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ON THE CHARACTERISTIC CAUCHY PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS

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We shall consider the differential operator

$$L = \sum_{j+\alpha_0 k \leq m} a_{j,k}(t) \frac{\partial^{j+k}}{\partial t^j \partial x^k} \qquad (0<\alpha_0<1,\ a_{m,0}(t)\equiv 1) \tag{0.1}$$

where m is a positive integer and $a_{j,k}(t)$ are complex-valued continuous functions on $0 \leqq t \leqq T$. If the plane $t=0$ is characteristic with respect to L, then we can suitably choose α_0 so that L contains at least one coefficient $\alpha_{j,k}(0) \neq 0$ for $j \neq m$ and $j+\alpha_0 k = m$. In this case, when the coefficients of L are constant, L. Hörmander [7] constructed non-trivial C^∞-functions $u=u(t, x)$ which satisfy $Lu=0$ in the whole (t, x)-space and vanish for $t \leqq 0$. On the other hand, I. M. Gelfand and G. E. Shilov [3] proved the uniqueness of the characteristic Cauchy problem for $Lu=0$ under some restrictions on the behavior of u as $|x| \to \infty$, and their results were improved by K. I. Babenko [1], B. L. Gurevich [5], [6] and A. Friedman [2].

In this note we shall show the existence of non-trivial functions of $G(\alpha_0, e^{-\gamma|x|^{1/(1-\alpha_0)}})$ (see Definition 1 and Lemma 1) and apply this result to prove a uniqueness theorem (Theorem 2) of the Cauchy problem without aid of generalized functions. To prove an existence theorem (Theorem 1) which is important for the proof of Theorem 2, we shall use the method of E. de Giorgi [4] which was also used by G. Talenti [9] and Tsutsumi [11]. For the case that $t=0$ is characteristic with respect to L with constant coefficients $a_{j,k}$, we shall show in Theorem 3 that the uniqueness in Theorem 1 and Theorem 2 does not hold for any $\varepsilon_0>0$ if we replace the factor $(n!)^{\alpha_0} g(x)$ in (2.3) by $(n!)^{\alpha_0+\varepsilon} g(x)$ and the factor $e^{\gamma|x|^{1/(1-\alpha_0)}}$ in (2.11) by $e^{\gamma|x|^{1/(1-\alpha_0)+\varepsilon_0}}$ respectively.

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1. We define the following class of functions for any postive continuous function $g(x)$ on $-\infty<x<+\infty$ and any α such that $0<\alpha<1$.

Definition 1. *We call $\varphi(x)$ a function of $G(\alpha, g(x))$ if $\varphi(x)$ is a C^∞-function on $-\infty<x<+\infty$ satisfying*

$$(1.1)\qquad |\varphi^{(n)}(x)|\leqq C^{n+1}(n!)^\alpha g(x)\qquad (n=0,1,2,\cdots;\ -\infty<x<+\infty)$$

for some constant C.

If $\varphi(x)$ ($\not\equiv 0$) is a function of $G(\alpha, g(x))$ (which is contained in the Gevray class of $\alpha<1$), it can be extended to be an entire function of order $1/(1-\alpha)$[1], so that for any $\varepsilon>0$ there exists a sequence $\{x_n\}_{n=1}^\infty$ such that $x_n\to+\infty$ and $|\varphi(x_n)|>e^{-x_n^{1/(1-\alpha)+\varepsilon}}$ ($n=1,2,\cdots$)[2]. Hence we know that for any $\gamma>0$ $G(\alpha, e^{-\gamma|x|^\beta})$ contains only one trivial function ($\equiv 0$) unless $\beta\leqq 1/(1-\alpha)$. Now we show the following lemma.

Lemma 1. *Let α and γ be any positive number and $0<\alpha<1$. Then we can construct a non-trivial entire function $F_{\alpha,\gamma}(z)$ of order $1/(1-\alpha)$ which satisfies*

$$(1.2)\qquad |F_{\alpha,\gamma}^{(n)}(x)|\leqq C^{n+1}(n!)^\alpha e^{-\gamma|x|^{1/(1-\alpha)}}\qquad (n=0,1,2,\cdots;\ -\infty<x<+\infty)$$

for some constant C depending only on α and γ.

Proof. For any positive number $\rho>1$ we consider a function

$$(1.3)\qquad f_\rho(x)=\int_{-\infty+i}^{+\infty+i} e^{i\alpha_\rho xw}\exp\{e^{i\theta_\rho}w^\rho\}\,dw\qquad (w=u+iv)$$

where w^ρ is defined by $e^{\rho\log w}$ such as $\log 1=0$ by setting the slit on the nonpositive imaginary axis, and θ_ρ is defined by

$$(1.4)\qquad \theta_\rho=(m+1-\rho/2)\pi\quad \textit{for}\quad 2m-1<\rho\leqq 2m+1\ (m=1,2,\cdots)$$

($\alpha_\rho>0$ will be defined later).

