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PRODUCTS OF TORSION THEORIES AND APPLICATIONS TO COALGEBRAS

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1. Introduction

Throughout this note R is a ring with 1. We shall write $I \leq R$ if I is a right ideal of R . A non-empty set of right ideals Γ of R is called a Gabriel filter if it satisfies

T1. If $I \in \Gamma$ and $r \in R$, then $(I:r) \in \Gamma$.

T2. If I is a right ideal and there exists $J \in \Gamma$ such that $(I:r) \in \Gamma$ for every $r \in J$, then $I \in \Gamma$.

It is well-known [4] that there is a one to one correspondence between Gabriel filters of R and hereditary torsion theories for the category of right R -modules. W. Schelter [3] investigated products of torsion theories or equivalently of Gabriel filters that for a family of pairs $\{(R_i, \Gamma_i), \Gamma_i: \text{Gabriel filter of } R_i\}$, $\Gamma_0 = \{D \leq \pi R_i \mid D \supseteq \sum_{\oplus} D_i, D_i \in \Gamma_i\}$ is a Gabriel filter of the product ring πR_i , furthermore the ring of right quotient of πR_i with respect to Γ_0 is isomorphic to the product of rings of right quotient of R_i with respect to $\Gamma_i: (\pi R_i)_{\Gamma_0} \cong \pi(R_i)_{\Gamma_i}$. This result generalizes one of Y. Utumi theorems [6]. In this paper these two sets $\Gamma_1 = \{D \leq \pi R_i \mid D \supseteq \pi D_i, D_i \in \Gamma_i\}$ and $\Gamma_2 = \{D \leq \pi R_i \mid D \supseteq \pi D_i, D_i \in \Gamma_i \text{ and almost all } D_i = R_i\}$ will be studied. Γ_1 does not always satisfy T2. A necessary and sufficient condition for Γ_1 to be a Gabriel filter is given. It follows that Γ_1 is a better notion of products of perfect torsion theories. However Γ_2 is a Gabriel filter of πR_i , and we use this fact to prove that over an algebraically closed field, cocommutative coalgebra has a torsion rat functor if and only if each space of primitives of its irreducible components is finitedimensional.

For a coalgebra (C, Δ, ε) over a field K , there exists a natural algebra structure on its dual space $C^* = \text{Hom}_K(C, K)$ induced by the diagonal map Δ and every left comodule (M, ϕ_M) over C can be defined as a right C^* -module by $m c^* = (c^* \otimes 1) \phi_M(m), m \in M, c^* \in C^*$. Moreover a right C^* -module M is called a rational module if it is a left comodule (M, ϕ_M) over C and its right C^* -module structure is derived in the way described above. With these observations we can embed the category of left C -comodules ${}^C\mathcal{M}$, as a full subcategory, into the category of right C^* -modules \mathcal{M}_{C^*} . A subspace I of C^* is called cofinite

closed if $I = V^\perp$ for some finite-dimensional subspace V of C .

We assume the reader is familiar with torsion theories of modules and elementary coalgebra theories. The terminology and notation are those of Stenstrom [4] and Sweedler [5].

2. Some properties

In this section we derive some properties of Γ_1 and Γ_2 . For convenience, we write a pair (R_i, Γ_i) as Γ_i is a Gabriel filter of R_i . The following are easily proved.

Lemma 1. *If I is a right ideal of R and there exists $J \in \Gamma$ such that $(I:r) \in \Gamma$ for r runs through a family of generators of J , then $I \in \Gamma$.*

Lemma 2. Γ_1, Γ_2 satisfy T1.

