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SINGULAR POINTS OF AFFINE ML-SURFACES

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Abstract

We give a geometric proof of the fact that any affine surface with trivial Makar-Limanov invariant has finitely many singular points. We deduce that a complete intersection surface with trivial Makar-Limanov invariant is normal.

1. Notation and introduction

Let us first fix some notation and recall some basic definitions. Throughout this paper, unless otherwise specified, $\mathbf{k}$ *will always denote a field of characteristic zero*. A domain means an integral domain. Given a domain R, $\operatorname{Frac} R$ denotes the field of fractions of R. By $\mathbf{k}^{[n]}$, we mean the polynomial ring in n variables over $\mathbf{k}$ and $\operatorname{Frac}(\mathbf{k}^{[n]})$ will be denoted by $\mathbf{k}^{(n)}$. The set of singular points of a variety X will be denoted by $\operatorname{Sing}(X)$.

DEFINITION 1.1. Given a $\mathbf{k}$-algebra B, a derivation $D\colon B \to B$ is *locally nilpotent* if for each $b \in B$, there exists a natural number n (depending on b) such that $D^n(b) = 0$. We use the following notations:

$$\operatorname{Der}(B) = \{D \mid D \text{ is a derivation of } B\},$$

$$\operatorname{LND}(B) = \{D \in \operatorname{Der}(B) \mid D \text{ is locally nilpotent}\},$$

$$\operatorname{KLND}(B) = \{\ker D \mid D \in \operatorname{LND}(B),\ D \neq 0\}.$$

Given a $\mathbf{k}$-domain B, one defines its *Makar-Limanov invariant* by

$$\operatorname{ML}(B) = \bigcap_{D \in \operatorname{LND}(B)} \ker D.$$

If $X = \operatorname{Spec} B$ is an affine $\mathbf{k}$-variety, define $\operatorname{ML}(X) = \operatorname{ML}(B)$. The Makar-Limanov invariant plays an important role in classifying and distinguishing affine varieties. We say that B has trivial Makar-Limanov invariant if $\operatorname{ML}(B) = \mathbf{k}$.

Affine spaces $\mathbb{A}^n_{\mathbf{k}}$ are the simplest examples of varieties with trivial Makar-Limanov invariant. While it is known that $\mathbb{A}^1_{\mathbf{k}}$ is the only affine curve which has trivial

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Makar-Limanov invariant, the class of affine surfaces with trivial Makar-Limanov invariant contains many more surfaces, some of which are not even normal. (See Example 5.4, for instance.)

Let $\mathcal{M}(\mathbf{k})$ denote the class of 2-dimensional affine $\mathbf{k}$-domains which have trivial Makar-Limanov invariant. We say that an affine surface $S = \operatorname{Spec} R$ belongs to the class $\mathcal{M}(\mathbf{k})$ if $R \in \mathcal{M}(\mathbf{k})$. Such a surface S is also called a ML-*surface*.

The following question arises naturally: *Classify all surfaces in the class* $\mathcal{M}(\mathbf{k})$.

In recent years, researchers including Bandman, Daigle, Dubouloz, Gurjar, Masuda, Makar-Limanov, Miyanishi, and Russell (see [1], [3], [6], [7], [9], [11]) have been actively investigating properties of normal (or smooth) surfaces belonging to the class $\mathcal{M}(\mathbf{k})$. However, it is desirable to understand what happens when we drop the assumption of normality. For instance, it is natural to ask *what are all hypersurfaces of the affine space* $\mathbb{A}^3_{\mathbf{k}}$ *with trivial Makar-Limanov invariant*, and it is not a priori clear that all those surfaces are normal: the fact that they are indeed normal is a consequence of the present paper.

In this paper, we prove that a surface in the class $\mathcal{M}(\mathbf{k})$ has only finitely many singular points. As an application, we prove that any complete intersection surface with trivial Makar-Limanov invariant is normal. Note that these results are valid over any field $\mathbf{k}$ of characteristic zero. The results of this paper will be used in a joint paper with D. Daigle [5], where we classify all hypersurfaces of $\mathbb{A}^3_{\mathbf{k}}$ (more generally, complete intersection surfaces over $\mathbf{k}$) with trivial Makar-Limanov invariant.

To understand the necessity of some of the arguments given in this paper, the reader should keep in mind certain pathologies that occur when $\mathbf{k}$ is not assumed to be algebraically closed. For instance, surfaces $S = \operatorname{Spec} R$ belonging to $\mathcal{M}(\mathbf{k})$ are not necessarily rational over $\mathbf{k}$ and may have very few $\mathbf{k}$-rational points; moreover, if $\bar{\mathbf{k}}$ is the algebraic closure of $\mathbf{k}$, then $\bar{\mathbf{k}} \otimes_{\mathbf{k}} R$ is not necessarily an integral domain.

2. Preliminaries

In this section, we gather some basic results and known facts.

2.1. Suppose that B is a $\mathbf{k}$-domain, let D be a nonzero locally nilpotent derivation of B, and let $A = \ker D$. The following are well-known definitions and facts about locally nilpotent derivations:

(i) A is *factorially closed* in B (i.e., the conditions $x, y \in B \setminus \{0\}$ and $xy \in A$ imply that $x, y \in A$). Consequently, A is algebraically closed in B.

