Title	Products of torsion theories and applications to coalgebras
Author(s)	Lin, I P'êng
Citation	Osaka Journal of Mathematics. 12(2) P.433-P.439
Issue Date	1975
Text Version	publisher
URL	https://doi.org/10.18910/12732
DOI	10.18910/12732
rights	
Note	

Lin, I.-P.
Osaka J. Math.
12 (1975), 433-439

PRODUCTS OF TORSION THEORIES AND APPLICATIONS TO COALGEBRAS

I-PENG LIN

(Received March 25, 1974)

1. Introduction

Throughout this note R is a ring with 1. We shall write $I \leqslant R$ if I is a right ideal of R. A non-empty set of right ideals Γ of R is called a Gabriel filter if it satisfies

T1. If $I \in \Gamma$ and $r \in R$, then $(I{:}r) \in \Gamma$.

T2. If I is a right ideal and there exists $J \in \Gamma$ such that $(I{:}r) \in \Gamma$ for every $r \in J$, then $I \in \Gamma$.

It is well-known [4] that there is a one to one correspondence between Gabriel filters of R and hereditary torsion theories for the category of right R-modules. W. Schelter [3] investigated products of torsion theories or equivalently of Gabriel filters that for a family of pairs $\{(R_i, \Gamma_i), \ \Gamma_i\colon$ Gabriel filter of $R_i\}$, $\Gamma_0 = \{D \leqslant \pi R_i \mid D \supseteq \sum_{\oplus} D_i, \ D_i \in \Gamma_i\}$ is a Gabriel filter of the product ring πR_i, furthermore the ring of right quotient of πR_i with respect to Γ_0 is isomorphic to the product of rings of right quoteint of R_i with respect to $\Gamma_i{:}(\pi R_i)_{\Gamma_0} \cong \pi(R_i)_{\Gamma_i}$ This result generalizes one of Y. Utumi theorems [6]. In this paper these two sets $\Gamma_1 = \{D \leqslant \pi R_i \mid D \supseteq \pi D_i, D_i \in \Gamma_i\}$ and $\Gamma_2 = \{D \leqslant \pi R_i \mid D \supseteq \pi D_i D_i \in \Gamma_i$ and almost all $D_i = R_i\}$ will be studied. Γ_1 does not always satisfy T2. A necessary and sufficient condition for Γ_1 to be a Gabriel filter is given. It follows that Γ_1 is a better notion of products of perfect torsion theories. However Γ_2 is a Gabriel filter of πR_i, and we use this fact to prove that over an algebraically closed field, cocommutative coalgebra has a torsion rat functor if and only if each space of primitives of its irreducible components is finitedimensional.

For a coalgebra (C, Δ, ε) over a field K, there exists a natural algebra structure on its dual space $C^* = \mathrm{Hom}_K$ (C, K) induced by the diagonal map Δ and every left comodule (M, ϕ_M) over C can be defined as a right C^*-module by $mc^* = (c^* \otimes 1)\phi_M(m), m \in M, \ c^* \in C^*$. Moroever a right C*-module M is called a rational module if it is a left comodule (M, ϕ_M) over C and its right C^*-module structure is derived in the way described above. With these observations we can embed the category of left C-comodules ${}^{C}\mathcal{M}$, as a full subcategory, into the category of right C*-modules $\mathcal{M}_{C^*}$. A subspace I of C* is callecd cofinite closed if $I=V^{\perp}$ for some finite-dimensional subspace V of C.

We assume the reader is familiar with torsion theories of modules and elementary coalgebra theories. The terminology and notation are those of Stenstrom [4] and Sweedler [5].

2. Some properties

In this section we derive some properties of Γ_1 and Γ_2. For convenience, we write a pair (R_i, Γ_i) as Γ_i is a Gabriel filter of R_i. The following are easily proved.

Lemma 1. *If I is a right ideal of R and there exists $J\in\Gamma$ such that $(I:r)\in\Gamma$ for r runs through a family of generators of J, then $I\in\Gamma$.*

Lemma 2. Γ_1, Γ_2 *satisfy* T1.

