. Ramamoorthi and Yamada [15]). Obviously, μ is pivotal if and only if

$$\sigma\left(\frac{dP}{d\mu},\ P\in\mathcal{P}\right)\wedge N(\mathcal{P})$$

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is smallest PSS. If $\mathcal{E}$ is majorized by a σ-finite measure then the sufficiency criterion of Halmos and Savage [6], Theorem 1, implies that each equivalent finite majorizing measure of the form $\sum_n c_n P_n$ with $c_n \geqq 0$ and $P_n \in \mathcal{P}$ is pivotal. From this theorem the Neyman factorization theorem easily follows. In order to prove extensions of these results for arbitrary majorized experiments, pivotal measures have been used by Ghosh et al. [5] and by Ramamoorthi and Yamada [15]. In [20] pivotal measures have been applied to construct common conditional probabilities in an extended form. We shall show that pivotal measures are closely related to maximal orthogonal systems in $L_m(\mathcal{E})$.

An orthogonal system W in a vector lattice V is a subset of $V^+\setminus\{0\}$ such that $u \wedge v = 0$ for all distinct members u and v of W. If $D \subset L_m(\mathcal{E})$ is a maximal orthogonal system, we define a measure ν_D on $\mathcal{A}$ by $\nu_D(A) = \sup \{\sum_{w \in F} w(A) : F \subset D,$ F finite$\}$. Notice that each maximal orthogonal system of $L_m(\mathcal{E})$ is also a maximal orthogonal system of the L-space of $\mathcal{E}$ and therefore ν_D is an equivalent majorizing measure for $\mathcal{E}$ (Luschgy and Mussmann [9], Theorem 1, see also Torgersen [19], p. 10). We shall prove the following results: $\sigma(dP/d\nu_D, P \in \mathcal{P}) \vee N(\mathcal{P})$ is a smallest PSS subfield and a pairwise smallest sufficient subfield in the sense of [5]. This implies that ν_D is pivotal. Conversely, each pivotal measure is of the type ν_D. $L_m(\mathcal{E})$ can be characterized as the set of all measures on $\mathcal{A}$ having $\sigma(dP/d\nu_D, P \in \mathcal{P}) \vee N(\mathcal{P})$-measurable densities with respect to ν_D. This generalizes a result by Torgersen [18], p. 47.

Furthermore, we discuss the relation between maximal orthogonal systems in $L_m(\mathcal{E})$ and maximal decompositions of X which have been used in the literature to prove the existence of pivotal measures (cf. Ramamoorthi and Yamada [15]). We need some more notations. Let μ be a measure on $\mathcal{A}$. $L^1(\mu)$ denotes the space of all μ-integrable functions. If $f \in L^1(\mu)$, then $f \cdot \mu$ is the bounded signed measure on $\mathcal{A}$ with μ-density f. Set $L(\mu) = \{f \cdot \mu : f \in L^1(\mu)\}$. The map from $L^1(\mu)$ onto $L(\mu)$ which carries $f \in L^1(\mu)$ into $f \cdot \mu$ is an isometric vector lattice isomorphism. This is easily seen by means of the Radon-Nikodym theorem since $\{f > 0\}$ has σ-finite μ-measure for each $f \in L^1(\mu)$.

2. Auxiliary Results

Put $\mathcal{E}^* = (X, \mathcal{A}, \mathcal{P}^*)$ where $\mathcal{P}^*$ is the set of all probability measures in $L_m(\mathcal{E})$. In the following we shall see that in some situations $\mathcal{E}$ can be replaced without loss of generality by $\mathcal{E}^*$.

Proposition 2.1. *Suppose $\mathcal{B} \subset \mathcal{A}$ is a subfield. Let W denote the set of all $w \in ca(\mathcal{A})$ of the form $w = f \cdot (\sum_n 2^{-n} P_n)$ where f is $\mathcal{B}$-measurable and $P_n \in \mathcal{P}$. The following assertions hold:*

a) *If each $P \in \mathcal{P}$ has an $\mathcal{E}$-support belonging to $\mathcal{B}$, then each $w \in W$ has an $\mathcal{E}$-*

support which belongs to $\mathcal{B}$.