(ii) Consider the multiplicative set $S = A \setminus \{0\}$ of B. We can extend D to an element $\mathfrak{D} \in \operatorname{LND}(S^{-1}B)$ defined by $\mathfrak{D}(b/s) = D(b)/s$. It is well-known that $S^{-1}B = (\operatorname{Frac} A)^{[1]}$.

(iii) For every $\lambda \in \mathbf{k}$, the map

$$e^{\lambda D}\colon B \to B, \quad b \mapsto \sum_{n=0}^{\infty} \lambda^n \frac{D^n(b)}{n!}$$

is a $\mathbf{k}$-algebra automorphism of B.
(iv) Let $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the canonical morphism induced by the inclusion map $A \hookrightarrow B$. Then there exists a nonempty open set $U \subseteq \operatorname{Spec} A$ such that

$$\pi^{-1}(\mathfrak{p}) \cong \mathbb{A}^1_{\kappa(\mathfrak{p})} \quad \text{for every } \mathfrak{p} \in U, \text{ where } \kappa(\mathfrak{p}) \text{ is the residue field } A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}.$$

Furthermore, if $\mathbf{k}$ is algebraically closed and A is $\mathbf{k}$-affine, then

$$\pi^{-1}(\mathfrak{m}) \cong \mathbb{A}^1_{\kappa(\mathfrak{m})} = \mathbb{A}^1_{\mathbf{k}} \quad \text{for every closed point } \mathfrak{m} \text{ of } U.$$

Lemma 2.2. *Given an affine* $\mathbf{k}$*-surface* $X = \operatorname{Spec} B$*, let* A_1 *and* A_2 *be two affine subalgebras of* B *of dimension* 1*. Set* $Y_i = \operatorname{Spec} A_i$ *and let* $Y_1 \xleftarrow{f_1} \operatorname{Spec} B \xrightarrow{f_2} Y_2$ *be the canonical morphisms determined by the inclusions* $A_i \hookrightarrow B$ *(for* $i = 1, 2$*). If* B *is algebraic over its subalgebra* $\mathbf{k}[A_1 \cup A_2]$*, then*

$$E = \{y \in Y_2 \mid f_1(f_2^{-1}(y)) \text{ is a point}\}$$

is not a dense subset of Y_2*, where by "*$y \in Y_2$*" we mean that* y *is a closed point of* Y_2*.*

We leave the proof of Lemma 2.2 to the reader, as it is basic algebraic geometry and is not directly related to the subject matter of this paper.

DEFINITION 2.3. A domain A of transcendence degree 1 over a field $\mathbf{k}$ is called a *polynomial curve* over $\mathbf{k}$ if it satisfies the following equivalent conditions:
(i) A is a subalgebra of $\mathbf{k}^{[1]}$.
(ii) $\operatorname{Frac} A = \mathbf{k}^{(1)}$ and A has one rational place at infinity.

NOTATION 2.4. Given a field extension $F/\mathbf{k}$, let $\mathbb{P}_{F/\mathbf{k}}$ be the set of valuation rings R of $F/\mathbf{k}$ such that $R \neq F$.

Lemma 2.5. *Let* A *be a* $\mathbf{k}$*-domain. If there exists an algebraic extension* $\mathbf{k}'$ *of* $\mathbf{k}$ *such that* $\mathbf{k}' \otimes_{\mathbf{k}} A$ *is a polynomial curve over* $\mathbf{k}'$*, then* A *is a polynomial curve over* $\mathbf{k}$*.*

Proof. We sketch a proof of this fact, as we were unable to find a suitable reference. It is easy to prove that A is affine. We may assume that $[\mathbf{k}' : \mathbf{k}] < \infty$. Let $F = \operatorname{Frac} A$ and $F' = \operatorname{Frac} A'$, where $A' = \mathbf{k}' \otimes_{\mathbf{k}} A$. Note that $[F' : F] = [\mathbf{k}' : \mathbf{k}]$ and $F' = \mathbf{k}'F$. In the terminology of [12], the function field $F'/\mathbf{k}'$ is an algebraic constant field extension of $F/\mathbf{k}$. By [12, Theorem III.6.3], $F'/\mathbf{k}'$ has same genus as $F/\mathbf{k}$ (hence, $F/\mathbf{k}$ has genus zero) and F'/F is unramified. It remains to prove that A has one rational place at infinity. Let

$$E = \{R \in \mathbb{P}_{F/\mathbf{k}} \mid A \not\subseteq R\} \quad \text{and} \quad E' = \{R' \in \mathbb{P}_{F'/\mathbf{k}'} \mid \mathbf{k}' \otimes_{\mathbf{k}} A \not\subseteq R'\}.$$

If R is any element of E, then every $R' \in \mathbb{P}_{F'/\mathbf{k}'}$ lying over R (i.e., satisfying $R' \cap F = R$) must belong to E'. But E' is a singleton, say $E' = \{R'\}$. It follows that E is a singleton, say $E = \{R\}$. Let κ' and κ be the residue fields of R' and R, respectively. Then $[F' : F] = ef$, where $f = [\kappa' : \kappa]$ and e is the ramification index of R' over R. As F'/F is unramified, we have $e = 1$. Since $\mathbf{k}' \otimes_{\mathbf{k}} A$ is a polynomial curve over $\mathbf{k}'$, $\kappa' = \mathbf{k}'$. Hence

$$[\mathbf{k}' : \mathbf{k}] = [F' : F] = ef = [\kappa' : \kappa] = [\mathbf{k}' : \kappa].$$

Thus, $\kappa = \mathbf{k}$ and A has one rational place at infinity. □

The following lemma can be obtained as an easy consequence of [4, Lemma 3.1].