Then, for the case $\rho=2m+1$ we have $\Re e\,(e^{i\theta_\rho}w^\rho)$[3] $=-\Im m\, w^\rho=-(2m+1)u^{2m}v(1+o(1))$ as $|u|\to\infty$ for any fixed $v>0$.
For $2m-1<\rho<2m+1$ we have

$$(1.5)\qquad \begin{cases} \pi-\frac{1}{2}\pi<\theta_\rho<\pi+\frac{1}{2}\pi \\ (2m+1)\pi-\frac{1}{2}\pi<\theta_\rho+\rho\pi<(2m+1)\pi+\frac{1}{2}\pi, \end{cases}$$

so that if we set

$$(1.6)\qquad \beta_\rho=\frac{\pi}{2}\,\mathrm{Min}\left\{\frac{(2m+1)-\rho}{2},\ \frac{\rho-(2m-1)}{2}\right\}$$

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1) [8], pp. 262–263. 2) [10], p. 273.

3) For a complex number w, $\Re e\,w$ means the real part of w and $\Im m\,w$ the imaginary part.

then, we have for $0\leqq\theta\leqq\beta_\rho$

$$(1.7)\qquad \begin{cases} \dfrac{\pi}{2}+\beta_\rho<\theta_\rho+\rho\theta<\dfrac{3}{2}\pi-\beta_\rho \\ \left(2m+\dfrac{1}{2}\right)\pi+\beta_\rho<\theta_\rho+\rho(\pi-\theta)<\left(2m+\dfrac{3}{2}\right)\pi-\beta_\rho \end{cases}$$

and consequently we have for $v>0$

$$(1.8)\qquad \Re e\,(e^{i\theta_\rho}w^\rho)\leqq-|w|^\rho\sin\beta_\rho\quad for\quad |u|\geqq v\cot\theta\,.$$

These facts mean that, for any $\rho>1$, we can extend $f_\rho(x)$ to an entire function $f_\rho(z)=f_\rho(x+iy)$ and we have

$$(1.9)\qquad f_\rho^{(n)}(x)=\int_{-\infty+i}^{+\infty+i}(ia_\rho w)^n e^{ia_\rho xw}\exp\{e^{i\theta_\rho}w^\rho\}\,dw\,.$$

Now we estimate the values of $f_\rho^{(n)}(x)$. First we do this for $\rho=2m+1$ which is fairly complicated. If we set

$$(1.10)\qquad \theta_0=\pi/(2\rho)\quad and\quad a_\rho=5\operatorname{cosec}\theta_0$$

we have, for $w=|w|e^{i\theta}$ and $w=|w|e^{i(\pi-\theta)}$ $(0\leqq\theta\leqq\theta_0)$,

$$(1.11)\quad \Re e\,(e^{i\theta_\rho}w^\rho)=-\Im m\,w^\rho=-|w|^\rho\sin(\rho\theta)\quad(0\leqq\rho\theta\leqq\rho\theta_0=\pi/2)\,.$$

In the case $x>0$, we deform the path of the integral (1.9) to $\Gamma=\mathfrak{C}_1\cup\mathfrak{C}_2\cup\mathfrak{C}_3$ where $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$ are the segments (of straight lines) from $+\infty e^{i(\pi-\theta_0)}$ to $x^\delta e^{i(\pi-\theta_0)}$, from $x^\delta e^{i(\pi-\theta_0)}$ to $x^\delta e^{i\theta_0}$, and from $x^\delta e^{i\theta_0}$ to $+\infty e^{i\theta_0}$ respectively by setting $\delta=1/(\rho-1)$. Let the values of the above integral on $\mathfrak{C}_j$ be denoted by $I_j(x)$ $(j=1,2,3)$. Then, by (1.10) and (1.11) we have

$$|I_1(x)|\leqq e^{-5x^{1+\delta}}\int_{x^\delta}^{+\infty}\{(a_\rho r)^n e^{-r^\rho/2}\}e^{-r^\rho/2}dr\,.$$

Now for positive δ and ρ we use the inequality

$$(1.12)\qquad r^n e^{-\delta r^\rho}\leqq C_{\delta,\rho}^n(n!)^{1/\rho}\qquad(r\geqq0\,,\ n=0,1,2,\cdots)$$

which will be often used in later discussions. Then, remarking $1+\delta=\rho'$ where $1/\rho+1/\rho'=1$, we have

$$(1.13)\qquad |I_1(x)|\leqq C_1^{n+1}(n!)^{1/\rho}e^{-5x^{\rho'}}.$$ [4]

For $I_2(x)$, remarking $\delta\rho=\rho'$ we have

4) In what follows in this section, C_i are constants depending only on ρ.

$$|I_2(x)| \leqq e^{-5x^{\rho'}}(a_\rho x^\delta)^n \int_{-x^\delta \cos\theta_0}^{x^\delta \cos\theta_0} e^{(x^\delta)^\rho} du$$
$$= e^{-2x^{\rho'}} a_\rho^n \{(x^\delta)^n e^{-(x^\delta)^\rho}\} \{2x^\delta \cos\theta_0 \cdot e^{-x^{\rho'}}\},$$

so that by (1.12) we have

$$(1.14) \qquad |I_2(x)| \leqq C_2^{n+1}(n!)^{1/\rho} e^{-2x^{\rho'}}.$$

Since $I_3(x)$ can be estimated in the same way as $I_1(x)$, we get by (1.13) and (1.14)

$$(1.15) \qquad |f_\rho^{(n)}(x)| \leqq C_3^{n+1}(n!)^{1/\rho} e^{-2x^{\rho'}} \qquad \textit{for } x>0.$$

