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A THEOREM OF CALABI-MATSUSHIMA'S TYPE

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1. Introduction

In this note, we introduce the concept of the Einstein condition for a special type of conformally Kähler manifolds, called multiplier Hermitian manifolds in [7]. Then for such manifolds, an analogue of the theorems of Calabi [1] and Matsushima [8] will be proved.

For an n-dimensional compact connected Kähler manifold M with Kähler form ω_0, let $\mathcal{K}$ denote the set of all Kähler forms on M cohomologous to ω_0. We write each $\omega \in \mathcal{K}$ as

$$\omega = \sqrt{-1} \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^{\alpha} \wedge dz^{\bar{\beta}}$$

by using a system $(z^1, z^2, \dots, z^n)$ of holomorphic local coordinates on M. Let $G := \mathrm{Aut}^0(M)$ denote the identity component of the group of all holomorphic automorphisms of M. For a holomorphic vector field X on M, we put

$$\mathcal{K}_X := \{\, \omega\, ;\, L_{X_{\mathbb{R}}} \omega = 0 \,\},$$

where $X_{\mathbb{R}} := X + \bar{X}$ denotes the real vector field on M associated to X. We say that X is *Hamiltonian* if in addition to $\mathcal{K}_X \neq \emptyset$, the holomorphic one-parameter subgroup

$$T := \{\, \exp(tX)\, ;\, t \in \mathbb{C} \,\}$$

of G generated by X sits in the linear algebraic part of G, i.e., for each $\omega \in \mathcal{K}_X$, the holomorphic vector field X is expressible as

$$\mathrm{grad}^{\mathbb{C}}_{\omega}\, u_{\omega} := \frac{1}{\sqrt{-1}} \sum_{\alpha,\beta} g^{\bar{\beta}\alpha} \frac{\partial u_{\omega}}{\partial z^{\bar{\beta}}} \frac{\partial}{\partial z^{\alpha}}$$

for some real-valued smooth function $u_{\omega} \in C^{\infty}(M)_{\mathbb{R}}$ on M. Here u_{ω} is always normalized by the condition $\int_M u_{\omega}\, \omega^n = 0$. Throughout this note, we fix once for all a Hamiltonian holomorphic vector field $X \neq 0$ on M. In view of the moment map associated to the T-action on M, both $l_0 := \min_M u_{\omega}$ and $l_1 := \max_M u_{\omega}$ are independent of the choice of ω in $\mathcal{K}_X$. We now fix a nonconstant real-valued smooth function

$$\sigma : [l_0, l_1] \to \mathbb{R}, \qquad s \mapsto \sigma(s),$$

and let $\dot\sigma = \dot\sigma(s)$ be its derivative $\dot\sigma := (\partial/\partial s)\,\sigma$. We further define a function ψ_ω in $C^\infty(M)_\mathbb{R}$ by setting $\psi_\omega := \sigma(u_\omega)$, and consider the operator

$$\tilde\Box_\omega \;:=\; \sum_{\alpha,\beta} g^{\bar\beta\alpha}\frac{\partial^2}{\partial z^\alpha \partial z^{\bar\beta}} \;-\; \sum_{\alpha,\beta} g^{\bar\beta\alpha}\frac{\partial\psi_\omega}{\partial z^\alpha}\frac{\partial}{\partial z^{\bar\beta}} \;=\; \Box_\omega \;+\; \sqrt{-1}\,\dot\sigma(u_\omega)\,\bar X, \qquad \omega \in \mathcal{K}_X,$$

where $\Box_\omega$ denotes the Laplacian of the Kähler manifold (M, ω). Then to each Kähler metric ω in $\mathcal{K}_X$, we associate a conformally Kähler metric

$$\tilde\omega \;:=\; \omega \,\exp\left(-\frac{\psi_\omega}{n}\right),$$

which is called a *multiplier Hermitian metric of type* σ. Here, a Hermitian form and the associated Hermitian metric are used interchangeably. The associated map

$$\psi : \mathcal{K}_X \to C^\infty(M)_\mathbb{R}, \qquad \omega \mapsto \psi_\omega,$$

is called a *multiplier*, and if in addition $\dot\sigma(s)$ is nowhere vanishing on the open interval (l_0, l_1), then ψ is said to be *nonsingular*. Among nonsingular multipliers, we say that ψ is a *Fubini-Study multiplier* or a *Euclidean multiplier*, according as $\sigma(s)$ is expressible as $-\log(s + C)$ or $-s + C$ for some real constant C on the whole closed interval $[l_0, l_1]$. Put

$$\text{(1.1)} \qquad \mathrm{Ric}^\sigma_X(\omega) \;:=\; \sqrt{-1}\,\bar\partial\partial \log \tilde\omega^n \;=\; \mathrm{Ric}(\omega) \;+\; \sqrt{-1}\,\partial\bar\partial\psi_\omega.$$

