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A CLUSTER OF SETS OF EXCEPTIONAL TIMES OF LINEAR BROWNIAN MOTION

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1. Introduction and the main theorems

Aspandiiarov-Le Gall [1] studied the following random closed sets K^- , K and K' : Let $(B_t; t \geq 0)$ be a linear standard Brownian motion starting at 0, and let

$$\begin{aligned} K^- &= \left\{ t \in [0, 1]; \int_s^t (B_u - B_t) du \leq 0 \text{ for every } s \in [0, t). \right\}, \\ K &= \left\{ t \in K^-; \int_t^s (B_u - B_t) du \leq 0 \text{ for every } s \in (t, 1]. \right\}, \\ K' &= \left\{ t \in K^-; \int_t^s (B_u - B_t) du \geq 0 \text{ for every } s \in (t, 1]. \right\}. \end{aligned}$$

They computed the Hausdorff dimension of K^- , K and K' .

Theorem ([1]). *It holds $\dim K^- = 3/4$, $\dim K = 1/2$ and $\dim K' \leq 1/2$ almost surely. The set K' is possibly empty or $\dim K' = 1/2$, both with positive probability. The same statements hold if the weak inequalities in the definition of K^- , K and K' are replaced by the strict inequalities.*

In this paper, we consider a cluster of random sets having various dimension. For $\alpha \geq 0$ and $c > 0$, we define the following functions $V(\alpha, c)$ increasing on \mathbb{R} :

$$V(\alpha, c; y) = y^\alpha \quad \text{for } y > 0; \quad V(\alpha, c; 0) = 0; \quad V(\alpha, c; y) = -\frac{|y|^\alpha}{c} \quad \text{for } y < 0.$$

Let $\alpha, \alpha_+, \alpha_- \geq 0$, $c, c_+, c_- > 0$ and write V for $V(\alpha, c)$, V_\pm for $V(\alpha_\pm, c_\pm)$. We define the random sets depending on the functions V , V_+ and V_- :

$$(1.1) \quad K^-(V) = \left\{ t \in [0, 1]; \int_s^t V(B_u - B_t) du \leq 0 \text{ for every } s \in [0, t). \right\},$$

$$(1.2) \quad K(V_-; V_+) = \left\{ t \in K^-(V_-); \int_t^s V_+(B_u - B_t) du \leq 0 \text{ for every } s \in (t, 1]. \right\},$$

$$(1.3) \quad K'(V_-; V_+) = \left\{ t \in K^-(V_-); \int_t^s V_+(B_u - B_t) du \geq 0 \text{ for every } s \in (t, 1]. \right\}.$$

These sets consist of exceptional times in the sense that $P[t \in K^-(V)] = 0$ for every $t \in (0, 1]$ and $P[t \in K(V_-; V_+)] = P[t \in K'(V_-; V_+)] = 0$ for every $t \in [0, 1]$.

Theorem 1. *We define $\nu = 1/(2 + \alpha)$, $\nu_- = 1/(2 + \alpha_-)$ and $\nu_+ = 1/(2 + \alpha_+)$. Let $\rho, \rho_-, \rho_+ \in (0, 1)$ be the unique solutions of the equations*

$$\begin{aligned} c^\nu \sin \pi \nu (1 - \rho) &= \sin \pi \nu \rho, \\ c_-^{\nu_-} \sin \pi \nu_- (1 - \rho_-) &= \sin \pi \nu_- \rho_-, \\ c_+^{\nu_+} \sin \pi \nu_+ (1 - \rho_+) &= \sin \pi \nu_+ \rho_+ \end{aligned}$$

respectively.

- (a) *For $V = V(\alpha, c)$, we have almost surely $\dim K^-(V) = 1 - \rho/2$. For $V_+ = V(\alpha_+, c_+)$ and $V_- = V(\alpha_-, c_-)$ we have (b) and (c):*
- (b) *$\dim K(V_-; V_+) \leq 1 - (\rho_- + \rho_+)/2$ almost surely and*

$$P \left[\dim K(V_-; V_+) \geq 1 - \frac{\rho_- + \rho_+}{2} \right] > 0.$$

- (c) *$\dim K'(V_-; V_+) \leq (1 - \rho_- + \rho_+)/2$ almost surely and*

$$P \left[\dim K'(V_-; V_+) \geq \frac{1 - \rho_- + \rho_+}{2} \right] > 0.$$

The behavior of V, V_+ and V_- outside any neighborhood of the origin have no influence on the Hausdorff dimension; We could state the theorem in that fashion. The parameters $\rho, \rho_-, \rho_+ \in (0, 1)$ in the statement of Theorem 1 are continuous and increasing in c, c_+, c_- and have the range $(0, 1)$ since $\lim_{c \rightarrow 0} \rho = 0$ and $\lim_{c \rightarrow \infty} \rho = 1$. In fact, they are equal to the probability of some event related to the parameters in the theorem, see the remark 4 in [4].

Note that for fixed α , it holds $\rho = 1/2$ if $c = 1$. Hence the statements in the theorem in [1] for K^- and K' can be included in Theorem 1 since $K^- = K^-(V(1, 1))$ and $K' = K'(V(1, 1); V(1, 1))$. The implication by Theorem 1 on K , however, is weaker than [1], since we have not obtained the almost sure estimate from below.

Let $\alpha, \tilde{\alpha} \geq 0$ and $c, \tilde{c} > 0$. If $V = V(\alpha, c)$ and $\tilde{V} = V(\tilde{\alpha}, \tilde{c})$, then there is no inclusion in general between $K^-(V)$ and $K^-(\tilde{V})$. However it is easy to see, for each α , that $K^-(V(\alpha, c)) \subset K^-(V(\alpha, \tilde{c}))$ if $\tilde{c} < c$. Hence we obtain a family

$$\{K^-(V(\alpha, c)); c \in (0, 1)\}$$

of decreasing random sets having strictly decreasing dimension.