In the case $x<0$, we deform the path of the integral (1.9) to $\Gamma' = \mathcal{C}_1' \cup \mathcal{C}_2' \cup \mathcal{C}_3' \cup \mathcal{C}_4' \cup \mathcal{C}_5'$ where $\mathcal{C}_1', \mathcal{C}_2', \mathcal{C}_3', \mathcal{C}_4', \mathcal{C}_5'$ are the segments from $+\infty e^{i(\pi-\theta_1)}$ to $b|x|^\delta e^{i(\pi-\theta_1)} \operatorname{cosec}\theta_1$, from $b|x|^\delta e^{i(\pi-\theta_1)} \operatorname{cosec}\theta_1$ to $b|x|^\delta e^{i(\pi-\theta_0)} \operatorname{cosec}\theta_0$, from $b|x|^\delta e^{i(\pi-\theta_0)} \operatorname{cosec}\theta_0$ to $b|x|^\delta e^{i\theta_0} \operatorname{cosec}\theta_0$, from $b|x|^\delta e^{i\theta_0} \operatorname{cosec}\theta_0$ to $b|x|^\delta e^{i\theta_1} \operatorname{cosec}\theta_1$, from $b|x|^\delta e^{i\theta_1} \operatorname{cosec}\theta_1$ to $+\infty e^{i\theta_1}$ respectively. ($0<b$ and $0<\theta_1<\theta_0$ will be determined later.) Let the values of the integral on $\mathcal{C}_j'$ be denoted by $I_j'(x)$ ($j=1, 2, 3, 4, 5$). For $I_3'(x)$ we have by (1.12)

$$(1.16) \qquad \begin{aligned} |I_3'(x)| &\leqq 2b|x|^\delta \cot\theta_0 (a_\rho b|x|^\delta \operatorname{cosec}\theta_0)^n e^{a_\rho b|x|^{1+\delta} + (b|x|^\delta \operatorname{cosec}\theta_0)^\rho} \\ &\leqq C_4^{n+1}(n!)^{1/\rho} e^{b(3a_\rho + b^{\rho-1}\operatorname{cosec}^\rho \theta_0)|x|^{\rho'}}, \end{aligned}$$

for $I_2'(x)$, remarking $\Re(e^{i\theta_\rho} w^\rho)<0$ by (1.11),

$$(1.17) \qquad \begin{aligned} |I_2'(x)| &\leqq 2b|x|^\delta \cot\theta_1 (a_\rho b|x|^\delta \operatorname{cosec}\theta_1)^n e^{a_\rho b|x|^{1+\delta}} \\ &\leqq C_5^{n+1}(n!)^{1/\rho} e^{3a_\rho b|x|^{\rho'}}, \end{aligned}$$

and for $I_1'(x)$ we have by (1.11)

$$|I_1'(x)| \leqq a_\rho^n \int_{b|x|^\delta \operatorname{cosec}\theta_1}^\infty r^n e^{a_\rho |x| r \sin\theta_1 - r^\rho \sin(\rho\theta_1)} dr.$$

If we determine θ_1 such as $a_\rho \leqq b^{\rho-1} \operatorname{cosec}^\rho \theta_1 \sin(\rho\theta_1)/2$ for fixed $b>0$ which is possible because of $\operatorname{cosec}^\rho \theta_1 \sin(\rho\theta_1) = \rho\theta_1^{-(\rho-1)}(1+o(1))$ as $\theta_1 \downarrow 0$, then we have $a_\rho |x| r \sin\theta_1 - r^\rho \sin(\rho\theta_1) \leqq -r^\rho \sin(\rho\theta_1)/2$ for $r \geqq b|x|^\delta \operatorname{cosec}\theta_1$. Hence we have by (1.12)

$$(1.18) \qquad \begin{aligned} |I_1'(x)| &\leqq a_\rho^n \int_0^\infty r^n e^{-\sin(\rho\theta_1) r^\rho/2} dr \\ &\leqq \left\{\int_0^\infty e^{-\sin(\rho\theta_1) r^\rho/4} dr\right\} C_{\rho,\theta_1}^{n+1} (n!)^{1/\rho}. \end{aligned}$$

For $I_4'(x)$ and $I_5'(x)$ we can get the same estimates as $I_2'(x)$ and $I_1'(x)$ respectively. Hence if we determine b such as $b(3a_\rho + b^{\rho-1}\operatorname{cosec}^\rho \theta_0) \leqq 1$ and $\theta_1>0$ such as $a_\rho \leqq b^{\rho-1} \operatorname{cosec}^\rho \theta_1 \sin(\rho\theta_1)/2$, we have

(1.19) $$|f_\rho^{(n)}(x)| \leqq C_6^{n+1}(n!)^{1/\rho} e^{|x|^{\rho'}} \qquad \textit{for } x<0$$

by (1.16), (1.17), (1.18). Consequently, if we set

$$F_{\alpha,\gamma}(z) = f_{1/\alpha}(\gamma^\alpha z)\cdot f_{1/\alpha}(-\gamma^\alpha z)$$

then, using Leibniz' formula and

$$\sum_{j=0}^{n}\binom{n}{j}(j!)^\alpha((n-j)!)^\alpha \leqq (n!)^\alpha \sum_{j=0}^{n}\binom{n}{j} = (n!)^\alpha 2^n,$$

we get (1.2) by (1.15) and (1.19).