Proposition 1. If $\{(R_i, \Gamma_i)_{i\in I}\}$ *is a family of pairs and each Γ_i has a cofinal family of n-generated right ideals (for a fixed integer ri), then $\Gamma_1=\{D\leqslant\underset{I}{\pi}R_i \mid D\supseteq\underset{I}{\pi}D_i,\ D_i\in\Gamma_i,\ \text{all } i\in I\}$ is a Gabriel filter of $\underset{I}{\pi}R_i$. Moreover* $(\pi R_i)_{\Gamma_i}\simeq\pi(R_i)_{\Gamma_i}$.

Proof. It only has to check T2 for Γ_1. Let $T\leqslant\pi R_i$ and $D\in\Gamma_1$ such that (T: rf)eΓi for every $d\in D$. We can assume $D=\pi D_i$, $D_i\in\Gamma_i$ and each D_i has n generators; x_i^1, • •-, x_i^n. Construct n elements of πD_i as $x^1=(x_i^1)$, $\cdots$, $x^n=(x_i^n)$, then we have $(T: x^j)\in\Gamma_1$. Therefore for each $j=1$, $\cdots$, n, there is $\pi D_i^{(j)}$ where $D_i^{(j)}\in\Gamma_i$ such that $x^j\pi D_i^{(j)}$ c T. Howevre for fixed i the finite sum $J_i=\sum_{j=1}^{n}x_i^jD_i^{(j)}\in\Gamma_i$ by Lemma 1 and $\pi J_i=x^1\pi D_i^{(1)}+\text{----}\vdash x^n\pi D_i^{(n)}$. This shows that $\pi J_i\subset T\in\Gamma_1$.

Next we find an isomorphism from $\pi(R_i)_{\Gamma_i}$ to $(\pi R_i)_{\Gamma_1}$. Let $([f_i])\in(\pi R_i)_{\Gamma_i}$, where $f_i\in\mathrm{Hom}_{R_i}$ $(D_i, R_i/(t_i(R_i))$ and $[f_i]$ is its equivalent class in $(R_i)_{\Gamma_i}$, and define a πR_i-homomorphism / from πD_i to $\pi R_i/t(\pi R_i)$ as $f_i((d_i))=(f_i(d_i))$. Since $t(\pi R_i)=\pi t(R_i)$, $\pi(R_i/t(R_i))\simeq\pi R_i/t(\pi R_i)$ we have a well-defined map α from $\pi(R_i)_{\Gamma_i}$ to $(\pi R_i)_{\Gamma_1}$, as $\alpha([f_i])=[/]$, for if f_i and f_i' agree on D_i for each i, then the corresponding f and f' agree on πD_i. It is routine to check that a is a one to one ring-homomorphism. Let f: $\pi D_i\to\pi R_i/t(\pi R_i)$ be a πR_i-homomorphism, $D\in\Gamma_i$ and define $f_i=\pi_i f e_i$, where e_i is the ith-inclusion, π_i is the ith-projection. Then $\alpha([f_i])=[f]$. Thus α is an isomorphism.

Note. (1) we agree that n generators of right ideals are not necessary distinct.

(2) In proposition 1, Γ_1 also has a cofinal family of n-generated right ideals.

Proposition 2. *If $\{(R_i, \Gamma_i), i\in I\}$ is a family of pairs, then* $\Gamma_2=\{I\leqslant\underset{I}{\pi}R_i | I\supseteq\underset{I}{\pi}D_i, D_i\in\Gamma_i$ *and almost all* $D_i=R_i\}$ *is a Gabriel filter of* $\underset{I}{\pi}R_i$.

Proof. Similarly it only has to check T_2 for Γ_2. Let $I\leqslant\pi R_i$ and $D\in\Gamma_2$ such that $(/: d)\in\Gamma_2$for all $d\in D$. We can assume $D=\pi D_i, D_i\in\Gamma_i$ and except for $D_{i_k}, k=I, \cdots, n$, all other D_i are equal to R_i. Let $e\in\pi D_i$be an element with i_k-th component$=0$, other component$=1$. It follows that there is a right ideal of the form πJ_i with $J_i\in\Gamma_i$ and almost all $Ji=Ri$ such that $I\supseteq e\pi J_i$. Also for each $d_{i_k}\in D_{i_k}$, there exists a right ideal $J_{i_k}^{(d)}\in\Gamma_{i_k}$ such that $I\supseteq e_{i_k}(d_{i_k}J_{i_k}^{(p)})$, where e_{i_k} is the i_k th inclusion. Now take $H_{i_k}=\sum d_{i_k}J_{i_k}^{(d)}$, the sum runs through all elements of D_{i_k}. We have $H_{i_k}\in\Gamma_{i_k}$and

$$(*)\ I\supseteq e\pi J_i+e_{i_1}(H_{i_1})+\cdots+e_{i_n}(H_{i_n})\,.$$

However the right side of (*) is of the form πJ_i with $J_i\in\Gamma_i$ and almost all $J_i=R_i$. Thus $I\in\Gamma_2$.

