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<th>A cluster of sets of exceptional times of linear Brownian motion</th>
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1. Introduction and the main theorems

Aspandiarov-Le Gall [1] studied the following random closed sets $K^-$, $K$ and $K'$: Let $(B_t; t \geq 0)$ be a linear standard Brownian motion starting at 0, and let

$$K^- = \left\{ t \in [0, 1]; \int_s^t (B_u - B_t)du \leq 0 \text{ for every } s \in [0, t) \right\},$$

$$K = \left\{ t \in K^-; \int_t^s (B_u - B_t)du \leq 0 \text{ for every } s \in (t, 1] \right\},$$

$$K' = \left\{ t \in K^-; \int_t^s (B_u - B_t)du \geq 0 \text{ for every } s \in (t, 1], \right\}.$$

They computed the Hausdorff dimension of $K^-$, $K$ and $K'$.

**Theorem** ([1]). It holds $\dim K^- = 3/4$, $\dim K = 1/2$ and $\dim K' \leq 1/2$ almost surely. The set $K'$ is possibly empty or $\dim K' = 1/2$, both with positive probability. The same statements hold if the weak inequalities in the definition of $K^-$, $K$ and $K'$ are replaced by the strict inequalities.

In this paper, we consider a cluster of random sets having various dimension.

For $\alpha \geq 0$ and $c > 0$, we define the following functions $V(\alpha, c)$ increasing on $\mathbb{R}$:

$$V(\alpha, c; y) = y^\alpha \text{ for } y > 0; V(\alpha, c; 0) = 0; V(\alpha, c; y) = -\frac{|y|^\alpha}{c} \text{ for } y < 0.$$

Let $\alpha, \alpha_+, \alpha_- \geq 0$, $c, c_+, c_- > 0$ and write $V$ for $V(\alpha, c)$, $V_+$ for $V(\alpha_+, c_+)$, and $V_-$ for $V(\alpha_-, c_-)$. We define the random sets depending on the functions $V$, $V_+$ and $V_-:

\begin{align*}
(1.1) \quad K^- (V) &= \left\{ t \in [0, 1]; \int_s^t V(B_u - B_t)du \leq 0 \text{ for every } s \in [0, t) \right\}, \\
(1.2) \quad K(V_-, V_+) &= \left\{ t \in K^- (V_-); \int_t^s V_+(B_u - B_t)du \leq 0 \text{ for every } s \in (t, 1] \right\}.
\end{align*}
These sets consist of exceptional times in the sense that \( P[t \in K^-(V)] = 0 \) for every \( t \in (0, 1) \) and \( P[t \in K'(V_\ldots; V_\ldots)] = P[t \in K'(V_-; V_\ldots)] = 0 \) for every \( t \in [0, 1] \).

**Theorem 1.** We define \( \nu = 1/(2 + \alpha), \nu_- = 1/(2 + \alpha_-) \) and \( \nu_+ = 1/(2 + \alpha_+) \).

Let \( \rho, \rho_-, \rho_+ \in (0, 1) \) be the unique solutions of the equations

\[
\begin{align*}
C^\nu \sin \pi \nu (1 - \rho) &= \sin \pi \nu \rho_+, \\
C_-^\nu \sin \pi \nu_-(1 - \rho_-) &= \sin \pi \nu_- \rho_-,
\end{align*}
\]

respectively.

(a) For \( V = V(\alpha, \ldots) \), we have almost surely \( \dim K^-(V) = 1 - \rho/2 \).

For \( V_\ldots = V(\alpha_+, \ldots) \) and \( V_- = V(\alpha_-, \ldots) \) we have

(b) \( \dim K(V_-; V_\ldots) \leq 1 - (\rho_- + \rho_+)/2 \) almost surely and

\[
P \left[ \dim K(V_-; V_\ldots) \geq 1 - \frac{\rho_- + \rho_+}{2} \right] > 0.
\]

(c) \( \dim K'(V_-; V_\ldots) \leq (1 - \rho_- + \rho_+)/2 \) almost surely and

\[
P \left[ \dim K'(V_-; V_\ldots) \geq \frac{1 - \rho_- + \rho_+}{2} \right] > 0.
\]

The behavior of \( V, V_\ldots \) and \( V_- \) outside any neighborhood of the origin have no influence on the Hausdorff dimension; We could state the theorem in that fashion. The parameters \( \rho, \rho_-, \rho_+ \in (0, 1) \) in the statement of Theorem 1 are continuous and increasing in \( \ldots, \ldots, \ldots \) and have the range \( (0, 1) \) since \( \lim_{\ldots \to 0} \rho = 0 \) and \( \lim_{\ldots \to \infty} \rho = 1 \). In fact, they are equal to the probability of some event related to the parameters in the theorem, see the remark 4 in [4].

Note that for fixed \( \alpha \), it holds \( \rho = 1/2 \) if \( \ldots = 1 \). Hence the statements in the theorem in [1] for \( K^- \) and \( K^\prime \) can be included in Theorem 1 since \( K^- = K^-(V(1, 1)) \) and \( K' = K'(V(1, 1); V(1, 1)) \). The implication by Theorem 1 on \( K \), however, is weaker than [1], since we have not obtained the almost sure estimate from below.