For $2m-1<\rho<2m+1$, we set $a_\rho = 5\operatorname{cosec}^\rho \beta_\rho$ with β_ρ of (1.6). For $x>0$, in order to estimate (1.9), we deform the path of the integral (1.9) to $\Gamma''=\mathcal{C}_1''\cup\mathcal{C}_2''\cup\mathcal{C}_3''$ where $\mathcal{C}_1''$, $\mathcal{C}_2''$, and $\mathcal{C}_3''$ are the segments from $-\infty+ix^\delta$ to $-x^\delta\cot\beta_\rho+ix^\delta$, from $-x^\delta\cot\beta_\rho+ix^\delta$ to $x^\delta\cot\beta_\rho+ix^\delta$, and from $x^\delta\cot\beta_\rho+ix^\delta$ to $+\infty+ix^\delta$ respectively. For $x<0$ we need not deform the path and we evaluate the integral on the intervals $(-\infty+i, -\cot\beta_\rho+i)$, $(-\cot\beta_\rho+i, \cot\beta_\rho+i)$, and $(\cot\beta_\rho+i, +\infty+i)$. Then, using (1.8), we get the same result as the case of $\rho=2m+1$.

Lemma 2. *Let $\varphi(x)$ be a function of $G(\alpha, g(x))$ satisfying* (1.1). *Then, for any real y, $\psi(x)\equiv e^{ixy}\varphi(x)$ is also of $G(\alpha, g(x))$ and satisfies*

$$|\psi^{(n)}(x)| \leqq (C+|y|)^{n+1}(n!)^\alpha g(x) \qquad (n=0, 1, 2, \cdots).$$

Proof. Omitted, as it is so easy.

2. For any function $f(t, x)$ such that $(\partial/\partial x)^{[m/\alpha_0]}f(t, x)$[5] is continuous on $\{0\leqq t\leqq T\}\times\{-\infty<x<+\infty\}$, we define $(Hf)(t, x)$ as follows.

(2.1) $$Hf = \sum_{j=1}^{m}\sum_{k=0}^{[j/\alpha_0]}(-1)^{j+k+1}H_{j,k}f$$

where

(2.2) $$(H_{j,k}f)(t, x) = \int_0^t \frac{(t-\tau)^{j-1}}{(j-1)!}a_{m-j,k}(\tau)\frac{\partial^k}{\partial x^k}f(\tau, x)d\tau.$$

Theorem 1. *Let $g(x)$ be any positive function on $-\infty<x<+\infty$ and $f(t, x)$ be a complex-valued function whose derivatives $\partial^n f/\partial x^n$ $(n=0, 1, \cdots)$ are continuous on $\{0\leqq t\leqq T\}\times\{-\infty<x<+\infty\}$ and satisfy*

(2.3) $$\left|\frac{\partial^n}{\partial x^n}f(t, x)\right| \leqq C^{n+1}(n!)^{\alpha_0}g(x) \quad (n=0, 1, \cdots; 0\leqq t\leqq T, -\infty<x<+\infty)$$

5) For a real number a, $[a]$ means the maximum of the integers which do not exceed a.

for some constant C. Then there exists only one function v(t, x) that satisfies

$$v-Hv = f \ (0\leqq t<T_0, \ -\infty<x<+\infty), \tag{2.4}$$

whose derivatives $\partial^n v/\partial x^n$ $(n=0, 1, 2, \cdots)$ *are continuous on* $\{0\leqq t<T_0\}\times\{-\infty<x<+\infty\}$ *and satisfy for some constant K*

$$\left|\frac{\partial^n}{\partial x^n}v(t, x)\right|\leqq K^{n+1}(n!)^{\alpha_0}g(x) \ (n=0, 1, \cdots; 0\leqq t<T_0, \ -\infty<x<+\infty) \tag{2.5}$$

where $T_0>0$ *depends only on* α_0, *C and*

$$A\equiv \text{Max}\,\{|a_{j,k}(t)|\,;\, j+\alpha_0 k\leqq m, \ 0\leqq t\leqq T\}. \tag{2.6}$$

Proof. Without loss of generality we assume $C>1$ and estimate $|H^n f|$. By (2.1), (2.2), (2.3), (2.6), we get easily

$$|(Hf)(t, x)|\leqq CA(m/\alpha_0+1)g(x)\sum_{j=1}^{m} C^{j/\alpha_0}([j/\alpha_0]!)^{\alpha_0}t^j/j!. \tag{2.7}$$

