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PRODUCTS OF TORSION THEORIES
AND APPLICATIONS TO
COALGEBRAS

I.-P. LIN

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1. Introduction

Throughout this note \( R \) is a ring with 1. We shall write \( / \unlhd \leq R \) if \( / \) is a right ideal of \( R \). A non-empty set of right ideals \( \Gamma \) of \( R \) is called a Gabriel filter if it satisfies

T1. If \( I \in \Gamma \) and \( r \in R \), then \( (I:r) \in \Gamma \).

T2. If \( / \) is a right ideal and there exists \( J \in \Gamma \) such that \( (I:r) \in \Gamma \) for every \( r \in I \), then \( I \in \Gamma \).

It is well-known [4] that there is a one to one correspondence between Gabriel filters of \( R \) and hereditary torsion theories for the category of right \( R \)-modules. W. Schelter [3] investigated products of torsion theories or equivalently of Gabriel filters that for a family of pairs \( \{(R_i, \Gamma_i), \Gamma_i: \text{Gabriel filter of } R_i\} \), \( \Gamma_0 = \{D \leq \pi R_i | D \supseteq \sum \sigma D \} \) is a Gabriel filter of the product ring \( \pi R_i \), furthermore the ring of right quotient of \( \pi R_i \) with respect to \( \Gamma_0 \) is isomorphic to the product of rings of right quotient of \( R_i \) with respect to \( \Gamma_i \). This result generalizes one of Y. Utumi theorems [6]. In this paper these two sets \( \Gamma_1 = \{D \leq \pi R_i | D \supseteq \pi D_i, D_i \in \Gamma_i \} \) and \( \Gamma_2 = \{D \leq \pi R_i | D \supseteq \pi D_i, D_i \in \Gamma_i \) and almost all \( D_i = R_i \} \) will be studied. \( \Gamma_1 \) does not always satisfy T2. A necessary and sufficient condition for \( \Gamma_1 \) to be a Gabriel filter is given. It follows that \( \Gamma_1 \) is a better notion of products of perfect torsion theories. However \( \Gamma_2 \) is a Gabriel filter of \( \pi R_i \), and we use this fact to prove that over an algebraically closed field, cocommutative coalgebra has a torsion rat functor if and only if each space of primitives of its irreducible components is finitedimensional.

For a coalgebra \( (C, \Delta, \epsilon) \) over a field \( K \), there exists a natural algebra structure on its dual space \( C^* = \text{Hom}_K (C, K) \) induced by the diagonal map \( \Delta \) and every left comodule \( (M, \phi_M) \) over \( C \) can be defined as a right \( C^* \)-module by \( mc^* = (c^* \otimes 1) \phi_M(M), m \in M, c^* \in C^* \). Moreover a right \( C^* \)-module \( M \) is called a rational module if it is a left comodule \( (M, \phi_M) \) over \( C \) and its right \( C^* \)-module structure is derived in the way described above. With these observations we can embed the category of left \( C \)-comodules \( \mathcal{C} \) as a full subcategory, into the category of right \( C^* \)-modules \( \mathcal{C}^* \). A subspace \( / \) of \( C^* \) is called cofinite
closed if $I = V^\perp$ for some finite-dimensional subspace $V$ of $C$.

We assume the reader is familiar with torsion theories of modules and elementary coalgebra theories. The terminology and notation are those of Stenstrom [4] and Sweedler [5].

2. Some properties

In this section we derive some properties of $\Gamma_1$ and $\Gamma_2$. For convenience, we write a pair $(R_i, \Gamma_i)$ as $\Gamma_i$ is a Gabriel filter of $R_i$. The following are easily proved.

Lemma 1. If $I$ is a right ideal of $R$ and there exists $J \in \Gamma$ such that $(I: r) \in \Gamma$ for $r$ runs through a family of generators of $J$, then $I \in \Gamma$.

Lemma 2. $\Gamma_1$, $\Gamma_2$ satisfy $T1$.

Proposition 1. If $\{(R_i, \Gamma_i)\}_{i=1}^n$ is a family of pairs and each $\Gamma_i$ has a cofinal family of $n$-generated right ideals (for a fixed integer $r_i$), then $\Gamma_i := \{D \leq \pi R_i | D \supset \pi D_i, D_i \in \Gamma_i, all \ i \in I\}$ is a Gabriel filter of $\pi R_i$. Moreover $(\pi R_i)_{\Gamma_i} \simeq \pi (R_i)_{\Gamma_i}$.

