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ON WEAK CONVERGENCE OF DIFFUSION PROCESSES GENERATED BY ENERGY FORMS

TOSHIHIRO UEMURA

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0. Introduction

Convergences of closed forms, energy forms or energy functions have been studied by many authors (see e.g. [1]-[3], [5]-[8]). It is important to know, given an energy form, if it can be approximated by “nice” ones, or, given a sequence of energy forms, what their “limit” is.

In this paper we consider a sequence of forms $\mathcal{E}^n(u,v) = \int_{\mathbb{R}^d} A_n(x) \nabla u(x) \cdot \nabla v(x) \, dx$ with certain domains on $L^2(\mathbb{R}^d, \phi_n^2 dx)$, where $\phi_n$ are locally bounded functions on $\mathbb{R}^d$, $A_n$ are $(d \times d)$ symmetric matrix valued functions on $\mathbb{R}^d$, $(\cdot, \cdot)$ means the inner product on $\mathbb{R}^d$ and $\nabla u = (\nabla_1 u, \nabla_2 u, \ldots, \nabla_d u)$ is the distributional (weak) derivative of $u$. Take strictly positive, bounded functions $f_n$ with $\int_{\mathbb{R}^d} f_n \phi_n^2 dx = 1$ and denote by $\{X_t, P^n_x, x \in \mathbb{R}^d\}$ the diffusion processes associated with the forms $\mathcal{E}^n$. We study the weak convergence of the probability measures $\{\mathcal{L}^n_{P^n_x}, \mathcal{L}^n_{P^n_x} \}$ with $d\mathbf{m}_n = f_n \phi_n^2 dx$, when the data $A_n, \phi_n$, and $f_n$ converge a.e. on $\mathbb{R}^d$, as $n \to \infty$.

Although our main result (see section 1) is similar to that of T.J. Lyons and T.S. Zhang [5], we assume only a certain local boundedness of $\phi_n$, while a uniform boundedness on the whole space is assumed in [5]. In order to obtain the result in [5], they generalized the theorem of Kato and Simon on monotone sequence of closed forms (see M. Reed and B. Simon [7]) used by S. Albeverio, R. Høegh-Krohn and L. Streit [1]. We will instead adopt the Mosco-convergence of closed forms (see U. Mosco [6]) to prove our theorem.

S. Albeverio, S. Kusuoka and L. Streit [2] obtained a semigroup convergence by imposing the regularity conditions that there exist $R>0$ and $C>0$ such that, the restrictions of $\phi_n$ to $\mathbb{R}^d - B_R$ is of class $C^2$ and the growth order of $x \cdot \nabla \phi_n / \phi_n$ is not greater than $C|x|^2$ on $\mathbb{R}^d - B_R$. No smoothness on $A_n, \phi_n$ is required in the present approach.

1. Statement of Theorem

Let $\phi_n(x), \phi(x)$ be measurable functions on $\mathbb{R}^d$ and $A_n(x), A(x)$ be $(d \times d)$ symmetric matrix valued functions on $\mathbb{R}^d$. Consider the following conditions:
(A.1) (i) there exists a constant $\delta > 0$ such that
\[ 0 \leq \frac{1}{\delta} |\xi|^2 \leq (A_n(x)\xi,\xi) \leq \delta |\xi|^2, \quad \text{for } dx\text{-a.e. } x \in \mathbb{R}^d, \xi \in \mathbb{R}^d, n \in \mathbb{N}. \]

(ii) for any relatively compact open set $G$ of $\mathbb{R}^d$, there exist constants $\lambda(G), \Lambda(G) > 0$ such that,
\[ 0 < \lambda(G) \leq \phi_n(x) \leq \Lambda(G), \quad \text{for } dx\text{-a.e. } x \in G, n \in \mathbb{N}, \]

(iii) $\phi_n(x) \to \phi(x)$, $dx$-a.e. on $\mathbb{R}^d$,

(iv) $A_n(x) \to A(x)$ in matrix norm, $dx$-a.e. on $\mathbb{R}^d$.