By Stirling's formula we have

$$\frac{([j/\alpha_0]!)^{\alpha_0}}{j!}\leqq K_0\frac{\{(j/\alpha_0)^{j/\alpha_0}\sqrt{2\pi j/\alpha_0}e^{-j/\alpha_0}\}^{\alpha_0}}{j^j\sqrt{2\pi j}\,e^{-j}}\leqq K_1\alpha_0^{-j} \quad (j=1, 2, \cdots) \tag{2.8}$$

for some constants K_0 and K_1. Hence, raplacing $([j/\alpha_0]!)^{\alpha_0}/j!$ of (2.7) by $K_1\alpha_0^{-j}$, we get

$$\begin{aligned}|Hf(t, x)|&\leqq CAK_1(m/\alpha_0+1)g(x)\sum_{j=1}^{m}(C^{1/\alpha_0}t/\alpha_0)^j\\ &\leqq CAK_1 m(m/\alpha_0+1)g(x)C^{1/\alpha_0}t/\alpha_0 \qquad \textit{for} \quad 0\leqq C^{1/\alpha_0}t/\alpha_0<1,\end{aligned}$$

so that

$$|Hf(t, x)|\leqq CK_1 g(x)t/(2T_0) \qquad \textit{for } 0\leqq t<2T_0$$

where

$$1/(2T_0) = \text{Max}\,\{m(m/\alpha_0+1)(A+1)C^{1/\alpha_0}/\alpha_0, \ 1/T\}. \tag{2.9}$$

Next we estimate $|H^2 f|$. Since by (2.2)

$$\begin{aligned}&(H_{h,l}\cdot H_{j,k}f)(t, x)\\ &=\int_0^t\frac{(t-\sigma)^{h-1}}{(h-1)!}a_{m-h,l}(\sigma)\int_0^\sigma\frac{(\sigma-\tau)^{j-1}}{(j-1)!}a_{m-j,k}(\tau)\frac{\partial^{l+k}}{\partial x^{l+k}}f(\tau, x)\,d\tau\, d\sigma\end{aligned}$$

we get, remarking $l+k\leqq[(h+j)/\alpha_0]$,

$$\begin{aligned}|H_{h,l}\cdot H_{j,k}f(t, x)|&\leqq A^2\int_0^t\frac{(t-\tau)^{h+j-1}}{(h+j-1)!}\left|\frac{\partial^{l+k}}{\partial x^{l+k}}f(\tau, x)\right|d\tau\\ &\leqq CA^2C^{(h+j)/\alpha_0}([(h+j)/\alpha_0]!)^{\alpha_0}g(x)t^{h+j}/(h+j)!.\end{aligned}$$

If we replace $([(h+j)/\alpha_0]!)^{\alpha_0}/(h+j)!$ by $K_1\alpha_0^{-(h+j)}$ by (2.8), then we get

$$|H^2f(t, x)| \leqq CK_1g(x)(t/(2T_0))^2 \qquad for\ 0\leqq t<2T_0.$$

Similarly we get for $n=3, 4, \cdots$

$$|H^nf(t, x)| \leqq CK_1g(x)(t/(2T_0))^n \qquad for\ 0\leqq t<2T_0.$$

Hence we can set

$$v(t, x) = \sum_{n=0}^{\infty}(H^nf)(t, x) \qquad for\ 0\leqq t<2T_0,$$

so that $v(t, x)$ is the unique solution of (2.4).

Next we estimate the value of $(\partial/\partial x)^nHf=H(\partial/\partial x)^nf$ $(n=0, 1, 2, \cdots)$. Remarking $2^{n+l}\geqq(n+l)!/(n!l!)$, we get in the same way as (2.7)

$$\begin{aligned}\left|\frac{\partial^n}{\partial x^n}Hf(t, x)\right| &\leqq CA(m/\alpha_0+1)g(x)\sum_{j=1}^{m}C^{j/\alpha_0+n}\{(n+[j/\alpha_0])!\}^{\alpha_0}t^j/j! \\ &\leqq CA(m/\alpha_0+1)g(x)\sum_{j=1}^{m}(2C)^{j/\alpha_0+n}(n!)^{\alpha_0}([j/\alpha_0]!)^{\alpha_0}t^j/j! \\ &\leqq (2C)^{n+1}(n!)^{\alpha_0}K_1g(x)t/T_0 \qquad for\ 0\leqq t<T_0.\end{aligned}$$

Similarly we get for $l=1, 2, \cdots$

$$\left|\frac{\partial^n}{\partial x^n}H^lf(t, x)\right| \leqq (2C)^{n+1}(n!)^{\alpha_0}K_1g(x)(t/T_0)^l \qquad (0\leqq t<T_0).$$

Hence we get

$$\left|\frac{\partial^n}{\partial x^n}v(t, x)\right| = \left|\sum_{l=0}^{\infty}\left(\frac{\partial}{\partial x}\right)^nH^lf(t, x)\right| \leqq (2C)^{n+1}(n!)^{\alpha_0}K_1g(x)T_0/(T_0-t)$$

for $0\leqq t<T_0$ so that (2.5) holds.

Theorem 2. *Let $u(t, x)$ be a solution of the Cauchy problem*

$$(2.10)\qquad \begin{aligned}&(Lu)(t, x) = 0 &&(0\leqq t\leqq T,\ -\infty<x<+\infty) \\ &\frac{\partial^j u}{\partial t^j}(0, x) = 0 &&(j=0, \cdots, m-1;\ -\infty<x<+\infty)\end{aligned}$$

whose derivatives contained in L are continuous and satisfy

$$(2.11)\qquad \left|\frac{\partial^{j+k}}{\partial t^j\partial x^k}u(t, x)\right| \leqq Ce^{\gamma|x|^{1/(1-\alpha_0)}} \qquad (j+\alpha_0k\leqq m,\ 0\leqq t\leqq T,\ -\infty<x<+\infty)$$

for some constants C and $\gamma>0$. Then $u(t, x)$ vanishes identically for $0\leqq t\leqq T$, $-\infty<x<+\infty$.