Proof. It only has to check $T2$ for $\Gamma_i$. Let $T \leq \pi R_i$ and $D \in \Gamma_i$ such that $(T: rf) \in \Gamma$ for every $d \in D$. We can assume $D = \pi D_i, D_i \in \Gamma_i$ and each $D_i$ has $n$ generators; $x_1^1, \cdots, x_1^n$. Construct $n$ elements of $\pi D_i$ as $x^1 = (x_1^1), \cdots, x^n = (x_1^n)$, then we have $(T: x_i^i) \in \Gamma_i$. Therefore for each $j = 1, \cdots, n$, there is $\pi D_i^{(j)}$ where $D_i^{(j)} \in \Gamma_i$ such that $x^j \pi D_i^{(j)} \in T$. However for fixed $i$ the finite sum $J_i = \sum_{j=1}^n x^j \pi D_i^{(j)} \in \Gamma_i$, by Lemma 1 and $\pi J_i = x^1 \pi D_i^{(1)} + \cdots + x^n \pi D_i^{(n)}$. This shows that $\pi J_i \in T \in \Gamma_i$.

Next we find an isomorphism from $\pi (R_i)_{\Gamma_i}$ to $(\pi R_i)_{\Gamma_i}$. Let $([f_i]) \in (\pi R_i)_{\Gamma_i}$, where $f_i \in \text{Hom}_{R_i}(D_i, R_i/[t_i(R_i)])$ and $[f_i]$ is its equivalent class in $(R_i)_{\Gamma_i}$, and define a $\pi R_i$-homomorphism $\psi$ from $\pi D_i$ to $\pi R_i/\pi (R_i)$ as $f_i((d_i)) = (f_i(d_i))$. Since $\psi((R_i)) = \psi(R_i)$, $\psi R_i/\pi (R_i) \psi R_i/\pi (R_i) \psi$ have a well-defined map $\alpha$ from $\pi (R_i)_{\Gamma_i}$ to $(\pi R_i)_{\Gamma_i}$, as $\alpha([f_i]) = [f]$, for if $f_i$ and $f'_i$ agree on $D_i$ for each $i$, then the corresponding $f$ and $f'$ agree on $\pi D_i$. It is routine to check that $\alpha$ is a one to one ring-homomorphism. Let $f: \pi D_i \rightarrow \pi R_i/\pi (R_i)$ a $\pi R_i$-homomorphism, $D \in \Gamma_i$ and define $f_i = \pi_i f e_i$, where $e_i$ is the ith-inclusion, $\pi_i$ is the ith-projection. Then $\alpha([f_i]) = [f]$. Thus $\alpha$ is an isomorphism.

Note. (1) we agree that $n$ generators of right ideals are not necessary distinct.

(2) In proposition 1, $\Gamma_1$ also has a cofinal family of $n$-generated right ideals.
Proposition 2. If \( \{(R_i, \Gamma_i), \ i \in I\} \) is a family of pairs, then \( \Gamma_2 = \{I \subseteq \pi R_i | I \supseteq \pi D_i, \ D_i \in \Gamma_i \text{ and almost all } D_i = R_i\} \) is a Gabriel filter of \( \pi R_i \).

Proof. Similarly it only has to check \( T_2 \) for \( \Gamma_2 \). Let \( I \subseteq \pi R_i \) and \( D \subseteq \Gamma_2 \) such that \( (/: d) \subseteq \Gamma_2 \) for all \( d \in D \). We can assume \( D = \pi D_i D_i \in \Gamma_i \) and except for \( D_i, k=1, \ldots, n \), all other \( D_i \) are equal to \( R_i \). Let \( e \in \pi D_i \) be an element with \( i_k \)-th component = 0, other component = 1. It follows that there is a right ideal of the form \( \pi J_i \) with \( J_i \in \Gamma_i \) and almost all \( J_i = R_i \) such that \( I \supseteq e \pi J_i \).