We consider the forms
\[ \mathcal{E}_n(u,v) = \int_{\mathbb{R}^d} (A_n(x)\nabla u(x),\nabla v(x))_d \phi^2_n(x) dx, \tag{1.1} \]
\[ \mathcal{F}^n = \{ u \in L^2(\mathbb{R}^d;\phi^2_n dx) : \nabla u \in L^2(\mathbb{R}^d;\phi^2_n dx), i = 1,2,\ldots,d \}, \]
for $n = 1,2,3,\ldots$,

\[ \mathcal{E}(u,v) = \int_{\mathbb{R}^d} (A(x)\nabla u(x),\nabla v(x))_d \phi^2(x) dx, \tag{1.2} \]
\[ \mathcal{F} = \{ u \in L^2(\mathbb{R}^d;\phi^2 dx) : \nabla u \in L^2(\mathbb{R}^d;\phi^2 dx), i = 1,2,\ldots,d \}, \]

Our assumption (A.1) implies that the forms (1.1) and (1.2) are regular local Dirichlet forms on $L^2(\mathbb{R}^d;\phi^2_n dx)$ and $L^2(\mathbb{R}^d;\phi^2 dx)$ (called "energy forms") respectively. It follows from M. Fukushima, Y. Oshima and M. Takeda [4] that there exist diffusion processes $M^n = \{X_t^n(x), P^n_x, x \in \mathbb{R}^d\}$ and $M^\phi = \{X_t^\phi(x), x \in \mathbb{R}^d\}$ associated with $\mathcal{E}^n$ and $\mathcal{E}$ respectively. Further, we consider the following condition:

(A.2) there exists a constant $c > 0$ such that $\sup_n \int_{B_r} \phi_n^2 dx \leq e^{cr^2}$, for all $r > 0$.

Then by condition (A.2) and Theorem 2.2 in M. Takeda [8], these processes are conservative. For every relatively compact open set $G$ of $\mathbb{R}^d$, we consider the Dirichlet forms of part on $G$ associated with (1.1) and (1.2):
\[ \mathcal{E}^n_G(u,v) = \int_G (A_n(x)\nabla u(x),\nabla v(x))_d \phi^2_n(x) dx, \tag{1.3} \]
\[ \mathcal{F}^n_G = H^1_0(G) \quad \text{on } L^2(G;\phi^2_n dx), \]
for \( n = 1, 2, 3, \ldots \),

\[
E^G(u, v) = \int_G (A(x)\nabla u(x), \nabla v(x))_x \phi^2(x) dx,
\]

\[
\mathcal{F}_G = H_0^1(G) \text{ on } L^2(G; \phi^2 dx),
\]

Now take strictly positive functions \( f_n \) of \( L^1(R^d, \phi^2 dx) \) and \( f \) of \( L^1(R^d, \phi^2 dx) \) and assume the conditions below:

\[\begin{align*}
(A.3) \text{ (i)} & \quad \int_{R^d} dm_n = \int_{R^d} dm = 1, \text{ where } dm_n = f_n \phi^2 dx \text{ and } dm = f \phi^2 dx, \\
& \quad \text{(ii) for any compact set } K, \sup_n \| f_n \|_{L^\infty(K; \phi^2 dx)} < \infty, \\
& \quad \text{(iii) } f_n(x) \to f(x), \, dx\text{-a.e. on } R^d.
\end{align*}\]

It follows from conditions (A.2), (A.3) and Theorem 3.1 in M. Takeda [8] that the sequence of probability measures \( \{ P_{m_n}, n = 1, 2, \ldots \} \) is tight on \( C([0, \infty) \to R^d) \). Moreover we can assert as follows:

**Theorem.** Assume the conditions (A.1)-(A.3). Then \( \{ P_{m_n}, n = 1, 2, \ldots \} \) converges weakly to \( P_m \) on \( C([0, \infty) \to R^d) \).

2. **Proof of Theorem**

In order to carry out the proof of Theorem, we need some lemmas and notations. Henceforth, for a form \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) on a Hilbert space \( \mathcal{H} \), we let \( \mathcal{E}(u, u) = \infty \) for every \( u \in \mathcal{H} \setminus \mathcal{D}(\mathcal{E}) \). Here a form means a non-negative definite symmetric form on \( \mathcal{H} \), not necessarily densely defined. As was mentioned in the introduction, we use the notion of the Mosco-convergence of forms, which is defined as follows:

**DEFINITION.** A sequence of forms \( \mathcal{E}^n \) on a Hilbert space \( \mathcal{H} \) is said to be Mosco-convergent to a form \( \mathcal{E} \) on \( \mathcal{H} \) if the following conditions are satisfied;

(M.1) for every sequence \( u_n \) weakly convergent to \( u \) in \( \mathcal{H} \),

\[\liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) \geq \mathcal{E}(u, u).\]

(M.2) for every \( u \) in \( \mathcal{H} \), there exists \( u_n \) converging to \( u \) in \( \mathcal{H} \), such that

\[\limsup_{n \to \infty} \mathcal{E}^n(u_n, u_n) \geq \mathcal{E}(u, u).\]

In [6], U. Mosco showed that a sequence of closed forms \( \mathcal{E}^n \) on a Hilbert space
\( \mathcal{H} \) is Mosco-convergent to a closed form \( \mathcal{E} \) on \( \mathcal{H} \) if and only if the resolvents associated with \( \mathcal{E}^n \) converges to the resolvent associated with \( \mathcal{E} \) strongly on \( \mathcal{H} \).