The estimate in Theorem 1 for $\dim K^-(V)$ is exhaustive in the following sense: Let H be the set of times t when B_t attains its past-maximum:

$$H := \left\{ t \in [0, 1]; B_t = \sup_{0 \leq s \leq t} B_s \right\}.$$

It is well known that $\dim H = 1/2$ a.s. Since $H \subset K^-(V(\alpha, c)) \subset [0, 1]$, we have $1/2 \leq \dim K^-(V) \leq 1$. The range of $1 - \rho/2$ is exactly $(1/2, 1)$ and the trivial case $K^-(V) = H$ or $K^-(V) = [0, 1]$ could be included if we allow $c = \infty$ or $c = 0$.

The estimate in Theorem 1 for $\dim K(V_-; V_+)$ is also exhaustive in the following sense: Let τ be the time when the maximum on $[0, 1]$ of B is attained: $B_\tau \geq B_t$ for every $t \in [0, 1]$. The inclusion $\{\tau\} \subset K(V_-; V_+) \subset [0, 1]$ implies $0 \leq \dim K(V_-; V_+) \leq 1$ and the range of the value $1 - (\rho_- + \rho_+)/2$ is exactly $(0, 1)$. The extreme cases could also be included here.

In the same sense as Aspdiiarov and LeGall [1] noted concerning $K', K'(V_-; V_+)$ can be interpreted as a weakened notion of the increasing points of Brownian motion and it is not straightforward to exhibit an element of $K'(V_-; V_+)$.

If both V_- and V_+ are $V(\alpha, c)$ then $(1 - \rho_- + \rho_+)/2 = 1/2$ irrespective of α and c . This motivates the next theorem, which could be a version of settlement of a conjecture at the end of [1]: $\dim K' = 1/2$ a.s. on the event $\{B_1 > 0\}$.

Theorem 2. *Let $\mathcal{V} = \{V : \mathbb{R} \rightarrow \mathbb{R}; V(0) = 0, V \text{ is strictly increasing}\}$.*

We define $\tilde{K}'(V; V)$ for $V \in \mathcal{V}$ in the same way as (1.3) replacing the weak inequalities by strict inequalities in the definition of $K'(V; V)$:

$$\tilde{K}'(V; V) = \left\{ t \in [0, 1]; \int_s^t V(B_u - B_t) du < 0 \text{ for every } s \in [0, t), \right. \\ \left. \text{and } \int_t^s V(B_u - B_t) du > 0 \text{ for every } s \in (t, 1]. \right\}.$$

Then we have $P[\dim \tilde{K}'(V; V) = 1/2] > 0, P[\tilde{K}'(V; V) \subset \{0, 1\}] > 0$ and

$$P \left[\dim \tilde{K}'(V; V) = \frac{1}{2} \text{ or } \tilde{K}'(V; V) \subset \{0, 1\} \right] = 1.$$

REMARK 1. When the set $\tilde{K}'(V; V)$ consists of exceptional times, we have the dichotomy that $\dim \tilde{K}'(V; V) = 1/2$ if it is not empty.

The result of Theorem 2 is stronger than Theorem 1(c) for each strictly increasing functions $V(\alpha, c)$, i.e. $\alpha > 0$, while Theorem 2 says nothing about $V(0, c)$.

Theorem 2 is in fact a corollary of the following Theorem 3 due essentially to Bertoin [3].

Let $V \in \mathcal{V}$, $x \in \mathbb{R}$ and $X = (X(t); t \geq 0)$ be a cadlag path with $\liminf_{t \rightarrow \infty} X(t) = +\infty$. We define, inspired by Bertoin [3],

$$K'_\infty(V, x, X) = \left\{ t \in [0, \infty); \int_s^t V(X_u - x) du \leq 0 \text{ for every } s \in [0, t), \right. \\ \left. \text{and } \int_t^s V(X_u - x) du \geq 0 \text{ for every } s \in (t, \infty). \right\}, \\ K'_1(V, x, X) = \left\{ t \in [0, 1]; \int_s^t V(X_u - x) du \leq 0 \text{ for every } s \in [0, t), \right. \\ \left. \text{and } \int_t^s V(X_u - x) du \geq 0 \text{ for every } s \in (t, 1]. \right\}.$$

It is then easy to see $\tilde{K}'(V; V) \cup \{0, 1\} = \cup_{\sharp K'_1(V, x, B)=1} K'_1(V, x, B)$.

In other words, $K'_\infty(V, x, X)$ and $K'_1(V, x, X)$ consist of the locations of the overall minimum of the function $s \mapsto \int_0^s V(X_u - x) du$ on $[0, \infty)$ or $[0, 1]$ respectively and $\tilde{K}'(V; V)$ is the collections of such t 's that the function $s \mapsto \int_0^s V(B_u - B_t) du$ has the unique minimum at $s = t$.

The following results are proven in Bertoin [3] in the case where $V(y) \equiv y = V(1, 1; y)$.