Proof. We write $\beta_0=1/(1-\alpha_0)$. By Lemma 1 we can take a non-trivial function $\varphi(x)$ of $G(\alpha_0, e^{-2\gamma|x|^{\beta_0}})$ satisfying

(2. 12) $$|\varphi^{(n)}(x)| \leqq C_1^{n+1}(n!)^{\alpha_0} e^{-2\gamma|x|^{\beta_0}} \qquad (n=0, 1, 2, \cdots)$$

for some constant C_1 depending only on α_0 and γ. We set

(2. 13) $$1/(2T_0) = \text{Max}\,\{m(m\alpha_0+1)(A+1)(1+C_1)^{1/\alpha_0}/\alpha_0,\ 1/T\}$$

where $A=\text{Max}\{|a_{j,k}(t)|;\ 0\leqq t\leqq T, j+\alpha_0 k\leqq m\}$ (cf. (2. 9)).

If the assertion of the theorem is not true, then, for some $u(t, x)$ satisfying (2. 10) and (2. 11), there exists t_0 such that $t_0=\inf\{0\leqq t\leqq T;\ u(t, x)\neq 0$ for some $x\}<T$. Taking τ such as $0<\tau<T'\equiv \text{Min}(T_0, T-t_0)$, we set $(-1)^j b_{j,k}(t)=a_{j,k}(t_0+\tau-t)$ $(j+\alpha_0 k\leqq m)$ and $w(t, x)=u(t_0+\tau-t, x)$ $(0\leqq t\leqq\tau,\ -\infty<x<+\infty)$.

Let L', H' and $H'_{j,k}$ be the operators given by (0. 1), (2. 1) and (2. 2) replacing $a_{j,k}(t)$ by $b_{j,k}(t)$ respectively. Then $w(t, x)$ is a solution of the Cauchy problem, $L'w=0$ $(0\leqq t\leqq\tau,\ -\infty<x<+\infty)$, $(\partial/\partial t)^j w(\tau, x)=0$ $(j=0, \cdots, m-1;\ -\infty<x<+\infty)$, satisfying the same condition as (2. 11). By Lemma 2, (2. 13) and the proof of Theorem 1, we obtain for $-1\leqq y\leqq 1$ the solution $v(t, x; y)$ of $v-H'v=e^{ixy}\varphi(x)$ on $\{0\leqq t\leqq\tau\}\times\{-\infty<x<+\infty\}$ where τ is independent of y. Furthermore

$$\left|\frac{\partial^n}{\partial x^n}v(t, x; y)\right|\leqq C_2^{n+1}(n!)^{\alpha_0}e^{-2\gamma|x|^{\beta_0}} \qquad (n=0, 1, 2, \cdots)$$

for some constant C_2. Integrating by parts, we get easily

$$(-1)^{j+k}\int_{-\infty}^{+\infty}\int_0^\tau (H'_{j,k}v)\frac{\partial^m w}{\partial t^m}dt\,dx = \int_{-\infty}^{+\infty}\int_0^\tau v b_{m-j,k}\left(\frac{\partial}{\partial t}\right)^{m-j}\left(\frac{\partial}{\partial x}\right)^k w\,dt\,dx\,,$$

and hence

$$0 = \int_{-\infty}^{+\infty}\int_0^\tau v(L'w)\,dt\,dx = \int_{-\infty}^{+\infty}\int_0^\tau (v-H'v)\frac{\partial^m w}{\partial t^m}dt\,dx$$

$$= \int_{-\infty}^{+\infty} e^{ixy}\varphi(x)\frac{\partial^{m-1}w}{\partial t^{m-1}}(0, x)\,dx \qquad \text{for } |y|\leqq 1\,.$$

Since $\int_{-\infty}^{+\infty} e^{ixy}\varphi(x)\frac{\partial^{m-1}w}{\partial t^{m-1}}(0, x)\,dx$ is an entire function of y by (2. 11) and (2. 12), this vanishes identically on $-\infty<y<+\infty$. Hence the integrable function $\varphi(x)(\partial^{m-1}w/\partial t^{m-1})(0, x)$ also vanishes identically on $-\infty<x<+\infty$. Since $\varphi(x)$ is a non-trivial entire function, we get

$$(-1)^{m-1}\frac{\partial^{m-1}w}{\partial t^{m-1}}(0, x) = \frac{\partial^{m-1}u}{\partial t^{m-1}}(t_0+\tau, x) = 0$$

for $0\leqq\tau<T'$, $-\infty<x<+\infty$.

By $\quad (\partial^j u/\partial t^j)(t_0, x)=0 \quad (j=0, \cdots, m-1;\ -\infty<x<+\infty)$

we get $\quad u(t, x)\equiv 0 \quad$ for $t_0\leqq t<t_0+T'$, $-\infty<x<+\infty$,

which is contradiction.