3. Products of perfect torsion theories

For a fixed ring R with a perfect Gabriel filter Γ, we will investigate the notion of their products.

The following two theorems (Chapt. 13, [4]) motivate our definition.

Theorem A. *The following properties of a pair* (R, Γ) *are equivalent*:

(1) $\mathrm{Ker}(M\to M\otimes_R R_\Gamma)=t(M)$ for all right R-module M.

(2) $\psi_R(I)R_\Gamma=R_\Gamma$for every $/\in\Gamma$.

Theorem B. *If* ϕ: $A\to B$ *is a ring homomorphism. The following statements are equivalent:*

(1) ϕ *is an epimorphismand makes* B *into a flat left* A*-module.*

(2) *The family* Γ *of right ideal* I *of* A *such that* $\phi(I)B=B$ *is a Gabriel filter, and there exists a ring isomorphism* $\sigma: B\to A_\Gamma$*such that* $\sigma\phi=\psi_A$.

(3) *The following two conditions are satisfied*;

(3a) *for every* $b\in B$, *there exists a finite subset* $T_n=\{(s_1,b_1), \cdots, (s_n, b_n)\}$ *of* AxB *such that* $b\phi(s_i)\in\phi(A)$ *aud* $\sum_i\phi(s)b_i=1$.

(3b) *if* $\phi(a)-0$, *then there exists a finite subset* $S_n=\{(s_1,b_1), \cdots, (s_n, b_n)\}$ *such that* as_i-0 *and* $\sum_i\phi(s_i)b_i=1$.

Note. A Gabriel filter Γ of a ring R is called perfect if it has properties listed in Theorem *A*. If Γ is perfect, then

(1) Γ has a cofinal family of finitely generated right ideals.

(2) $\Gamma=\{I\leqslant R | \psi_R(I)R_\Gamma=R_\Gamma\}$.

DEFINITION. If Γ is a perfect Gabriel filter of *R,* for each $b\in R_\Gamma$define Ind $b=\inf|T_n|$, T_n runs through all subsets of $R\times R_\Gamma$ that satisfy Theorem *B*, 3(a).

If $\psi_R(r)=0$, define $\mathrm{Ind}\, r = \mathrm{Inf}\ |S_n|, S_n$ runs through all subsets of $R\times R_\Gamma$ that satisfy Theorem B, (3b). Then let

$$\mathrm{Ind}\, R_\Gamma = \mathrm{Max}\,\{\sup_{b\in R_\Gamma}(\mathrm{Ind}\, b),\ \sup_{\psi_R(r)=0}(\mathrm{Ind}\, r)\}\ .$$

Theorem 3. *The following statements are* ***equivalent*** *for a perfect Gabriel filter* Γ *of* R.

(1) Γ *has a confinalfamily of* ***n-generated*** *right ideals.*

(2) $\Gamma_1=\{I\leqslant \pi R\,|\, I\supseteq \pi D_i,\ D_i\in\Gamma\}$ *ιs a Gabriel filter of* πR, *for any direct product of* R.

(3) *I ιd* R_Γ *is finite.*

Proof. (1)⇒(2). By Proposition 1.

(2)⇒(3). If Γ_1 is a Gabriel filter, then it is perfect. Suppose there is a sequence $\{b_1, b_2, \cdots, b_n, \cdots\,|\, b_n\in R_\Gamma\}$such that $\mathrm{Ind}\, b_n > \mathrm{Ind}\, b_{n-1}$. Consider the countable product πR of R and the element $x=(b_1, b_2, \cdots)$. Then we have $s_1, \cdots, s_t\in\pi R, x_1, \cdots$, tf,e(ττ,R)$_\Pi$^7rJ?$_\Gamma$ such that $x\psi(s_i)\in\psi\pi R$and $\sum\psi(s_i)x_i=1$ $\cdot$ Projecting to each component, $\mathrm{Ind}\, b_n\leqslant t$ for each n. This is a contradiction. Similary we can prove that $\sup_{\psi_R(r)=0}\{\mathrm{Ind}\, r\}$ is finite.

