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ON WEAK CONVERGENCE OF DIFFUSION PROCESSES GENERATED BY ENERGY FORMS

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0. Introduction

Convergences of closed forms, energy forms or energy functions have been studied by many authors (see e.g. [1]-[3], [5]-[8]). It is important to know, given an energy form, if it can be approximated by "nice" ones, or, given a sequence of energy forms, what their "limit" is.

In this paper we consider a sequence of forms $\mathscr{E}^n(u,v)=\int_{\boldsymbol{R}^d}(A_n(x)\nabla u(x), \nabla v(x))_d\phi_n^2(x)dx$ with certain domains on $L^2(\boldsymbol{R}^d;\phi_n^2dx)$, where ϕ_n are locally bounded functions on $\boldsymbol{R}^d$, A_n are $(d\times d)$ symmetric matrix valued functions on $\boldsymbol{R}^d$, $(\cdot,\cdot)_d$ means the inner product on $\boldsymbol{R}^d$ and $\nabla u=(\nabla_1 u,\nabla_2 u,\cdots,\nabla_d u)$ is the distributional (weak) derivative of u. Take strictly positive, bounded functions f_n with $\int_{\boldsymbol{R}^d}f_n\phi_n^2dx=1$ and denote by $\{X_t,P_x^n,x\in\boldsymbol{R}^d\}$ the diffusion processes associated with the forms $\mathscr{E}^n$. We study the weak convergence of the probability measures $\{P_{m_n}^n,n=1,2,\cdots\}$ with $dm_n=f_n\phi_n^2dx$, when the date A_n, ϕ_n, and f_n converge *a.e.* on $\boldsymbol{R}^d$, as $n\to\infty$.

Although our main result (see section 1) is similar to that of T.J. Lyons and T.S. Zhang [5], we assume only a certain local boundedness of ϕ_n, while a uniform boundedness on the whole space is assumed in [5]. In order to obtain the result in [5], they generalized the theorem of Kato and Simon on monotone sequence of closed forms (see M. Reed and B. Simon [7]) used by S. Albeverio, R. Høegh-Krohn and L. Streit [1]. We will instead adopt the Mosco-convergence of closed forms(see U. Mosco [6]) to prove our theorem.

S. Albeverio, S. Kusuoka and L. Streit [2] obtained a semigroup convergence by imposing the regularity conditions that there exist $R>0$ and $C>0$ such that, the restrictions of ϕ_n to $\boldsymbol{R}^d-B_R$ is of class C^2 and the growth order of $x\cdot\nabla\phi_n/\phi_n$ is not greater than $C|x|^2$ on $\boldsymbol{R}^d-B_R$. No smoothness on A_n, ϕ_n is required in the present approach.

1. Statement of Theorem

Let $\phi_n(x)$, $\phi(x)$ be measurable functions on $\boldsymbol{R}^d$ and $A_n(x)$, $A(x)$ be $(d\times d)$ symmetric matrix valued functions on $\boldsymbol{R}^d$. Consider the following conditions:

(A.1) (i) there exists a constant $\delta>0$ such that

$$0\le\frac{1}{\delta}|\xi|^2\le(A_n(x)\xi,\xi)_d\le\delta|\xi|^2, \qquad \text{for } dx\text{-}a.e.x\in \boldsymbol{R}^d,\xi\in\boldsymbol{R}^d,n\in\boldsymbol{N}.$$

(ii) for any relatively compact open set G of $\boldsymbol{R}^d$, there exist constants $\lambda(G)$, $\Lambda(G)>0$ such that,

$$0<\lambda(G)\le\phi_n(x)\le\Lambda(G), \qquad \text{for } dx\text{-}a.e.x\in G,n\in\boldsymbol{N},$$

(iii) $\phi_n(x)\to\phi(x)$, dx-$a.e.$ on $\boldsymbol{R}^d$,

(iv) $A_n(x)\to A(x)$ in matrix norm, dx-$a.e.$ on $\boldsymbol{R}^d$.