Proposition 1. *If $\{(R_i, \Gamma_i)_{i \in I}\}$ is a family of pairs and each Γ_i has a cofinal family of n -generated right ideals (for a fixed integer n), then $\Gamma_1 = \{D \leq \pi R_i \mid D \supseteq \pi D_i, D_i \in \Gamma_i, \text{ all } i \in I\}$ is a Gabriel filter of πR . Moreover $(\pi R_i)_{\Gamma_1} \cong \pi(R_i)_{\Gamma_i}$.*

Proof. It only has to check T2 for Γ_1 . Let $T \leq \pi R$ and $D \in \Gamma_1$ such that $(T:rf) \in \Gamma_1$ for every $d \in D$. We can assume $D = \pi D_i, D_i \in \Gamma_i$ and each D_i has n generators; x^1_i, \dots, x^n_i . Construct n elements of πD_i as $x^1 = (x^1_i), \dots, x^n = (x^n_i)$, then we have $(T: x^j) \in \Gamma_1$. Therefore for each $j=1, \dots, n$, there is $\pi D_i^{(j)}$ where $D_i^{(j)} \in \Gamma_i$ such that $x^j \pi D_i^{(j)} \subset T$. However for fixed i the finite sum $J_i = \sum_{j=1}^n x^j D_i^{(j)} \in \Gamma_i$ by Lemma 1 and $\pi J_i = x^1 \pi D_i^{(1)} + \dots + x^n \pi D_i^{(n)}$. This shows that $\pi J_i \subset T \in \Gamma_1$.

Next we find an isomorphism from $\pi(R_i)_{\Gamma_i}$ to $(\pi R_i)_{\Gamma_1}$. Let $([f_i]) \in (\pi R_i)_{\Gamma_i}$, where $f_i \in \text{Hom}_{R_i}(D_i, R_i/t_i(R_i))$ and $[f_i]$ is its equivalent class in $(R_i)_{\Gamma_i}$, and define a πR_i -homomorphism $/$ from πD_i to $\pi R_i/t(\pi R_i)$ as $f_i((d_i)) = (f_i(d_i))$. Since $t(\pi R_i) = \pi t(R_i)$, $\pi(R_i/t(R_i)) \cong \pi R_i/t(\pi R_i)$, we have a well-defined map α from $\pi(R_i)_{\Gamma_i}$ to $(\pi R_i)_{\Gamma_1}$, as $\alpha([f_i]) = [/math>, for if f_i and f'_i agree on D_i for each i , then the corresponding f and f' agree on πD_i . It is routine to check that α is a one to one ring-homomorphism. Let $f: \pi D_i \rightarrow \pi R_i/t(\pi R_i)$ be a πR_i -homomorphism, $D \in \Gamma_1$ and define $f_i = \pi_i f e_i$, where e_i is the i th-inclusion, π_i is the i th-projection. Then $\alpha([f_i]) = [f]$. Thus α is an isomorphism.$

Note. (1) we agree that n generators of right ideals are not necessary distinct.

(2) In proposition 1, Γ_1 also has a cofinal family of n -generated right ideals.

Proposition 2. *If $\{(R_i, \Gamma_i), i \in I\}$ is a family of pairs, then $\Gamma_2 = \{I \leq \pi R_i \mid I \supseteq \pi D_i, D_i \in \Gamma_i \text{ and almost all } D_i = R_i\}$ is a Gabriel filter of πR_i .*

Proof. Similarly it only has to check T_2 for Γ_2 . Let $I \leq \pi R_i$ and $D \in \Gamma_2$ such that $(/: d) \in \Gamma_2$ for all $d \in D$. We can assume $D = \pi D_i, D_i \in \Gamma_i$ and except for $D_{i_k}, k = 1, \dots, n$, all other D_i are equal to R_i . Let $e \in \pi D_i$ be an element with i_k -th component = 0, other component = 1. It follows that there is a right ideal of the form πJ_i with $J_i \in \Gamma_i$ and almost all $J_i = R_i$ such that $I \supseteq e \pi J_i$. Also for each $d_{i_k} \in D_{i_k}$, there exists a right ideal $J_{i_k}^{(a)} \in \Gamma_{i_k}$ such that $I \supseteq e_{i_k}(d_{i_k} J_{i_k}^{(a)})$, where e_{i_k} is the i_k th inclusion. Now take $H_{i_k} = \sum d_{i_k} J_{i_k}^{(a)}$, the sum runs through all elements of D_{i_k} . We have $H_{i_k} \in \Gamma_{i_k}$ and

$$(*) \quad I \supseteq e \pi J_i + e_{i_1}(H_{i_1}) + \dots + e_{i_n}(H_{i_n}).$$

However the right side of (*) is of the form πJ_i with $J_i \in \Gamma_i$ and almost all $J_i = R_i$. Thus $I \in \Gamma_2$.