Lemma 2.6. *Let B be a $\mathbf{k}$-algebra and $f(T) \in B[T]$, where T is an indeterminate.*
(a) *If $f(T)$ has infinitely many roots in $\mathbf{k}$, then $f(T) = 0$.*
(b) *If J is an ideal of B and $f(\lambda) \in J$ for infinitely many $\lambda \in \mathbf{k}$, then $f(T) \in J[T]$.*

DEFINITION 2.7. Let R be a ring and $D \in \operatorname{Der}(R)$. An ideal I of R is called an *integral ideal* for D if $D(I) \subseteq I$.

Lemma 2.8. *Let R be a $\mathbf{k}$-domain, and let I be a nonzero ideal of R. If $A \in$ KLND(R), then the following statements are equivalent*:
(1) $I \cap A \neq (0)$.
(2) *There exists $D \in$ LND(R) such that* $\ker D = A$ *and I is an integral ideal for D.*

Proof. Assume that (1) holds. Let $0 \neq a \in I \cap A$, and let $E \in$ LND(R) be such that $A = \ker E$. Since $a \in A$, $aE \in$ LND(R) and aE has kernel A. Moreover, as $a \in I$, $(aE)(b) = a(Eb) \in I$ for all $b \in I$. So $(aE)(I) \subseteq I$, and hence $D := aE$ is the required locally nilpotent derivation of R proving assertion (2).

In the other direction, assume that $0 \neq D \in$ LND(R), $\ker D = A$, and $D(I) \subseteq I$. Choose any $b \in I$, $b \neq 0$. Then the set $\{b, Db, D^2b, \dots\}$ is included in I and contains a nonzero element of A. □

The following is an easy consequence of [2, Lemma 2.10].

Lemma 2.9. *Let R be a noetherian $\mathbf{k}$-algebra, and let $D \in \operatorname{Der}(R)$. If I is an integral ideal for D, so is every minimal prime-over ideal of I.*

Lemma 2.10. *Let B be a $\mathbf{k}$-algebra, J an ideal of B, and $D \in$ LND(B). If $e^{tD}(J) \subseteq J$ for some nonzero $t \in \mathbf{k}$, then J is an integral ideal for D.*

Proof. First observe that if $e^{tD}(J) \subseteq J$ for some nonzero $t \in \mathbf{k}$, then $e^{tD}(J) \subseteq J$ for infinitely many $t \in \mathbf{k}$. Let $f \in J$. We will show that $D(f) \in J$. Let $n = \deg_D(f)$, i.e., n is the maximum nonnegative integer such that $D^n(f) \neq 0$. Define a polynomial $P(T) \in B[T]$ by

$$P(T) = f + D(f)T + \frac{D^2(f)T^2}{2!} + \cdots + \frac{D^n(f)T^n}{n!}.$$

Then for infinitely many $t \in \mathbf{k}$,

$$P(t) = f + D(f)t + \frac{D^2(f)t^2}{2!} + \cdots + \frac{D^n(f)t^n}{n!} = e^{tD}(f) \in J.$$

By Lemma 2.6, all coefficients of $P(T)$ belong to J, so $D(f) \in J$. $\square$

Lemma 2.11. *Let B be an affine $\mathbf{k}$-domain, and let $D \in \text{LND}(B)$. If $\tilde{B}$ denotes the normalization of B, then there exists $\tilde{D} \in \text{LND}(\tilde{B})$ such that* $\ker \tilde{D} \cap B = \ker D$.

Proof. We recall the well-known argument. Write $A = \ker D$ and let $S = A \setminus \{0\}$. Then D extends to a locally nilpotent derivation $\mathfrak{D}$ of $S^{-1}B$ such that $B \cap \ker \mathfrak{D} = A$. As $S^{-1}B$ is a polynomial ring over the field $S^{-1}A$, it is normal, and consequently $B \subseteq \tilde{B} \subseteq S^{-1}B$. It follows that there exists $s \in S$ such that the locally nilpotent derivation $s\mathfrak{D}\colon S^{-1}B \to S^{-1}B$ maps $\tilde{B}$ into itself. The restriction $\tilde{D}\colon \tilde{B} \to \tilde{B}$ of $s\mathfrak{D}$ satisfies $\ker \tilde{D} \cap B = \ker D$. $\square$

Lemma 2.12. *For a two-dimensional affine $\mathbf{k}$-domain R,*

$$|\text{KLND}(R)| > 1 \quad \textit{if and only if} \quad \text{ML}(R) \textit{ is algebraic over } \mathbf{k}.$$