(3)⇒(1). If $\mathrm{Ind}\, R_\Gamma$ is finite, then any direct product πR_Γ of R_Γ satisfies Theorem B, (3). So the product $\pi_{|\Gamma|} R_\Gamma$ is a ring of right quotient of $\pi_{|\Gamma|} R$ with respect to this perfect Gabriel filter $\Gamma=\{D\leqslant \pi_{|\Gamma|} R\,|\,\phi(D)\pi R_\Gamma=\pi R_\Gamma.\}$ Applying the well-ordering theorem to the family Γ, the right ideal $\pi_{D_i\in\Gamma} D_i$is in Γ. So $\pi_{D_i\in\Gamma} D_i$ contains a n-generated right ideal $J\in\Gamma$. For each i J_i, the i-th projection of J, is contained in D_i. Since $\psi_R(J_i)R_\Gamma=R_\Gamma, J_i\in\Gamma$. This shows that Γ has a cofinal family of n-generated right ideals.

EXAMPLE. Let Z be the ring of integers, $\Gamma=\{$all non-zero ideals of $Z\}$, take a countable product πZ of Z, then $\Gamma_0=\{I\leqslant\pi Z\,|\, I\supseteq \sum_\oplus D_i,\ D_i\in\Gamma\}$ is not a perfact Gabriel filter. However $\Gamma_1=\{/\leqslant \pi Z\setminus I\supseteq \pi D_i,\ D_i\in\Gamma\}$ is perfact.

4. Applications to coalgebras

In this section we consider a subfunctor of the identity for the category of right C*-module $\mathcal{M}_{C^*}$ and study when this subfunctor defines a hereditary torsion theory. The main effect is to classify some types of cocommutative coalgebras. If C is a coalgebra, for a right C*-module M there is a unique maximal rational submodule M^{rat} of M. Actually $M^{\mathrm{rat}}=\{m\in M\,|\,\mathrm{Ann}(m)$ is cofinite closed in $C^*\}$. There are some properties of $\mathcal{M}_{C^*}$.

(1) If (M, ϕ_M) is a left C-comodule, M can be considered as a right C^*-module by $mc^*=(c^*\otimes 1)\phi_M(m)$. Then $(M_{C^*})^{\mathrm{rat}}=M$.

(2) Direct sum of rational C^*-modules is rational.

(3) $(C^{**})^{\mathrm{rat}}=C$.

(4) For a submodule N of a C*-module M, $N^{\mathrm{rat}}=N\cap M^{\mathrm{rat}}$.

(5) Homomorphic image of a rational module is rational.

So we have a subfunctor rat of the identity on $\mathcal{M}_{C^*}$ just assigned each C*-module M the maximal rational submodule M^{rat} and each homomorphism /: $M\rightarrow N$ the restriction map $\bar{f}$: $M^{\mathrm{rat}}\rightarrow N^{\mathrm{rat}}$.

DEFINITION. A coalgebra C is said to have torsion rat functor if rat is a left exact radical of $\mathcal{M}_{C^*}$.

Note. If C has the torsion rat functor, then

(1) the category of left C-comodules or equivalently of rational right C^*-modules is the torsion class.

(2) the corresponding Gabriel filter is

$$\Gamma = \{I\leqslant C^* \quad I \text{ is cofinite closed in } C^*\} .$$

EXAMPLE. Let V be an infinite dimensional vector space and $C=C(V)$ denote the connected coalgebra K© V with

$$\Delta(v) = 1\otimes v+v\otimes 1 \qquad \forall v\in V$$
$$6(1) = 1$$
$$\varepsilon(v) = 0 \qquad \forall v\in V .$$

Take a basis $\{v_i| i\in I\}$ of V and let $\{v_i^*|i\in\tilde{I}\}$ be its dual independent set in V^*. Extending this set to a basis $\{v_i^*|i\in I\}$ of V^*. We construct a linear map / from C^* to K as

$$\begin{cases} f(v_i^*) = 1 \text{ if } i\in I \\ f(1) = 1 , \end{cases}$$

this element $f\notin C= C^{**\mathrm{rat}}$, however $fv^*= f(v^*)1\in C$ for any $v^*\in V^*$. So $(C^{**}/C^{**\mathrm{rat}})^{\mathrm{rat}}\neq 0$.