We consider the forms

$$\begin{aligned}\mathscr{E}^n(u,v)&=\int_{\boldsymbol{R}^d}(A_n(x)\nabla u(x),\nabla v(x))_d\phi_n^2(x)dx,\\ \mathscr{F}^n&=\{u\in L^2(\boldsymbol{R}^d;\phi_n^2dx):\nabla_iu\in L^2(\boldsymbol{R}^d;\phi_n^2dx),i=1,2,\cdots,d\},\end{aligned}\tag{1.1}$$

for $n=1,2,3,\cdots$,

$$\begin{aligned}\mathscr{E}(u,v)&=\int_{\boldsymbol{R}^d}(A(x)\nabla u(x),\nabla v(x))_d\phi^2(x)dx,\\ \mathscr{F}&=\{u\in L^2(\boldsymbol{R}^d;\phi^2dx):\nabla_iu\in L^2(\boldsymbol{R}^d;\phi^2dx),i=1,2,\cdots,d\},\end{aligned}\tag{1.2}$$

Our assumption (A.1) implies that the forms (1.1) and (1.2) are regular local Dirichlet forms on $L^2(\boldsymbol{R}^d;\phi_n^2dx)$ and $L^2(\boldsymbol{R}^d;\phi^2dx)$ (called "energy forms") respectively. It follows from M. Fukushima, Y. Oshima and M. Takeda [4] that there exist diffusion processes $M^n=\{X_t,P_x^n,x\in\boldsymbol{R}^d\}$ and $M^n=\{X_t,P_x,x\in\boldsymbol{R}^d\}$ associated with $\mathscr{E}^n$ and $\mathscr{E}$ respectively. Further, we consider the following condition:

(A.2) there exists a constant $c>0$ such that $\sup_n\int_{B_r}\phi_n^2dx\le e^{cr^2}$, for all $r>0$.

Then by condition (A.2) and Theorem 2.2 in M. Takeda [8], these processes are conservative. For every relatively compact open set G of $\boldsymbol{R}^d$, we consider the Dirichlet forms of part on G associated with (1.1) and (1.2):

$$\begin{aligned}\mathscr{E}^{n,G}(u,v)&=\int_G(A_n(x)\nabla u(x),\nabla v(x))_d\phi_n^2(x)dx,\\ \mathscr{F}_G^n&=H_0^1(G)\quad\text{on } L^2(G;\phi_n^2dx),\end{aligned}\tag{1.3}$$

for $n=1,2,3,\cdots$,

$$\mathscr{E}^G(u,v)=\int_G (A(x)\nabla u(x),\nabla v(x))_d\phi^2(x)dx,$$
$$\mathscr{F}_G=H_0^1(G) \quad \text{on } L^2(G;\phi^2 dx), \tag{1.4}$$

Now take strictly positive functions f_n of $L^1(\boldsymbol{R}^d;\phi_n^2dx)$ and f of $L^1(\boldsymbol{R}^d;\phi^2dx)$ and assume the conditions below:

(A.3) (i) $\int_{\boldsymbol{R}^d} dm_n=\int_{\boldsymbol{R}^d} dm=1$, where $dm_n=f_n\phi_n^2dx$ and $dm=f\phi^2dx$,

(ii) for any compact set K, $\sup_n \|f_n\|_{L^\infty(K;\phi_n^2dx)}<\infty$,

(iii) $f_n(x)\to f(x)$, dx-*a.e.* on $\boldsymbol{R}^d$.

It follows from conditions (A.2), (A.3) and Theorem 3.1 in M. Takeda [8] that the sequence of probability measures $\{P^n_{m_n}, n=1,2,\cdots\}$ is tight on $C([0,\infty)\to\boldsymbol{R}^d)$. Moreover we can assert as follows:

Theorem. *Assume the conditions* (A.1)-(A.3). *Then* $\{P^n_{m_n}, n=1,2,\cdots\}$ *converges weakly to* P_m *on* $C([0,\infty)\to\boldsymbol{R}^d)$.