3. Products of perfect torsion theories

For a fixed ring R with a perfect Gabriel filter Γ , we will investigate the notion of their products.

The following two theorems (Chapt. 13, [4]) motivate our definition.

Theorem A. *The following properties of a pair (R, Γ) are equivalent:*

- (1) $\text{Ker}(M \rightarrow M \otimes_R R_\Gamma) = t(M)$ for all right R -module M .
- (2) $\psi_R(I)R_\Gamma = R_\Gamma$ for every $I \in \Gamma$.

Theorem B. *If $\phi: A \rightarrow B$ is a ring homomorphism. The following statements are equivalent:*

- (1) ϕ is an epimorphism and makes B into a flat left A -module.
- (2) The family Γ of right ideal I of A such that $\phi(I)B = B$ is a Gabriel filter, and there exists a ring isomorphism $\sigma: B \rightarrow A_\Gamma$ such that $\sigma\phi = \psi_A$.
- (3) The following two conditions are satisfied;
 - (3a) for every $b \in B$, there exists a finite subset $T_n = \{(s_1, b_1), \dots, (s_n, b_n)\}$ of $A \times B$ such that $b\phi(s_i) \in \phi(A)$ and $\sum_i \phi(s_i)b_i = 1$.
 - (3b) if $\phi(a) = 0$, then there exists a finite subset $S_n = \{(s_1, b_1), \dots, (s_n, b_n)\}$ such that $as_i = 0$ and $\sum_i \phi(s_i)b_i = 1$.

Note. A Gabriel filter Γ of a ring R is called perfect if it has properties listed in Theorem A. If Γ is perfect, then

- (1) Γ has a cofinal family of finitely generated right ideals.
- (2) $\Gamma = \{I \leq R \mid \psi_R(I)R_\Gamma = R_\Gamma\}$.

DEFINITION. If Γ is a perfect Gabriel filter of R , for each $b \in R_\Gamma$ define $\text{Ind } b = \inf |T_n|$, T_n runs through all subsets of $R \times R_\Gamma$ that satisfy Theorem B, 3(a).

If $\psi_R(r)=0$, define $\text{Ind } r = \text{Inf } |S_n|, S_n$ runs through all subsets of $R \times R_\Gamma$ that satisfy Theorem B, (3b). Then let

$$\text{Ind } R_\Gamma = \text{Max } \{ \sup_{b \in R_\Gamma} (\text{Ind } b), \sup_{\psi_R(r)=0} (\text{Ind } r) \} .$$

Theorem 3. *The following statements are equivalent for a perfect Gabriel filter Γ of R .*

- (1) Γ has a cofinal family of n -generated right ideals.
- (2) $\Gamma_1 = \{ I \leq \pi R \mid I \supseteq \pi D_i, D_i \in \Gamma \}$ is a Gabriel filter of πR , for any direct product of R .
- (3) $\text{Ind } R_\Gamma$ is finite.

Proof. (1) \Rightarrow (2). By Proposition 1.

(2) \Rightarrow (3). If Γ_1 is a Gabriel filter, then it is perfect. Suppose there is a sequence $\{b_1, b_2, \dots, b_n, \dots \mid b_n \in R_\Gamma\}$ such that $\text{Ind } b_n > \text{Ind } b_{n-1}$. Consider the countable product πR of R and the element $x = (b_1, b_2, \dots)$. Then we have $s_1, \dots, s_t \in \pi R, x_1, \dots, t, e(\tau, R)_\Gamma \wedge \tau \Gamma_\Gamma$ such that $x \psi(s_i) \in \psi \pi R$ and $\sum \psi(s_i) x_i = 1$. Projecting to each component, $\text{Ind } b_n \leq t$ for each n . This is a contradiction. Similarly we can prove that $\text{Sup}_{\psi_R(r)=0} \{ \text{Ind } r \}$ is finite.