b) *If $\mathcal{B}$ is pairwise sufficient for $\mathcal{P}$, then W is a closed vector sublattice of $ca(\mathcal{A})$ and $\mathcal{B}$ is pairwise sufficient for the set of all probability measures from W.*

Proof.

a) If f is $\mathcal{B}$-measurable, $w=f\cdot(\sum_n 2^{-n}P_n)$, $w\in ca(\mathcal{A})$, and $T_n\in\mathcal{B}$ is an $\mathcal{E}$-support for P_n for each positive integer n, then

$$\{f>0\}\cap(\bigcup_n T_n)\in\mathcal{B}$$

is an $\mathcal{E}$-support for w.

b) If $\mathcal{B}$ is pairwise sufficient for $\mathcal{P}$, it is also sufficient for each dominated subset of $\mathcal{P}$. Therefore, by a theorem of Halmos and Savage [6], Theorem 1, we can assume that $dP/d(\sum_n 2^{-n}P_n)$ is $\mathcal{B}$-measurable whenever $P\in\mathcal{P}$ is absolutely continuous with respect to $\sum_n 2^{-n}P_n$. From this we obtain that for each sequence (w_m) in W there is a sequence (f_m) of $\mathcal{B}$-measurable functions and a sequence (P_n) in $\mathcal{P}$ such that $w_m=f_m\cdot(\sum_n 2^{-n}P_n)$ for all m. By means of these representations it is easily shown that W is a closed vector sublattice. Furthermore, we see that it only remains to prove that $\mathcal{B}$ is sufficient for subsets of probability measures $w\in W$ of the form $w=f\cdot(\sum_n 2^{-n}P_n)$ where f is $\mathcal{B}$-measurable and the sequence (P_n) is fixed. For such a $w\in W$ we get

$$\int_B 1_A dw=\int_B 1_A f d(\sum_n 2^{-n}P_n)=\int_B E_\bullet(1_A\,|\,\mathcal{B})fd(\sum_n 2^{-n}P_n)$$
$$=\int_B E_\bullet(1_A\,|\,\mathcal{B})\,\mathrm{d}w \quad \textit{for all} \quad A\in\mathcal{A} \quad \textit{and} \quad B\in\mathcal{B}$$

where $E_\bullet(1_A\,|\,\mathcal{B})$ is a common conditional expectation for the sequence (P_n): $E_\bullet(1_A\,|\,\mathcal{B})=E_{P_n}(1_A\,|\,\mathcal{B})P_n$-a.e. for all n. □

Corollary 2.2. *Suppose $\mathcal{B}\subset\mathcal{A}$ is a subfield. Then the following assertions hold:*

a) *If each $P\in\mathcal{P}$ has an $\mathcal{E}$-support belonging to $\mathcal{B}$, then each $Q\in\mathcal{P}^*$ has an $\mathcal{E}$-support belonging to $\mathcal{B}$.*

b) *If $\mathcal{B}$ is pairwise sufficient for $\mathcal{E}$, then $\mathcal{B}$ is pairwise sufficient for $\mathcal{E}^*$.*

An inspection of the proof of Proposition 2.1 shows that Corollary 2.2.b holds if "pairwise sufficient" is replaced by "sufficient". Notice that here and in Proposition 2.1.b and Corollary 2.2.b we do not use the assumption that $\mathcal{E}$ is majorized, From [9], Lemma 1, we see that $\mathcal{E}^*$ is majorized by a measure μ whenever $\mathcal{E}$ is majorized by μ.

Lemma 2.3. *Suppose $\mathcal{E}$ is majorized by μ. Then we have*

$$\sigma\left(\frac{dQ}{d\mu}, Q\in\mathcal{P}^*\right)\vee N(\mathcal{P}) = \sigma\left(\frac{dP}{d\mu}, P\in\mathcal{P}\right)\vee N(\mathcal{P}).$$

Proof. The set of all $f\in L^1(\mu)$ such that f is $\sigma(dP/d\mu, P\in\mathcal{P})\vee N(\mathcal{P})$-measurable is a closed vector sublattice of $L^1(\mu)$. Because of the vector lattice isomorphism between $L^1(\mu)$ and $L(\mu)$ (see Section 1), the proposition easily follows. □

Now we need a lemma which we shall use for the calculation of the densities if the majorizing measure is of the form ν_D (see Section 1). If V is an L-space, we define $\pi_x(y)=\sup_n (y\wedge nx)$ for all $x, y\in V^+$; $\pi_x(y)$ is the projection of y onto the band generated by x ([16], Proposition II.2.11 and Corollary 2). Note that for every L-space there exist maximal orthogonal systems by Zorn's lemma.