DEFINITION. A multiplier Hermitian metric $\tilde\omega$ of type σ on the Kähler manifold (M, ω) is said to satisfy the *Einstein condition* if $\mathrm{Ric}^\sigma_X(\omega) = k\,\omega$ for some real constant k. If (M, ω) satisfies the Einstein condition, then $\mathrm{Ric}(\omega)$ is cohomologous to $k\,\omega$, and by replacing ω by its suitable positive constant multiple, we may assume without loss of generality that k is either 0 or ± 1. According as k is 1, 0, or -1, the Kähler class of $\mathcal{K}$ is $2\pi c_1(M)$, 0 or $-2\pi c_1(M)$. Then only the situation $k = 1$ can occur by our assumption $X \neq 0$.

Hence, we assume the Kähler class of $\mathcal{K}$ to be $2\pi c_1(M)$ until the end of this note. In particular, $c_1(M) > 0$. Then for Fubini-Study multipliers and Euclidean multipliers, the set $\mathcal{E}^\sigma_X$ of all $\omega \in \mathcal{K}_X$ satisfying $\mathrm{Ric}^\sigma_X(\omega) = \omega$ is characterized as follows:

Theorem A. (1) *Suppose that* ψ *is a Fubini-Study multiplier. Then* $\omega \in \mathcal{E}^\sigma_X$ *if and only if the pair* (ω, X) *satisfies the following conditions*:

(a) *ω is a "Kähler-Einstein metric" in the sense of* [6];
(b) *$-X$ coincides, up to a positive constant multiple, with the extremal Kähler vector field* (cf. [4]) *on the Kähler manifold (M, ω).*
(2) *Suppose that ψ is an Euclidean multiplier. Then $\omega \in \mathcal{E}_X^\sigma$ if and only if the pair (ω, X) is a Kähler Ricci soliton in a strong sense on the Kähler manifold (M, ω).*

Let $\mathcal{E}_{\mathrm{KE}}$ denote the set of all "Kähler-Einstein metrics" in the sense of [6] in the class $2\pi c_1(M)_{\mathbb{R}}$ on M. Note that G is a linear algebraic group by $c_1(M) > 0$ (see for instance [2]). Then as an easy corollary of [7] together with Theorem A above, we obtain

Corollary B. *Let $\mathcal{E}_{\mathrm{KE}}$ be nonempty. Then $\mathcal{E}_{\mathrm{KE}}$ consists of a single G-orbit under the natural action of G.*

Corollary C. *If $\omega \in \mathcal{E}_{\mathrm{KE}}$, then the diameter* $\mathrm{Diam}(M, \omega)$ *of the Kähler manifold (M, ω) satisfies* $\mathrm{Diam}(M, \omega) \leq 2\pi(2n - 1 + 4\gamma_M)^{1/2}$, *where $\gamma_M > 0$ is a holomorphic invariant of M defined in* §2 *below.*

It is easily seen that by the same arguments, a similar diameter bound exists also for Kähler Ricci solitons in a strong sense.

For the Lie algebra $\mathfrak{g} := H^0(M, \mathcal{O}(\mathcal{TM}))$ of G, let $Z_{\mathfrak{g}}(X)$ and $\mathfrak{u}$ be respectively the centralizer of X in $\mathfrak{g}$ and the Lie subalgebra of $\mathfrak{g}$ associated to the unipotent radical of G. Then in §3, we shall see that the nonsingularity of multipliers plays a crucial role in the proof of the following analogue of the theorems of Calabi [1] and Matsushima [8]. (For partial results, see also [6] and [9].)