Theorem 3. *Let $V \in \mathcal{V}$ and X be a Lévy process with no positive jump such that $\liminf_{t \rightarrow \infty} X(t) = +\infty$ a.s. Let $a(x)$ be the rightmost element of $K'_\infty(V, x, X)$.*

(a) *$\{a(x) - a(0); x \geq 0\}$ and the process $T^X(x) := \inf\{t \geq 0; X_t \geq x\}$ have the same law.*

(b) *For every fixed $x \in \mathbb{R}$, $P^X[\sharp K'_\infty(V, x, X) = 1] = 1$.*

(c) *Let $g(0) = \sup\{t \geq 0; X(t) \leq 0\}$ be the last exit time from $(-\infty, 0]$. If $V \in \mathcal{V}$ satisfies $V(y) = -V(-y)$, then $a(0)$ and $g(0) - a(0)$ are independent and have the same law.*

(d) *If X is a Brownian motion with unit drift, then $\{a(x) - a(0); x \geq 0\}$ has the Lévy measure $(2\pi)^{-1/2} y^{-3/2} e^{-y/2} dy$ on $(0, \infty)$. If, moreover, $V \in \mathcal{V}$ satisfies $V(y) = -V(-y)$, then the density of the common law of $a(0)$ and $g(0) - a(0)$ is $2^{-1/4} \Gamma(1/4)^{-1} y^{-3/4} e^{-y/2} dy$ on $(0, \infty)$.*

REMARK 2. The statement (a) and the first sentence in (d) hold for nondecreasing V satisfying $V(0) = 0$. The second sentence in (d) was known to Jean Bertoin (private communication).

This paper is organized as follows: We prove Theorem 1 in Section 2 using Theorem 4, which contains an asymptotic estimate for some fluctuating additive functionals. Theorems 2 and 3 are proven in Section 3. We prove Theorem 4 in Section 4 using a theorem in [4].

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2. Proof of Theorem 1

The argument here mimics that of Aspandiarov and Le Gall [1] line by line. We first state Theorem 4, an estimate for the distribution of the first hitting time of $\int_0^t V(B_u)du$, next define suitable approximations of $K^-(V)$, $K(V_-; V_+)$ and $K'(V_-; V_+)$ and obtain some preliminary estimates. From that point on, we only need the straightforward changes.

Theorem 4. *Let $\alpha \geq 0$, $c > 0$, $V = V(\alpha, c)$, $\nu = 1/(2 + \alpha)$ and $\rho \in (0, 1)$ be the solution of $c^\nu \sin \pi\nu(1 - \rho) = \sin \pi\nu\rho$. We denote by $p(t, x, y; V)$ the probability $P[\forall s \in [0, t], x + \int_0^s V(y + B_u)du \leq 0]$.*

For any $t > 0$, $x < 0$, $y \in \mathbb{R}$ and there exist constants $C_0(t, x, y; V) > 0$, $C_1(\alpha, c) > 0$ and $\tilde{C}(x, y) > 0$ such that it holds

$$(2.4) \quad \sup_{\sigma > 0} \sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V) = C_0(t, x, y; V),$$

$$(2.5) \quad \lim_{\sigma \rightarrow 0} \sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V) = C_1(\alpha, c) t^{-\rho/2} \tilde{C}(x, y),$$

Moreover it holds

$$(2.6) \quad C_0(t, x, y; V) \leq \text{const } t^{-\rho/2} (|x|^{\nu\rho} \vee |y|^{-\rho}),$$

DEFINITION. Let $\varepsilon \in [0, 1/2]$, $a \in [0, 1 - \varepsilon]$ and $b \in [\varepsilon, 1]$. For $V, V_+, V_- \in \cup_{\alpha \geq 0, c > 0} \{V(\alpha, c)\}$ we define

$$K_{\varepsilon,a}^-(V) = \left\{ t \in [a + \varepsilon, 1]; \int_s^t V(B_u - B_t)du \leq 0 \text{ for every } s \in [a, t - \varepsilon] \right\},$$

$$K_{\varepsilon,b}^+(V) = \left\{ t \in [0, b - \varepsilon]; \int_t^s V(B_u - B_t)du \leq 0 \text{ for every } s \in [t + \varepsilon, b] \right\},$$

$$K_{\varepsilon,b}^*(V) = \left\{ t \in [0, b - \varepsilon]; \int_t^s V(B_u - B_t)du \geq 0 \text{ for every } s \in [t + \varepsilon, b] \right\},$$

$$K_{\varepsilon,a,b}(V_-; V_+) = K_{\varepsilon,a}^-(V_-) \cap K_{\varepsilon,b}^+(V_+),$$

$$K'_{\varepsilon,a,b}(V_-; V_+) = K_{\varepsilon,a}^-(V_-) \cap K_{\varepsilon,b}^*(V_+).$$

We also define

$$(2.7) \quad K_\varepsilon^-(V) = K_{\varepsilon,0}^-(V), \quad K^-(V) = K_0^-(V),$$

$$(2.8) \quad K_\varepsilon(V_-; V_+) = K_{\varepsilon,0,1}(V_-; V_+), \quad K(V_-; V_+) = K_0(V_-; V_+),$$

$$(2.9) \quad K'_\varepsilon(V_-; V_+) = K'_{\varepsilon,0,1}(V_-; V_+), \quad K'(V_-; V_+) = K'_0(V_-; V_+).$$

Lemma 5. *Let $\alpha, \alpha_+, \alpha_- \geq 0$, $c, c_+, c_- > 0$ and let ρ, ρ_+, ρ_- be defined in the statement of Theorem 1.*

(a) *For any $V = V(\alpha, c)$, $0 < \varepsilon < 1/2$ and $t > a$, it holds*

$$\left(\frac{t-a}{\varepsilon}\right)^{\rho/2} P[t \in K_{\varepsilon,a}^-(V)] < \text{const.}$$

There exists a constant $C_3(V) > 0$ such that it holds

$$P[t \in K_{\varepsilon,a}^-(V)] \sim C_3(V) \left(\frac{\varepsilon}{t-a}\right)^{\rho/2}$$

as $\varepsilon \searrow 0$ for every t .