3. In this section we shall consider a differential operator with constant coefficients

$$(3.1)\qquad L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) = \sum_{j+\alpha_0 k \leqq m} a_{j,k} \frac{\partial^{j+k}}{\partial t^j \partial x^k} \qquad (0<\alpha_0<1\,,\ a_{m,0} = 1)$$

containing at least one coefficient $a_{j_0,k_0} \neq 0$ with $j_0 \neq m$ and $j_0 + \alpha_0 k_0 = m$.

Lemma 3. *Let* $L(-i\lambda, -i\eta) = \sum_{j+\alpha_0 k \leqq m} a_{j,k}(-i\lambda)^j(-i\eta)^k$ *be the differential polynomial corresponding to* (3.1). *Then we have an analytic function* $\eta = \eta(\lambda)$ *defined on* $\mathfrak{Im}\,\lambda \geqq K$ *for some constant* $K>0$ *such that* $L(-i\lambda, -i\eta(\lambda))=0$ *and* $|\eta(\lambda)| \leqq A|\lambda|^{\alpha_0}$ *for some constant* A.

Proof. We write

$$\begin{aligned} L(-i\lambda, -i\eta) &= (-i)^m \prod_{j=1}^{m} (\lambda - \lambda_j(\eta)) \\ &= (-i)^m(\lambda^m + Q_1(\eta)\lambda^{m-1} + \cdots + Q_{m-j_0}(\eta)\lambda^{j_0} + \cdots + Q_m(\eta))\,. \end{aligned}$$

Then $\lambda_j(\eta)$ have Puiseux series expansions at infinity[6)]

$$\lambda_j(\eta) = \sum_{n=-\infty}^{l_j} \alpha_{j,n}(\eta^{1/p_j})^n \qquad (\alpha_{j,l_j} \neq 0\,,\ j = 1, \cdots, m)$$

on $|\eta| \geqq K'$ for some constant $K'>0$. Hence $\lambda_j(\eta) = \alpha_{j,l_j}\eta^{l_j/p_j}(1+o(1))$ as $\eta \to \infty$. If we would assume $l_j/p_j < 1/\alpha_0$ for all $j=1, \cdots, m$ then on $|\eta| \geqq K'$

$$(3.2)\qquad |Q_{m-j_0}(\eta)| = \Big|\sum_{j_1,\cdots,j_{m-j_0}} \lambda_{j_1}(\eta)\cdots\lambda_{j_{m-j_0}}(\eta)\Big| \leqq C|\eta|^{(m-j_0)/\alpha_0 - \varepsilon}$$

for some C, $\varepsilon>0$. On the other hand

$$Q_{m-j_0}(\eta) = a\eta^{(m-j_0)/\alpha_0} + \cdots \qquad (a = i^{m-j_0-k_0}a_{j_0,k_0} \neq 0)\,,$$

so that $Q_{m-j_0}(\eta) = a\eta^{(m-j_0)/\alpha_0}(1+o(1))$ as $\eta \to \infty$. This contradicts to (3.2). Consequently we have an analytic function

$$\lambda(\eta) = \sum_{n=-\infty}^{l} a_n(\eta^{1/p})^n \qquad (a_l \neq 0\,,\ l/p \geqq 1/\alpha_0)$$

on $|\eta| \geqq K'$ such that $L(-i\lambda(\eta), -i\eta)=0$. Since $\lambda(\eta) = \alpha_l\eta^{l/p}(1+o(1))$ as $\eta \to \infty$, we have an analytic function $\eta = \eta(\lambda)$ defined on $\mathfrak{Im}\,\lambda \geqq K$ for some canstant $K \geqq 1$ such that $L(-i\lambda, -i\eta(\lambda))=0$ and $|\eta(\lambda)| \leqq A|\lambda|^{p/l} \leqq A|\lambda|^{\alpha_0}$ for some constant A.

Theorem 3. *Let* $L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)$ *be a differential operator of the form*

6) [12], pp. 50–55.

(3. 1) *containing at least one coefficient* $a_{j_0k_0}\neq 0$ *with* $j_0\neq m$ *and* $j_0+\alpha_0 k_0=m$. *Then for any* $\varepsilon_0>0$, *we can construct a* C^∞*-solution* $U(t, x)$ *of*

$$L\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\Big)U(t, x) = 0 \qquad (-\infty<t<+\infty,\ -\infty<x<+\infty)$$

such that

$$\text{(3. 3)}\qquad \begin{cases} U(t, x) \not\equiv 0 & \text{for } (-\infty<t<+\infty,\ -\infty<x<+\infty), \\ U(t, x) = 0 & \text{for } (-\infty<t\leqq 0,\ -\infty<x<+\infty), \end{cases}$$

and

$$\text{(3. 4)}\qquad \left|\frac{\partial^{j+k}}{\partial t^j\partial x^k}U(t, x)\right| \leqq C_1^{1+j+k}(j!)^{1+\varepsilon_0}(k!)^{\alpha_0+\varepsilon_0}e^{c_2(|x|^{1/(1-\alpha_0)+\varepsilon_0}+1)t}$$
$$(-\infty<t<+\infty,\ -\infty<x<+\infty)$$

for some positive constants C_1 *and* C_2 *depending only on* ε_0.