Let \( \alpha, \ldots \geq 0 \) and \( \ldots > 0 \). If \( V = V(\alpha, \ldots) \) and \( \ldots = V(\ldots, \ldots) \), then there is no inclusion in general between \( K^-(V) \) and \( K^-(\ldots) \). However it is easy to see, for each \( \alpha \), that \( K^-(V(\alpha, \ldots)) \subset K^-(V(\alpha, \ldots)) \) if \( \ldots < \ldots \). Hence we obtain a family

\[\{K^-(V(\alpha, \ldots); \ldots \in (0, 1)\}\]

of decreasing random sets having strictly decreasing dimension.
The estimate in Theorem 1 for \( \dim K^-(V) \) is exhaustive in the following sense: Let \( H \) be the set of times \( t \) when \( B_t \) attains its past-maximum:

\[
H := \left\{ t \in [0, 1]; B_t = \sup_{0 \leq s \leq t} B_s \right\}.
\]

It is well known that \( \dim H = 1/2 \) a.s. Since \( H \subset K^-(V(\alpha, c)) \subset [0, 1] \), we have \( 1/2 \leq \dim K^-(V) \leq 1 \). The range of \( 1 - \rho/2 \) is exactly \( (1/2, 1) \) and the trivial case \( K^-(V) = H \) or \( K^-(V) = [0, 1] \) could be included if we allow \( c = \infty \) or \( c = 0 \).

The estimate in Theorem 1 for \( \dim K(V_-; V_+) \) is also exhaustive in the following sense: Let \( \tau \) be the time when the maximum on \( [0, 1] \) of \( B \) is attained: \( B_{\tau} \geq B_t \) for every \( t \in [0, 1] \). The inclusion \( \{\tau\} \subset K(V_-; V_+) \subset [0, 1] \) implies \( 0 \leq \dim K(V_-; V_+) \leq 1 \) and the range of the value \( 1 - (\rho_- + \rho_+)/2 \) is exactly \( (0, 1) \). The extreme cases could also be included here.

In the same sense as Aspandiiarov and Le Gall [1] noted concerning \( K', K'(V_-; V_+) \) can be interpreted as a weakened notion of the increasing points of Brownian motion and it is not straightforward to exhibit an element of \( K'(V_-; V_+) \).

If both \( V_- \) and \( V_+ \) are \( V(\alpha, c) \) then \( (1 - \rho_- + \rho_+)/2 = 1/2 \) irrespective of \( \alpha \) and \( c \). This motivates the next theorem, which could be a version of settlement of a conjecture at the end of [1]: \( \dim K' = 1/2 \) a.s. on the event \( \{B_1 > 0\} \).

**Theorem 2.** Let \( \mathcal{V} = \{V : \mathbb{R} \to \mathbb{R}; V(0) = 0, \ V \text{ is strictly increasing}\} \).

We define \( \tilde{K}'(V; V) \) for \( V \in \mathcal{V} \) in the same way as (1.3) replacing the weak inequalities by strict inequalities in the definition of \( K'(V; V) \):

\[
\tilde{K}'(V; V) = \left\{ t \in [0, 1]; \int_s^t V(B_u - B_t) \, du < 0 \quad \text{for every } s \in [0, t), \right. \\
\left. \quad \text{and} \int_t^s V(B_u - B_t) \, du > 0 \quad \text{for every } s \in (t, 1]. \right\}
\]

Then we have \( P[\dim \tilde{K}'(V; V) = 1/2] > 0 \), \( P[\tilde{K}'(V; V) \subset \{0, 1\}] > 0 \) and

\[
P \left[ \dim \tilde{K}'(V; V) = \frac{1}{2} \quad \text{or} \quad \tilde{K}'(V; V) \subset \{0, 1\} \right] = 1.
\]

**Remark 1.** When the set \( \tilde{K}'(V; V) \) consists of exceptional times, we have the dichotomy that \( \dim \tilde{K}'(V; V) = 1/2 \) if it is not empty.

The result of Theorem 2 is stronger than Theorem 1(c) for each strictly increasing functions \( V(\alpha, c) \), i.e. \( \alpha > 0 \), while Theorem 2 says nothing about \( V(0, c) \).

Theorem 2 is in fact a corollary of the following Theorem 3 due essentially to Bertoin [3].
Let \( V \in \mathcal{V}, x \in \mathbb{R} \) and \( X = (X(t); t \geq 0) \) be a cadlag path with \( \lim \inf_{t \to -\infty} X(t) = +\infty \). We define, inspired by Bertoin [3],

\[
K'_\infty(V, x, X) = \left\{ t \in [0, \infty); \int_s^t V(X_u - x)du \leq 0 \text{ for every } s \in [0, t), \right. \\
\left. \quad \text{and } \int_t^\infty V(X_u - x)du \geq 0 \text{ for every } s \in (t, \infty) \right\},
\]

\[
K'_1(V, x, X) = \left\{ t \in [0, 1]; \int_s^t V(X_u - x)du \leq 0 \text{ for every } s \in [0, t), \right.
\]
\[
\left. \quad \text{and } \int_t^1 V(X_u - x)du \geq 0 \text{ for every } s \in (t, 1]. \right\}
\]

It is then easy to see \( \tilde{K}'(V; V) \cup \{0, 1\} = \cup_{K'_\infty(V, x, x, B) \in K'_1(V, x, x, B)} \).

In other words, \( K'_\infty(V, x, X) \) and \( K'_1(V, x, X) \) consist of the locations of the overall minimum of the function \( s \mapsto \int_0^s V(X_u - x)du \) on \([0, \infty)\) or \([0, 1]\) respectively and \( \tilde{K}'(V; V) \) is the collection of such \( t \)'s that the function \( s \mapsto \int_0^s V(B_u - B_1)du \) has the unique minimum at \( s = t \).

The following results are proven in Bertoin [3] in the case where \( V(y) \equiv y = V(1, 1; y) \).

**Theorem 3.** Let \( V \in \mathcal{V} \) and \( X \) be a Lévy process with no positive jump such that \( \lim \inf_{t \to -\infty} X(t) = +\infty \) a.s. Let \( a(x) \) be the rightmost element of \( K'_\infty(V, x, X) \).

(a) \( \{a(x) - a(0); x \geq 0\} \) and the process \( T^X(x) := \inf\{t \geq 0; X_t \geq x\} \) have the same law.

(b) For every fixed \( x \in \mathbb{R} \), \( P^Y[X'_\infty(V, x, X) = 1] = 1 \).

