SINGULAR POINTS OF AFFINE ML-SURFACES

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Abstract

We give a geometric proof of the fact that any affine surface with trivial Makar-Limanov invariant has finitely many singular points. We deduce that a complete intersection surface with trivial Makar-Limanov invariant is normal.

1. Notation and introduction

Let us first fix some notation and recall some basic definitions. Throughout this paper, unless otherwise specified, \( k \) will always denote a field of characteristic zero. A domain means an integral domain. Given a domain \( R \), \( \text{Frac} R \) denotes the field of fractions of \( R \). By \( k[[n]] \), we mean the polynomial ring in \( n \) variables over \( k \) and \( \text{Frac}(k[[n]]) \) will be denoted by \( k^{(n)} \). The set of singular points of a variety \( X \) will be denoted by \( \text{Sing}(X) \).

Definition 1.1. Given a \( k \)-algebra \( B \), a derivation \( D: B \to B \) is locally nilpotent if for each \( b \in B \), there exists a natural number \( n \) (depending on \( b \)) such that \( D^n(b) = 0 \). We use the following notations:

\[
\text{Der}(B) = \{ D \mid D \text{ is a derivation of } B \},
\]

\[
\text{LND}(B) = \{ D \in \text{Der}(B) \mid D \text{ is locally nilpotent} \},
\]

\[
\text{KLND}(B) = \{ \ker D \mid D \in \text{LND}(B), D \neq 0 \}.
\]

Given a \( k \)-domain \( B \), one defines its Makar-Limanov invariant by

\[
\text{ML}(B) = \bigcap_{D \in \text{KLND}(B)} \ker D.
\]

If \( X = \text{Spec} B \) is an affine \( k \)-variety, define \( \text{ML}(X) = \text{ML}(B) \). The Makar-Limanov invariant plays an important role in classifying and distinguishing affine varieties. We say that \( B \) has trivial Makar-Limanov invariant if \( \text{ML}(B) = k \).

Affine spaces \( \mathbb{A}^n_k \) are the simplest examples of varieties with trivial Makar-Limanov invariant. While it is known that \( \mathbb{A}^1_k \) is the only affine curve which has trivial

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Makar-Limanov invariant, the class of affine surfaces with trivial Makar-Limanov invariant contains many more surfaces, some of which are not even normal. (See Example 5.4, for instance.)

Let $\mathcal{M}(k)$ denote the class of 2-dimensional affine $k$-domains which have trivial Makar-Limanov invariant. We say that an affine surface $S = \text{Spec } R$ belongs to the class $\mathcal{M}(k)$ if $R \in \mathcal{M}(k)$. Such a surface $S$ is also called a ML-surface.

The following question arises naturally: Classify all surfaces in the class $\mathcal{M}(k)$.

In recent years, researchers including Bandman, Daigle, Dubouloz, Gurjar, Masuda, Makar-Limanov, Miyanishi, and Russell (see [1], [3], [6], [7], [9], [11]) have been actively investigating properties of normal (or smooth) surfaces belonging to the class $\mathcal{M}(k)$. However, it is desirable to understand what happens when we drop the assumption of normality. For instance, it is natural to ask what are all hypersurfaces of the affine space $\mathbb{A}^3_k$ with trivial Makar-Limanov invariant, and it is not a priori clear that all those surfaces are normal: the fact that they are indeed normal is a consequence of the present paper.

In this paper, we prove that a surface in the class $\mathcal{M}(k)$ has only finitely many singular points. As an application, we prove that any complete intersection surface with trivial Makar-Limanov invariant is normal. Note that these results are valid over any field $k$ of characteristic zero. The results of this paper will be used in a joint paper with D. Daigle [5], where we classify all hypersurfaces of $\mathbb{A}^3_k$ (more generally, complete intersection surfaces over $k$) with trivial Makar-Limanov invariant.

To understand the necessity of some of the arguments given in this paper, the reader should keep in mind certain pathologies that occur when $k$ is not assumed to be algebraically closed. For instance, surfaces $\tilde{S} = \text{Spec } R$ belonging to $\mathcal{M}(k)$ are not necessarily rational over $k$ and may have very few $k$-rational points; moreover, if $\tilde{k}$ is the algebraic closure of $k$, then $\tilde{k} \otimes_k R$ is not necessarily an integral domain.

2. Preliminaries

In this section, we gather some basic results and known facts.

2.1. Suppose that $B$ is a $k$-domain, let $D$ be a nonzero locally nilpotent derivation of $B$, and let $A = \ker D$. The following are well-known definitions and facts about locally nilpotent derivations:

(i) $A$ is factorially closed in $B$ (i.e., the conditions $x, y \in B \setminus \{0\}$ and $xy \in A$ imply that $x, y \in A$). Consequently, $A$ is algebraically closed in $B$.