Theorem D. *Let $\omega \in \mathcal{E}_X^\sigma$, and assume that the associated multiplier is nonsingular. Then according as* (i) $\dot\sigma(s) > 0$ *or* (ii) $\dot\sigma(s) < 0$ *on (l_0, l_1), there exists a sequence of real numbers* (i) $0 = \lambda_0 < \lambda_1 < \lambda_2 \cdots < \lambda_r$ *or* (ii) $0 = \lambda_0 > \lambda_1 > \lambda_2 \cdots > \lambda_r$ *for a nonnegative integer r such that*
(a) $\mathfrak{k}^{\mathbb{C}} = \mathfrak{g}(\lambda_0) = Z_{\mathfrak{g}}(X)$;
(b) *$\mathfrak{g}$ is, as a vector space, written as a direct sum* $\oplus_{i=0}^r \mathfrak{g}(\lambda_i)$;
(c) $\mathfrak{u} = \oplus_{i=1}^r \mathfrak{g}(\lambda_i)$,
where we put $\mathfrak{g}(\mu) := \{ Y \in \mathfrak{g} ; [\sqrt{-1}\, X, Y] = \mu Y \}$ *for each real number μ, and $\mathfrak{k}^{\mathbb{C}}$ denotes the complexification in $\mathfrak{g}$ of the space $\mathfrak{k}$ of the Killing vector fields on the Kähler manifold (M, ω).*

2. Proof of Theorem A

Let $\omega \in \mathcal{K}_X$. Since the Kähler class of $\mathcal{K}$ is $2\pi c_1(M)$, there exists a unique function f_ω in $C^\infty(M)_{\mathbb{R}}$ such that $\mathrm{Ric}(\omega) = \omega + \sqrt{-1}\,\partial\bar\partial f_\omega$ and that $\int_M (e^{f_\omega} - 1)\,\omega^n = 0$.

Put

$$\tilde{f}_\omega := f_\omega + \psi_\omega + C_\omega = f_\omega + \sigma(u_\omega) + C_\omega, \tag{2.1}$$

where $C_\omega := \log(\int_M \tilde{\omega}^n / \int_M \omega^n) \in \mathbb{R}$. Then by (1.1), we have $\mathrm{Ric}^\sigma_X(\omega) = \omega + \sqrt{-1}\,\partial\bar{\partial}\tilde{f}_\omega$. By the definition of $\mathcal{E}^\sigma_X$, we see that $\omega \in \mathcal{E}^\sigma_X$ if and only if $\tilde{f}_\omega$ is a constant. Hence, by (2.1), ω belongs to $\mathcal{E}^\sigma_X$ if and only if

$$f_\omega + \sigma(u_\omega) \text{ is a constant.} \tag{2.2}$$

DEFINITION 2.3 (cf. [6]). An element ω in $\mathcal{K}$ is a "*Kähler-Einstein metric*" in the sense of [6], if $X_\omega := \mathrm{grad}^{\mathbb{C}}_\omega (e^{f_\omega} - 1)$ is a holomorphic vector field on M. Then $-X_\omega$ is called the *extremal Kähler vector field* on the Kähler manifold (M, ω) (see also [4]).

DEFINITION 2.4 (cf. [5], [9]). For ω in $\mathcal{K}$, we put $W := \mathrm{grad}^{\mathbb{C}}_\omega f_\omega$ and $V := \sqrt{-1}\,W/2$. Then the pair (ω, W) is called a *Kähler-Ricci soliton in a strong sense*, if W is a holomorphic vector field on M. Hence, if (ω, W) is a Kähler-Ricci soliton in a strong sense, the real vectotr field $V_{\mathbb{R}} := V + \bar{V}$ associated to the holomorphic vector field V satisfies $L_{V_{\mathbb{R}}}\omega = \sqrt{-1}\,\partial\bar{\partial} f_\omega = \mathrm{Ric}(\omega) - \omega$.

Proof of (1) of Theorem A. Since $\sigma(s) = -\log(s + C)$ for some real constant C, the statement (2.2) is equivalent to $u_\omega = e^{-C_0} e^{f_\omega} - C$ for some real constant C_0. Then by $\int_M u_\omega\, \omega^n = \int_M (e^{f_\omega} - 1)\,\omega^n = 0$, this is further equivalent to the following statement:

$$u_\omega \quad \text{and} \quad e^{f_\omega} - 1 \ \text{ coincide up to a positive constant multiple.} \tag{2.5}$$

Hence, by Definition 2.3 above, the conditions (a) and (b) are satisfied.