(c) Let \( g(0) = \sup\{t \geq 0; X(t) \leq 0\} \) be the last exit time from \( (-\infty, 0] \). If \( V \in \mathcal{V} \) satisfies \( V(y) = -V(-y) \), then \( a(0) \) and \( g(0) - a(0) \) are independent and have the same law.

(d) If \( X \) is a Brownian motion with unit drift, then \( \{a(x) - a(0); x \geq 0\} \) has the Lévy measure \( (2\pi)^{-1/2}y^{-3/2}e^{-y/2}dy \) on \((0, \infty)\). If, moreover, \( V \in \mathcal{V} \) satisfies \( V(y) = -V(-y) \), then the density of the common law of \( a(0) \) and \( g(0) - a(0) \) is \( 2^{-1/4}\Gamma(1/4)^{-1}y^{-3/4}e^{-y/2}dy \) on \((0, \infty)\).

**Remark 2.** The statement (a) and the first sentence in (d) hold for nondecreasing \( V \) satisfying \( V(0) = 0 \). The second sentence in (d) was known to Jean Bertoin(private communication).

This paper is organized as follows: We prove Theorem 1 in Section 2 using Theorem 4, which contains an asymptotic estimate for some fluctuating additive functionals. Theorems 2 and 3 are proven in Section 3. We prove Theorem 4 in Section 4 using a theorem in [4].
\textbf{Acknowledgement.} The author would express his gratitude to Professor Shin’ichi Kotani for his careful reading of a draft of this paper and for his helpful comment. Thanks also goes to the anonymous referee for advice on improving the presentation.

2. Proof of Theorem 1

The argument here mimics that of Aspandiiarov and Le Gall [1] line by line. We first state Theorem 4, an estimate for the distribution of the first hitting time of $\int_0^1 V(B_u)du$, next define suitable approximations of $K^-(V)$, $K(V_-; V_+)$ and $K'(V_-; V_+)$ and obtain some preliminary estimates. From that point on, we only need the straightforward changes.

\textbf{Theorem 4.} Let $\alpha \geq 0$, $c > 0$, $V = V(\alpha, c)$, $\nu = 1/(2 + \alpha)$ and $\rho \in (0, 1)$ be the solution of $c' \sin \pi \nu (1 - \rho) = \sin \pi \nu \rho$. We denote by $P(t, x, y; V)$ the probability $P[\exists s \in [0, t], x + \int_0^s V(y + B_u)du \leq 0]$.

For any $t > 0$, $x < 0$, $y \in \mathbb{R}$ and there exist constants $C_0(t, x, y; V) > 0$, $C_1(\alpha, c) > 0$ and $\tilde{C}(x, y) > 0$ such that it holds

\begin{equation}
\sup_{\sigma > 0} \sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V) = C_0(t, x, y; V),
\end{equation}

\begin{equation}
\lim_{\sigma \to 0} \sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V) = C_1(\alpha, c)t^{-\rho/2}\tilde{C}(x, y),
\end{equation}

Moreover it holds

\begin{equation}
C_0(t, x, y; V) \leq \text{const } t^{-\rho/2(|x|^{\nu/\rho} \vee |y|^{-\rho})},
\end{equation}

\textbf{Definition.} Let $\varepsilon \in [0, 1/2]$, $a \in [0, 1 - \varepsilon]$ and $b \in [\varepsilon, 1]$. For $V$, $V_+$, $V_- \in \cup_{\alpha \geq 0, c > 0} \{V(\alpha, c)\}$ we define

\begin{align*}
K_{\varepsilon, a}(V) &= \left\{ t \in [a + \varepsilon, 1]; \int_s^t V(B_u - B_t)du \leq 0 \text{ for every } s \in [a, t - \varepsilon] \right\}, \\
K_{\varepsilon, a}(V) &= \left\{ t \in [0, b - \varepsilon]; \int_t^s V(B_u - B_t)du \leq 0 \text{ for every } s \in [t + \varepsilon, b] \right\}, \\
K_{\varepsilon, b}(V) &= \left\{ t \in [0, b - \varepsilon]; \int_t^s V(B_u - B_t)du \geq 0 \text{ for every } s \in [t + \varepsilon, b] \right\}, \\
K_{\varepsilon, a,b}(V_-; V_+) &= K_{\varepsilon, a}(V_-) \cap K_{\varepsilon, b}(V_+), \\
K_{\varepsilon, a,b}'(V_-; V_+) &= K_{\varepsilon, a}(V_-) \cap K_{\varepsilon, b}(V_+),
\end{align*}