Lemma 2.4. *Suppose V is an L-space. Then the following assertions hold:*

a) *If $(x_i, i\in I)$ is an increasing net in V^+ with $\sup_i \|x_i\|<\infty$, then $\lim_i x_i$ exists and $\lim_i x_i=\sup_i x_i$.*

b) *If D is a maximal orthogonal system in V and $y\in V^+$, then*

$$y = \sup_\Lambda \sum_{u\in\Lambda} \pi_u(y) = \lim_\Lambda \sum_{u\in\Lambda} \pi_u(y)$$

where Λ ranges through finite nonempty subsets of D. The set $\{u\in D: \pi_u(y)\neq 0\}$ is countable.

Proof.

a) See [3], proof of Theorem 26 B.

b) By [7], Lemma 3.5, we get

$$\sum_{u\in\Lambda} \pi_u(y) = \pi_{\sum_{u\in\Lambda} u}(y)\leqq y.$$

Then a) implies

$$\sup_\Lambda \sum_{u\in\Lambda} \pi_u(y) = \lim_\Lambda \sum_{u\in\Lambda} \pi_u(y).$$

By the Riesz decomposition theorem ([16], Theorem II.2.10), the band in V generated by D is equal to V. From [16], Proposition II.2.11, we get $\sup_\Lambda \pi_{\sum_{u\in\Lambda} u}(y)=y$. If $\pi_u(y)\neq 0$ for all u from an uncountable subset of D, then there is an $\varepsilon>o$ such that $\{u\in D: \|\pi_u(y)\|\geqq\varepsilon\}$ is infinite. Because $\|y\|\geqq \sum_{u\in\Lambda} \|\pi_u(y)\|$ for all Λ, we get a contradiction. □

If $u, v\in L_m(\mathcal{E})^+$, then $\pi_v(u)\in L_m(\mathcal{E})$ by Lemma 2.4.a. Using the vector lattice isomorphism between $L^1(\mu)$ and $L(\mu)$ with $\mu=u+v$, we get

$$\mu_v(u) = \left(\frac{du}{d\mu} 1_{\{dv/d\mu>0\}}\right) \cdot \mu \, .$$

Proposition 2.5. *Suppose $D \subset L_m(\mathcal{E})$ is a maximal orthogonal system. Then for each $v \in L_m(\mathcal{E})^+$ there is a countable subset $D' \subset D$ such that $\sum_{w \in D'} 1_{S_w} \frac{d\pi_w(v)}{dw}$ is a density of v with respect to ν_D.*

Proof. By Lemma 2.4.b, there is a countable subset $D' \subset D$ such that $v = \sum_{w \in D'} \pi_w(v)$. Using properties of the $\mathcal{E}$-supports S_w of w, we see that

$$f = \sum_{w \in D'} 1_{S_w} \frac{d\pi_w(v)}{dw}$$

is a density of v with respect to ν_D. □

Proposition 2.5 is essentially known. The above form of the density is given by Torgersen [19], p. 10, for $v \in \mathcal{P}$.

EXAMPLE 2.6.

a) Suppose $\mathcal{E}$ is majorized by a σ-finite measure. Then there is a majorizing measure of the form $\nu = \sum_n 2^{-n} P_n$, $P_n \in \mathcal{P}$. The set $D = \{\nu\}$ is a maximal orthogonal system in $L_m(\mathcal{E})$ and $\nu = \nu_D$.

b) If $\mathcal{A}$ is the power set of X and if $\mathcal{P}$ contains all Dirac measures, then the subset D of all Dirac measures is a maximal orthogonal system in $L_m(\mathcal{E})$ and ν_D is the counting measure.

3. Main Results

In the situation of Example 2.6.a it is known that $\sigma(dP/d\nu_D, P \in \mathcal{P}) \vee N(\mathcal{P})$ is a smallest sufficient subfield. For an arbitrary majorized experiment we shall show in Theorem 3.1 that a subfield of this form is not dependent on the special maximal orthogonal system D and that it is smallest PSS.