(b) *For any $V_+ = V(\alpha_+, c_+)$, $V_- = V(\alpha_-, c_-)$, $0 < \varepsilon < 1/2$ and $t \in (a, b)$,*

$$\begin{aligned} \left(\frac{t-a}{\varepsilon}\right)^{\rho_-/2} \left(\frac{b-t}{\varepsilon}\right)^{\rho_+/2} P[t \in K_{\varepsilon,a,b}(V_-; V_+)] &< \text{const.}, \\ \left(\frac{t-a}{\varepsilon}\right)^{\rho_-/2} \left(\frac{b-t}{\varepsilon}\right)^{(1-\rho_+)/2} P[t \in K'_{\varepsilon,a,b}(V_-; V_+)] &< \text{const.} \end{aligned}$$

We denote by $V_+(-\cdot)$ the function $y \mapsto V_+(-y)$. It holds as $\varepsilon \searrow 0$

$$\begin{aligned} P[t \in K_{\varepsilon,a,b}(V_-; V_+)] &\sim C_3(V_-)C_3(V_+) \left(\frac{\varepsilon}{t-a}\right)^{\rho_-/2} \left(\frac{\varepsilon}{b-t}\right)^{\rho_+/2}, \\ P[t \in K'_{\varepsilon,a,b}(V_-; V_+)] &\sim C_3(V_-)C_3(V_+(-\cdot)) \left(\frac{\varepsilon}{t-a}\right)^{\rho_-/2} \left(\frac{\varepsilon}{b-t}\right)^{(1-\rho_+)/2} \end{aligned}$$

Proof. We only prove (a) since the statement (b) follows by time-reversal $\tilde{B}_s = B_{1-s}$ and by reflection $\tilde{B}_s = -B_s$.

Let $P_{(x,y)}^V$ be the law of the following two-dimensional diffusion $(X(t), Y(t))$:

$$Y(t) = y + B(t), \quad X(t) = x + \int_0^t V(Y(s))ds.$$

By the strong Markov property,

$$P[t \in K_{\varepsilon,a}^-(V)] = E_{(0,0)}^V[p(t-a-\varepsilon, X(\varepsilon), Y(\varepsilon); V)]$$

Under $P_{(0,0)}^V$, the law of $(X(\varepsilon), Y(\varepsilon))$ is the same as that of $(\varepsilon^{1/2\nu}X(1), \varepsilon^{1/2}Y(1))$. By (2.4) and (2.6), we have for any $\varepsilon > 0$,

$$\begin{aligned} &\varepsilon^{-\rho/2} p(t - a - \varepsilon, \varepsilon^{1/2\nu} X(1), \varepsilon^{1/2} Y(1); V) \\ &< \text{const}(t - a - \varepsilon)^{-\rho/2} (|X(1)|^{-\nu\rho} \vee |Y(1)|^{-\rho}) \\ &< \text{const}(t - a)^{-\rho/2} (|X(1)|^{-\nu\rho} \vee |Y(1)|^{-\rho}). \end{aligned}$$

The quantity $((t - a)/\varepsilon)^{\rho/2} P[t \in K_{\varepsilon,a}^-(V)]$ is hence bounded. This bound also enables us to prove the second sentence of (a) with $C_3(V) = C_1(\alpha, c) E_{(0,0)}^V[\tilde{C}(X(1), Y(1))]$. □

Lemma 6. *We use the same notations as the previous lemma. It holds for any $\varepsilon \in (0, 1/2)$ and $0 < s < t < 1$,*

$$(2.10) \quad P[\{s, t\} \subset K_{\varepsilon,a}^-(V)] \leq \text{const} \frac{\varepsilon^\rho}{s^{\rho/2}(t-s)^{\rho/2}},$$

$$(2.11) \quad P[\{s, t\} \subset K_{\varepsilon,a,b}(V_-; V_+)] \leq \frac{\text{const} \varepsilon^{\rho_+ + \rho_-}}{s^{\rho_-/2}(t-s)^{(\rho_- + \rho_+)/2}(1-t)^{\rho_+/2}},$$

$$(2.12) \quad P[\{s, t\} \subset K'_{\varepsilon,a,b}(V_-; V_+)] \leq \frac{\text{const} \varepsilon^{\rho_- + (1-\rho_+)}}{s^{\rho_-/2}(t-s)^{(\rho_- + 1 - \rho_+)/2}(1-t)^{(1-\rho_+)/2}}.$$

The constants here depend on $\alpha, \alpha_+, \alpha_-$ and c, c_+, c_- .

Proof. This can be done using Lemma 5. See the proof of Proposition 4 in [1]. □

Lemma 7. *Let $\mathcal{F}_{a,b}$ be the σ -field $\sigma(B_u - B_a; u \in [a, b])$ for $0 \leq a < b \leq 1$.*

For any $\alpha \geq 0, c > 0$ and $V = V(\alpha, c)$ there exist $\mathcal{F}_{a,b}$ -measurable random variables $U_{a,b,-}, U_{a,b,+}$ and $U_{a,b,}$ such that*

$$(2.13) \quad P[K^-(V) \cap [a, b] \neq \emptyset \mid \mathcal{F}_{a,1}] \leq (b - a)^{\rho/2} U_{a,b,-},$$

$$(2.14) \quad P[K^+(V) \cap [a, b] \neq \emptyset \mid \mathcal{F}_{0,b}] \leq (b - a)^{\rho/2} U_{a,b,+},$$

$$(2.15) \quad P[K^*(V) \cap [a, b] \neq \emptyset \mid \mathcal{F}_{0,b}] \leq (b - a)^{(1-\rho)/2} U_{a,b,*},$$

and $E_0[(U_{a,b,-})^2] \leq \text{const} a^{-\rho}, E_0[(U_{a,b,+})^2] \leq \text{const}(1 - b)^{-\rho}, E_0[(U_{a,b,*})^2] \leq \text{const}(1 - b)^{-1+\rho}$. The constants here depend on α and c .

Proof. We prove (2.14) since (2.13), (2.15) and the corresponding moment estimates follow by time-reversal $\tilde{B}_s = B_{1-s}$ and by reflection $\tilde{B}_s = -B_s$.