Proof. Assume that $\text{ML}(R)$ is algebraic over $\mathbf{k}$. Since $\operatorname{trdeg}_{\mathbf{k}} A = 1$ for any $A \in \text{KLND}(R)$, it follows that $|\text{KLND}(R)| > 1$. In the other direction, let A and A' be distinct elements of $\text{KLND}(R)$. As $\operatorname{trdeg}_{\mathbf{k}} A = 1 = \operatorname{trdeg}_{\mathbf{k}} A'$ and $A \cap A'$ is algebraically closed in R, it follows that $A \cap A'$ is algebraic over $\mathbf{k}$. Hence $\text{ML}(R)$ is algebraic over $\mathbf{k}$. $\square$

Corollary 2.13. *If $R \in \mathcal{M}(\mathbf{k})$, then $\tilde{R} \in \mathcal{M}(\mathbf{k}')$ for some algebraic field extension $\mathbf{k}' \supseteq \mathbf{k}$ such that $\mathbf{k}' \subset \tilde{R}$. In particular, if $\mathbf{k}$ is algebraically closed, then* $\text{ML}(\tilde{R}) = \mathbf{k}$.

Proof. As $R \in \mathcal{M}(\mathbf{k})$, we get $|\text{KLND}(R)| > 1$ by Lemma 2.12. Let A_1 and A_2 be distinct elements of $\text{KLND}(R)$. There exist $\tilde{A}_1, \tilde{A}_2 \in \text{KLND}(\tilde{R})$ satisfying $\tilde{A}_i \cap R = A_i$ (cf. Lemma 2.11), so $|\text{KLND}(\tilde{R})| > 1$. Hence $\text{ML}(\tilde{R})$ is algebraic over $\mathbf{k}$ and is a field, say, $\text{ML}(\tilde{R}) = \mathbf{k}'$. Then clearly, $\mathbf{k} \subseteq \mathbf{k}' \subset \tilde{R}$ and $\mathbf{k}'$ is algebraic over $\mathbf{k}$. $\square$

Lemma 2.14. *Let $B \in \mathcal{M}(\mathbf{k})$. If B is normal and $A \in \text{KLND}(B)$, then $A \cong \mathbf{k}^{[1]}$.*

Proof. This result is well-known when $\mathbf{k}$ is algebraically closed. (See [6, 2.3], for instance.) To prove the general case, denote the algebraic closure of $\mathbf{k}$ by $\bar{\mathbf{k}}$. Let $A \in \text{KLND}(B)$ and note that A is a 1-dimensional noetherian normal domain. To prove that $A \cong \mathbf{k}^{[1]}$, it suffices to check that $A \subseteq \mathbf{k}^{[1]}$. By [3, Lemma 3.7], $\mathcal{B} := \bar{\mathbf{k}} \otimes_{\mathbf{k}} B$ is an integral domain and $\text{ML}(\mathcal{B}) = \bar{\mathbf{k}}$. If $\tilde{\mathcal{B}}$ denotes the normalization of $\mathcal{B}$, then $\text{ML}(\tilde{\mathcal{B}}) = \bar{\mathbf{k}}$ by Corollary 2.13. Note that each element of $\text{KLND}(\tilde{\mathcal{B}})$ is isomorphic to $\bar{\mathbf{k}}^{[1]}$. Given $A \in \text{KLND}(B)$, $\bar{\mathbf{k}} \otimes_{\mathbf{k}} A \in \text{KLND}(\mathcal{B})$ and there exists $D \in \text{LND}(\tilde{\mathcal{B}})$ such that $\ker D \cap \mathcal{B} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} A$ (cf. Lemma 2.11). As $\ker D \cong \bar{\mathbf{k}}^{[1]}$, it follows that $\bar{\mathbf{k}} \otimes_{\mathbf{k}} A \subseteq \bar{\mathbf{k}}^{[1]}$. Then $A \subseteq \mathbf{k}^{[1]}$ by Lemma 2.5. □

3. Completion of surfaces and fibrations

Throughout Section 3, we fix $\mathbf{k}$ to be an algebraically closed field of characteristic zero. All varieties are assumed to be $\mathbf{k}$-varieties. In this section, we state some properties of affine normal surfaces, fibrations on such surfaces, and completions of such surfaces. The material of this section is well-known.

3.1. Let S be a complete normal surface. By an *SNC-divisor* on S, we mean a Weil divisor $D = \sum_{i=1}^{n} C_i$ where $C_1, \dots, C_n$ are distinct irreducible curves on S satisfying the following conditions:
(i) $\text{Supp}(D) = \bigcup_{i=1}^{n} C_i$ is included in $S \setminus \text{Sing}(S)$.
(ii) Each irreducible component C_i of D is isomorphic to $\mathbb{P}^1$.
(iii) If $i \neq j$ then $C_i \cap C_j \leq 1$.
(iv) If i, j, k are distinct then $C_i \cap C_j \cap C_k = \varnothing$.