(3) \Rightarrow (1). If $\text{Ind } R_\Gamma$ is finite, then any direct product πR_Γ of R_Γ satisfies Theorem B, (3). So the product πR_Γ is a ring of right quotient of πR with respect to this perfect Gabriel filter $\Gamma = \{ D \leq \pi R \mid \phi(D) \pi R_\Gamma = \pi R_\Gamma \}$. Applying the well-ordering theorem to the family Γ , the right ideal πD_i is in Γ . So πD_i contains a n -generated right ideal $J \in \Gamma$. For each i , J_i , the i -th projection of J , is contained in D_i . Since $\psi_R(J_i) R_\Gamma = R_\Gamma, J_i \in \Gamma$. This shows that Γ has a cofinal family of n -generated right ideals.

EXAMPLE. Let Z be the ring of integers, $\Gamma = \{ \text{all non-zero ideals of } Z \}$, take a countable product πZ of Z , then $\Gamma_0 = \{ I \leq \pi Z \mid I \supseteq \sum_{\oplus} D_i, D_i \in \Gamma \}$ is not a perfect Gabriel filter. However $\Gamma_1 = \{ I \leq \pi Z \mid I \supseteq \pi D_i, D_i \in \Gamma \}$ is perfect.

4. Applications to coalgebras

In this section we consider a subfunctor of the identity for the category of right C^* -module \mathcal{M}_{C^*} and study when this subfunctor defines a hereditary torsion theory. The main effect is to classify some types of cocommutative coalgebras. If C is a coalgebra, for a right C^* -module M there is a unique maximal rational submodule M^{rat} of M . Actually $M^{\text{rat}} = \{ m \in M \mid \text{Ann}(m) \text{ is cofinite closed in } C^* \}$. There are some properties of \mathcal{M}_{C^*} .

- (1) If (M, ϕ_M) is a left C -comodule, M can be considered as a right C^* -module by $m c^* = (c^* \otimes 1) \phi_M(m)$. Then $(M_{C^*})^{\text{rat}} = M$.

- (2) Direct sum of rational C^* -modules is rational.
- (3) $(C^{**})^{\text{rat}} = C$.
- (4) For a submodule N of a C^* -module M , $N^{\text{rat}} = N \cap M^{\text{rat}}$.
- (5) Homomorphic image of a rational module is rational.

So we have a subfunctor rat of the identity on \mathcal{M}_{C^*} just assigned each C^* -module M the maximal rational submodule M^{rat} and each homomorphism $f: M \rightarrow N$ the restriction map $\bar{f}: M^{\text{rat}} \rightarrow N^{\text{rat}}$.

DEFINITION. A coalgebra C is said to have torsion rat functor if rat is a left exact radical of \mathcal{M}_{C^*} .

Note. If C has the torsion rat functor, then

- (1) the category of left C -comodules or equivalently of rational right C^* -modules is the torsion class.
- (2) the corresponding Gabriel filter is

$$\Gamma = \{I \leq C^* \mid I \text{ is cofinite closed in } C^*\} .$$

EXAMPLE. Let V be an infinite dimensional vector space and $C = C(V)$ denote the connected coalgebra $K \otimes V$ with

$$\begin{aligned} \Delta(v) &= 1 \otimes v + v \otimes 1 & \forall v \in V \\ \delta(1) &= 1 \\ \varepsilon(v) &= 0 & \forall v \in V . \end{aligned}$$

Take a basis $\{v_i \mid i \in I\}$ of V and let $\{v_i^* \mid i \in \tilde{I}\}$ be its dual independent set in V^* . Extending this set to a basis $\{v_i^* \mid i \in I\}$ of V^* . We construct a linear map f from C^* to K as

$$\begin{cases} f(v_i^*) = 1 \text{ if } i \in I \\ f(1) = 1 , \end{cases}$$

this element $f \in C = C^{**\text{rat}}$, however $f v^* = f(v^*)1 \in C$ for any $v^* \in V^*$. So $(C^{**}/C^{**\text{rat}})^{\text{rat}} \neq 0$.