On the other hand, let the pair (ω, X) satisfy the conditions (a) and (b). By (b) together with (a), there exists a positive constant $C_1 > 0$ such that

$$\left(e^{f_\omega} - 1\right) - C_1 u_\omega$$

is a constant. Since $\int_M u_\omega\, \omega^n = \int_M (e^{f_\omega} - 1)\,\omega^n = 0$, we now obtain (2.5), and hence $\omega \in \mathcal{E}^\sigma_X$, as required. This now completes the proof of (1) of Theorem A. □

Proof of (2) of Theorem A. Since $\sigma(s) = -s + C$ for some real constant C, the statement (2.2) is equivalent to

$$u_\omega \text{ and } f_\omega \text{ coincide up to an additive real constant.} \tag{2.6}$$

Hence, by using the notation in Definition 2.4, $W = \mathrm{grad}^{\mathbb{C}}_\omega f_\omega = \mathrm{grad}^{\mathbb{C}}_\omega u_\omega = X$ is holomorphic. Thus, the pair (ω, X) is a Kähler-Ricci soliton in a strong sense.

Next, let (ω, X) be a Kähler-Ricci soliton in a strong sense. Then by Definition 2.4, X coincides with $\mathrm{grad}^{\mathbb{C}}_{\omega} f_{\omega}$, and is a holomorphic vector field on M. On the other hand, by $X = \mathrm{grad}^{\mathbb{C}}_{\omega} u_{\omega}$, we obtain (2.6), and hence $\omega \in \mathcal{E}^{\sigma}_{X}$, as required. This now completes the proof of (2) of Theorem A. □

Proof of Corollary B. For an element ω in $\mathcal{E}_{\mathrm{KE}}$, by Definition 2.3, $-X_{\omega}$ is the associated extremal Kähler vector field. Now, let ω', $\omega'' \in \mathcal{E}_{\mathrm{KE}}$. By [4], there exists $\hat{g} \in G$ such that

$$X'' = \mathrm{Ad}(\hat{g})X' = \hat{g}_* X', \tag{2.7}$$

where we put $X' := X_{\omega'}$ and $X'' := X_{\omega''}$ for simplicity. Write $X' = \mathrm{grad}^{\mathbb{C}}_{\omega'} u'$ and $X'' = \mathrm{grad}^{\mathbb{C}}_{\omega''} u''$, where $u' := e^{f_{\omega'}} - 1$ and $u'' := e^{f_{\omega''}} - 1$. Since ω' and $\hat{g}^*\omega''$ are in the same Kähler class, by [6] and $\int_M u'(\omega')^n = \int_M u''(\omega'')^n = 0$, we obtain

$$\begin{cases} \max_M u' \ \left(= \max_M \hat{g}^* u''\right) = \max_M u'' = \ \beta_M, \\ \min_M u' \ \left(= \min_M \hat{g}^* u''\right) = \min_M u'' = -\alpha_M, \end{cases} \tag{2.8}$$

for some nonnegative real constants α_M, β_M satisfying $\alpha_M < 1$. Then by setting $\sigma(s) := -\log(s+1)$, we see from Theorem A that

$$\omega' \in \mathcal{E}^{\sigma}_{X'} \quad \text{and} \quad \omega'' \in \mathcal{E}^{\sigma}_{X''}.$$

Now by (2.7), $\mathcal{E}^{\sigma}_{X'} = \hat{g}^*\mathcal{E}^{\sigma}_{X''}$ and hence both ω' and $\hat{g}^*\omega''$ belongs to $\mathcal{E}^{\sigma}_{X'}$. On the other hand, by [7, Theorem C], $\mathcal{E}^{\sigma}_{X'}$ consists of a single $Z^0(X')$-orbit, where $Z^0(X')$ denotes the identity component of the subgroup

$$Z(X') := \{\, g \in G\,;\, \mathrm{Ad}(g)\, X' = X' \,\}$$

of G. Then $\omega' = g^*(\hat{g}^*\omega'') = (\hat{g}g)^*\omega''$ for some $g \in Z^0(X')$. We now conclude that $\mathcal{E}_{\mathrm{KE}}$ consists of a single G-orbit, as required. □