We also define

\begin{align*}
K_{\varepsilon}^{-}(V) &= K_{\varepsilon, 0}^{-}(V), & K^{-}(V) &= K_{0}^{-}(V), \\
K_{\varepsilon}(V_-; V_+) &= K_{\varepsilon, 0, 1}(V_-; V_+), & K(V_-; V_+) &= K_{0}(V_-; V_+),
\end{align*}
Lemma 5. Let \(\alpha, \alpha_+, \alpha_- \geq 0\), \(c, c_+, c_- > 0\) and let \(\rho, \rho_+, \rho_-\) be defined in the statement of Theorem 1.
(a) For any \(V = V(\alpha, c), 0 < \varepsilon < 1/2\) and \(t > a\), it holds
\[
\left(\frac{t - a}{\varepsilon}\right)^{\rho/2} P[t \in K_{\varepsilon a}(V)] < \text{const}.
\]
There exists a constant \(C_3(V) > 0\) such that it holds
\[
P[t \in K_{\varepsilon a}(V)] \sim C_3(V) \left(\frac{\varepsilon}{t - a}\right)^{\rho/2}
\]
as \(\varepsilon \searrow 0\) for every \(t\).
(b) For any \(V_+ = V(\alpha_+, c_+), V_- = V(\alpha_-, c_-), 0 < \varepsilon < 1/2\) and \(t \in (a, b)\),
\[
\left(\frac{t - a}{\varepsilon}\right)^{\rho_-/2} \left(\frac{b - t}{\varepsilon}\right)^{\rho_+/2} P[t \in K_{\varepsilon a b}(V_-; V_+)] < \text{const},
\]
\[
\left(\frac{t - a}{\varepsilon}\right)^{\rho_-/2} \left(\frac{b - t}{\varepsilon}\right)^{(1 - \rho_+)/2} P[t \in K_{\varepsilon a b}'(V_-; V_+)] < \text{const}.
\]
We denote by \(V_+(- \cdot)\) the function \(y \mapsto V_+(-y)\). It holds as \(\varepsilon \searrow 0\)
\[
P[t \in K_{\varepsilon a b}(V_-; V_+)] \sim C_3(V_-) C_3(V_+) \left(\frac{\varepsilon}{t - a}\right)^{\rho_-/2} \left(\frac{\varepsilon}{b - t}\right)^{\rho_+/2},
\]
\[
P[t \in K_{\varepsilon a b}'(V_-; V_+)] \sim C_3(V_-) C_3(V_+(- \cdot)) \left(\frac{\varepsilon}{t - a}\right)^{\rho_-/2} \left(\frac{\varepsilon}{b - t}\right)^{(1 - \rho_+)/2}
\]
Proof. We only prove (a) since the statement (b) follows by time-reversal \(\tilde{B}_s = B_{t-s}\) and by reflection \(\tilde{B}_s = -B_s\).
Let \(P_{(x,y)}^V\) be the law of the following two-dimensional diffusion \((X(t), Y(t)):\)
\[
Y(t) = y + B(t), \quad X(t) = x + \int_0^t V(Y(s)) ds.
\]
By the strong Markov property,
\[
P[t \in K_{\varepsilon a}(V)] = \mathbb{E}_{(0,0)}^V[p(t - a - \varepsilon, X(\varepsilon), Y(\varepsilon); V)]
\]
Under \(P_{(0,0)}^V\), the law of \((X(\varepsilon), Y(\varepsilon))\) is the same as that of \((\varepsilon^{1/2} X(1), \varepsilon^{1/2} Y(1))\). By (2.4) and (2.6), we have for any \(\varepsilon > 0\),...
\[ \varepsilon^{-\rho/2} P(t - a - \varepsilon, \varepsilon^{1/2} \nu \cdot X(1), \varepsilon^{1/2} \nu \cdot Y(1); V) \]
\[ < \text{const}(t - a - \varepsilon)^{-\rho/2} \left( |X(1)|^{-\rho} \lor |Y(1)|^{-\rho} \right) \]
\[ < \text{const}(t - a)^{-\rho/2} \left( |X(1)|^{-\rho} \lor |Y(1)|^{-\rho} \right). \]

The quantity \((t - a)/\varepsilon)^{\rho/2} P(t \in K_{\varepsilon a}^{-}(V))\) is hence bounded. This bound also enables us to prove the second sentence of (a) with \(C_{3}(V) = C_{1}(\alpha, c) E_{0}(\nu_{0}) \mu(X(1), Y(1)).\)

**Lemma 6.** We use the same notations as the previous lemma. It holds for any \(\varepsilon \in (0, 1/2)\) and \(0 < s < t < 1,\)

\[ P\{s, t \in K_{\varepsilon a}^{-}(V)\} \leq \frac{\varepsilon^{\rho}}{s^{\rho/2}(t - s)^{\rho/2}}, \]

\[ P\{s, t \in K_{\varepsilon a,b}^{-}(V; V_{+})\} \leq \frac{\text{const} \varepsilon^{\rho - \rho_{a,c}}}{s^{\rho/2}(t - s)^{(\rho_{a,c} + 1)(1 - t)^{\rho_{a,c}/2}}}, \]

\[ P\{s, t \in K_{\varepsilon a,b}^{-}(V_{-}; V_{+})\} \leq \frac{\text{const} \varepsilon^{\rho - \rho_{a,c}}}{s^{\rho/2}(t - s)^{(\rho_{a,c} + 1)(1 - t)^{\rho_{a,c}/2}}}. \]

The constants here depend on \(\alpha, \alpha_{+}, \alpha_{-}\) and \(c, c_{+}, c_{-}.\)

Proof. This can be done using Lemma 5. See the proof of Proposition 4 in [1].

**Lemma 7.** Let \(F_{a,b}\) be the \(\sigma\)-field \(\sigma(B_{t} - B_{t}; u \in [a, b])\) for \(0 \leq a < b \leq 1.\)

For any \(\alpha \geq 0, c > 0\) and \(V = V(\alpha, c)\) there exist \(F_{a,b}\)-measurable random variables \(U_{a,b,\varepsilon}, U_{a,b,\varepsilon}^{+}\) and \(U_{a,b,\varepsilon}^{-}\) such that

\[ P\{K^{-}(V) \cap [a, b] \neq \emptyset | F_{a,1}\} \leq (b - a)^{\rho/2} U_{a,b,\varepsilon}^{-}, \]

\[ P\{K^{+}(V) \cap [a, b] \neq \emptyset | F_{b,0}\} \leq (b - a)^{\rho/2} U_{a,b,\varepsilon}^{+}, \]

\[ P\{K^{+}(V) \cap [a, b] \neq \emptyset | F_{b,0}\} \leq (b - a)^{(1 - \rho)/2} U_{a,b,\varepsilon}^{-}, \]

and \(E_{0}[(U_{a,b,\varepsilon}^{+})^{2}] \leq \text{const} a^{-\rho}, E_{0}[(U_{a,b,\varepsilon}^{-})^{2}] \leq \text{const}(1 - b)^{-\rho}, E_{0}[(U_{a,b,\varepsilon}^{-})^{2}] \leq \text{const}(1 - b)^{-1\rho}.\) The constants here depend on \(\alpha\) and \(c.\)