(ii) Consider the multiplicative set $S = A \setminus \{0\}$ of $B$. We can extend $D$ to an element $\mathcal{D} \in \text{LND}(S^{-1}B)$ defined by $\mathcal{D}(b/s) = D(b)/s$. It is well-known that $S^{-1}B = (\text{Frac } A)^{[1]}$.

(iii) For every $\lambda \in k$, the map

$$e^{\lambda \mathcal{D}} : B \to B, \quad b \mapsto \sum_{n=0}^{\infty} \frac{\lambda^n D^n(b)}{n!}$$

is a \( k \)-algebra automorphism of \( B \).

(iv) Let \( \pi : \text{Spec} \ B \to \text{Spec} \ A \) be the canonical morphism induced by the inclusion map \( A \hookrightarrow B \). Then there exists a nonempty open set \( U \subseteq \text{Spec} \ A \) such that

\[
\pi^{-1}(p) \cong \mathbb{A}^1_{\kappa(p)} \quad \text{for every } p \in U, \quad \text{where } \kappa(p) \text{ is the residue field } A_p/pA_p.
\]

Furthermore, if \( k \) is algebraically closed and \( A \) is \( k \)-affine, then

\[
\pi^{-1}(m) \cong \mathbb{A}^1_{\kappa(m)} = \mathbb{A}^1_k \quad \text{for every closed point } m \text{ of } U.
\]

**Lemma 2.2.** Given an affine \( k \)-surface \( X = \text{Spec} \ B \), let \( A_1 \) and \( A_2 \) be two affine subalgebras of \( B \) of dimension 1. Set \( Y_i = \text{Spec} \ A_i \) and let \( Y_1 \xleftarrow{f_1} \text{Spec} \ B \xrightarrow{f_2} Y_2 \) be the canonical morphisms determined by the inclusions \( A_i \hookrightarrow B \) (for \( i = 1, 2 \)). If \( B \) is algebraic over its subalgebra \( k[A_1 \cup A_2] \), then

\[
E = \{ y \in Y_2 \mid f_1(f_2^{-1}(y)) \text{ is a point} \}
\]

is not a dense subset of \( Y_2 \), where by “\( y \in Y_2 \)” we mean that \( y \) is a closed point of \( Y_2 \).

We leave the proof of Lemma 2.2 to the reader, as it is basic algebraic geometry and is not directly related to the subject matter of this paper.

**Definition 2.3.** A domain \( A \) of transcendence degree 1 over a field \( k \) is called a polynomial curve over \( k \) if it satisfies the following equivalent conditions:

(i) \( A \) is a subalgebra of \( k^1 \).

(ii) \( \text{Frac} \ A = k^1 \) and \( A \) has one rational place at infinity.

**Notation 2.4.** Given a field extension \( F/k \), let \( \mathbb{P}_{F/k} \) be the set of valuation rings \( R \) of \( F/k \) such that \( R \neq F \).

**Lemma 2.5.** Let \( A \) be a \( k \)-domain. If there exists an algebraic extension \( k' \) of \( k \) such that \( k' \otimes_k A \) is a polynomial curve over \( k' \), then \( A \) is a polynomial curve over \( k \).

**Proof.** We sketch a proof of this fact, as we were unable to find a suitable reference. It is easy to prove that \( A \) is affine. We may assume that \( [k' : k] < \infty \). Let \( F = \text{Frac} \ A \) and \( F' = \text{Frac} \ A' \), where \( A' = k' \otimes_k A \). Note that \( [F' : F] = [k' : k] \) and \( F' = k'F \). In the terminology of [12], the function field \( F'/k' \) is an algebraic constant field extension of \( F/k \). By [12, Theorem III.6.3], \( F'/k' \) has same genus as \( F/k \) (hence, \( F/k \) has genus zero) and \( F'/F \) is unramified. It remains to prove that \( A \) has one rational place at infinity. Let

\[
E = \{ R \in \mathbb{P}_{F/k} \mid A \not\subseteq R \} \quad \text{and} \quad E' = \{ R' \in \mathbb{P}_{F'/k'} \mid k' \otimes_k A \not\subseteq R' \}.
\]
If $R$ is any element of $E$, then every $R' \in \mathbb{P}_{/k'}$ lying over $R$ (i.e., satisfying $R' \cap F = R$) must belong to $E'$. But $E'$ is a singleton, say $E' = \{ R' \}$. It follows that $E$ is a singleton, say $E = \{ R \}$. Let $k'$ and $k$ be the residue fields of $R'$ and $R$, respectively. Then $[F' : F] = ef$, where $f = [k' : k]$ and $e$ is the ramification index of $R'$ over $R$. As $F'/F$ is unramified, we have $e = 1$. Since $k' \otimes_k A$ is a polynomial curve over $k'$, $k' = k$. Hence
\[
[k' : k] = [F' : F] = ef = [k' : k] = [k : k].
\]
Thus, $k = k$ and $A$ has one rational place at infinity.