Let $\eta_{a,b}$ be the amplitude of B_s on $[a, b]$. Note that V is increasing. By modifying the argument in the proof of Lemma 7 in [1], we can take

$$U_{a,b,+} = (b - a)^{-\rho/2} p(1 - b, (b - a)V(-\eta_{a,b}), -\eta_{a,b}; V).$$

The bound of the moment follows by (2.6) and by the fact that $\eta_{a,b}$ has the same law as $(b - a)^{1/2}\eta_{0,1}$. □

Proof of Theorem 1. The upper estimates for the Hausdorff dimension is obtained by the argument in the proof Proposition 6 in [1].

To obtain the lower estimates, we define the normalized Lebesgue measures: For any Borel subset F of $[0, 1]$, let

$$\begin{aligned}\mu_{\varepsilon}^{-}(F) &= \varepsilon^{-\rho/2} |F \cap K_{\varepsilon}^{-}(V)|, \\ \mu_{\varepsilon}(F) &= \varepsilon^{-(\rho-\rho_+)/2} |F \cap K_{\varepsilon}(V_-; V_+)|, \\ \mu'_{\varepsilon}(F) &= \varepsilon^{-(\rho-+1-\rho_+)/2} |F \cap K'_{\varepsilon}(V_-; V_+)|.\end{aligned}$$

We denote by \mathcal{M}_f the Polish space of all finite measures on $[0, 1]$ equipped with the topology of weak convergence, and by $C([0, 1])$ the Banach space of all continuous map from $[0, 1]$ to \mathbb{R} .

Let (ε_n) be a sequence strictly decreasing to 0. We define the random variables ζ^n taking values in $\mathcal{M}_f \times C([0, 1])$ by $\zeta^n = (\mu_{\varepsilon_n}, (B_t; 0 \leq t \leq 1))$. We define $\zeta^{-,n}$ and $\zeta^{',n}$ in the same way using $\mu_{\varepsilon_n}^{-}$ and μ'_{ε_n} . The argument in [1] ensures that we may assume the sequence (ζ^n) is weakly convergent by extracting a subsequence. Skorohod's representation theorem says that there is a probability space carrying a sequence of random variables $\overline{\zeta}^n = (\mu^n, (B_t^n; 0 \leq t \leq 1))$ and a random variable $\overline{\zeta}^{\infty} = (\mu^{\infty}, (B_t^{\infty}; 0 \leq t \leq 1))$ such that $\overline{\zeta}^n$ and ζ^n have the same law and $\overline{\zeta}^n$ converges to $\overline{\zeta}^{\infty}$ almost surely.

Let $K(V_-; V_+; B^{\infty})$ be defined in the same way as $K(V_-; V_+)$ replacing B by B^{∞} . To prove that μ^{∞} is a.s. supported on $K(V_-; V_+; B^{\infty})$, we change the definition of $G(\eta, \gamma)$ appearing in the proof of Lemma 9 in [1].

$$G(\eta, \gamma) = \left\{ t < 1 - \eta; \sup_{t+\eta < s \leq 1} \int_t^s V_+(B_u^{\infty} - B_t^{\infty}) du > \gamma \right\}.$$

Since V_+ has no discontinuities of the second kind, it is locally bounded and hence we can deduce, from the occupation time formula, that $G(\eta, \gamma)$ is an open set.

On the other hand, μ^n is a.s. supported on

$$\left\{ t \leq 1 - \varepsilon_n; \sup_{t+\varepsilon_n < s \leq 1} \int_t^s V_+(B_u^n - B_t^n) du \leq 0 \right\}.$$

To deduce that $\mu^{\infty}(G(\eta, \gamma)) = 0$ and μ^{∞} is a.s. supported on $K(V_-; V_+; B^{\infty})$ by the argument in the proof of Lemma 9 in [1], we need only to prove the following:

$$(2.16) \quad \text{For fixed } s \text{ and } t, \int_t^s V_+(B_u^n - B_t^n) du \rightarrow \int_t^s V_+(B_u^{\infty} - B_t^{\infty}) du \quad \text{as } n \rightarrow \infty.$$

To prove (2.16), let $\varepsilon, \varepsilon'$ be arbitrary positive numbers and let

$$R^{\infty}(\varepsilon', s) := \{x \in \mathbb{R}; \exists u < s, |x - B_u^{\infty}| < 2\varepsilon'\}.$$

Since V_+ has discontinuity only at the origin (when $\alpha = 0$), there exists $0 < \delta < \varepsilon'$ such that for any $x, y \in R^\infty(\varepsilon', s)$ satisfying $|x - y| < \delta$ and $|x| > \varepsilon'$, it holds $|V_+(x) - V_+(y)| < \varepsilon$.

We can make $\int_t^s 1_{\{|B_u^\infty - B_t^\infty| \leq 3\varepsilon'\}} du$ arbitrarily small by taking ε' small, and hence $\int_t^s V_+(B_u^n - B_t^n) 1_{\{|B_u^n - B_t^n| \leq \varepsilon'\}} du$ is also small if $\|B^n - B^\infty\| < \varepsilon'$, since V_+ is bounded on $R^\infty(\varepsilon', s)$.

For $u \in [t, s]$ satisfying $|B_u^\infty - B_t^\infty| > \varepsilon'$, we have $|V_+(B_u^n - B_t^n) - V_+(B_u^\infty - B_t^\infty)| < \varepsilon$ if $\|B^n - B^\infty\| < \delta/2$, which is satisfied for all large n .

We have thus proven (2.16).