Proof. We prove (2.14) since (2.13), (2.15) and the corresponding moment estimates follow by time-reversal \(\tilde{B}_{t} = B_{1-t}\) and by reflection \(\tilde{B}_{t} = -B_{t}.\)

Let \(\eta_{a,b}\) be the amplitude of \(\tilde{B}_{t}\) on \([a, b].\) Note that \(V\) is increasing. By modifying the argument in the proof of Lemma 7 in [1], we can take

\[ U_{a,b,\varepsilon}^{+} = (b - a)^{-\rho/2} (1 - b, (b - a)V(-\eta_{a,b}, -\eta_{a,b}; V). \]

The bound of the moment follows by (2.6) and by the fact that \(\eta_{a,b}\) has the same law as \((b - a)^{1/2} \eta_{0,1}.\)
Proof of Theorem 1. The upper estimates for the Hausdorff dimension is obtained by the argument in the proof Proposition 6 in [1].

To obtain the lower estimates, we define the normalized Lebesgue measures: For any Borel subset $F$ of $[0, 1]$, let

\[
\mu_e^-(F) = \varepsilon^{-\rho/2}|F \cap K_e^-(V)|,
\]

\[
\mu_e(F) = \varepsilon^{-(\rho + \rho_1)/2}|F \cap K_e(V^-; V^+)|,
\]

\[
\mu'_e(F) = \varepsilon^{-(\rho - 1 - \rho_1)/2}|F \cap K'_e(V^-; V^+)|.
\]

We denote by $\mathcal{M}_f$ the Polish space of all finite measures on $[0, 1]$ equipped with the topology of weak convergence, and by $C([0, 1])$ the Banach space of all continuous map from $[0, 1]$ to $\mathbb{R}$.

Let $(\varepsilon_n)$ be a sequence strictly decreasing to 0. We define the random variables $\zeta^n$ taking values in $\mathcal{M}_f \times C([0, 1])$ by $\zeta^n = (\mu_{\varepsilon_n}, (B_t: 0 \leq t \leq 1))$. We define $\zeta^{-n}$ and $\zeta'^n$ in the same way using $\mu_{\varepsilon_n}$ and $\mu'_{\varepsilon_n}$. The argument in [1] ensures that we may assume the sequence $(\zeta^n)$ is weakly convergent by extracting a subsequence. Skorohod's representation theorem says that there is a probability space carrying a sequence of random variables $\overline{\zeta^n} = (\mu_{\varepsilon_n}, (B_t^\infty; 0 \leq t \leq 1))$ and a random variable $\overline{\zeta^\infty} = (\mu^\infty, (B_t^\infty; 0 \leq t \leq 1))$ such that $\overline{\zeta^n}$ and $\overline{\zeta^\infty}$ have the same law and $\overline{\zeta^n}$ converges to $\overline{\zeta^\infty}$ almost surely.

Let $K(V_-, V_+; B^\infty)$ be defined in the same way as $K(V_-, V_+)$ replacing $B$ by $B^\infty$. To prove that $\mu^\infty$ is a.s. supported on $K(V_-, V_+; B^\infty)$, we change the definition of $G(\eta, \gamma)$ appearing in the proof of Lemma 9 in [1].

\[
G(\eta, \gamma) = \left\{ t < 1 - \eta; \sup_{t + \eta < s \leq 1} \int_t^s V_s(B_u^\infty - B_t^\infty) du > \gamma \right\}.
\]

Since $V_+$ has no discontinuities of the second kind, it is locally bounded and hence we can deduce, from the occupation time formula, that $G(\eta, \gamma)$ is an open set.

On the other hand, $\mu^\infty$ is a.s. supported on

\[
\left\{ t \leq 1 - \varepsilon_n; \sup_{t + \varepsilon_n < s \leq 1} \int_t^s V_s(B_u^\infty - B_t^\infty) du \leq 0 \right\}.
\]

To deduce that $\mu^\infty(G(\eta, \gamma)) = 0$ and $\mu^\infty$ is a.s. supported on $K(V_-, V_+; B^\infty)$ by the argument in the proof of Lemma 9 in [1], we need only to prove the following:

\[\tag{2.16} \text{For fixed } s \text{ and } t, \int_t^s V_s(B_u^\infty - B_t^\infty) du \to \int_t^s V_s(B_u^\infty - B_t^\infty) du \quad \text{as } n \to \infty.\]

To prove (2.16), let $\varepsilon, \varepsilon'$ be arbitrary positive numbers and let

\[R^\infty(\varepsilon', s) := \{ x \in \mathbb{R}; \exists u < s, |x - B_u^\infty| < 2\varepsilon' \}.\]
Since $V_\varepsilon$ has discontinuity only at the origin (when $\alpha = 0$), there exists $0 < \delta < \varepsilon'$ such that for any $x, y \in \mathbb{R}^{\infty}(\varepsilon', s)$ satisfying $|x - y| < \delta$ and $|x| > \varepsilon'$, it holds $|V_\varepsilon(x) - V_\varepsilon(y)| < \varepsilon$.

We can make \( \int_0^s 1 \{|B^n_u - B^n_v| \leq 3\varepsilon'\}du \) arbitrarily small by taking $\varepsilon'$ small, and hence \( \int_0^s V_\varepsilon(B^n_u - B^n_v)1 \{|B^n_u - B^n_v| \leq 3\varepsilon'\}du \) is also small if $\|B^n - B^\infty\| < \varepsilon'$, since $V_\varepsilon$ is bounded on $\mathbb{R}^{\infty}(\varepsilon', s)$.