2. Proof of Theorem

In order to carry out the proof of Theorem, we need some lemmas and notations.

Henceforth, for a form $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ on a Hilbert space $\mathscr{H}$, we let $\mathscr{E}(u,u)=\infty$ for every $u\in\mathscr{H}-\mathscr{D}(\mathscr{E})$. Here a form means a non-negative definite symmetric form on $\mathscr{H}$, not necessarily densely defined. As was mentioned in the introduction, we use the notion of the Mosco-convergence of forms, which is defined as follows:

DEFINITION. A sequence of forms $\mathscr{E}^n$ on a Hilbert space $\mathscr{H}$ is said to be Mosco-convergent to a form $\mathscr{E}$ on $\mathscr{H}$ if the following conditions are satisfied;

(M.1) for every sequence u_n weakly convergent to u in $\mathscr{H}$,

$$\liminf_{n\to\infty}\mathscr{E}^n(u_n,u_n)\geq\mathscr{E}^n(u,u).$$

(M.2) for every u in $\mathscr{H}$, there exists u_n converging to u in $\mathscr{H}$, such that

$$\limsup_{n\to\infty}\mathscr{E}^n(u_n,u_n)\geq\mathscr{E}(u,u).$$

In [6], U. Mosco showed that a sequence of closed forms $\mathscr{E}^n$ on a Hilbert space

$\mathscr{H}$ is Mosco-convergent to a closed form $\mathscr{E}$ on $\mathscr{H}$ if and only if the resolvents associated with $\mathscr{E}^n$ converges to the resolvent associated with $\mathscr{E}$ strongly on $\mathscr{H}$.

In order to use Mosco's theorem, we introduce related forms:

$$\begin{aligned} \mathscr{A}^{n,G}(u,v) &= \mathscr{E}^{n,G}\left(\frac{u}{\phi_n},\frac{v}{\phi_n}\right)=\int_G \left(A_n(x)\nabla\frac{u(x)}{\phi_n(x)},\nabla\frac{v(x)}{\phi_n(x)}\right)_d \phi_n^2(x)dx, \\ \mathscr{D}(\mathscr{A}^{n,G}) &= \{u\in L^2(G;dx): u/\phi_n \in H_0^1(G)\} \end{aligned} \tag{2.1}$$

for $n=1,2,3,\cdots$,

$$\begin{aligned} \mathscr{A}^{G}(u,v) &= \mathscr{E}^{G}\left(\frac{u}{\phi},\frac{v}{\phi}\right)=\int_G \left(A(x)\nabla\frac{u(x)}{\phi(x)},\nabla\frac{v(x)}{\phi(x)}\right)_d \phi^2(x)dx, \\ \mathscr{D}(\mathscr{A}^{G}) &= \{u\in L^2(G;dx): u/\phi \in H_0^1(G)\}. \end{aligned} \tag{2.2}$$

By the unitary map $f\mapsto \phi_n^{-1}f$ between $L^2(G;dx)$ and $L^2(G;\phi_n^2dx)$ and by the condition (A.1), the forms $\mathscr{A}^{n,G}$ and $\mathscr{A}^G$ are closed on $L^2(G;dx)$.

Lemma. *Assume the condition* (A.1). *Then the forms $\mathscr{A}^{n,G}$ is Mosco-convergent to the form $\mathscr{A}^G$ on $L^2(G;dx)$.*

Proof. We have to check the conditions (M.1) and (M.2).

First we note that, from the condition (A.1), there exist $(d\times d)$ symmetric matrix valued functions $\sqrt{A_n(x)}=(\sigma_{ij}^n(x))$ and $\sqrt{A(x)}=(\sigma_{ij}(x))$ defined on $\boldsymbol{R}^d$ with the following properties:

(i) $A_n(x)=(\sqrt{A_n(x)})^2$, $A(x)=(\sqrt{A(x)})^2$,

(ii) $\sqrt{A_n(x)}\to\sqrt{A(x)}$ in matrix norm, dx-*a.e.* on $\boldsymbol{R}^d$.