Using Lemma 5 and the weak convergence we have

$$\begin{aligned}
 E[\mu^{-, \infty}([0, 1])] &= \int_0^1 dt t^{-\rho/2} C_3(V) > 0, \\
 E[\mu^\infty([0, 1])] &= \int_0^1 dt t^{-\rho_-/2} C_3(V_-)(1-t)^{-\rho_+/2} C_3(V_+) > 0, \\
 E[\mu'^{\infty}([0, 1])] &= \int_0^1 dt t^{-\rho_-/2} C_3(V_-)(1-t)^{-(1-\rho_+)/2} C_3(V_+(-\cdot)) > 0.
 \end{aligned}$$

The positivity of these values is, through Frostman’s lemma, related to the positivity of $P[\dim K^-(V) \leq 1 - \rho/2]$ and its companions; The a.s. estimate from below follows by the scaling property of Brownian motion as in [1]. □

3. Proof of Theorems 3 and 2

In this section, V is an strictly increasing function with $V(0) = 0$ and $a(x)$ is the rightmost element in $K'_\infty(V, x, X)$.

Lemma 8. (a) *If $x_0 < x_1$ and there exists a triple (t_0, t_1, t_2) such that*

$$\begin{aligned}
 t_0 &\in K'_\infty(V, x_0, X) \setminus K'_\infty(V, x_1, X), \\
 t_1 &\in K'_\infty(V, x_0, X) \cap K'_\infty(V, x_1, X), \\
 t_2 &\in K'_\infty(V, x_1, X) \setminus K'_\infty(V, x_0, X),
 \end{aligned}$$

then it holds $t_0 < t_1 < t_2$.

(b) *The cardinality of $K'_\infty(V, x_0, X) \cap K'_\infty(V, x_1, X)$ are 0 or 1 for all $x_0 < x_1$. For all but countable x ’s, the cardinality of $K'_\infty(V, x, X)$ ’s are 1.*

(c) *If $\int_0^t V(X_u - x) du$ is continuous in t and x , then $a(x)$ is right continuous.*

Proof. We first note that for $s < t$, $\int_s^t V(X_u - x) du$ is strictly decreasing in x .

(a) Assume $t_1 < t_0$. We then have $\int_{t_1}^{t_0} V(X_u - x_0) du = 0$ and $\int_{t_1}^{t_0} V(X_u - x_1) du > 0$, which is a contradiction. We can prove $t_1 < t_2$ by the same argument and time-reversal.

(b) If both t_0 and t_1 with $t_0 < t_1$ belong to $K'_\infty(V, x_0, X) \cap K'_\infty(V, x_1, X)$ then we have $\int_{t_0}^{t_1} V(X_u - x_0)du = 0 = \int_{t_0}^{t_1} V(X_u - x_1)du$, which provides a contradiction.

By (a) and the first part of (b), we have for any $x_0 < x_1$, $t_0 \in K'_\infty(V, x_0, X)$ and $t_1 \in K'_\infty(V, x_1, X)$, $t_1 - t_0 \geq \sum_{x \in (x_0, x_1)} \text{diam } K'_\infty(V, x, X)$. Hence at most countably many x 's admit $\text{diam } K'_\infty(V, x, X) > 0$.

(c) For any sequence $t_n \rightarrow t_\infty$ and $x_n \rightarrow x_\infty$ such that $t_n \in K'_\infty(V, x_n, X)$, we prove $t_\infty \in K'_\infty(V, x_\infty, X)$. If s is greater than t_∞ , then eventually $s > t_n$. By the definition of $t_n \in K'_\infty(V, x_n, X)$,

$$0 \leq \int_{t_n}^s V(X_r - x_n)dr \rightarrow \int_{t_\infty}^s V(X_r - x_\infty)dr.$$

If $s < t_\infty$, $\int_s^{t_\infty} V(X_r - x_\infty)dr \leq 0$ by the same argument and this establishes $t_\infty \in K'_\infty(V, x_\infty, X)$.

We have thus proven that $a(x+) \equiv \lim_{\delta \searrow 0} a(x + \delta)$ is in $K'_\infty(V, x, X)$. It follows from (a) that $a(x+)$ dominates every element in $K'_\infty(V, x, X)$ and hence $a(x+) = a(x)$. \square

Lemma 9. *If X is a Lévy process with no positive jumps which satisfies $\lim_{t \rightarrow \infty} X_t = \infty$, then for any $x \geq 0$, the two processes $(X_t - x; 0 \leq t \leq a(x))$ and $(X - x) \circ \theta_{a(x)} \equiv (X_{a(x)+t} - x; t \geq 0)$ are independent. Moreover, the law of the latter process does not depend on x .*

Proof. It can be proved by the same argument in Bertoin [3].

We define $I_s^x = \int_0^s V(X_u - x)du$ and $m_s^x = \inf_{0 \leq t \leq s} I_t^x$. Then $a(x)$ is the last exit time for the process $(X_t - x, I_t^x - m_t^x)$ from the point $(0, 0)$, which is finite almost surely. It can also be proved $X_{a(x)} = x$. This enables us to apply the result by Gettoor on the last exit decomposition as in Bertoin [3]. \square

Proof of Theorem 3(a). To use Lemma 8(c), we first show that $f(x, t) = \int_0^t V(X_u - x)du$ is jointly continuous in t and x . Fix an $\tau > 0$ and $\xi > 0$. The set

$$R(\tau, \xi) = \{X_t - x; 0 \leq t \leq \tau, |x| < \xi\}$$

is bounded and so is its image by $V(\cdot)$. This implies $f(x, t)$ is uniformly continuous in t on the rectangle $\{0 \leq t \leq \tau, |x| < \xi\}$.

Single point sets are not essentially polar for a Lévy process with no positive jump diverging to $+\infty$. There exist local times $L_t(\cdot)$, the sojourn time density, so that

$$f(x, t) = \int_{R(\tau, \xi)} V(y)L_t(y+x)dy$$

for $0 \leq t \leq \tau$ and $|x| < \xi$. See e.g. Bertoin [2]. Let a and x' be two points such that $|x| < \xi$, $|x'| < \xi$. By making x' arbitrarily close to x , the \mathcal{L}^1 -norm of $L_t(y+x') -$

$L_t(y+x)$ with respect to dy can be made arbitrarily small since $L_t(\cdot)$ is integrable. The boundedness of V on $R(\tau, \xi)$ enables us to conclude that $f(x, t)$ is continuous in x . Local uniform continuity in t combined with this implies continuity in two variables.