In order to use Mosco’s theorem, we introduce related forms:

\[
\mathcal{A}^n_G(u,v) = \mathcal{E}^n_G \left( \frac{u}{\phi_n}, \frac{v}{\phi_n} \right) = \int_G \left( A_n(x) \nabla \frac{u(x)}{\phi_n(x)} \cdot \nabla \frac{v(x)}{\phi_n(x)} \right) \phi_n^2(x) dx,
\]

for \( n = 1, 2, 3, \ldots \),

\[
\mathcal{D}(\mathcal{A}^n_G) = \{ u \in L^2(G;dx) : u / \phi_n \in H^1_0(G) \}
\]

(2.1)

\[
\mathcal{D}(\mathcal{A}^G) = \{ u \in L^2(G;dx) : u / \phi \in H^1_0(G) \}.
\]

By the unitary map \( f \mapsto \phi^{-1}_n f \) between \( L^2(G;dx) \) and \( L^2(G;\phi_n^2 dx) \) and by the condition (A.1), the forms \( \mathcal{A}^n_G \) and \( \mathcal{A}^G \) are closed on \( L^2(G;dx) \).

**Lemma.** Assume the condition (A.1). Then the forms \( \mathcal{A}^n_G \) is Mosco-convergent to the form \( \mathcal{A}^G \) on \( L^2(G;dx) \).

**Proof.** We have to check the conditions (M.1) and (M.2).

First we note that, from the condition (A.1), there exist \((d \times d)\) symmetric matrix valued functions \( \sqrt{A_n}(x) = (\sigma_{ijn}(x)) \) and \( \sqrt{A}(x) = (\sigma_{ij}(x)) \) defined on \( \mathbb{R}_d \) with the following properties:

(i) \( A_n(x) = (\sqrt{A_n}(x))^2, A(x) = (\sqrt{A}(x))^2, \)

(ii) \( \sqrt{A_n}(x) \to \sqrt{A}(x) \) in matrix norm, \( dx \)-a.e. on \( \mathbb{R}_d \).

In particular, \( |\sqrt{A_n}(x) \xi| \leq \delta n \xi dx - a.e. x \in \mathbb{R}_d, \xi \in \mathbb{R}_d, n \in \mathbb{N} \). Hence \( \sigma_{ijn}(x) \) is uniformly bounded on \( \mathbb{R}_d \) and converges to \( \sigma_{ij}(x) dx \)-a.e. as \( n \to \infty \) for each \( ij \).

**Proof of (M.1).** Suppose \( u_n \to u \) weakly in \( L^2(G;dx) \). We may assume

\[
\liminf_{n \to \infty} \mathcal{A}^n_G(u_n, u_n) < \infty.
\]

Then we have

\[
+ \infty > \liminf_{n \to \infty} \mathcal{A}^n_G(u_n, u_n)
\]

\[
= \liminf_{n \to \infty} \int_G \left( A_n(\nabla \frac{u_n}{\phi_n}) \cdot (\nabla \frac{u_n}{\phi_n}) \phi_n^2 dx
\]

\[
\geq \frac{\lambda(G)^2}{\delta} \liminf_{n \to \infty} \int_G |\nabla \frac{u_n}{\phi_n}|^2 dx,
\]
and we can take a subsequence \( \{n_k\} \) such that \( \nabla_i \left( \frac{u_{n_k}}{\phi_{n_k}} \right) \) is weakly convergent to an element \( h_i \in L^2(G;dx) \) for each \( i = 1, 2, \cdots, d \) and \( \lim \inf_{n \to \infty} \mathscr{A}^{n_i, G}(u_{n_k}, u_{n_k}) = \lim_{k \to \infty} A^{n_k, G}(u_{n_k}, u_{n_k}) \).