Also for each \( d_i k \subseteq D_i \), there exists a right ideal \( J_{i_k} \subseteq \Gamma_i \) such that \( I \supseteq e_{i_k} J_{i_k}^{(p)} \), where \( e_{i_k} \) is the \( i_k \) th inclusion. Now take \( H_{i_k} = \sum d_i J_{i_k}^{(p)} \), the sum runs through all elements of \( D_k \). We have \( H_{i_k} \subseteq \Gamma_i \) and

\[
(\ast) \quad I \supseteq e \pi J_i + e_{i_k}(H_{i_k}) + \cdots + e_{i_k}(H_{i_k}).
\]

However the right side of \( (\ast) \) is of the form \( \pi J_i \) with \( J_i \in \Gamma_i \) and almost all \( J_i = R_i \). Thus \( I \subseteq \Gamma_2 \).

3. Products of perfect torsion theories

For a fixed ring \( R \) with a perfect Gabriel filter \( \Gamma \), we will investigate the notion of their products.

The following two theorems (Chapt. 13, [4]) motivate our definition.

Theorem A. The following properties of a pair \((R, \Gamma)\) are equivalent:
1. \( \text{Ker}(M \to M \otimes_R \pi R) = \tau(M) \) for all right \( R \)-module \( M \).
2. \( \psi_R(I) R = R \) for every \( I \in \Gamma \).

Theorem B. If \( \phi: A \to B \) is a ring homomorphism. The following statements are equivalent:
1. \( \phi \) is an epimorphism and makes \( B \) into a flat left \( A \)-module.
2. The family \( \Gamma \) of right ideal \( I \) of \( A \) such that \( \phi(I) B = B \) is a Gabriel filter, and there exists a ring isomorphism \( \sigma: B \to A \) such that \( \sigma \phi = \psi_A \).
3. The following two conditions are satisfied;
   3a) for every \( b \in B \), there exists a finite subset \( T_n = \{(s_1, b_1), \ldots, (s_n, b_n)\} \) of \( A \times B \) such that \( b \phi(s_i) \subseteq \phi(A) \) and \( \sum_i \phi(s_i)b_i = 1 \).
   3b) if \( \phi(a) = 0 \), then there exists a finite subset \( S_n = \{(s_1, b_1), \ldots, (s_n, b_n)\} \) such that \( a_i = 0 \) and \( \sum_i \phi(s_i)b_i = 1 \).

Note. A Gabriel filter \( \Gamma \) of a ring \( R \) is called perfect if it has properties listed in Theorem A. If \( \Gamma \) is perfect, then
1. \( \Gamma \) has a cofinal family of finitely generated right ideals.
2. \( \Gamma = \{I \subseteq \pi R | \psi_R(I) R = R\} \).

DEFINITION. If \( \Gamma \) is a perfect Gabriel filter of \( R \), for each \( b \in R \), define \( \text{Ind} b = \inf |T_n| \), \( T_n \) runs through all subsets of \( R \times R \) that satisfy Theorem B, 3(a).
If \( \psi^r(\tau) = 0 \), define \( \text{Ind } r = \inf |S_n| \in S_n \), runs through all subsets of \( R \times R \), that satisfy Theorem \( B \), (3b). Then let

\[
\text{Ind } R_\Gamma = \max \{ \sup_{r \in R_\Gamma} (\text{Ind } \psi^r), \sup_{r \in R_\Gamma} (\text{Ind } r) \}.
\]

**Theorem 3.** The following statements are equivalent for a perfect Gabriel filter \( \Gamma \) of \( R \).

1. \( \Gamma \) has a cofinal family of \( n \)-generated right ideals.
2. \( \Gamma = \{ I \subseteq \pi R | I \supseteq \pi D_i, D_i \in \Gamma \} \) is a Gabriel filter of \( \pi R \), for any direct product of \( R \).
3. \( \text{Id } R_\Gamma \) is infinite.

**Proof.** (1)\( \Rightarrow \) (2). By Proposition 1.

(2)\( \Rightarrow \) (3). If \( \Gamma_i \) is a Gabriel filter, then it is perfect. Suppose there is a sequence \( \{ b_1, b_2, \cdots, b_n, \cdots \} \), such that \( \text{Ind } b_n > \text{Ind } b_{n-1} \). Consider the countable product \( \pi R \) of \( R \) and the element \( x = (b_1, b_2, \cdots) \). Then we have \( s_1, s_2, \cdots, s_i, s_{i+1} \in \pi R \times \gamma_1, \cdots, s_{i+1} \in \gamma_i \), such that \( \gamma_i(s_i) \in \pi R \) and \( \sum \psi^r(s_i) = 1 \). Projecting to each component, \( \text{Ind } b_n \leq t \) for each \( n \). This is a contradiction. Similarly, we can prove that \( \sup_{r \in R_\Gamma} (\text{Ind } r) \) is finite.