In particular, $|\sqrt{A_n(x)}\xi|\le\sqrt{\delta}|\xi|, dx-a.e. x\in\boldsymbol{R}^d, \xi\in\boldsymbol{R}^d, n\in\boldsymbol{N}$. Hence $\sigma_{ij}^n(x)$ is uniformly bounded on $\boldsymbol{R}^d$ and converges to $\sigma_{ij}(x) dx$-*a.e.* as $n\to\infty$ for each i,j.

Proof of (M.1). Suppose $u_n\to u$ weakly in $L^2(G;dx)$. We may assume

$$\liminf \mathscr{A}^{n,G}(u_n,u_n)<\infty.$$

Then we have

$$\begin{aligned} +\infty &> \liminf_{n\to\infty}\mathscr{A}^{n,G}(u_n,u_n) \\ &= \liminf_{n\to\infty}\int_G \left(A_n\nabla\left(\frac{u_n}{\phi_n}\right),\nabla\left(\frac{u_n}{\phi_n}\right)\right)_d\phi_n^2dx \\ &\ge \frac{\lambda(G)^2}{\delta}\liminf_{n\to\infty}\int_G \left|\nabla\left(\frac{u_n}{\phi_n}\right)\right|^2dx, \end{aligned}$$

and we can take a subsequence $\{n_k\}$ such that $\nabla_i\left(\frac{u_{n_k}}{\phi_{n_k}}\right)$ is weakly convergent to an element $h_i \in L^2(G;dx)$ for each $i=1,2,\cdots,d$ and $\liminf_{n\to\infty}\mathscr{A}^{n,G}(u_n,u_n)=\lim_{k\to\infty}A^{n_k,G}(u_{n_k},u_{n_k})$.

On the other hand, for all $\eta\in C_0^\infty(G)$, $\int_G \nabla_i\left(\frac{u_{n_k}}{\phi_{n_k}}\right)\eta dx=-\int_G\frac{u_{n_k}}{\phi_{n_k}}\nabla_i\eta dx$, and u_n/ϕ_n converges to u/ϕ weakly in $L^2(G;dx)$, because ϕ_n^{-1} is uniformly bounded and converges to ϕ^{-1} *dx-a.e.* on G. This shows that

$$\int_G h_i\eta dx=-\int_G\frac{u}{\phi}\nabla_i\eta dx, \qquad \text{for all } \eta\in C_0^\infty(G).$$

Thus we have $h_i=\nabla_i\left(\frac{u}{\phi}\right)$, $i=1,2,\cdots,d$, and in particular $u\in\mathscr{D}(\mathscr{A}^G)$.

Furthermore $\sum_{j=1}^d\sigma_{ij}^{n_k}\nabla_j\left(\frac{u_{n_k}}{\phi_{n_k}}\right)\phi_{n_k}$ converges to $\sum_{j=1}^d\sigma_{ij}\nabla_j\left(\frac{u}{\phi}\right)\phi$ weakly in $L^2(G;dx)$, since $\sigma_{ij}^n\phi_n$ is uniformly bounded and converges to $\sigma_{ij}\phi$ *dx-a.e.* on G as $n\to\infty$ for each i, j.
Consequently,

$$\begin{aligned}
\liminf_{n\to\infty}\mathscr{A}^{n,G}(u_n,u_n)&=\lim_{n_k\to\infty}\mathscr{A}^{n_k,G}(u_{n_k},u_{n_k})\\
&=\lim_{n_k\to\infty}\int_G\left(A_{n_k}\nabla\left(\frac{u_{n_k}}{\phi_{n_k}}\right),\nabla\left(\frac{u_{n_k}}{\phi_{n_k}}\right)\right)_d\phi_{n_k}^2dx\\
&=\lim_{n_k\to\infty}\int_G\left|\sqrt{A_{n_k}}\nabla\left(\frac{u_{n_k}}{\phi_{n_k}}\right)\right|^2\phi_{n_k}^2dx\\
&=\lim_{n_k\to\infty}\sum_{i=1}^d\left\|\sum_{j=1}^d\sigma_{ij}^{n_k}\nabla_j\left(\frac{u_{n_k}}{\phi_{n_k}}\right)\phi_{n_k}\right\|_{L^2(G;dx)}^2\\
&\geq\sum_{i=1}^d\left\|\sum_{j=1}^d\sigma_{ij}\nabla_j\left(\frac{u}{\phi}\right)\phi\right\|_{L^2(G;dx)}^2=\mathscr{A}^G(u,u).
\end{aligned}$$