On the other hand, for all \( \eta \in C_0^\infty(G) \), \( \int_G \nabla_i \left( \frac{u_{n_k}}{\phi_{n_k}} \right) \eta dx = -\int_G \frac{u_{n_k} \nabla \eta}{\phi_{n_k}} dx \), and \( u_n/\phi_n \) converges to \( u/\phi \) weakly in \( L^2(G;dx) \), because \( \phi_n^{-1} \) is uniformly bounded and converges to \( \phi^{-1} \) \( dx \)-a.e. on \( G \). This shows that

\[
\int_G h_i \eta dx = -\int_{\phi} \nabla \eta dx, \quad \text{for all } \eta \in C_0^\infty(G).
\]

Thus we have \( h_i = \nabla_i \left( \frac{u}{\phi} \right) \), \( i = 1, 2, \cdots, d \), and in particular \( u \in \mathcal{D} \mathcal{(A^G)} \).

Furthermore \( \Sigma_{j=1}^d \sigma_{ij}^n \nabla_i \left( \frac{u_{n_k}}{\phi_{n_k}} \right) \phi_{n_k} \) converges to \( \Sigma_{j=1}^d \sigma_{ij} \nabla_i \left( \frac{u}{\phi} \right) \phi \) weakly in \( L^2(G;dx) \), since \( \sigma_{ij}^n \phi_n \) is uniformly bounded and converges to \( \sigma_{ij} \phi dx \)-a.e. on \( G \) as \( n \to \infty \) for each \( i, j \).

Consequently,

\[
\lim \inf_{n \to \infty} \mathscr{A}^{n_i, G}(u_{n_k}, u_{n_k}) = \lim_{n_k \to \infty} \mathscr{A}^{n_k, G}(u_{n_k}, u_{n_k})
\]

\[
= \lim_{n_k \to \infty} \int_G \left( A_{n_k} \nabla \left( \frac{u_{n_k}}{\phi_{n_k}} \right) \nabla \left( \frac{u_{n_k}}{\phi_{n_k}} \right) \right) dx
\]

\[
= \lim_{n_k \to \infty} \int_G \left( \sqrt{A_{n_k}} \nabla \left( \frac{u_{n_k}}{\phi_{n_k}} \right) \right)^2 \phi_{n_k}^2 dx
\]

\[
= \lim_{n_k \to \infty} \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij}^n \nabla_i \left( \frac{u_{n_k}}{\phi_{n_k}} \right) \phi_{n_k} \|L^2(G;dx)
\]

\[
\geq \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} \nabla_i \left( \frac{u}{\phi} \right) \phi \|L^2(G;dx) = \mathcal{A}^G(u, \mu).
\]

Proof of (M.2). Let \( u \) be in \( \mathcal{D}(A^G) \), that is, \( u \in L^2(G;dx) \) and \( u/\phi \in H_0^1(G) \).

Accordingly there exists a sequence \( \{n_k\} \) in \( C_0^\infty(G) \) such that \( \|u/\phi - n_k\|_{H_0^1(G)} \) converges to 0 as \( n \to \infty \). Put \( u_n = \phi_n n_k \). Then we can see that \( u_n \to u \) in \( L^2(G;dx) \). Further, using again the property of the sequence \( \sigma_{ij} \phi_n \) observed above, we get that

\[
\sum_{j=1}^d \sigma_{ij}^n \nabla_j n_k \phi_n \to \sum_{j=1}^d \sigma_{ij} \nabla_j \left( \frac{u}{\phi} \right) \phi \text{ in } L^2(G;dx), \quad \text{for } i = 1, 2, \cdots, d.
\]

Therefore we have
This lemma shows that, if we let \( H^{n,G} \) and \( H^G \) be the selfadjoint operators associated with the forms \( \mathcal{A}^{n,G} \) and \( \mathcal{A}^G \) respectively, then \( H^{n,G} \) converges to \( H^G \) in the strong resolvent sense, hence, in the semigroup sense on \( L^2(G;dx) \) by Mosco's theorem.

Let \( H^{n,G}_{\Phi} \) and \( H^G_{\Phi} \) also denote the selfadjoint operators associated with the forms \( \Phi^{n,G} \) and \( \Phi^G \) respectively. Then by the unitary map \( f \mapsto \Phi^{-1}_n f \) between \( L^2(G;dx) \) and \( L^2(G_\Phi^G;dx) \), \( H^{n,G}_{\Phi} = \Phi_n H^{n,G}_{\Phi} \Phi_n^{-1} \).