For $u \in [t, s]$ satisfying $|B^n_u - B^n_v| > \varepsilon'$, we have $|V_\varepsilon(B^n_u - B^n_v) - V_\varepsilon(B^n_v - B^n_u)| < \varepsilon$ if $\|B^n - B^\infty\| < \delta/2$, which is satisfied for all large $n$.

We have thus proven (2.16).

Using Lemma 5 and the weak convergence we have

\[
E[\mu^{-\infty}(0, 1)] = \int_0^1 dt t^{-\rho/2}C_3(V) > 0,
\]

\[
E[\mu^{\infty}(0, 1)] = \int_0^1 dt t^{-\rho/2}C_3(V_-)(1 - t)^{-\rho/2}C_3(V_+). > 0,
\]

\[
E[\mu^{\infty}(0, 1)] = \int_0^1 dt t^{-\rho/2}C_3(V_-)(1 - t)^{-2(1 - \rho)/2}C_3(V_-(\cdot)). > 0.
\]

The positivity of these values is, through Frostman’s lemma, related to the positivity of $P(\dim K^-(V) \leq 1 - \rho/2)$ and its companions; The a.s. estimate from below follows by the scaling property of Brownian motion as in [1].

\[\square\]

### 3. Proof of Theorems 3 and 2

In this section, $V$ is an strictly increasing function with $V(0) = 0$ and $a(x)$ is the rightmost element in $K'_{\infty}(V, x, X)$.

**Lemma 8.** (a) If $x_0 < x_1$ and there exists a triple $(t_0, t_1, t_2)$ such that

\[
t_0 \in K'_{\infty}(V, x_0, X) \setminus K'_{\infty}(V, x_1, X),
\]

\[
t_1 \in K'_{\infty}(V, x_0, X) \cap K'_{\infty}(V, x_1, X),
\]

\[
t_2 \in K'_{\infty}(V, x_1, X) \setminus K'_{\infty}(V, x_0, X),
\]

then it holds $t_0 < t_1 < t_2$.

(b) The cardinality of $K'_{\infty}(V, x_0, X) \cap K'_{\infty}(V, x_1, X)$ are 0 or 1 for all $x_0 < x_1$. For all but countable $x$‘s, the cardinality of $K'_{\infty}(V, x, X)$‘s are 1.

(c) If \( \int_0^t V(X_u - x)du \) is continuous in $t$ and $x$, then $a(x)$ is right continuous.

**Proof.** We first note that for $s < t$, \( \int_s^t V(X_u - x)du \) is strictly decreasing in $x$.

(a) Assume $t_1 < t_0$. We then have \( \int_{t_1}^t V(X_u - x_0)du = 0 \) and \( \int_{t_1}^t V(X_u - x_1)du > 0 \), which is a contradiction. We can prove $t_1 < t_2$ by the same argument and time-reversal.
(b) If both \( t_0 \) and \( t_1 \) with \( t_0 < t_1 \) belong to \( K'_\infty(V, x_0, X) \cap K'_\infty(V, x_1, X) \) then we have \( \int_{t_0}^{t_1} V(X_u - x_0)du = 0 = \int_{t_0}^{t_1} V(X_u - x_1)du \), which provides a contradiction.

By (a) and the first part of (b), we have for any \( x_0 < x_1 \), \( t_0 \in K'_\infty(V, x_0, X) \) and \( t_1 \in K'_\infty(V, x_1, X) \), \( t_1 - t_0 \geq \sum_{x \in \{x_0, x_1\}} \text{diam} K'_\infty(V, x, X) \). Hence at most countably many \( x \)'s admit diam \( K'_\infty(V, x, X) > 0 \).

(c) For any sequence \( t_n \to t_\infty \) and \( x_n \to x_\infty \) such that \( t_n \in K'_\infty(V, x_n, X) \), we prove \( t_\infty \in K'_\infty(V, x_\infty, X) \). If \( s \) is greater than \( t_\infty \), then eventually \( s > t_n \). By the definition of \( t_n \),

\[
0 \leq \int_{t_n}^{s} V(X_r - x_n)dr \to \int_{t_\infty}^{s} V(X_r - x_\infty)dr.
\]

If \( s < t_\infty \), \( \int_{t_\infty}^{s} V(X_r - x_\infty)dr \leq 0 \) by the same argument and this establishes \( t_\infty \in K'_\infty(V, x_\infty, X) \).

We have thus proven that \( a(x+) \equiv \lim_{\delta \to 0} a(x + \delta) \) is in \( K'_\infty(V, x, X) \). It follows from (a) that \( a(x+) \) dominates every element in \( K'_\infty(V, x, X) \) and hence \( a(x+) = a(x) \).

\[ \square \]

**Lemma 9.** If \( X \) is a Lévy process with no positive jumps which satisfies \( \lim_{t \to \infty} X_t = \infty \), then for any \( x \geq 0 \), the two processes \( (X_t - x; 0 \leq t \leq a(x)) \) and \( (X - x) \circ \theta_{a(x)} \equiv (X_{a(x),t} - x; t \geq 0) \) are independent. Moreover, the law of the latter process does not depend on \( x \).

**Proof.** It can be proved by the same argument in Bertoin [3].

We define \( I_t^x = \int_0^t V(X_u - x)du \) and \( m_t^x = \inf_{0 \leq t \leq s} I_t^x \). Then \( a(x) \) is the last exit time for the process \( (X_t - x, I_t^x - m_t^x) \) from the point \((0, 0)\), which is finite almost surely. It can also be proved \( X_{a(x)} = x \). This enables us to apply the result by Getoor on the last exit decomposition as in Bertoin [3].

\[ \square \]

**Proof of Theorem 3(a).** To use Lemma 8(c), we first show that \( f(x, t) = \int_0^t V(X_u - x)du \) is jointly continuous in \( t \) and \( x \). Fix an \( \tau > 0 \) and \( \xi > 0 \). The set

\[
R(\tau, \xi) = \{X_t - x; 0 \leq t \leq \tau, |x| < \xi\}
\]

is bounded and so is its image by \( V(\cdot) \). This implies \( f(x, t) \) is uniformly continuous in \( t \) on the rectangle \( \{0 \leq t \leq \tau, |x| < \xi\} \).