The following lemma can be obtained as an easy consequence of [4, Lemma 3.1].

**Lemma 2.6.** Let $B$ be a $k$-algebra and $f(T) \in B[T]$, where $T$ is an indeterminate.
(a) If $f(T)$ has infinitely many roots in $k$, then $f(T) = 0$.
(b) If $J$ is an ideal of $B$ and $f(\lambda) \in J$ for infinitely many $\lambda \in k$, then $f(T) \in J[T]$.

**Definition 2.7.** Let $R$ be a ring and $D \in \text{Der}(R)$. An ideal $I$ of $R$ is called an integral ideal for $D$ if $D(I) \subseteq I$.

**Lemma 2.8.** Let $R$ be a $k$-domain, and let $I$ be a nonzero ideal of $R$. If $A \in \text{KLND}(R)$, then the following statements are equivalent:
(1) $I \cap A \neq (0)$.
(2) There exists $D \in \text{LND}(R)$ such that $\ker D = A$ and $I$ is an integral ideal for $D$.

Proof. Assume that (1) holds. Let $0 \neq a \in I \cap A$, and let $E \in \text{LND}(R)$ be such that $A = \ker E$. Since $a \in A$, $aE \in \text{LND}(R)$ and $aE$ has kernel $A$. Moreover, as $a \in I$, $(aE)(b) = a(Eb) \in I$ for all $b \in I$. So $(aE)(I) \subseteq I$, and hence $D := aE$ is the required locally nilpotent derivation of $R$ proving assertion (2).
In the other direction, assume that $0 \neq D \in \text{LND}(R)$, ker $D = A$, and $D(I) \subseteq I$. Choose any $b \in I$, $b \neq 0$. Then the set $\{ b, Db, D^2b, \ldots \}$ is included in $I$ and contains a nonzero element of $A$.

The following is an easy consequence of [2, Lemma 2.10].

**Lemma 2.9.** Let $R$ be a noetherian $k$-algebra, and let $D \in \text{Der}(R)$. If $I$ is an integral ideal for $D$, so is every minimal prime-over ideal of $I$.

**Lemma 2.10.** Let $B$ be a $k$-algebra, $J$ an ideal of $B$, and $D \in \text{LND}(B)$. If $e^t_D(J) \subseteq J$ for some nonzero $t \in k$, then $J$ is an integral ideal for $D$.

Proof. First observe that if $e^t_D(J) \subseteq J$ for some nonzero $t \in k$, then $e^t_D(J) \subseteq J$ for infinitely many $t \in k$. Let $f \in J$. We will show that $D(f) \in J$. Let $n = \deg_D(f)$,
i.e., \( n \) is the maximum nonnegative integer such that \( D^n(f) \neq 0 \). Define a polynomial \( P(T) \in B[T] \) by
\[
P(T) = f + D(f)T + \frac{D^2(f)T^2}{2!} + \cdots + \frac{D^n(f)T^n}{n!}.
\]
Then for infinitely many \( t \in k \),
\[
P(t) = f + D(f)t + \frac{D^2(f)t^2}{2!} + \cdots + \frac{D^n(f)t^n}{n!} = e^{tD}(f) \in J.
\]
By Lemma 2.6, all coefficients of \( P(T) \) belong to \( J \), so \( D(f) \in J \).

**Lemma 2.11.** Let \( B \) be an affine \( k \)-domain, and let \( D \in \text{LND}(B) \). If \( \tilde{B} \) denotes the normalization of \( B \), then there exists \( \tilde{D} \in \text{LND}(\tilde{B}) \) such that \( \ker \tilde{D} \cap B = \ker D \).

Proof. We recall the well-known argument. Write \( A = \ker D \) and let \( S = A \setminus \{0\} \). Then \( D \) extends to a locally nilpotent derivation \( D \) of \( S^{-1}B \) such that \( B \cap \ker D = A \). As \( S^{-1}B \) is a polynomial ring over the field \( S^{-1}A \), it is normal, and consequently \( B \subseteq \tilde{B} \subseteq S^{-1}B \). It follows that there exists \( s \in S \) such that the locally nilpotent derivation \( sD : S^{-1}B \to S^{-1}B \) maps \( \tilde{B} \) into itself. The restriction \( \tilde{D} : \tilde{B} \to \tilde{B} \) of \( sD \) satisfies \( \ker \tilde{D} \cap B = \ker D \).