DEFINITION 3.2. An $\mathbb{A}^1$-*fibration* (respectively, a $\mathbb{P}^1$-*fibration*) on a surface S is a surjective morphism $\rho\colon S \to Z$ on a nonsingular curve Z whose general fibres are isomorphic to $\mathbb{A}^1$ (respectively, to $\mathbb{P}^1$). For our purposes, we will always consider $\mathbb{A}^1$-fibrations whose codomain Z is $\mathbb{A}^1$.

DEFINITION 3.3. Let S be an affine normal surface and $\rho\colon S \to \mathbb{A}^1$ an $\mathbb{A}^1$-fibration. By a *completion of the pair* (S, ρ), we mean a commutative diagram of morphisms of algebraic varieties

(1)
$$\begin{array}{ccc} S & \hookrightarrow & \bar{S} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\bar{\rho}} \\ \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \end{array}$$

such that the "$\hookrightarrow$" are open immersions, $\bar{S}$ is a complete normal surface, and $\bar{S} \setminus S$ is the support of an SNC-divisor of $\bar{S}$.

It is well-known that given any affine normal surface S and an $\mathbb{A}^1$-fibration $\rho\colon S \to \mathbb{A}^1$, there exists a completion of (S, ρ).

SETUP 3.4. Throughout Paragraph 3.4, we assume:
(i) S is an affine normal surface.
(ii) $\rho\colon S \to \mathbb{A}^1$ is an $\mathbb{A}^1$-fibration.
(iii) $(\bar{S}, \bar{\rho})$ is a completion of (S, ρ), with notation as in Diagram (1); we let D be the SNC-divisor of $\bar{S}$ whose support is $\bar{S} \setminus S$.

As $\bar{S}$ is complete, $\bar{\rho}$ is closed. So given any curve $C \subset \bar{S}$, $\bar{\rho}(C)$ is either a point or all of $\mathbb{P}^1$. Accordingly we have:

DEFINITION 3.4.1. A curve $C \subset \bar{S}$ is said to be *$\bar{\rho}$-vertical* if $\bar{\rho}(C)$ is a point. Otherwise, we say that the curve is *$\bar{\rho}$-horizontal*. Thus $C \subset \bar{S}$ is $\bar{\rho}$-horizontal if and only if $\bar{\rho}(C) = \mathbb{P}^1$.

Lemma 3.4.2. *Let the setup be as in* Setup 3.4.
(a) *For a general point $z \in \mathbb{P}^1$, $\bar{\rho}^{-1}(z) \cong \mathbb{P}^1$ and $\bar{\rho}^{-1}(z) \cap S \cong \mathbb{A}^1$. In particular, $\bar{\rho}\colon \bar{S} \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration.*
(b) *Exactly one irreducible component of D is $\bar{\rho}$-horizontal.*

Proof. As these facts are well-known, we only sketch the proof. By commutativity of Diagram (1), $\bar{\rho}^{-1}(z) \cap S = \rho^{-1}(z) \cong \mathbb{A}^1$ for general $z \in \mathbb{P}^1$. Assertion (a) follows from this. It also follows that the general fibre $\bar{\rho}^{-1}(z)$ meets D in *exactly* one point, and this implies that D has exactly one horizontal component. □

4. Geometry of surfaces in the class $\mathcal{M}(\mathbf{k})$

In this section, $\mathbf{k}$ is an arbitrary field of characteristic zero (except in Setup 4.1 and Corollary 4.3, where it is assumed to be algebraically closed).

SETUP 4.1. The following assumptions and notations are valid throughout Paragraph 4.1. Suppose that $\mathbf{k}$ is algebraically closed. Fix $B \in \mathcal{M}(\mathbf{k})$, suppose that B is normal, and let $S = \operatorname{Spec} B$. Consider distinct elements $A_1, A_2 \in \text{KLND}(B)$ and recall from Lemma 2.14 that $A_i \cong \mathbf{k}^{[1]}$ for $i = 1, 2$. Let $\rho_i\colon S \to \mathbb{A}^1$ be the morphism determined by the inclusion $A_i \hookrightarrow B$ for $i = 1, 2$. It follows from Paragraph 2.1 (iv) that ρ_1 and ρ_2 are $\mathbb{A}^1$-fibrations, and Lemma 2.2 implies that ρ_1 and ρ_2 have distinct general fibres. Choose a complete normal surface $\bar{S}$ and morphisms $\bar{\rho}_1, \bar{\rho}_2\colon \bar{S} \to \mathbb{P}^1$ such that, for each $i = 1, 2$, $(\bar{S}, \bar{\rho}_i)$ is a completion of (S, ρ_i) in the sense of Definition 3.3. We also consider the following diagram:

$$\begin{array}{ccc} S & \hookrightarrow & \bar{S} \\ \rho_1 \downarrow\;\downarrow \rho_2 & & \bar{\rho}_1 \downarrow\;\downarrow \bar{\rho}_2 \\ \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1. \end{array} \tag{2}$$

Let ∞ be such that $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ in Diagram (2). For $i = 1, 2$, let H_i be the unique irreducible component of $D = \bar{S} \setminus S$ which is $\bar{\rho}_i$-horizontal. (See Lemma 3.4.2.)