(3)\( \Rightarrow \) (1). If \( \text{Ind } R_\Gamma \) is finite, then any direct product \( \pi R_\Gamma \) of \( R_\Gamma \) satisfies Theorem \( B \), (3). So the product \( \pi R_\Gamma \) is a ring of right quotient of \( \pi R \) with respect to this perfect Gabriel filter \( \Gamma = \{ D \subseteq \pi R | \phi(D) \pi R_\Gamma = \pi R_\Gamma \} \). Applying the well-ordering theorem to the family \( \Gamma \), the right ideal \( \pi D_i \) is in \( \Gamma \). So \( \pi D_i \) contains a \( n \)-generated right ideal \( J = r \Gamma \). For each \( i, j \), the \( i \)-th projection of \( J_i \) is contained in \( D_i \). Since \( \psi^r(J_i) \pi R_\Gamma = r \Gamma \gamma_i \in \Gamma \), this shows that \( \Gamma \) has a cofinal family of \( n \)-generated right ideals.

**EXAMPLE.** Let \( Z \) be the ring of integers, \( \Gamma = \{ \text{all non-zero ideals of } Z \} \), take a countable product \( \pi Z \) of \( Z \), then \( \Gamma_\Gamma = \{ I \subseteq \pi Z | I \supseteq \sum D_i, D_i \in \Gamma \} \) is not a perfect Gabriel filter. However \( \Gamma = \{ I \subseteq \pi Z | I \supseteq \pi D_i, D_i \in \Gamma \} \) is perfect.

4. **Applications to coalgebras**

In this section we consider a subfunctor of the identity for the category of right \( C^* \)-module \( \mathcal{M}_{C^*} \) and study when this subfunctor defines a hereditary torsion theory. The main effect is to classify some types of cocommutative coalgebras. If \( C \) is a coalgebra, for a right \( C^* \)-module \( M \) there is a unique maximal rational submodule \( M_{\text{rat}} \) of \( M \). Actually \( M_{\text{rat}} = \{ m \in M | \text{Ann}(m) \) is cofinite \} closed in \( C^* \). There are some properties of \( \mathcal{M}_{C^*} \).

1. If \( (M, \phi_M) \) is a left \( C \)-comodule, \( M \) can be considered as a right \( C^* \)-module by \( mc^* = (c^* \otimes 1) \phi_M(m) \). Then \( (M_{C^*})_{\text{rat}} = M \).
(2) Direct sum of rational $C^*$-modules is rational.

(3) $(C^{**})^{rat} = C$.

(4) For a submodule $N$ of a $C^*$-module $M$, $N^{rat} = N \cap M^{rat}$.

(5) Homomorphic image of a rational module is rational.

So we have a subfunctor $rat$ of the identity on $\mathcal{M}_{C^*}$ just assigned each $C^*$-module $M$ the maximal rational submodule $M^{rat}$ and each homomorphism $f: M \to N$ the restriction map $\bar{f}: M^{rat} \to N^{rat}$.

**DEFINITION.** A coalgebra $C$ is said to have torsion $rat$ functor if $rat$ is a left exact radical of $\mathcal{M}_{C^*}$.

Note. If $C$ has the torsion $rat$ functor, then

(1) the category of left $C$-comodules or equivalently of rational right $C^*$-modules is the torsion class.