We define a subfield $\mathcal{A}_m \subset \mathcal{A}$ by

$$\mathcal{A}_m = \sigma\left(\frac{dP}{d\nu_D}, P \in \mathcal{P}\right) \vee N(\mathcal{P}) \, ,$$

where $D \subset L_m(\mathcal{E})$ is a maximal orthogonal system. We use the terms "pairwise smallest sufficient" and "smallest pairwise sufficient containing supports" smallest PSS, for short) in the sense of [5].

Theorem 3.1. *The subfield $\mathcal{A}_m$ is pairwise smallest sufficient and smallest PSS for $\mathcal{E}$. Especially, ν_D is a pivotal measure for each maximal orthogonal system $D \subset L_m(\mathcal{E})$.*

Proof. Without loss of generality we assume $\mathcal{E}=\mathcal{E}^*$ (see Proposition 2.1). Obviously, $\mathcal{A}_m$ contains an $\mathcal{E}$-support for each $P\in\mathcal{P}$. Next we show that $\mathcal{A}_m$ is pairwise sufficient. Suppose $P_1, P_2\in\mathcal{P}$. Put $\mu=P_1+P_2\in L_m(\mathcal{E})$. There are $\mathcal{A}_m$-measurable versions of $dP_i/d\mu$ since

$$\frac{dP_i}{d\mu}=\frac{dP_i}{d\nu_D}1_{\{d\mu/d\nu_D>0\}}\left(\frac{d\mu}{d\nu_D}\right)^{-1}\ \mu\text{-a.e}\ .$$

Thus $\mathcal{A}_m$ is sufficient for $\{P_1, P_2\}$ because of [6], Theorem 1. It remains to investigate the minimality of $\mathcal{A}_m$. Let $\mathcal{S}\subset\mathcal{A}$ be a pairwise sufficient subfield. For $P\in\mathcal{P}$ there is a countable subset $D'\subset D$ such that $P=\sum_{w\in D'}\pi_w(P)$ (see Lemma 2.4.b). Let $\kappa\in ca(\mathcal{A})$ be of the form $\kappa=\sum_{w\in D'}c_w w$, $c_w\geqq 0$. Since $\mathcal{E}=\mathcal{E}^*$ and since in the dominated case pairwise sufficiency implies sufficiency, $\mathcal{S}$ is sufficient for

$$\{\|w\|^{-1}w\colon w\in D'\}\cup\{\|\pi_w P\|^{-1}\pi_w P\colon \pi_w P\neq 0,\ w\in D'\}\ .$$

By [6], Theorem 1, we may assume that $d\pi_w(P)/d\kappa$ and $dw/d\kappa$ are $\mathcal{S}$-measurable for all $w\in D'$. Furthermore,

$$\frac{dP}{d\nu_D}=\sum_{w\in D'}\frac{d\pi_w(P)}{dw}1_{S_w}=\sum_{w\in D'}\left(\frac{d\pi_w(P)}{d\kappa}\Big/\frac{dw}{d\kappa}\right)1_{\{\frac{dw}{d\kappa}>0\}}\ \kappa\text{-a.e.}$$

Hence $dP/d\nu_D$ is $\mathcal{S}\vee N(\kappa)$-measurable. For fixed $P_1, P_2\in\mathcal{P}$ we may suppose that $P_1, P_2\ll\kappa$ holds in the above calculation. Therefore $dP/d\nu_D$ is $\mathcal{S}\vee N(P_1+P_2)$-measurable for all $P_1, P_2\in P$, and $\mathcal{A}_m$ is pairwise smallest sufficient. If $\mathcal{S}$ contains an $\mathcal{E}$-support for each $P\in\mathcal{P}$, then $\{dw/d\nu_D>0\}\in\mathcal{S}cvN(\mathcal{P})$ for all $w\in D$ and

$$\frac{dP}{d\nu_D}=\sum_{w\in D'}\left(\frac{d\pi_w(P)}{d\kappa}\Big/\frac{dw}{d\kappa}\right)1_{\{\frac{dw}{d\nu_D}>0\}}\ \nu_D\text{-a.e.}$$

Hence $dP/d\nu_D$ is $\mathcal{S}\vee N(\mathcal{P})$-measurable. We conclude that $\mathcal{A}_m$ is smallest PSS and ν_D is a pivotal measure. □

The existence of a smallest PSS subfield has been proved by Ghosh et at. [5], Theorem 5. A detailed discussion of the smallest PSS subfield can be found in Fujii and Morimoto [4], Theorem 5. Pairwise smallest sufficiency is treated in Siebert [17] (see also [5], Theorem 5) and using invariance considerations in [11]. In Theorem 3.4 we shall see that any piovtal measure can be represented by means of a suitable maximal orthogonal system in $L_m(\mathcal{E})$. First we give a more concrete representation of $L_m(\mathcal{E})$.