Single point sets are not essentially polar for a Lévy process with no positive jump diverging to \( +\infty \). There exist local times \( L_t(\cdot) \), the sojourn time density, so that

\[
f(x, t) = \int_{R(\tau, \xi)} V(y)L_t(y + x)dy
\]

for \( 0 \leq t \leq \tau \) and \( |x| < \xi \). See e.g. Bertoin [2]. Let \( a \) and \( x' \) be two points such that \( |x| < \xi, |x'| < \xi \). By making \( x' \) arbitrarily close to \( x \), the \( L^1 \)-norm of \( L_t(y + x') - L_t(y) \) is

\[ \square \]
with respect to $dy$ can be made arbitrarily small since $L_x(\cdot)$ is integrable. The boundedness of $V$ on $R(\tau, \xi)$ enables us to conclude that $f(x, t)$ is continuous in $x$.

Local uniform continuity in $t$ combined with this implies continuity in two variables.

Hence right continuity of $a(x)$ follows from Lemma 8(c). Let $\tilde{a}(y)$ be the rightmost location of the overall minimum of $\int_0^t V(X_{a(x)} + y - x)ds$. By Lemma 8(a), we have $a(x + y) = a(x) + \tilde{a}(y)$ for $x \geq 0$ and $y > 0$. The rest can be done just like the proof of Theorem 1 in Bertoin [3].

Proof of Theorem 3(b). For any $0 \leq x < x_1$, the event $\{t K'_\infty(V, x_1, X) \geq 2\}$ is independent of $(X_t - x; 0 \leq t \leq a(x))$ because it is the event that $\int_0^t V(X_{a(x)} + y - x)dt$ attains its overall minimum at least twice. Hence $P^X[\exists K'_\infty(V, x, X) \geq 2]$ is the same value for all $x \geq 0$. If it is positive, then with a positive probability, $\{x \in [0, \infty); \exists K'_\infty(V, x, X) \geq 2\}$ has positive mass with respect to the Lebesgue measure.

This contradicts Lemma 8(b).

In the case where $x < 0$, we just condition on the event that $I_1^x$ hits 0. We resort to the strong Markov property at the first time $X_t = 0$ after $I_1^x = 0$ and finally use $P^X[\exists K'_\infty(V, 0, X) = 1] = 1$.

Proof of Theorem 3(c). We follow the argument by Bertoin [3]. Independence is proven in Lemma 9. By (b), $a(0)$ is the unique location of the overall minimum of $\int_0^t V(X_u)du$. We define a new process $\tilde{X}$ by $\tilde{X}_t = -X_{g(0) - t - 0}$ for $0 \leq t \leq g(0)$, $\tilde{X}_t = X_t$ for $t > g(0)$. It is known that $\tilde{X}$ and $X$ have the same law. Since $V$ is an odd function,

$$\tilde{\theta}_t = \int_0^t V(\tilde{X}_u)du = \int_{g(0) - t}^{g(0)} V(-X_u)du = I_{g(0) - t} - \int_0^{g(0)} V(X_u)du.$$

The unique location of the minimum of $\tilde{\theta}_t$ is $g(0) - a(0)$ and has the same law as that of $a(0)$.

Proof of Theorem 3(d). This is proven in the same way as the final part of Theorem 1 in [3].

Now we restate Theorem 2 as the following Lemma. Note that $K'(V, B_1(2, B)) \subset (0, 1)$ if $B_1 > 0$ and the following lemma implies $\dim \tilde{K}'(V; V) = 1/2$ a.s. on the event $\{B_1 > 0\}$.

**Lemma 10.** Let $a_1(x)$ be the rightmost element in $K'_1(V, x, B)$. It holds

$$\dim \tilde{K}'(V; V) = 1/2 \text{ a.s. on } \{\exists x, K'_1(V, x, B) \subset (0, 1)\} = \{\exists x, 0 < a_1(x) < 1\}, \text{ and}$$

$$\tilde{K}'(V; V) \subset (0, 1) \text{ a.s. on } \{\forall x, K'_1(V, x, B) = \{0\} \text{ or } 1 \in K'_1(V, x, B)\} = \{\forall x, a_1(x) = 0 \text{ or } 1\}.$$
Proof. We first note that, by the continuity of $B(t)$, $B(a_t(x)) = x$ if $0 < a_t(x) < 1$ and hence $\tilde{K}'(V; V) \cup \{0, 1\} = \{a_t(x) ; \tilde{K}'(V, x, B) = 1\}$. The symmetric difference of $\tilde{K}'(V; V)$ and $\{a_t(x) ; x \in \mathbb{R}\}$ is at most a countable set, which has no effect on the Hausdorff dimension.

We define the event for $\xi \in \mathbb{R}$, $\eta > 0$, $x \in R$ and $\epsilon > 0$,

$$E(\xi, \eta, x, \epsilon) = \{K'_1(V, x, B) \subset (0, 1), B_1 - x - \epsilon \geq \xi, \eta \geq \epsilon, I^{x+\epsilon} - m^{x+\epsilon} \geq \eta\}.$$

Let $\tilde{B}$ be the process conditioned on $E(\xi, \eta, x, \epsilon)$. Since $P[E(\xi, \eta, x, \epsilon)] > 0$, the law of $\tilde{B}$ is absolutely continuous with respect to the law of a standard Brownian motion on $[0, 1]$, and hence to the law of a Brownian motion on $[0, 1]$ with unit drift.