On the other hand, let \( M_\{X_n, P^{n,G} \} \) and \( M_\{X, P^G \} \) be the diffusion processes associated with the forms \( \Phi^{n,G} \) and \( \Phi^G \) respectively. Because \( \Phi^{n,G} \) is the part of \( \Phi \) on \( G \) as we have already noted, the behaviour of the process \( \{ X_{n, P^{n,G}_X}, x \in G \} \) is the same as that of \( \{ X_{n, P^G_X}, x \in G \} \) before it leaves \( G \) for each \( n \).

Now we can give the proof of Theorem:

Proof of Theorem. By Lemma and the argument following it, we see that \( \Phi_n e^{-tH^{n,G} \Phi_n} \) converges to \( \Phi e^{-tH^G \Phi} \) strongly on \( L^2(G;dx) \). Here \( e^{-tH^{n,G} \Phi_n} \) and \( e^{-tH^G \Phi} \) denotes the semigroups associated with \( \Phi^{n,G} \) and \( \Phi^G \) respectively. Therefore, by virtue of Theorem 7 in [1], \( P^{n,G}_m \) converges to \( P^G \) in the finite dimensional distribution sense.

On the other hand, one has from condition (A.2) and Lemma 2.1 in [8] that

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} (|X_d| - |X_0|) = 0, \quad \text{for all } R > 0, T > 0.
\]

Then, for any \( 0 < t_1 < t_2 < \cdots < t_p \), \( A_i \in \mathcal{A}(R^d) \), \( i = 1, 2, \ldots, p \) and \( \epsilon > 0 \), there exists an \( r > 0 \) such that \( \sup_n P^\{t_1 \leq \tau_r \} \leq \epsilon / 2 \). Moreover, we can see that \( P_m(t_p \geq \tau_r) \leq \epsilon / 2 \). Here \( \tau_r \) denotes the exit time for the open ball \( B_r \) with radius \( r \) and center \( O \).

Let \( \Lambda = \{ X_{t_1} \in A_1, X_{t_2} \in A_2, \ldots, X_{t_p} \in A_p \} \). Then we see that

\[
|P^\{t_1 \leq \tau_r \} - P^\{t_1 \leq \tau_r \}| \leq |P^\{t_1 \leq \tau_r \} - P^\{t_1 \leq \tau_r \} - P^\{t_1 \leq \tau_r \} - P^\{t_1 \leq \tau_r \}|
\]

q.e.d.
\[ \leq P_{m_n}(t_p \geq \tau_r) + P_m(t_p \geq \tau_r) \]
\[ + |P_{m_n}(\Lambda \cap \{ t_p < \tau_r \}) - P_m(\Lambda \cap \{ t_p < \tau_r \})| \]

The first and second term of the right hand side are less than \( \epsilon \). Since the last term is the finite dimensional distribution of \( M^{n,B_r} \) and \( M^{B_r} \), we conclude that \( P_{m_n} \) converges to \( P_m \) in the finite dimensional distribution sense.

We have already noted the tightness of \( \{ P_{m_n} \} \) on \( C([0,\infty) \to \mathbb{R}^d) \). Thus the proof of Theorem is completed.

Example. Let \( f \) be a locally bounded measurable function on \( \mathbb{R}^d \), and consider a mollifier, e.g., \( j(x) = \gamma \exp(-1/1 - |x|^2) \) for \( |x| < 1 \), \( j(x) = 0 \) for \( |x| \geq 1 \), where \( \gamma \) is a constant to make \( \int_{\mathbb{R}^d} j(x) \, dx = 1 \). We put \( f_\varepsilon(x) = j(x/\varepsilon) / \varepsilon^d \), \( f_\varepsilon(x) = \int_{\mathbb{R}^d} j_\varepsilon(x-y) f(y) \, dy \), for any \( \varepsilon > 0 \). Since \( f_\varepsilon \) converges to \( f \) in \( L^2(G, dx) \) for each relatively compact open set \( G \), we can take a sequence \( \varepsilon_n \) converging to 0 such that \( f_{\varepsilon_n} \) converges to \( f \), \( dx \)-a.e. on \( \mathbb{R}^d \). Thus if we set \( \phi_n(x) = \exp f_{\varepsilon_n}(x) \), \( \phi(x) = \exp f(x) \), and assume that there exists a constant \( c > 0 \) with \( \int_{\mathbb{R}^d} e^{2f(x)} \, dx \leq e^{cr^2} \), for \( r > 0 \), then \( \phi_n \), \( \phi \) satisfies the conditions (A.1) and (A.2). Therefore we have the weak convergence statement for the processes associated with \( \phi_n \), \( \phi \) and \( A_n = A = \text{identity matrix} \).

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