Theorem 3.2. *Suppose $D\subset L_m(\mathcal{E})$ is a maximal orthogonal system. Then*

$$L_m(\mathcal{E}) = \{f \cdot \nu_D : f \in L^1(\nu_D | \mathcal{A}_m)\} .$$

Proof. It suffices to prove that the inclusion $\supset$ holds. Because of the form of the densities given in Proposition 2.5, it is enough to show for each fixed $w \in D$ that for each $f \in L^1(\pi_D | \mathcal{A}_m)$ there is a $u \in L_m(\mathcal{E})$ with

$$f \cdot 1_{S_m} = \frac{du}{d\nu_D} \cdot 1_{S_w} \ \nu_D\text{-a.e.}$$

This follows from [14], proof of Proposition I-1-1, since $\nu_D(\cdot \cap S_w) = w$ is a finite measure. □

In Theorem 3.2 $\mathcal{A}_m$ can be replaced by any pairwise smallest sufficient subfield. This theorem generalizes a result of Torgersen [18], p. 47, for dominated experiments and of Mussmann [13], Proposition 2.1 and Proposition 2.5, for weakly dominated experiments. The latter paper also gives a characterization of the smallest sufficient subfield.

Theorem 3,3. *If μ is a pivotal measure for $\mathcal{E}$, then $\mu = \nu_G$ for some maximal orthogonal system G in $L_m(\mathcal{E})$.*

Proof. Let $D \subset L_m(\mathcal{E})$ be a maximal orthogonal system. Put $S_w = \{dw/d\mu > 0\}$ for all $w \in D$. S_w is an $\mathcal{E}$-support for w. By Lemma 2.3, $dw/d\mu$ is $\mathcal{A}_m$-measurable. Hence $S_w \in \mathcal{A}_m$ and there is a countable set K_w and a pairwise disjoint family $(S_{wk}, k \in K_w)$ in $\mathcal{A}_m$ with $S_w = \bigcup_{k \in K_w} S_{wk}$ and $0 < \mu(S_{wk}) < \infty$ for all $k \in K_w$. By Theorem 3.2, the measures $1_{S_{wk}} \cdot w$, $w \in D$ and $k \in K_w$, also define a maximal orthogonal system in $L_m(\mathcal{E})$. Therefore we shall assume without loss of generality that $0 < \mu(S_w) < \infty$ holds for all $w \in D$. Put $v_x = \mu(\cdot \cap S_w)$ for all $w \in D$. $v_w \in L_m(\mathcal{E})$ because of

$$v_w = ((dw/d\mu)^{-1} \cdot 1_{S_w}) \cdot \nu_D$$

and Theorem 3.2. Furthermorc, v_w and w are equivalent for all $w \in D$. Therefore $G = (v_w, w \in D)$ is a maximal orthogonal system in $L_m(\mathcal{E})$. It is easily shown that $\mu = \nu_G$ since μ is semi-finite, that is $\mu(A) = \sup\{\mu(F): F \subset A, F \in A$, and $\mu(F) < \infty\}$. □

EXAMPLE 3.4. Suppose X is the unit interval, $\mathcal{A}$ the corresponding Borel sets, and $\mathcal{P} = \{\lambda\}$ where λ is the Lebesgue measure on X. Then λ is also pivotal for $\mathcal{E}$ since $\{\emptyset, X\}$ is a smallest sufficient subfield. We have $L_m(\mathcal{E}) = \{v: v = \alpha\lambda$ for some real $\alpha\}$. Because $\lambda = \sum_{i \in J} \lambda(\cdot \cap A_i)$ for any countable measurable partition $\{A_i, i \in J\}$ of X, we see that the pivotal measure λ can be represented as a sum of orthogonal measures which are not from $L_m(\mathcal{E})$.