Proof of Corollary C. Note that α_M and β_M in (2.8) are holomorphic invariants of M. Put

$$\gamma_M := \max\{\log(1+\beta_M), -\log(1-\alpha_M)\} > 0,$$

which is also a holomorphic invariant of M. Since the function $\sigma(s) = -\log(s+1)$ is considered on the interval $I = [-\alpha_M, \beta_M]$, and since $\max_{s\in I} |\sigma(s)| = \gamma_M$, we now apply the diameter estimate in [7, Theorem B] to the case $(\nu, c) = (1, \gamma_M)$, we obtain $\mathrm{Diam}(M, \omega) \leq 2\pi(2n - 1 + 4\gamma_M)^{1/2}$, as required. □

REMARK 2.9. Only in this remark, we get rid of the assumption that M is compact. In order to see why the terminology "Fubini-Study" or "Euclidean" is used, we consider the noncompact case where $M = \mathbb{C}^n = \{\, z = (z^1, z^2, \dots, z^n) \in \mathbb{C}^n \,\}$. Put

$$\omega = \sqrt{-1} \sum_{\alpha} dz^{\alpha} \wedge dz^{\bar{\alpha}}, \quad u_{\omega} := \sum_{\alpha} |\, z^{\alpha}|^2, \quad X := \frac{1}{\sqrt{-1}} \sum_{\alpha} z^{\alpha} \frac{\partial}{\partial z^{\alpha}}.$$

(a) If $\sigma(s) = -\log(s + C)$ with $C > 0$, then $e^{-\psi_{\omega}}\omega^n = e^{-\sigma(u_{\omega})}\omega^n = n!\,(u_{\omega} + C)\,\Pi_{\alpha}(\sqrt{-1}\,dz^{\alpha} \wedge dz^{\bar{\alpha}})$, and the corresponding $\mathrm{Ric}^{\sigma}_{X}(\omega)$ is given by

$$-\ \mathrm{Ric}^{\sigma}_{X}(\omega) = \sqrt{-1}\,\partial\bar{\partial}\log(u_{\omega} + C) = \sqrt{-1}\,\partial\bar{\partial}\log(\Sigma_{\alpha}\,|\, z^{\alpha}|^2 + C),$$

and by letting $C \to 0$, this converges to the pullback of the Fubibi-Study form by the natural projection of $\mathbb{C}^n \setminus \{0\}$ onto $\mathbb{P}^{n-1}(\mathbb{C})$. This is why multipliers associated to $\sigma(s) = -\log(s + C)$ are called Fubini-Study multipliers.
(b) If $\sigma(s) = -s + C$, then $e^{-\psi_{\omega}}\omega^n = e^{-\sigma(u_{\omega})}\omega^n = n!\,e^{u_{\omega} - C}\,\Pi_{\alpha}(\sqrt{-1}\,dz^{\alpha} \wedge dz^{\bar{\alpha}})$, and the corresponding $\mathrm{Ric}^{\sigma}_{X}(\omega)$ is given by

$$-\ \mathrm{Ric}^{\sigma}_{X}(\omega) = \sqrt{-1}\,\partial\bar{\partial}u_{\omega} = \omega,$$

which is the Kähler form associated to the standard Euclidean metric on $\mathbb{C}^n$. Therefore, multipliers associated to $\sigma(s) = -s + C$ are called Euclidean multipliers.

3. Proof of Theorem D

The purpose of this section is to prove Theorem D. Since G is a linear algebraic group, every element in $\mathfrak{g}$ is uniquely written as $\mathrm{grad}^{\mathbb{C}}_{\omega}\,\varphi$ for some $\varphi \in C^{\infty}(M)_{\mathbb{C}}$ satisfying $\int_M \varphi\,\omega^n = 0$, where ω is as in Theorem D. Before getting into the proof of Theorem D, we give the following remark:

REMARK 3.1. In Theorem D, let K denote the connected Lie subgroup of G generated by $\mathfrak{k}$. Then K is easily shown to be a maximal compact subgroup in G as follows: Take an arbitrary compact subgroup K' in G such that $K \subset K'$, and the proof is reduced to showing $K' = K$. We first observe that, by the below proof of Theorem D, the group K coincides with the connected component of K'. Hence, it suffices to show that K' is connected. Let U be the unipotent subgroup of G generated by $\mathfrak{u}$, and consider the connected reductive algebraic subgroup $K^{\mathbb{C}}$ of G obtained as the complexification of K in G. Then by the Chevalley decomposition, we can write G as a semidirect product

$$G = K^{\mathbb{C}} \ltimes U.$$

Let $\rho : G \to G/U\,(= K^{\mathbb{C}})$ be the natural quotient homomorphism. Since the image $\rho(K')$ is a compact group containing K, and since K is a maximal compact subgroup of $K^{\mathbb{C}}$, the groups $\rho(K')$ and K coincide. In particular, $\rho(K')$ is connected. On the other hand, the kernel of the restriction

$$\rho_{|K'} : K' \to K^{\mathbb{C}}$$

is a compact subgroup of U, and is a trivial group. Hence, $\rho_{|K'}$ is injective, and we now conclude that K' is connected, as required.

Proof of Theorem D. As to the sign of the function $\dot{\sigma}$ on (l_0, l_1), it is easily seen that the proof for $\dot{\sigma} > 0$ and that for $\dot{\sigma} < 0$ are similar. Hence, we may assume $\dot{\sigma} > 0$ on the open interval (l_0, l_1) without loss of generality. By $X \in \mathfrak{k}$, we see that $\mathfrak{g}$ has an $\mathrm{ad}(X)$-invariant $\mathbb{C}$-linear subspace $\mathfrak{m}$ containing $\mathfrak{u}$ such that $\mathfrak{g}$ is a direct sum $\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}$ of vector spaces. There exist sequences of real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ and $\mu_1 < \mu_2 < \cdots < \mu_m$ such that

$$\mathfrak{m} = \bigoplus_{i=1}^{r} \mathfrak{g}(\lambda_i) \quad \text{and} \quad \mathfrak{k}^{\mathbb{C}} = \bigoplus_{j=1}^{m} \mathfrak{g}(\mu_j).$$

Then the proof is reduced to showing $\lambda_1 > 0 = \mu_1$ and $m = 1$, Because if we can show these, then (a) and (b) follow immediately, and an argument in [1, p. 109] shows that $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup of G, which together with $\mathfrak{u} \subset \mathfrak{m}$ implies the equality $\mathfrak{u} = \mathfrak{m}$ and (c) above. Now by (2.2), our assumption $\omega \in \mathcal{E}_X^{\sigma}$ allows us to write $f_\omega = -\psi_\omega + C$ for some real constant C. Hence,

$$\tilde{\Box}_\omega = \Box_\omega + \sum_{\alpha,\beta} g(\omega)^{\bar{\beta}\alpha} \frac{\partial f_\omega}{\partial z^\alpha} \frac{\partial}{\partial z^\beta}.$$

Let $\tilde{\mathfrak{g}}$ (resp. $\tilde{\mathfrak{k}}$) denote the space $\mathrm{Ker}_{\mathbb{C}}(\tilde{\Box}_\omega + 1)$ (resp. $\mathrm{Ker}_{\mathbb{R}}(\tilde{\Box}_\omega + 1)$) of all complex-valued (resp. real-valued) C^∞ functions u on M such that $(\tilde{\Box}_\omega + 1)u = 0$. Put $\tilde{\mathfrak{k}}^{\mathbb{C}} := \tilde{\mathfrak{k}} + \sqrt{-1}\,\tilde{\mathfrak{k}}$. By [3, p. 41], we have an isomorphism $\tilde{\mathfrak{g}} \cong \mathfrak{g}$ (resp. $\tilde{\mathfrak{k}}^{\mathbb{C}} \cong \mathfrak{k}^{\mathbb{C}}$) of complex Lie algebras by sending each u in $\tilde{\mathfrak{g}}$ (resp. $\tilde{\mathfrak{k}}^{\mathbb{C}}$) to $\mathrm{grad}^{\mathbb{C}}_\omega u$ in $\mathfrak{g}$ (resp. $\mathfrak{k}^{\mathbb{C}}$). The preimage of $\mathfrak{m}$ under the isomorphism $\tilde{\mathfrak{g}} \cong \mathfrak{g}$ will be denoted by $\tilde{\mathfrak{m}}$. Let v be a nontrivial element of $\tilde{\mathfrak{g}}$. Then