**Lemma 2.12.** For a two-dimensional affine \( k \)-domain \( R \),
\[
|\text{KLND}(R)| > 1 \quad \text{if and only if} \quad \text{ML}(R) \quad \text{is algebraic over} \quad k.
\]

Proof. Assume that \( \text{ML}(R) \) is algebraic over \( k \). Since \( \text{trdeg}_k A = 1 \) for any \( A \in \text{KLND}(R) \), it follows that \( |\text{KLND}(R)| > 1 \). In the other direction, let \( A \) and \( A' \) be distinct elements of \( \text{KLND}(R) \). As \( \text{trdeg}_k A = 1 = \text{trdeg}_k A' \) and \( A \cap A' \) is algebraically closed in \( R \), it follows that \( A \cap A' \) is algebraic over \( k \). Hence \( \text{ML}(R) \) is algebraic over \( k \).

**Corollary 2.13.** If \( R \in \mathcal{M}(k) \), then \( \tilde{R} \in \mathcal{M}(k') \) for some algebraic field extension \( k' \supseteq k \) such that \( k' \subset \tilde{R} \). In particular, if \( k \) is algebraically closed, then \( \text{ML}(\tilde{R}) = k \).

Proof. As \( R \in \mathcal{M}(k) \), we get \( |\text{KLND}(R)| > 1 \) by Lemma 2.12. Let \( A_1 \) and \( A_2 \) be distinct elements of \( \text{KLND}(R) \). There exist \( \tilde{A}_1, \tilde{A}_2 \in \text{KLND}(\tilde{R}) \) satisfying \( \tilde{A}_i \cap R = A_i \) (cf. Lemma 2.11), so \( |\text{KLND}(\tilde{R})| > 1 \). Hence \( \text{ML}(\tilde{R}) \) is algebraic over \( k \) and is a field, say, \( \text{ML}(\tilde{R}) = k' \). Then clearly, \( k \subseteq k' \subset \tilde{R} \) and \( k' \) is algebraic over \( k \).

**Lemma 2.14.** Let \( B \in \mathcal{M}(k) \). If \( B \) is normal and \( A \in \text{KLND}(B) \), then \( A \cong k^{[1]} \).
Proof. This result is well-known when \( k \) is algebraically closed. (See [6, 2.3], for instance.) To prove the general case, denote the algebraic closure of \( k \) by \( \bar{k} \). Let \( A \in \text{KLND}(B) \) and note that \( A \) is a 1-dimensional noetherian normal domain. To prove that \( A \cong \bar{k}^{[1]} \), it suffices to check that \( A \subseteq \bar{k}^{[1]} \). By [3, Lemma 3.7], \( B := \bar{k} \otimes_k B \) is an integral domain and \( \text{ML}(B) = \bar{k} \) by Corollary 2.13. Note that each element of \( \text{KLND}(B) \) is isomorphic to \( \bar{k}^{[1]} \). Given \( A \in \text{KLND}(B) \), \( \bar{k} \otimes_k A \in \text{KLND}(B) \) and there exists \( D \in \text{LND}(\bar{B}) \) such that \( \ker D \cap B = \bar{k} \otimes_k A \) (cf. Lemma 2.11). As \( \ker D \cong \bar{k}^{[1]} \), it follows that \( \bar{k} \otimes_k A \subseteq \bar{k}^{[1]} \). Then \( A \subseteq \bar{k}^{[1]} \) by Lemma 2.5.

3. Completion of surfaces and fibrations

Throughout Section 3, we fix \( k \) to be an algebraically closed field of characteristic zero. All varieties are assumed to be \( k \)-varieties. In this section, we state some properties of affine normal surfaces, fibrations on such surfaces, and completions of such surfaces. The material of this section is well-known.

3.1. Let \( S \) be a complete normal surface. By an SNC-divisor on \( S \), we mean a Weil divisor \( D = \sum_{i=1}^n C_i \) where \( C_1, \ldots, C_n \) are distinct irreducible curves on \( S \) satisfying the following conditions:
(i) \( \text{Supp}(D) = \bigcup_{i=1}^n C_i \) is included in \( S \setminus \text{Sing}(S) \).
(ii) Each irreducible component \( C_i \) of \( D \) is isomorphic to \( \mathbb{P}^1 \).
(iii) If \( i \neq j \) then \( C_i \cap C_j \leq 1 \).
(iv) If \( i, j, k \) are distinct then \( C_i \cap C_j \cap C_k = \emptyset \).

DEFINITION 3.2. An \( \mathbb{A}^1 \)-fibration (respectively, a \( \mathbb{P}^1 \)-fibration) on a surface \( S \) is a surjective morphism \( \rho: S \rightarrow Z \) on a nonsingular curve \( Z \) whose general fibres are isomorphic to \( \mathbb{A}^1 \) (respectively, to \( \mathbb{P}^1 \)). For our purposes, we will always consider \( \mathbb{A}^1 \)-fibrations whose codomain \( Z \) is \( \mathbb{A}^1 \).