Lemma 4.1.1. *We have $\bar{\rho}_1(H_2) = \{\infty\}$ and $\bar{\rho}_2(H_1) = \{\infty\}$. In particular, $H_1 \neq H_2$.*

Proof. Recall that $H_i \subseteq D$ and $\bar{\rho}_i(H_i) = \mathbb{P}^1$ for each $i = 1, 2$. For a general $z_1 \in \mathbb{P}^1$, $(\bar{\rho}_1)^{-1}(z_1) = C_1$, where C_1 is an irreducible curve of $\bar{S}$ which intersects H_1 in a unique point, say Q. As ρ_1 and ρ_2 have distinct general fibres, we choose z_1 so that $\rho_2(\rho_1^{-1}(z_1))$ is not a point. Then $\bar{\rho}_2(C_1)$ is not a point, so $\bar{\rho}_2(C_1) = \mathbb{P}^1$. Choose $Q_1 \in C_1$ such that $\bar{\rho}_2(Q_1) = \{\infty\}$. Clearly, $Q_1 \in D$. Since C_1 meets D in exactly one point, $C_1 \cap D = \{Q_1\}$. Consequently, $\{Q\} = C_1 \cap H_1 \subseteq C_1 \cap D = \{Q_1\}$. It follows that $\{Q_1\} = C_1 \cap H_1$. Repeating this process for infinitely many points z_i of $\mathbb{P}^1$, we get infinitely many points $Q_i \in H_1$ satisfying $\bar{\rho}_1(Q_i) = z_i$ and $\bar{\rho}_2(Q_i) = \{\infty\}$. Hence we conclude that $\bar{\rho}_2(H_1) = \{\infty\}$. Similarly, we can prove that $\bar{\rho}_1(H_2) = \{\infty\}$. As $\bar{\rho}_1(H_1) = \mathbb{P}^1 = \bar{\rho}_2(H_2)$, it follows immediately that H_1 and H_2 are distinct. □

Proposition 4.1.2. *There does not exist an irreducible curve $C \subset S$ such that $\rho_1(C)$ and $\rho_2(C)$ are points.*

Proof. By contradiction, suppose that there exists an irreducible curve C_0 of S such that $\rho_1(C_0) = a_1$ and $\rho_2(C_0) = a_2$ for some points $a_i \in \mathbb{A}^1$. Consider $C := \bar{C}_0$, the closure of C_0 in $\bar{S}$. Then C is a curve in $\bar{S}$ such that $C \cap D \neq \varnothing$, $\bar{\rho}_1(C) = a_1$, and $\bar{\rho}_2(C) = a_2$ (where $a_1, a_2 \in \mathbb{P}^1 \setminus \{\infty\}$). Since D is connected, there is an integer $k \geq 1$ and a sequence $D_1, \ldots, D_k$ of irreducible components of D satisfying:

- For each $1 \leq i < k$, D_i is $\bar{\rho}_1$-vertical and $\bar{\rho}_2$-vertical, and $D_k \in \{H_1, H_2\}$.
- $C \cap D_1 \neq \varnothing$, and $D_i \cap D_{i+1} \neq \varnothing$ (for $1 \leq i < k$).

Note that $\bar{\rho}_j(D_k) = \infty$ for some $j \in \{1, 2\}$. Since $C \cup D_1 \cup \cdots \cup D_k$ is connected, it follows that $\bar{\rho}_j(C \cup D_1 \cup \cdots \cup D_k)$ is connected and is a finite set of points, i.e., is one point. But $a_j, \infty \in \bar{\rho}_j(C \cup D_1 \cup \cdots \cup D_k)$, so we obtain a contradiction. □

For the remainder of this paper, we assume that **k** is an arbitrary field of characteristic zero.

DEFINITION 4.2. Let B be an integral domain of characteristic zero. We say that B has property $(*)$ if B has no height 1 proper ideal I which intersects two distinct elements $A_1, A_2 \in \mathrm{KLND}(B)$ nontrivially. That is, B has property $(*)$ if $I \cap A_1 = 0$ or $I \cap A_2 = 0$ for all height 1 proper ideals I of B and all distinct $A_1, A_2 \in \mathrm{KLND}(B)$.

Our next goal is to prove Theorem 4.6. We do this in several steps, as follows.

Corollary 4.3. *Suppose that* $\mathbf{k}$ *is algebraically closed and that* $B \in \mathcal{M}(\mathbf{k})$ *is normal. Then* B *has property* $(*)$.

Proof. By contradiction, suppose that there exist distinct $A_1, A_2 \in \mathrm{KLND}(B)$ and a height 1 ideal I of B such that $I \cap A_i \neq 0$ for $i = 1, 2$. Pick a height 1 prime ideal $\mathfrak{p}$ of B such that $\mathfrak{p} \supseteq I$, and note that $\mathfrak{p} \cap A_i \neq 0$ for $i = 1, 2$. So the irreducible curve $C = V(\mathfrak{p}) \subset \operatorname{Spec} B$ is mapped to a point by each canonical morphism $\rho_i\colon \operatorname{Spec} B \to \operatorname{Spec} A_i$ $(i = 1, 2)$. This contradicts Proposition 4.1.2. $\square$

NOTATION 4.4. Let $B \subseteq B'$ be integral domains of characteristic zero. We write $B \triangleleft B'$ to indicate that B' is integral over B and that, for each $A \in \mathrm{KLND}(B)$, there exists $A' \in \mathrm{KLND}(B')$ such that $A' \cap B = A$. Clearly, $\triangleleft$ is a transitive relation.