Proof of (M.2). Let u be in $\mathscr{D}(\mathscr{A}^G)$, that is, $u\in L^2(G;dx)$ and $u/\phi\in H_0^1(G)$. Accordingly there exists a sequence $\{\eta_n\}$ in $C_0^\infty(G)$ such that $\|u/\phi-\eta_n\|_{H^1(G)}$ converges to 0 as $n\to\infty$. Put $u_n=\phi_n\eta_n$. Then we can see that $u_n\to u$ in $L^2(G;dx)$. Further, using again the property of the sequence $\sigma_{ij}^n\phi_n$ observed above, we get that

$$\sum_{j=1}^d\sigma_{ij}^n(\nabla_j\eta_n)\phi_n\to\sum_{j=1}^d\sigma_{ij}\nabla_j\left(\frac{u}{\phi}\right)\phi \text{ in } L^2(G;dx), \quad \text{for } i=1,2,\cdots,d.$$

Therefore we have

$$\mathscr{A}^{n,G}(u_n,u_n)=\int_G (A_n\nabla\left(\frac{u_n}{\phi_n}\right),\nabla\left(\frac{u_n}{\phi_n}\right))_d\phi_n^2dx$$

$$=\int_G |\sqrt{A_n}\nabla\eta_n|^2\phi_n^2dx=\sum_{i=1}^d \|\sum_{j=1}^d \sigma_{ij}^n(\nabla_j\eta_n)\phi_n\|^2_{L^2(G;dx)}$$

$$\to \sum_{i=1}^d \|\sum_{j=1}^d \sigma_{ij}\nabla_j\left(\frac{u}{\phi}\right)\phi\|^2_{L^2(G;dx)}=\mathscr{A}^G(u,u), \quad n\to\infty.$$

q.e.d.

This lemma shows that, if we let $H^{n,G}$ and H^G be the selfadjoint operators associated with the forms $\mathscr{A}^{n,G}$ and $\mathscr{A}^G$ respectively, then $H^{n,G}$ converges to H^G in the strong resolvent sense, hence, in the semigroup sense on $L^2(G;dx)$ by Mosco's theorem.

Let $H^{n,G}_{\phi_n}$ and H^G_ϕ also denote the selfadjoint operators associated with the forms $\mathscr{E}^{n,G}$ and $\mathscr{E}^G$ respectively. Then by the unitary map $f\mapsto\phi_n^{-1}f$ between $L^2(G;dx)$ and $L^2(G;\phi_n^2dx)$, $H^{n,G}=\phi_nH^{n,G}_{\phi_n}\phi_n^{-1}$.

On the other hand, let $M^{n,G}=\{X_t,P^{n,G}_x,x\in G\}$ and $M^G=\{X_t,P^G_x,x\in G\}$ be the diffusion processes associated with the forms $\mathscr{E}^{n,G}$ and $\mathscr{E}^G$ respectively. Because $\mathscr{E}^{n,G}$ is the part of $\mathscr{E}^n$ on G as we have already noted, the behaviour of the process $\{X_t,P^n_x,x\in \boldsymbol{R}^d\}$ is the same as that of $\{X_t,P^{n,G}_x,x\in G\}$ before it leaves G for each n.