Proof. Consider the equation $L(-i\lambda, -i\eta)=0$. Then by Lemma 3, we have an analytic root $\eta(\lambda)$ defined on $\Im m\,\lambda\geqq K$ for some constant $K>0$ which satisfies

$$\text{(3. 5)}\qquad |\eta(\lambda)|\leqq A|\lambda|^{\alpha_0} \qquad (\Im m\,\lambda\geqq K)$$

for some constant A. Take ρ such that $0<\alpha_0<\rho<1$ and consider the function

$$\text{(3. 6)}\qquad U_\rho(t, x) = \int_{-\infty+i\tau}^{+\infty+i\tau} e^{-i(t\lambda+x\eta(\lambda))-(\lambda/i)^\rho}d\lambda^{7)} \qquad (\tau\geqq K)$$

where $(\lambda/i)^\rho$ is defined such that $(\lambda/i)^\rho>0$ on the positive imaginary axis. Since $\Re e\,(\lambda/i)^\rho\geqq|\lambda|^\rho\cos(\rho\pi/2)$, by (3. 5) we can see that $U_\rho(t, x)\in C^\infty(-\infty<t<+\infty,\ -\infty<x<+\infty)$, $L\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\Big)U_\rho(t, x)=0$, and that U_ρ does not depend on $\tau(\geqq K)$. For any fixed x, if we set $C_\rho=\{2A\cos(\rho\pi/2)\}^{1/(\rho-\alpha_0)}$, then for $\Im m\,\lambda=\tau\geqq(C_\rho|x|^{1/(\rho-\alpha_0)}+K)$ we have

$$\text{(3. 7)}\qquad \begin{aligned} |x\eta(\lambda)|-|\lambda|^\rho\cos(\rho\pi/2) &\leqq A|\lambda|^{\alpha_0}|x|-|\lambda|^\rho\cos(\rho\pi/2)\,. \\ &\leqq -\frac{1}{2}|\lambda|^\rho\cos(\rho\pi/2)\,. \end{aligned}$$

Taking $t<0$ and x arbitrarily, we have

$$\begin{aligned} |U_\rho(t, x)| &\leqq e^{t\tau}\int_{-\infty+i\tau}^{+\infty+i\tau} e^{-|\lambda|^\rho\cos(\rho\pi/2)/2}d|\lambda| \\ &\leqq e^{t\tau}\int_{-\infty}^{+\infty} e^{-|y|^\rho\cos(\rho\pi/2)/2}dy\to 0 \qquad (\tau\to\infty) \end{aligned}$$

7) [7], p. 122.

for $\tau \geqq C_\rho |x|^{1/(\rho-\alpha_0)}+K$. Hence $U_\rho(t, x)=0$ for $-\infty<t<0$ and $-\infty<x<+\infty$. Making x fixed, we can write

$$U_\rho(t, x) = e^{t\tau}\int_{-\infty}^{+\infty} e^{-ity}e^{-ix\eta(x+i\tau)-(-iy+\tau)^\rho}dy,$$

so that we have $U_\rho(t, x)\not\equiv 0$ as a function of t on $(-\infty, \infty)$. We get (3.3) by these arguments.

Next, for the derivatives of $U_\rho(t, x)$, we have by (3.5)

$$\left|\frac{\partial^{j+k}}{\partial t^j \partial x^k} U_\rho(t, x)\right| \leqq A^k e^{\tau t}\int_{-\infty+i\tau}^{+\infty+i\tau} |\lambda|^j |\lambda|^{\alpha_0 k} e^{-|\lambda|^\rho \cos(\rho\pi/2)} d|\lambda| \tag{3.8}$$

where $\tau=C_\rho|x|^{1/(\rho-\alpha_0)}+K$. If we write the integrand of the right hand side of (3.8) as

$$(|\lambda|^j e^{-|\lambda|^\rho \cos(\rho\pi/2)/6})(|\lambda|^{\alpha_0 k} e^{-(|\lambda|^{\alpha_0})^{\rho/\alpha_0}\cos(\rho\pi/2)/6})(e^{-|\lambda|^\rho \cos(\rho\pi/2)/6})$$

then by (1.12) we have

$$\begin{aligned}\left|\frac{\partial^{j+k}}{\partial t^j \partial x^k} U_\rho(t, x)\right| &\leqq A^k e^{\tau t} C_\rho^j (j!)^{1/\rho} C_{\rho,\alpha_0}^k (k!)^{\alpha_0/\rho}\int_{-\infty+i\tau}^{+\infty+i\tau} e^{-|\lambda|^\rho \cos(\rho\pi/2)/6} d|\lambda| \\ &\leqq C^{1+j+k} e^{(C_\rho|x|^{1/(\rho-\alpha_0)}+K)t}(j!)^{1/\rho}(k!)^{\alpha_0/\rho}\int_{-\infty}^{+\infty} e^{-|y|^\rho \cos(\rho\pi/2)/6}dy\end{aligned}$$

where C is a constant depending on A, ρ and α_0. We fix ρ $(\alpha_0<\rho<1)$ such as $1/\rho\leqq 1+\varepsilon_0$, $\alpha_0/\rho\leqq\alpha_0+\varepsilon_0$, and $1/(\rho-\alpha_0)\leqq 1/(1-\alpha_0)+\varepsilon_0$. Hence we get (3.4).

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