Lemma 4.5. *Let* B, B' *be integral domains of characteristic zero such that* $B \triangleleft B'$. *If* B' *has property* $(*)$, *then so does* B.

Proof. Let $I \neq B$ be a height 1 ideal of B and let $A_1, A_2 \in \mathrm{KLND}(B)$ satisfy $I \cap A_i \neq 0$. As B' is integral over B, $IB' \neq B'$ and $\operatorname{ht} IB' = 1$. Since $B \triangleleft B'$, there exist $A_1', A_2' \in \mathrm{KLND}(B')$ such that $A_i' \cap B = A_i$ for $i = 1,2$. Moreover, $A_i' \cap IB' \supset A_i \cap I \neq 0$. Since B' has property $(*)$, it follows that $A_1' = A_2'$. Consequently, $A_1 = A_2$. $\square$

Recall that $\mathbf{k}$ is an arbitrary field of characteristic zero.

Theorem 4.6. *Each element* B *of* $\mathcal{M}(\mathbf{k})$ *has property* $(*)$.

Proof. If $\tilde{B}$ denotes the normalization of B, $B \triangleleft \tilde{B}$ follows by Lemma 2.11. Moreover, Corollary 2.13 implies that $\tilde{B} \in \mathcal{M}(\mathbf{k}')$ for some field $\mathbf{k}'$. As $B \triangleleft \tilde{B}$, it suffices to prove the theorem when B is normal by Lemma 4.5.

If B is normal, $\mathcal{B} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} B$ is an integral domain and $\mathrm{ML}(\mathcal{B}) = \bar{\mathbf{k}}$ by [3, Lemma 3.7]. Then the normalization $\tilde{\mathcal{B}} \in \mathcal{M}(\bar{\mathbf{k}})$ by Corollary 2.13, so $\tilde{\mathcal{B}}$ has property $(*)$ by Corollary 4.3. It suffices to prove that $B \triangleleft \tilde{\mathcal{B}}$ because then the result follows by Lemma 4.5.

As $\bar{\mathbf{k}}$ is integral over $\mathbf{k}$, it follows that $\bar{\mathbf{k}} \otimes_{\mathbf{k}} B$ is integral over $\mathbf{k} \otimes_{\mathbf{k}} B \cong B$. Furthermore, given $A \in \mathrm{KLND}(B)$, $\bar{A} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} A$ belongs to $\mathrm{KLND}(\mathcal{B})$ and $\bar{A} \cap (\mathbf{k} \otimes_{\mathbf{k}} B) = A$. This proves that $B \triangleleft \mathcal{B}$. Finally, $\mathcal{B} \triangleleft \tilde{\mathcal{B}}$ and $\triangleleft$ is transitive, so it follows that $B \triangleleft \tilde{\mathcal{B}}$. $\square$

REMARK 4.7. Every two-dimensional affine **k**-domain has property $(*)$. Indeed, let B be such a ring. If $|\text{KLND}(B)| \leq 1$, then it is trivial that B has property $(*)$. If $|\text{KLND}(B)| > 1$ then $B \in \mathcal{M}(\mathbf{k}')$ for some field $\mathbf{k}'$, where $\mathbf{k}'$ is algebraic over $\mathbf{k}$ (cf. Lemma 2.12). Then the result follows from Theorem 4.6.

DEFINITION 4.8. An affine scheme Spec A is *regular in codimension* 1 if and only if $A_{\mathfrak{p}}$ is regular for every height 1 prime ideal $\mathfrak{p}$ of A.

Theorem 4.9 ([10, Theorem 73, p. 246]). *Let A an affine domain containing a field. Then*

$$U = \{\mathfrak{p} \in \operatorname{Spec} A \mid A_{\mathfrak{p}} \textit{ is a regular local ring}\}$$

is a nonempty open subset of the affine scheme $X = \operatorname{Spec} A$.

Proposition 4.10. *Let B be an affine* **k**-*domain. If $\mathfrak{p}$ is a height* 1 *prime ideal of B such that $B_{\mathfrak{p}}$ is not regular, then $D(\mathfrak{p}) \subseteq \mathfrak{p}$ for every $D \in \text{LND}(B)$.*

Proof. The set $T = \{\mathfrak{p} \in \operatorname{Spec} B \mid B_{\mathfrak{p}} \text{ is not regular}\}$ is a closed and proper subset of $X := \operatorname{Spec} B$. For every $\mathfrak{p} \in T$ satisfying $\operatorname{ht} \mathfrak{p} = 1$, the closure $\overline{\{\mathfrak{p}\}}$ is an irreducible component of T and $\mathfrak{p}$ is the unique generic point of that component. As T has only finitely many irreducible components, it follows that T contains only finitely many prime ideals of height 1. Denote these prime ideals by $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.