Hence right continuity of $a(x)$ follows from Lemma 8(c). Let $\tilde{a}(y)$ be the rightmost location of the overall minimum of $\int_0^t V(X_{a(x)+s} - x - y)ds$. By Lemma 8(a), we have $a(x + y) = a(x) + \tilde{a}(y)$ for $x \geq 0$ and $y > 0$. The rest can be done just like the proof of Theorem 1 in Bertoin [3]. \square

Proof of Theorem 3(b). For any $0 \leq x < x_1$, the event $\{\#K'_\infty(V, x_1, X) \geq 2\}$ is independent of $(X_t - x; 0 \leq t \leq a(x))$ because it is the event that $\int_0^s V(X_{a(x)+t} - x_1)dt$ attains its overall minimum at least twice. Hence $P^X[\#K'_\infty(V, x, X) \geq 2]$ is the same value for all $x \geq 0$. If it is positive, then with a positive probability, $\{x \in [0, \infty); \#K'_\infty(V, x, X) \geq 2\}$ has positive mass with respect to the Lebesgue measure. This contradicts Lemma 8(b).

In the case where $x < 0$, we just condition on the event that I_t^x hits 0. We resort to the strong Markov property at the first time $X_t = 0$ after $I_t^x = 0$ and finally use $P^X[\#K'_\infty(V, 0, X) = 1] = 1$. \square

Proof of Theorem 3(c). We follow the argument by Bertoin [3]. Independence is proven in Lemma 9. By (b), $a(0)$ is the unique location of the overall minimum of $\int_0^t V(X_s)ds$. We define a new process \hat{X} by $\hat{X}_t = -X_{g(0)-t-0}$ for $0 \leq t \leq g(0)$, $\hat{X}_t = X_t$ for $t > g(0)$. It is known that \hat{X} and X have the same law. Since V is an odd function,

$$\hat{I}_t = \int_0^t V(\hat{X}_u)du = \int_{g(0)-t}^{g(0)} V(-X_u)du = I_{g(0)-t} - \int_0^{g(0)} V(X_u)du.$$

The unique location of the minimum of \hat{I}_t is $g(0) - a(0)$ and has the same law as that of $a(0)$. \square

Proof of Theorem 3(d). This is proven in the same way as the final part of Theorem 1 in [3]. \square

Now we restate Theorem 2 as the following Lemma. Note that $K'(V, B_1/2, B) \subset (0, 1)$ if $B_1 > 0$ and the following lemma implies $\dim \tilde{K}'(V; V) = 1/2$ a.s. on the event $\{B_1 > 0\}$.

Lemma 10. *Let $a_1(x)$ be the rightmost element in $K'_1(V, x, B)$. It holds $\dim \tilde{K}'(V; V) = 1/2$ a.s. on $\{\exists x, K'_1(V, x, B) \subset (0, 1)\} = \{\exists x, 0 < a_1(x) < 1\}$, and $\tilde{K}'(V; V) \subset \{0, 1\}$ a.s. on $\{\forall x, K'_1(V, x, B) = \{0\} \text{ or } 1 \in K'_1(V, x, B)\} = \{\forall x, a_1(x) = 0 \text{ or } 1\}$.*

Proof. We first note that, by the continuity of $B(t)$, $B(a_1(x)) = x$ if $0 < a_1(x) < 1$ and hence $\tilde{K}'(V; V) \cup \{0, 1\} = \{a_1(x); \#K'_1(V, x, B) = 1\}$. The symmetric difference of $\tilde{K}'(V; V)$ and $\{a_1(x); x \in \mathbb{R}\}$ is at most a countable set, which has no effect on the Hausdorff dimension.

We define the event for $\xi \in \mathbb{R}$, $\eta > 0$, $x \in R$ and $\varepsilon > 0$,

$$E(\xi, \eta, x, \varepsilon) = \{K'_1(V, x, B) \subset (0, 1), B_1 - x - \varepsilon \geq \xi, I_1^{x+\varepsilon} - m_1^{x+\varepsilon} \geq \eta\}.$$

Let \tilde{B} be the process conditioned on $E(\xi, \eta, x, \varepsilon)$. Since $P[E(\xi, \eta, x, \varepsilon)] > 0$, the law of \tilde{B} is absolutely continuous with respect to the law of a standard Brownian motion on $[0, 1]$, and hence to the law of a Brownian motion on $[0, 1]$ with unit drift.

If X is a Brownian motion on $[0, \infty)$ with unit drift independent of B , then

$$P \left[\forall t \geq 0, \eta + \int_0^t V(X_u + \xi) du > 0 \right] > 0.$$

Let \tilde{X} be the conditioned process on this event.

Now we define Z by $Z_t = \tilde{B}_t$ for $t \in [0, 1]$ and $Z_t = \tilde{B}_1 + \tilde{X}_{t-1}$ for $t > 1$. The law of Z is absolutely continuous with respect to the law of a Brownian motion on $[0, \infty)$ with unit drift. For all $x' < x + \varepsilon$, it follows from definition $K'_1(V, x', \tilde{B}) \equiv K'_\infty(V, x', Z) \subset (0, 1)$ and hence $0 < a_1(x') = a(x') < 1$.