(2) the corresponding Gabriel filter is

$$\Gamma = \{ I \subseteq C^* \mid I \text{ is cofinite closed in } C^* \}.$$

**EXAMPLE.** Let $V$ be an infinite dimensional vector space and $C = C(V)$ denote the connected coalgebra $K \otimes V$ with

$$\Delta(v) = 1 \otimes v + v \otimes 1 \quad \forall v \in V,$$

$$\delta(1) = 1,$$

$$\epsilon(v) = 0 \quad \forall v \in V.$$

Take a basis $\{v_i \mid i \in \mathbb{I}\}$ of $V$ and let $\{v_i^* \mid i \in \mathbb{I}\}$ be its dual independent set in $V^*$. Extending this set to a basis $\{v_i^* \mid i \in \mathbb{I}\}$ of $V^*$. We construct a linear map $f$ from $C^*$ to $K$ as

$$f(v_i^*) = \begin{cases} 1 & \text{if } i \in \mathbb{I} \\ 0 & \text{otherwise} \end{cases} \quad f(1) = 1,$$

this element $f \in C = C^{**}{rat}$, however $f v^* = f(v^*)1 \in C$ for any $v^* \in V^*$. So $(C^{**}/C^{**}{rat})^{rat} \neq 0$.

The following proposition is proved in [2, p. 521].

**Proposition.** Suppose $C$ is a coalgebra and $0 \to M' \to M \to M'' \to 0$ is an exact sequence of right $C^*$-modules with $M'$ and $M''$ rational. If the annihilator of each $m'' \in M''$ is a finitely generated right ideal, then $M$ is rational.

Note. From the proposition, we see that if $C^*$ is a right Noetherian, then $C$ has the torsion $rat$ functor. In particular the universal cocommutative pointed irreducible coalgebra $B(V)$ over a finite dimensional vector space $V$ has the torsion $rat$ functor.
Proposition 4. If $D$ is a subcoalgebra of $C$, then $D$ has the torsion rat functor provided $C$ has.

Proof. There exists a ring epimorphism $\pi: C^* \to D^*$. Every $D^*$-module $M$ is a $C^*$-module by $mc^* = m\pi(c^*)$. Thus $(M_D^*)_\text{rat} = (M_C^*)_\text{rat}$ and $(M_D^*/M_D^*_\text{rat})_{\text{rat}} = (M_C^*/M_C^*_\text{rat})_{\text{rat}} = 0$.

Corollary 5. For any pointed irreducible cocommutative coalgebra $C$, it has the torsion rat functor if and only if its space of primitive elements $P(C)$ is finite-dimensional.

Proof. If $P(C)$ is infinite-dimensional, the connected sub-coalgebra $D = \mathbb{K}@P(C)$ of $C$ does not have the torsion rat functor. Conversely if $P(C)$ is finite-dimensional there is an inclusion map from $C$ to the universal cocommutative pointed irreducible coalgebra over $P(C)$. So by Proposition 4 $C$ has the torsion rat functor.

Theorem 6. (*) If $\{C_i | i \in I\}$ is a family of coalgebras and $C_i$ has the torsion rat functor for each $i \in I$. Then the direct sum $C = \sum_\oplus C_i$ has the torsion rat functor.

Proof. Let $\Gamma_i = \{D_i \leq C | D_i \text{ is cofinite closed in } C_i^*\}$, and $\Gamma = \{D \leq C^* = \pi C^* | D \text{ is cofinite closed in } C^*\}$. By proposition 2 $\Gamma_2 = \{I \leq \pi C^* | I \geq \pi D_i, D_i \in \Gamma_i \text{ and almost all } D_i = C_i^*\}$ is a Gabriel filter of $C^* = \pi C^*$. Hence it is sufficient to show that $\Gamma = \Gamma_2$. If $D \in \Gamma$, then $D = V^\perp$ for a finite dimensional subspace $V = V_0 \oplus C_i \oplus \cdots \oplus C_i^n$ for some $n$.

For each $i$, let $V_i$ be the projection of $V$ to $C_i$. Then $V_i$ is a finite-dimensional subspace, almost all $V_i = 0$ and $V \subseteq \pi V_i$. Hence we have $\pi V_i \subseteq V \subseteq D \in \Gamma_2$. Conversely suppose $I \in \Gamma_2$, since $I$ contains a cofinite closed subspace $\pi D_i$, so $I$ is also cofinite closed. Thus $\Gamma = \Gamma_2$.

Corollary 7. Over an algebraically closed field, a cocommutative coalgebra has the torsion rat functor if and only if each space of primitives of its irreducible components is finite-dimensional.

Proof. Over an algebraically closed field, a cocommutative coalgebra is a direct sum of its pointed irreducible components. So by Theorem 6 and Corollary 5, we have this result.

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(*) This theorem also appeared in [1], here we use the notion of products of torsion theories to give a different proof.
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