Pick $\mathfrak{p} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $D \in \text{LND}(B)$. We will prove that $D(\mathfrak{p}) \subseteq \mathfrak{p}$. In view of Lemma 2.10, it is enough to show that

$$e^{\lambda D}(\mathfrak{p}) \subseteq \mathfrak{p} \quad \text{for some nonzero} \quad \lambda \in \mathbf{k}. \tag{3}$$

As the group $\operatorname{Aut}(B)$ acts on the set T, it follows that it acts on $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Furthermore, $\mathbf{k} = \bigcup_{i=1}^{n} \{\lambda \in \mathbf{k} \mid e^{\lambda D}(\mathfrak{p}) = \mathfrak{p}_i\}$. Since $\mathbf{k}$ is infinite, there exists $i \in \{1, \ldots, n\}$ such that $\Omega := \{\lambda \in \mathbf{k} \mid e^{\lambda D}(\mathfrak{p}) = \mathfrak{p}_i\}$ is infinite. Pick distinct elements λ_1, λ_2 of Ω. Then $e^{(-\lambda_2+\lambda_1)D}(\mathfrak{p}) \subseteq \mathfrak{p}$. So (3) is true. $\square$

Corollary 4.11. *If $B \in \mathcal{M}(\mathbf{k})$ and $X = \operatorname{Spec} B$, then the set*

$$\operatorname{Sing}(X) = \{\mathfrak{p} \in \operatorname{Spec} B \mid B_{\mathfrak{p}} \textit{ is not a regular local ring}\}$$

is finite. Consequently, B is regular in codimension 1.

Proof. The set $T = \operatorname{Sing}(X)$ is a proper closed subset of X, so $\dim T \leq 1$. It follows by Proposition 4.10 that given a height 1 prime ideal $\mathfrak{p}$ of B belonging to T, $D(\mathfrak{p}) \subseteq \mathfrak{p}$ for every $D \in \text{LND}(B)$. Then Lemma 2.8 implies that $\mathfrak{p} \cap \ker D \neq 0$ for every $D \in \text{LND}(B)$. Since B has property $(*)$ by Theorem 4.6, we obtain that the set

KLND(B) is a singleton, a contradiction. So T contains no height 1 prime ideal; consequently, B is regular in codimension 1. This also proves that $\dim T = 0$. So T is a finite set of maximal ideals. □

5. An application to complete intersections

DEFINITION 5.1. Let A be a domain containing a field $\mathbf{k}$. We say that A is a *complete intersection over* $\mathbf{k}$ if it is isomorphic to a quotient

$$\mathbf{k}[X_1, \ldots, X_n]/(f_1, \ldots, f_p)$$

for some $n, p \in \mathbb{N}$, where $(f_1, \ldots, f_p)$ is a prime ideal of $\mathbf{k}[X_1, \ldots, X_n]$ of height p. If R is a complete intersection over $\mathbf{k}$, we also call Spec R a complete intersection over $\mathbf{k}$.

Recall the following criterion for noetherian normal rings due to Serre.

Theorem 5.2 (Serre). *A noetherian ring A is normal if and only if it satisfies*
(R_1) $A_{\mathfrak{p}}$ *is regular for all* $\mathfrak{p} \in \operatorname{Spec} A$ *with* $\operatorname{ht}\mathfrak{p} \leq 1$, *and*
(S_2) $\operatorname{depth} A_{\mathfrak{p}} \geq \min(\operatorname{ht}\mathfrak{p}, 2)$ *for all* $\mathfrak{p} \in \operatorname{Spec} A$.

Corollary 5.3. *Let $B \in \mathcal{M}(\mathbf{k})$. If B satisfies Serre's condition (S_2), then B is normal. In particular, complete intersection surfaces in the class $\mathcal{M}(\mathbf{k})$ are normal.*

Proof. Consider $B \in \mathcal{M}(\mathbf{k})$ and suppose that B satisfies (S_2). To show that B is normal, it suffices to prove that B satisfies (R_1). So let $\mathfrak{p} \in \operatorname{Spec} B$. If $\operatorname{ht}\mathfrak{p} = 0$, then clearly $B_{\mathfrak{p}}$ is regular. If $\operatorname{ht}\mathfrak{p} = 1$, $B_{\mathfrak{p}}$ is regular by Corollary 4.11.

If B is a complete intersection, then B is Cohen–Macaulay (cf. [8, Proposition 18.13]), and so it satisfies (S_2) (cf. [10, 17.I, p. 125]). Then the result follows by the previous case. □

EXAMPLE 5.4. Let $B = \mathbf{k}[x, xy, y^2, y^3]$. Then $D = x\,\partial/\partial y$, $E = y^2\,\partial/\partial x$ are two nonzero locally nilpotent derivations of B and $\operatorname{ML}(B) = \mathbf{k}$. Note that B is not normal. So by Corollary 5.3, Spec B is not a complete intersection surface over $\mathbf{k}$. By similar arguments, we can prove that $S := \operatorname{Spec} \mathbf{k}[x^2, x^3, y^3, y^4, y^5, xy, x^2y, xy^2, xy^3]$ is a ML-surface which is not a complete intersection surface over $\mathbf{k}$.

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