By a general theory for subordinators, for every $\varepsilon > 0$, $\dim\{a(x'); x < x' < x + \varepsilon\} = 1/2$ a.s. on the event $\{0 < a(x) < a(x + \varepsilon) < 1\}$. See e.g. Bertoin [2] Theorem III.15. Now we have a.s. on $E(\xi, \eta, x, \varepsilon)$,

$$\frac{1}{2} = \dim\{a(x'); x < x' < x + \varepsilon\} = \dim\{a_1(x'); x < x' < x + \varepsilon\}.$$

Let

$$F(\xi, \eta, x, \varepsilon) := \{a_1(x'); x < x' < x + \varepsilon, E(\xi, \eta, x, \varepsilon) \text{ occurs}\},$$

a random subset which is nonempty only on the event $E(\xi, \eta, x, \varepsilon)$. Since $\tilde{K}'(V; V) \setminus \{0, 1\}$ is the same as a countable union of the random subsets of the form $F(\xi, \eta, x, \varepsilon)$, the dichotomy that $\dim(\tilde{K}'(V; V) \setminus \{0, 1\}) = 1/2$ or $\tilde{K}'(V; V) \setminus \{0, 1\} = \emptyset$ holds.

Finally, if $K'_1(V, x, B) \neq \{0\}$ and $1 \notin K'_1(V, x, B)$ for some x , then there exists an x' such that $K'_1(V, x', B) \subset (0, 1)$ by the continuity of $\int_0^t V(B_s - x') ds$ in x' . \square

4. Proof of Theorem 4

We quote a theorem in [4] and the proof of Theorem 4 is based on it. We fix $\alpha \geq 0$, $c > 0$ and write $V(y)$ for $V(\alpha, c; y)$. Throughout this section we set $\nu = 1/(\alpha + 2)$ and let $\rho \in (0, 1)$ be the unique solution of $c^\nu \sin \pi\nu(1 - \rho) = \sin \pi\nu\rho$ and $\tilde{C}(x, y)$ be

defined for $x \leq 0, y \in \mathbb{R}$ by

$$(4.17) \quad \tilde{C}(x, y) = \Gamma(\nu)^{-1} |x|^{1-\nu+\nu\rho} \exp \left\{ \frac{-2\nu^2(y^+)^{1/\nu}}{|x|} \right\} \\ \times \int_0^\infty dt e^{-t} \left(|x|t + 2\nu^2 c^{-1} |y^-|^{1/\nu} \right)^{\nu\rho} \left(|x|t + 2\nu^2 (y^+)^{1/\nu} \right)^{-1+\nu-\nu\rho}.$$

Now we have

Theorem 11 ([4]). *For $\mu \geq 0, V = V(\alpha, c)$ there exists a constant $C_4(\alpha, c) > 0$ such that it holds*

$$(4.18) \quad \lim_{\sigma \rightarrow 0} \int_0^\infty dt \mu e^{-\mu t} \sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V) = C_4(\alpha, c) \mu^{\rho/2} \tilde{C}(x, y).$$

Proof of Theorem 4. Since the integrand above, $\sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V)$, is decreasing in t ,

$$\limsup_{\sigma \rightarrow 0} \sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V)$$

must be finite for every $t > 0$ and it is trivial that $\sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V) < \sigma^{-\rho} < 1$ for large σ . Hence we know the overall supremum is finite, verifying (2.4), and we denote it by $C_0(t, x, y; V)$, which is clearly monotone decreasing in t and inherits the scaling property from $p(t, x, y, V)$:

$$C_0(t, x, y; V) = \sigma^{-\rho} C_0(t, \sigma^{1/\nu} x, \sigma y; V) = \sigma^{-\rho} C_0(\sigma^{-2} t, x, y; V).$$

It is sufficient to prove (2.6) when $x < 0$ and $y < 0$. We deduce from the scaling property and the monotonicity that

$$C_0(t, x, y; V) = |x|^{\nu\rho} C_0 \left(t, -1, \frac{y}{|x|^\nu}; V \right) \\ \leq |x|^{\nu\rho} C_0 \left(t, (-1) \wedge \frac{-|y|^{-1/\nu}}{|x|}, (-1) \wedge \frac{y}{|x|^\nu}; V \right) \\ = C_0 \left(t, x \wedge (-|y|^{-1/\nu}), (-|x|^\nu) \wedge y; V \right) \\ = (|x|^{\nu\rho} \vee |y|^\rho) C_0(t, -1, -1; V).$$

Combining this with $C_0(t, x, y; V) = t^{-\rho/2} C_0(1, x, y; V)$, we obtain (2.6).

To prove (2.5), we note that the family $\{\sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V); \sigma > 0\}$ of decreasing functions has an upper bound $C_0(t, x, y; V)$, which satisfies

$$\int_0^\infty dt \mu e^{-\mu t} C_0(t, x, y; V) < \text{const} \int_0^\infty dt \mu e^{-\mu t} t^{-\rho/2} < \infty.$$

Given any sequence $\sigma_n \searrow 0$, we can choose a subsequence σ'_n such that the functions $(\sigma'_n)^{-\rho} p(t, (\sigma'_n)^{1/\nu} x, \sigma'_n y; V)$ converge pointwise on a dense subset of $\{t > 0\}$ and that

$$\begin{aligned} & \int_0^\infty dt \mu e^{-\mu t} (\sigma'_n)^{-\rho} p(t, (\sigma'_n)^{1/\nu} x, \sigma'_n y; V) \\ & \rightarrow \int_0^\infty dt \mu e^{-\mu t} \lim_{n \rightarrow \infty} (\sigma'_n)^{-\rho} p(t, (\sigma'_n)^{1/\nu} x, \sigma'_n y; V). \end{aligned}$$

By uniqueness of the Laplace transform, we deduce, for any $t > 0$,

$$\lim_{\sigma \rightarrow 0} \sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V) = \frac{C_4(\alpha, c) \tilde{C}(x, y) t^{-\rho/2}}{\Gamma(1 - \rho/2)}. \quad \square$$

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