DEFINITION 3.3. Let \( S \) be an affine normal surface and \( \rho: S \rightarrow \mathbb{A}^1 \) an \( \mathbb{A}^1 \)-fibration. By a completion of the pair \((S, \rho)\), we mean a commutative diagram of morphisms of algebraic varieties

\[
\begin{array}{ccc}
S & \xrightarrow{\rho} & \tilde{S} \\
\downarrow & & \downarrow \tilde{\rho} \\
\mathbb{A}^1 & \xleftarrow{\rho} & \mathbb{P}^1 \\
\end{array}
\]

such that the \( \xleftarrow{\rho} \) are open immersions, \( \tilde{S} \) is a complete normal surface, and \( \tilde{S} \setminus S \) is the support of an SNC-divisor of \( \tilde{S} \).
It is well-known that given any affine normal surface $S$ and an $\mathbb{A}^1$-fibration $\rho: S \to \mathbb{A}^1$, there exists a completion of $(S, \rho)$.

**Setup 3.4.** Throughout Paragraph 3.4, we assume:
(i) $S$ is an affine normal surface.
(ii) $\rho: S \to \mathbb{A}^1$ is an $\mathbb{A}^1$-fibration.
(iii) $(\tilde{S}, \bar{\rho})$ is a completion of $(S, \rho)$, with notation as in Diagram (1); we let $D$ be the SNC-divisor of $\tilde{S}$ whose support is $\tilde{S} \setminus S$.

As $\tilde{S}$ is complete, $\bar{\rho}$ is closed. So given any curve $C \subset \tilde{S}$, $\bar{\rho}(C)$ is either a point or all of $\mathbb{P}^1$. Accordingly we have:

**Definition 3.4.1.** A curve $C \subset \tilde{S}$ is said to be $\bar{\rho}$-vertical if $\bar{\rho}(C)$ is a point. Otherwise, we say that the curve is $\bar{\rho}$-horizontal. Thus $C \subset \tilde{S}$ is $\bar{\rho}$-horizontal if and only if $\bar{\rho}(C) = \mathbb{P}^1$.

**Lemma 3.4.2.** Let the setup be as in Setup 3.4.
(a) For a general point $z \in \mathbb{P}^1$, $\bar{\rho}^{-1}(z) \cong \mathbb{P}^1$ and $\bar{\rho}^{-1}(z) \cap S \cong \mathbb{A}^1$. In particular, $\bar{\rho}: \tilde{S} \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration.
(b) Exactly one irreducible component of $D$ is $\bar{\rho}$-horizontal.

Proof. As these facts are well-known, we only sketch the proof. By commutativity of Diagram (1), $\bar{\rho}^{-1}(z) \cap S = \rho^{-1}(z) \cong \mathbb{A}^1$ for general $z \in \mathbb{P}^1$. Assertion (a) follows from this. It also follows that the general fibre $\bar{\rho}^{-1}(z)$ meets $D$ in exactly one point, and this implies that $D$ has exactly one horizontal component.

**4. Geometry of surfaces in the class $\mathcal{M}(k)$**

In this section, $k$ is an arbitrary field of characteristic zero (except in Setup 4.1 and Corollary 4.3, where it is assumed to be algebraically closed).

**Setup 4.1.** The following assumptions and notations are valid throughout Paragraph 4.1. Suppose that $k$ is algebraically closed. Fix $B \in \mathcal{M}(k)$, suppose that $B$ is normal, and let $S = \text{Spec } B$. Consider distinct elements $A_1, A_2 \in \text{KLnD}(B)$ and recall from Lemma 2.14 that $A_i \cong k[i]$ for $i = 1, 2$. Let $\rho_i: S \to \mathbb{A}^1$ be the morphism determined by the inclusion $A_i \hookrightarrow B$ for $i = 1, 2$. It follows from Paragraph 2.1 (iv) that $\rho_1$ and $\rho_2$ are $\mathbb{A}^1$-fibrations, and Lemma 2.2 implies that $\rho_1$ and $\rho_2$ have distinct general fibres. Choose a complete normal surface $\tilde{S}$ and morphisms $\bar{\rho}_1, \bar{\rho}_2: \tilde{S} \to \mathbb{P}^1$ such that,
for each $i = 1, 2$, $(\bar{S}, \bar{\rho}_i)$ is a completion of $(S, \rho_i)$ in the sense of Definition 3.3. We also consider the following diagram:

\[
\begin{array}{c}
S \\ \rho_1 \downarrow \rho_1 \quad \rho_2 \downarrow \bar{\rho}_2 \\
\mathbb{A}^1 \downarrow \mathbb{A}^1 \\
\end{array}
\]

Let $\infty$ be such that $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ in Diagram (2). For $i = 1, 2$, let $H_i$ be the unique irreducible component of $D = \bar{S} \setminus S$ which is $\bar{\rho}_i$-horizontal. (See Lemma 3.4.2.)