$$\int_M \bar{v}\, e^{f_\omega} \omega^n = \overline{\int_M v\, e^{f_\omega} \omega^n} = -\overline{\int_M (\tilde{\Box}_\omega v)\, e^{f_\omega} \omega^n} = 0,$$

where all eigenvalues of $-\tilde{\Box}_\omega$ are nonnegative real numbers and its first positive eigenvalue is 1 (cf. [3]). Hence,

$$\text{(3.2)} \qquad \begin{cases} \displaystyle\int_M (-\tilde{\Box}_\omega \bar{v}) v\, e^{f_\omega} \omega^n > \int_M |v|^2 e^{f_\omega} \omega^n & \text{if } v \in \tilde{\mathfrak{m}}; \\ \displaystyle\int_M (-\tilde{\Box}_\omega \bar{v}) v\, e^{f_\omega} \omega^n = \int_M |v|^2 e^{f_\omega} \omega^n & \text{if } v \in \tilde{\mathfrak{k}}^{\mathbb{C}}. \end{cases}$$

On the other hand, by $(\tilde{\square}_\omega + 1)\, v = 0$,

$$\int_M (-\tilde{\square}_\omega v)\, \bar{v}\, e^{f_\omega} \omega^n = \int_M |v|^2 e^{f_\omega} \omega^n. \tag{3.3}$$

Subtracting (3.3) from (3.2), we see that $\int_M \{ (-\tilde{\square}_\omega \bar{v}) v + (\tilde{\square}_\omega v)\, \bar{v} \}\, e^{f_\omega} \omega^n$ is positive or zero, according as $v \in \tilde{\mathfrak{m}}$ or $v \in \tilde{\mathfrak{k}}^{\mathbb{C}}$. Then we obtain

$$\begin{cases} \displaystyle\int_M 2\sqrt{-1}\, \{ (\operatorname{Im} \tilde{\square}_\omega)\, v \}\, \bar{v}\, e^{f_\omega} \omega^n > 0, & \text{if } v \in \tilde{\mathfrak{m}}; \\ \displaystyle\int_M 2\sqrt{-1}\, \{ (\operatorname{Im} \tilde{\square}_\omega)\, v \}\, \bar{v}\, e^{f_\omega} \omega^n = 0, & \text{if } v \in \tilde{\mathfrak{k}}^{\mathbb{C}}, \end{cases} \tag{3.4}$$

where $\operatorname{Re} \tilde{\square}_\omega$ (resp. $\operatorname{Im} \tilde{\square}_\omega$) are the real (resp. imaginary) part of $\tilde{\square}_\omega$, so that $\tilde{\square}_\omega = \operatorname{Re} \tilde{\square}_\omega + \sqrt{-1} \operatorname{Im} \tilde{\square}_\omega$. In view of $f_\omega = -\sigma(u_\omega) + C$, we here observe that

$$2\sqrt{-1}\, \{ (\operatorname{Im} \tilde{\square}_\omega)\, v \} = (\sqrt{-1})^{-1} [\, f_\omega,\, v\,] = \dot{\sigma}(u_\omega)\, [\, \sqrt{-1}\, u_\omega,\, v\,], \tag{3.5}$$

where the Poisson bracket is defined as in [4]. If $0 \neq \operatorname{grad}^{\mathbb{C}}_\omega v \in \mathfrak{g}(\lambda_1)$, then we have $[\, \sqrt{-1}\, u_\omega,\, v\,] = \lambda_1 v$, and by the positivity of $\dot{\sigma}(s)$ on (l_0, l_1), (3.5) together with the first line of (3.4) implies $\lambda_1 > 0$. Next, we consider the case $0 \neq \operatorname{grad}^{\mathbb{C}}_\omega v \in \mathfrak{g}(\mu_j)$. Then $[\, \sqrt{-1}\, u_\omega,\, v\,] = \mu_j v$, and by the positivity of $\dot{\sigma}(s)$ on (l_0, l_1), (3.5) and the second line of (3.4) show that $\mu_j = 0$, which implies the equalities $m = 1$ and $\mu_1 = 0$, as required. □

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