The following proposition is proved in [2, p. 521].

Proposition. *Suppose C is a coalgebra and $0\rightarrow M'\rightarrow M\rightarrow M''\rightarrow 0$ is an exact sequence of right C^*-modules with M' and M'' rational. If the annihilator of each m''*EΞ M'' *is a finitely generated right ideal, then M is rational.*

Note. From the proposition, we see that if C* is a right Noetherian, then C has the torsion rat functor. In particular the universal cocommutative pointed irreducible coalgebra $B(V)$ over a finite dimensional vector space V has the torsion rat functor.

Proposition 4. *If D is a subcoalgebra of C, then D has the torsion rat functor provided C has.*

Proof. There exists a ring epimorphism π: $C^* \twoheadrightarrow D^*$. Every D^*-module M is a C^*-module by $mc^*=m\pi(c^*)$. Thus $(M_{D^*})^{\mathrm{rat}}=(M_{C^*})^{\mathrm{rat}}$ and $(M_{D^*}/M_{D^*}{}^{\mathrm{rat}})^{\mathrm{rat}}=(M_{C^*}/M_{C^*}{}^{\mathrm{rat}})^{\mathrm{rat}}=0$.

Corollary 5. *For any pointed irreducible cocommutative coalgebra C, it has the torsion rat functor if and only if its space of primitive elements $P(C)$ is finite-dimensional.*

Proof. If *P(C)* is infinite-dimensional, the connected sub-coalgebra $D=K\oplus P(C)$ of C does not have the torsion rat functor. Conversely if *P(C)* is finite-dimensional there is an inclusion map from C to the universal cocommutative pointed irreducible coalgebra over *P(C)*. So by Proposition 4 C has the torsion rat functor.

Theorem 6. (*) *If $\{C_i \mid i\in I\}$ is a family of coalgebras and C_i has the torsion rat functor for each $i\in I$. Then the direct sum $C=\sum_\oplus C_i$ has the torsion rat functor.*

Proof. Let $\Gamma_i = \{D_i \leqslant Cf \mid D_i$ is cofinite closed in $C_i^*\}$, and $\Gamma=\{D\leqslant C^* = \pi C_i^* D$ is cofinite closed in $C^*\}$. By proposition 2 $\Gamma_2=\{I\leqslant \pi C_i^* \mid I\geqslant \pi D_i,\ D_i\in\Gamma_i$ and almost all $D_i=C_i^*\}$ is a Gabriel filter of $C^*=\pi C_i^*$. Hence it is sufficient to show that $\Gamma=\Gamma_2$. If $D\in\Gamma$, then $D=V^\perp$ for a finite dimensional subspace $V$of$C=\sum_\oplus C_i$, and so $V\subseteq C_i\oplus \cdots \oplus C_{i_n}$ for some n.
For each $\imath$, let V_i be the projection of V to C_i. Then V_i is a finite-dimensional subspace, almost all $V_i=0$ and $V\subseteq \pi V_i$. Hence we have $\pi V_i^\perp \subseteqq V^\perp = D\in\Gamma_2$. Conversely suppose $I\in\Gamma_2$, since / contains a cofinite closed subspace πD_i, so / is also cofinite closed. Thus $\Gamma=\Gamma_2$.

Corollary 7. *Over an algebraically closed field, a cocommutative coalgebra has the torsion rat functor if and only if each space of primitives of its irreducible components is finite-dimensional.*

Proof. Over an algebrically closed field, a cocommutative coalgebra is a direct sum of its pointed irreducible components. So by Theorem 6 and Corollary 5, we have this result.

NATIONAL TAIWAN UNIVERSITY

(*) This theorem also appeared in [1], here we use the notion of products of torsion theories to give a different proof.

References

[1] I-Peng Lin: *On homological properties of coalgebras*, Thesis, University of Illinois at Chicago Circle, 1973.
[2] D. Radford: *Coreflexive coalgebras*, J. Algebra 26 (1973), 512-535.
[3] W. Schelter: *Products of torsion theories*, Arch. Math. 22 (1971) 590-596.
[4] B. Stenstrom: Rings and Modules of Quotients, Lecture Notes in Math. 237, Berlin 1971.
[5] M. Sweedler: Hopf Algebra, Benjamin, W. A., New York, 1969.
[6] Y. Utumi: *On quotient rings*, Osaka J. Math. 8 (1956), 1-18.