**Lemma 4.1.1.** We have $\bar{\rho}_1(H_2) = \{\infty\}$ and $\bar{\rho}_2(H_1) = \{\infty\}$. In particular, $H_1 \neq H_2$.

**Proof.** Recall that $H_i \subseteq D$ and $\bar{\rho}_i(H_i) = \mathbb{P}^1$ for each $i = 1, 2$. For a general $z_1 \in \mathbb{P}^1$, $(\bar{\rho}_1)^{-1}(z_1) = C_1$, where $C_1$ is an irreducible curve of $\bar{S}$ which intersects $H_1$ in a unique point, say $Q$. As $\rho_1$ and $\rho_2$ have distinct general fibres, we choose $z_1$ so that $\rho_2(\rho_1^{-1}(z_1))$ is not a point. Then $\bar{\rho}_2(C_1)$ is not a point, so $\bar{\rho}_2(C_1) = \mathbb{P}^1$. Choose $Q_1 \in C_1$ such that $\bar{\rho}_2(Q_1) = \{\infty\}$. Clearly, $Q_1 \in D$. Since $C_1$ meets $D$ in exactly one point, $C_1 \cap D = \{Q_1\}$. Consequently, $\{Q\} = C_1 \cap H_1 \subseteq C_1 \cap D = \{Q_1\}$. It follows that $\{Q\} = C_1 \cap H_1$. Repeating this process for infinitely many points $z_i$ of $\mathbb{P}^1$, we get infinitely many points $Q_i \in H_1$ satisfying $\bar{\rho}_1(Q_i) = z_i$ and $\bar{\rho}_2(Q_i) = \{\infty\}$. Hence we conclude that $\bar{\rho}_2(H_1) = \{\infty\}$. Similarly, we can prove that $\bar{\rho}_1(H_2) = \{\infty\}$. As $\bar{\rho}_1(H_1) = \mathbb{P}^1 = \bar{\rho}_2(H_2)$, it follows immediately that $H_1$ and $H_2$ are distinct. \qed

**Proposition 4.1.2.** There does not exist an irreducible curve $C \subseteq S$ such that $\rho_1(C)$ and $\rho_2(C)$ are points.

**Proof.** By contradiction, suppose that there exists an irreducible curve $C_0$ of $S$ such that $\rho_1(C_0) = a_1$ and $\rho_2(C_0) = a_2$ for some points $a_i \in \mathbb{A}^1$. Consider $C := \bar{C}_0$, the closure of $C_0$ in $\bar{S}$. Then $C$ is a curve in $\bar{S}$ such that $C \cap D \neq \emptyset$, $\bar{\rho}_1(C) = a_1$, and $\bar{\rho}_2(C) = a_2$ (where $a_1, a_2 \in \mathbb{P}^1 \setminus \{\infty\}$). Since $D$ is connected, there is an integer $k \geq 1$ and a sequence $D_1, \ldots, D_k$ of irreducible components of $D$ satisfying:

- For each $1 \leq i < k$, $D_i$ is $\bar{\rho}_1$-vertical and $\bar{\rho}_2$-vertical, and $D_k \in \{H_1, H_2\}$.
- $C \cap D_1 \neq \emptyset$, and $D_i \cap D_{i+1} \neq \emptyset$ (for $1 \leq i < k$).

Note that $\bar{\rho}_j(D_k) = \infty$ for some $j \in \{1, 2\}$. Since $C \cup D_1 \cup \cdots \cup D_k$ is connected, it follows that $\bar{\rho}_j(C \cup D_1 \cup \cdots \cup D_k)$ is connected and is a finite set of points, i.e., is one point. But $a_j, \infty \in \bar{\rho}_j(C \cup D_1 \cup \cdots \cup D_k)$, so we obtain a contradiction. \qed

For the remainder of this paper, we assume that $\mathbf{k}$ is an arbitrary field of characteristic zero.
DEFINITION 4.2. Let $B$ be an integral domain of characteristic zero. We say that $B$ has property (*) if $B$ has no height 1 proper ideal $I$ which intersects two distinct elements $A_1, A_2 \in \text{KLND}(B)$ nontrivially. That is, $B$ has property (*) if $I \cap A_1 = 0$ or $I \cap A_2 = 0$ for all height 1 proper ideals $I$ of $B$ and all distinct $A_1, A_2 \in \text{KLND}(B)$.

Our next goal is to prove Theorem 4.6. We do this in several steps, as follows.