The following proposition is proved in [2, p. 521].

Proposition. *Suppose C is a coalgebra and $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of right C^* -modules with M' and M'' rational. If the annihilator of each $m'' \in M''$ is a finitely generated right ideal, then M is rational.*

Note. From the proposition, we see that if C^* is a right Noetherian, then C has the torsion rat functor. In particular the universal cocommutative pointed irreducible coalgebra $B(V)$ over a finite dimensional vector space V has the torsion rat functor.

Proposition 4. *If D is a subcoalgebra of C , then D has the torsion rat functor provided C has.*

Proof. There exists a ring epimorphism $\pi: C^* \rightarrow D^*$. Every D^* -module M is a C^* -module by $m c^* = m \pi(c^*)$. Thus $(M_{D^*})^{\text{rat}} = (M_{C^*})^{\text{rat}}$ and $(M_{D^*}/M_{D^*}^{\text{rat}})^{\text{rat}} = (M_{C^*}/M_{C^*}^{\text{rat}})^{\text{rat}} = 0$.

Corollary 5. *For any pointed irreducible cocommutative coalgebra C , it has the torsion rat functor if and only if its space of primitive elements $P(C)$ is finite-dimensional.*

Proof. If $P(C)$ is infinite-dimensional, the connected sub-coalgebra $D = K \oplus P(C)$ of C does not have the torsion rat functor. Conversely if $P(C)$ is finite-dimensional there is an inclusion map from C to the universal cocommutative pointed irreducible coalgebra over $P(C)$. So by Proposition 4 C has the torsion rat functor.

Theorem 6. (*) *If $\{C_i | i \in I\}$ is a family of coalgebras and C_i has the torsion rat functor for each $i \in I$. Then the direct sum $C = \sum_{\oplus} C_i$ has the torsion rat functor.*

Proof. Let $\Gamma_i = \{D_i \leq C_i | D_i \text{ is cofinite closed in } C_i^*\}$, and $\Gamma = \{D \leq C^* = \pi C_i^* | D \text{ is cofinite closed in } C^*\}$. By proposition 2 $\Gamma_2 = \{I \leq \pi C_i^* | I \geq \pi D_i, D_i \in \Gamma_i \text{ and almost all } D_i = C_i^*\}$ is a Gabriel filter of $C^* = \pi C_i^*$. Hence it is sufficient to show that $\Gamma = \Gamma_2$. If $D \in \Gamma$, then $D = V^\perp$ for a finite dimensional subspace V of $C = \sum_{\oplus} C_i$, and so $V \subseteq C_i \oplus \dots \oplus C_{i_n}$ for some n .

For each i , let V_i be the projection of V to C_i . Then V_i is a finite-dimensional subspace, almost all $V_i = 0$ and $V \subseteq \pi V_i$. Hence we have $\pi V_i^\perp \subseteq V^\perp = D \in \Gamma_2$. Conversely suppose $I \in \Gamma_2$, since I contains a cofinite closed subspace πD_i , so I is also cofinite closed. Thus $\Gamma = \Gamma_2$.

Corollary 7. *Over an algebraically closed field, a cocommutative coalgebra has the torsion rat functor if and only if each space of primitives of its irreducible components is finite-dimensional.*

Proof. Over an algebraically closed field, a cocommutative coalgebra is a direct sum of its pointed irreducible components. So by Theorem 6 and Corollary 5, we have this result.

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(*) This theorem also appeared in [1], here we use the notion of products of torsion theories to give a different proof.

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