Now we can give the proof of Theorem:

Proof of Theorem. By Lemma and the argument following it, we see that $\phi_ne^{-tH^{n,G}_{\phi_n}}\phi_n^{-1}$converges to $\phi e^{-tH^G_\phi}\phi^{-1}$ strongly on $L^2(G;dx)$. Here $e^{-tH^{n,G}_{\phi_n}}$ and $e^{-tH^G_\phi}$ denotes the semigroups associated with $\mathscr{E}^{n,G}$ and $\mathscr{E}^G$ respectively. Therefore, by virtue of Theorem 7 in [1], $P^{n,G}_{m_n}$ converges to P^G_m in the finite dimensional distribution sense.

On the other hand, one has from condition (A.2) and Lemma 2.1 in [8] that

$$\lim_{r\to\infty}\sup_n P^n_{I_{B_R}\phi_n^2dx}\left(\sup_{0\le t\le T}(|X_t|-|X_0|)\ge r\right)=0,\quad \text{for all } R>0,T>0.$$

Then, for any $0<t_1<t_2\cdots<t_p$, $A_i\in\mathscr{B}(\boldsymbol{R}^d)$, $i=1,2,\cdots,p$ and $\varepsilon>0$, there exists an $r>0$ such that $\sup_nP^n_{m_n}(t_p\ge\tau_r)<\varepsilon/2$. Moreover, we can see that $P_m(t_p\ge\tau_r)<\varepsilon/2$. Here τ_r denotes the exit time for the open ball B_r with radius r and center O.

Let $\Lambda=\{X_{t_1}\in A_1,X_{t_2}\in A_2,\cdots,X_{t_p}\in A_p\}$. Then we see that

$$|P^n_{m_n}(\Lambda)-P_m(\Lambda)|\le|P^n_{m_n}(\Lambda)-P^n_{m_n}(\Lambda\cap\{t_p<\tau_r\})|$$
$$+|P^n_{m_n}(\Lambda\cap\{t_p<\tau_r\})-P_m(\Lambda\cap\{t_p<\tau_r\})|$$
$$+|P_m(\Lambda\cap\{t_p<\tau_r\})-P_m(\Lambda)|$$

$$\leq P^n_{m_n}(t_p \geq \tau_r) + P_m(t_p \geq \tau_r)$$
$$+ |P^n_{m_n}(\Lambda \cap \{t_p < \tau_r\}) - P_m(\Lambda \cap \{t_p < \tau_r\})|.$$

The first and second term of the right hand side are less than ε. Since the last term is the finite dimensional distribution of M^{n,B_r} and M^{B_r}, we conclude that $P^n_{m_n}$ converges to P_m in the finite dimensional distribution sense.

We have already noted the tightness of $\{P^n_{m_n}\}$ on $C([0,\infty) \to \boldsymbol{R}^d)$. Thus the proof of Theorem is completed. q.e.d.

Example. Let f be a locally bounded measurable function on $\boldsymbol{R}^d$, and consider a mollifier, e.g., $j(x) = \gamma \exp(-1/1-|x|^2)$ for $|x|<1$, $j(x)=0$ for $|x| \geq 1$, where γ is a constant to make $\int_{\boldsymbol{R}^d} j(x)dx = 1$. We put $j_\varepsilon(x) = j(x/\varepsilon)/\varepsilon^d, f_\varepsilon(x) = \int_{\boldsymbol{R}^d} j_\varepsilon(x-y)f(y)dy$, for any $\varepsilon > 0$. Since f_ε converges to f in $L^2(G;dx)$ for each relatively compact open set G, we can take a sequence ε_n converging to 0 such that f_{ε_n} converges to f, dx-*a.e.* on $\boldsymbol{R}^d$. Thus if we set $\phi_n(x) = \exp f_{\varepsilon_n}(x), \phi(x) = \exp f(x)$, and assume that there exists a constant $c>0$ with $\int_{B_r} e^{2f(x)}dx \leq e^{cr^2}$, for $r>0$, then ϕ_n, ϕ satisfies the conditions (A.1) and (A.2). Therefore we have the weak convergence statement for the processes associated with ϕ_n, ϕ and $A_n = A =$ identity matrix.

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