For each $P \in \mathcal{P}$ let $S_p \in \mathcal{A}_m$ be an $\mathcal{E}$-support for P. A subset $\mathcal{F} \subset \mathcal{A}_m$ is called a maximal $\mathcal{E}|\mathcal{A}_m$-decomposition if $F_1\cap F_2\in N(\mathcal{P})$ for distinct members $F_1, F_2\in\mathcal{F}_1$, for each $F\in\mathcal{F}$ there is a $P_F\in\mathcal{P}$ with $P_F(F)>0$ and $F\backslash S_{P_F}\in N(\mathcal{P})$, and each $B\in\mathcal{A}_m$ such that $B\backslash S_Q\in N(\mathcal{P})$ for some $Q\in\mathcal{P}$ and $B\cap F\in N(\mathcal{P})$ for all $F\in\mathcal{F}$ is in $N(\mathcal{P})$. Such an $\mathcal{F}$ exists by Zorn's lemma (cf. [15], p. 171). In [15], Proposition 3, it is shown that $\sum_{F\in\mathcal{F}} P_F(\cdot\cap F)$ defines a pivotal measure. This will also follow from Theorem 3.1 and our next theorem, where the relation between maximal $\mathcal{E}^*|\mathcal{A}_m$-decompositions and maximal orthogonal systems in $L_m(\mathcal{E})$ is exhibited. It easily follows from the definition that each maximal $\mathcal{E}|\mathcal{A}_m$-decomposition is a maximal $\mathcal{E}^*|\mathcal{A}_m$-decomposition.

Theorem 3.5. *The following assertions hold*

a) *If $\mathcal{F}$ is a maximal $\mathcal{E}^*|\mathcal{A}_m$-decomposition, then $\{P_F(\cdot\cap F): F\in\mathcal{F}\}$ is a maximal orthogonal system in $L_m(\mathcal{E})$.*

b) *If D is a maximal orthogonal system in $L_m(\mathcal{E})$ and $S_w\in\mathcal{A}_m$ is an $\mathcal{E}$-support for each $w\in D$, then $\{S_w: w\in D\}$ is a maximal $\mathcal{E}^*|\mathcal{A}_m$-decomposition.*

Proof.

a) Let D be a maximal orthogonal system in $L_m(\mathcal{E})$. By Theorem 3.2, for each $F\in\mathcal{F}$ there is an $\mathcal{A}_m$-meesurable g_F such that

$$P_F(\cdot\cap F) = (1_F g_F)\cdot\nu_D\in L_m(\mathcal{E})\,.$$

We conclude that $\{P_F(\cdot\cap F\}): F\in\mathcal{F}\}$ is an orthogonal system in $L_m(\mathcal{E})$. Suppose $v\in L_m(\mathcal{E})^+$ and $v\wedge P_F(\cdot\cap F)=0$ for all $F\in\mathcal{F}$. By Theorem 3.2, $v=f\cdot\nu_D$ for some $\mathcal{A}_m$-measurable f. We get

$$0=\nu_D(\{f>0\}\cap\{1_F g_F>0\})=\nu_D(\{f>0\}\cap F)\qquad\text{for all}\quad F\in\mathcal{F}\,.$$

The definition of $\mathcal{F}$ implies $\nu_D(\{f>0\})=0$. Hence $v=0$, and the maximality of $\{P_F(\cdot\cap F): F\in\mathcal{F}\}$ follows.

b) Suppose $B\in\mathcal{A}_m$, $B\backslash S_0\in N(\mathcal{P})$ for some $Q\in\mathcal{P}^*$, and $B\cap S_w\in N(\mathcal{P})$for all $w\in D$. By Theorem 3.2, $Q(\cdot\cap B)=(1_B f)\cdot\nu_D$ for some $\mathcal{A}_m$-measurable f. We conclude $Q(\cdot\cap B)\in L_m(\mathcal{E})$ and $Q(\cdot\cap B)\wedge w=0$ for all $w\in D$. Maximality of D implies $Q(\cdot\cap B)=0$. Hence $B\in N(\mathcal{P})$. Now it is easily seen that $\{S_w: w\in D\}$ is a maximal $\mathcal{P}^*|\mathcal{A}_m$-decomposition. □

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