**Corollary 4.3.** Suppose that $k$ is algebraically closed and that $B \in \mathcal{M}(k)$ is normal. Then $B$ has property (*).

**Proof.** By contradiction, suppose that there exist distinct $A_1, A_2 \in \text{KLND}(B)$ and a height 1 ideal $I$ of $B$ such that $I \cap A_i \neq 0$ for $i = 1, 2$. Pick a height 1 prime ideal $p$ of $B$ such that $p \supseteq I$, and note that $p \cap A_i \neq 0$ for $i = 1, 2$. So the irreducible curve $C = V(p) \subset \text{Spec } B$ is mapped to a point by each canonical morphism $\rho_i : \text{Spec } B \rightarrow \text{Spec } A_i$ ($i = 1, 2$). This contradicts Proposition 4.1.2.

**Notation 4.4.** Let $B \subseteq B'$ be integral domains of characteristic zero. We write $B \triangleleft B'$ to indicate that $B'$ is integral over $B$ and that, for each $A \in \text{KLND}(B)$, there exists $A' \in \text{KLND}(B')$ such that $A' \cap B = A$. Clearly, $\triangleleft$ is a transitive relation.

**Lemma 4.5.** Let $B, B'$ be integral domains of characteristic zero such that $B \triangleleft B'$. If $B'$ has property (*), then so does $B$.

**Proof.** Let $I \neq B$ be a height 1 ideal of $B$ and let $A_1, A_2 \in \text{KLND}(B)$ satisfy $I \cap A_i \neq 0$. As $B'$ is integral over $B$, $IB' \neq B'$ and $\text{ht } IB' = 1$. Since $B \triangleleft B'$, there exist $A'_1, A'_2 \in \text{KLND}(B')$ such that $A'_i \cap B = A_i$ for $i = 1, 2$. Moreover, $A'_i \cap IB' \supseteq A_i \cap I \neq 0$. Since $B'$ has property (*), it follows that $A'_1 = A'_2$. Consequently, $A_1 = A_2$.

Recall that $k$ is an arbitrary field of characteristic zero.

**Theorem 4.6.** Each element $B$ of $\mathcal{M}(k)$ has property (*).

**Proof.** If $\tilde{B}$ denotes the normalization of $B$, $B \triangleleft \tilde{B}$ follows by Lemma 2.11. Moreover, Corollary 2.13 implies that $\tilde{B} \in \mathcal{M}(k')$ for some field $k'$. As $B \triangleleft \tilde{B}$, it suffices to prove the theorem when $B$ is normal by Lemma 4.5.

If $B$ is normal, $\overline{B} = \tilde{k} \otimes_k B$ is an integral domain and $\text{ML}(\overline{B}) = \tilde{k}$ by [3, Lemma 3.7]. Then the normalization $\tilde{B} \in \mathcal{M}(\tilde{k})$ by Corollary 2.13, so $\tilde{B}$ has property (*) by Corollary 4.3. It suffices to prove that $B \triangleleft \tilde{B}$ because then the result follows by Lemma 4.5.

As $\tilde{k}$ is integral over $k$, it follows that $\tilde{k} \otimes_k B$ is integral over $k \otimes_k B \cong B$. Furthermore, given $A \in \text{KLND}(B)$, $\tilde{A} = \tilde{k} \otimes_k A$ belongs to $\text{KLND}(\tilde{B})$ and $\tilde{A} \cap (k \otimes_k B) = A$. This proves that $B \triangleleft \tilde{B}$. Finally, $B \triangleleft \tilde{B}$ and $\triangleleft$ is transitive, so it follows that $B \triangleleft \tilde{B}$. 

Remark 4.7. Every two-dimensional affine $k$-domain has property (*). Indeed, let $B$ be such a ring. If $|\text{Klnd}(B)| \leq 1$, then it is trivial that $B$ has property (*). If $|\text{Klnd}(B)| > 1$ then $B \in \mathcal{M}(k')$ for some field $k'$, where $k'$ is algebraic over $k$ (cf. Lemma 2.12). Then the result follows from Theorem 4.6.

Definition 4.8. An affine scheme $\text{Spec } A$ is regular in codimension 1 if and only if $A_p$ is regular for every height 1 prime ideal $p$ of $A$.

Theorem 4.9 ([10, Theorem 73, p. 246]). Let $A$ an affine domain containing a field. Then

$$U = \{ p \in \text{Spec } A \mid A_p \text{ is a regular local ring} \}$$

is a nonempty open subset of the affine scheme $X = \text{Spec } A$.

Proposition 4.10. Let $B$ be an affine $k$-domain. If $p$ is a height 1 prime ideal of $B$ such that $B_p$ is not regular, then $D(p) \subseteq p$ for every $D \in \text{Lnd}(B)$.

Proof. The set $T = \{ p \in \text{Spec } B \mid B_p \text{ is not regular} \}$ is a closed and proper subset of $X := \text{Spec } B$. For every $p \in T$ satisfying $\text{ht } p = 1$, the closure $\overline{\{p\}}$ is an irreducible component of $T$ and $p$ is the unique generic point of that component. As $T$ has only finitely many irreducible components, it follows that $T$ contains only finitely many prime ideals of height 1. Denote these prime ideals by $p_1, \ldots, p_n$.

Pick $p \in \{p_1, \ldots, p_n\}$ and $D \in \text{Lnd}(B)$. We will prove that $D(p) \subseteq p$. In view of Lemma 2.10, it is enough to show that

$$e^{\lambda}D(p) \subseteq p \quad \text{for some nonzero } \lambda \in k.$$  

As the group $\text{Aut}(B)$ acts on the set $T$, it follows that it acts on $\{p_1, \ldots, p_n\}$. Furthermore, $k = \bigcup_{i=1}^n \{ \lambda \in k \mid e^{\lambda}D(p) = p_i \}$. Since $k$ is infinite, there exists $i \in \{1, \ldots, n\}$ such that $\Omega := \{ \lambda \in k \mid e^{\lambda}D(p) = p_i \}$ is infinite. Pick distinct elements $\lambda_1, \lambda_2$ of $\Omega$. Then $e^{(\lambda_2 + \lambda_1)}D(p) \subseteq p$. So (3) is true.

Corollary 4.11. If $B \in \mathcal{M}(k)$ and $X = \text{Spec } B$, then the set

$$\text{Sing}(X) = \{ p \in \text{Spec } B \mid B_p \text{ is not a regular local ring} \}$$

is finite. Consequently, $B$ is regular in codimension 1.

Proof. The set $T = \text{Sing}(X)$ is a proper closed subset of $X$, so $\dim T \leq 1$. It follows by Proposition 4.10 that given a height 1 prime ideal $p$ of $B$ belonging to $T$, $D(p) \subseteq p$ for every $D \in \text{Lnd}(B)$. Then Lemma 2.8 implies that $p \cap \ker D \neq 0$ for every $D \in \text{Lnd}(B)$. Since $B$ has property (*) by Theorem 4.6, we obtain that the set
KLND(B) is a singleton, a contradiction. So T contains no height 1 prime ideal; consequently, B is regular in codimension 1. This also proves that dim T = 0. So T is a finite set of maximal ideals.

5. An application to complete intersections

**Definition 5.1.** Let A be a domain containing a field k. We say that A is a complete intersection over k if it is isomorphic to a quotient

\[ k[X_1, \ldots, X_n]/(f_1, \ldots, f_p) \]

for some \( n, p \in \mathbb{N} \), where \( (f_1, \ldots, f_p) \) is a prime ideal of \( k[X_1, \ldots, X_n] \) of height \( p \). If R is a complete intersection over k, we also call Spec R a complete intersection over k.

Recall the following criterion for noetherian normal rings due to Serre.

**Theorem 5.2 (Serre).** A noetherian ring A is normal if and only if it satisfies

- \( (R_1) \) \( A_p \) is regular for all \( p \in \text{Spec} A \) with \( \text{ht} p \leq 1 \), and
- \( (S_2) \) depth \( A_p \) \( \geq \min(\text{ht} p, 2) \) for all \( p \in \text{Spec} A \).

**Corollary 5.3.** Let \( B \in \mathcal{M}(k) \). If B satisfies Serre’s condition \( (S_2) \), then B is normal. In particular, complete intersection surfaces in the class \( \mathcal{M}(k) \) are normal.

**Proof.** Consider \( B \in \mathcal{M}(k) \) and suppose that B satisfies \( (S_2) \). To show that B is normal, it suffices to prove that B satisfies \( (R_1) \). So let \( p \in \text{Spec} B \). If \( \text{ht} p = 0 \), then clearly \( B_p \) is regular. If \( \text{ht} p = 1 \), \( B_p \) is regular by Corollary 4.11.

If B is a complete intersection, then B is Cohen–Macaulay (cf. [8, Proposition 18.13]), and so it satisfies \( (S_2) \) (cf. [10, 17.I, p. 125]). Then the result follows by the previous case.

**Example 5.4.** Let \( B = k[x, xy, y^2, y^3] \). Then \( D = x \partial / \partial y \), \( E = y^2 \partial / \partial x \) are two nonzero locally nilpotent derivations of B and ML(B) = k. Note that B is not normal. So by Corollary 5.3, Spec B is not a complete intersection surface over k. By similar arguments, we can prove that \( S := \text{Spec} k[x^2, x^3, y^3, y^4, xy, x^2y, xy^2, xy^3] \) is a ML-surface which is not a complete intersection surface over k.

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References


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