If $X$ is a Brownian motion on $[0, \infty)$ with unit drift independent of $B$, then

$$P \left[ \forall t \geq 0, \eta + \int_0^t V(X_u + \xi)du > 0 \right] > 0.$$

Let $\tilde{X}$ be the conditioned process on this event.

Now we define $Z$ by $Z_t = \tilde{B}_t$ for $t \in [0, 1]$ and $Z_t = \tilde{B}_t + \tilde{X}_{t-1}$ for $t > 1$. The law of $Z$ is absolutely continuous with respect to the law of a Brownian motion on $[0, \infty)$ with unit drift. For all $x' < x + \epsilon$, it follows from definition $K'_1(V, x', B) \equiv K'_1(V, x', Z) \subset (0, 1)$ and hence $0 < a_t(x') = a(x') < 1$.

By a general theory for subordinators, for every $\epsilon > 0$, $\dim\{a(x'); x < x' < x + \epsilon\} = 1/2$ a.s. on the event $\{0 < a(x) < a(x + \epsilon) < 1\}$. See e.g. Bertoin [2] Theorem III.15. Now we have a.s. on $E(\xi, \eta, x, \epsilon)$,

$$\frac{1}{2} = \dim\{a(x'); x < x' < x + \epsilon\} = \dim\{a_t(x'); x < x' < x + \epsilon\}.$$

Let

$$F(\xi, \eta, x, \epsilon) := \{a_t(x'); x < x' < x + \epsilon, E(\xi, \eta, x, \epsilon) \text{ occurs}\} ,$$

a random subset which is nonempty only on the event $E(\xi, \eta, x, \epsilon)$. Since $\tilde{K}'(V; V) \setminus \{0, 1\}$ is the same as a countable union of the random subsets of the form $F(\xi, \eta, x, \epsilon)$, the dichotomy that $\dim(\tilde{K}'(V; V) \setminus \{0, 1\}) = 1/2$ or $\tilde{K}'(V; V) \setminus \{0, 1\} = \emptyset$ holds.

Finally, if $K'_1(V, x, B) \neq \emptyset$ and $1 \notin K'_1(V, x, B)$ for some $x$, then there exists an $x'$ such that $K'_1(V, x', B) \subset (0, 1)$ by the continuity of $\int_0^t V(B_s - x')ds$ in $x'$. \qed

4. Proof of Theorem 4

We quote a theorem in [4] and the proof of Theorem 4 is based on it. We fix $\alpha \geq 0$ $c > 0$ and write $V(y)$ for $V(\alpha, c; y)$. Throughout this section we set $\nu = 1/(\alpha + 2)$ and let $\rho \in (0, 1)$ be the unique solution of $e^\nu \sin \pi \nu (1 - \rho) = \sin \pi \nu \rho$ and $\tilde{C}(x, y)$ be
defined for $x \leq 0$, $y \in \mathbb{R}$ by

\[
(4.17) \quad \tilde{C}(x, y) = \Gamma(\nu)^{-1}|x|^{1-\nu+\rho \mu} \exp \left\{ -\frac{2\nu^2(y^+)^{1/\nu}}{|x|} \right\} \times \int_0^\infty dt e^{-t} \left( |x| t + 2\nu^2 c^{-1} |y^{-1/\nu}| \right)^{\nu \rho} \left( |x| t + 2\nu^2 (y^+)^{1/\nu} \right)^{-1+\nu-\nu \rho}.
\]

Now we have

**Theorem 11** ([4]). For $\mu \geq 0$, $V = V(\alpha, c)$ there exists a constant $C_4(\alpha, c) > 0$ such that it holds

\[
(4.18) \quad \lim_{\sigma \to 0} \int_0^\infty dt \mu e^{-\mu t} \sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V) = C_4(\alpha, c) \mu^{\rho/2} \tilde{C}(x, y).
\]

**Proof of Theorem 4.** Since the integrand above, $\sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V)$, is decreasing in $t$,

\[
\limsup_{\sigma \to 0} \sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V)
\]

must be finite for every $t > 0$ and it is trivial that $\sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V) < \sigma^{-\rho} < 1$ for large $\sigma$. Hence we know the overall supremum is finite, verifying (2.4), and we denote it by $C_0(t, x, y; V)$, which is clearly monotone decreasing in $t$ and inherits the scaling property from $p(t, x, y, V)$:

\[
C_0(t, x, y; V) = \sigma^{-\rho} C_0(t, \sigma^{1/\nu} x, \sigma y; V) = \sigma^{-\rho} C_0(\sigma^{-2} t, x, y; V).
\]

It is sufficient to prove (2.6) when $x < 0$ and $y < 0$. We deduce from the scaling property and the monotonicity that

\[
C_0(t, x, y; V) = |x|^{\rho \mu} C_0 \left( t, -1, \frac{y}{|x|^{\nu}}; V \right)
\]

\[
\leq |x|^{\rho \mu} C_0 \left( t, (-1) \wedge \frac{-|y|^{-1/\nu}}{|x|}, (-1) \wedge \frac{y}{|x|^{\nu}}; V \right)
\]

\[
= C_0 \left( t, x \wedge (-|y|^{-1/\nu}), (-|x|^{\nu}) \wedge y; V \right)
\]

\[
= (|x|^{\rho \mu} \vee |y|^{\rho}) C_0(t, -1, -1; V).
\]

Combining this with $C_0(t, x, y; V) = t^{-\rho/2} C_0(1, x, y; V)$, we obtain (2.6).

To prove (2.5), we note that the family $\{\sigma^{-\rho} p(t, \sigma^{1/\nu} x, \sigma y; V); \sigma > 0\}$ of decreasing functions has an upper bound $C_0(t, x, y; V)$, which satisfies

\[
\int_0^\infty dt \mu e^{-\mu t} C_0(t, x, y; V) < \text{const} \int_0^\infty dt \mu e^{-\mu t} t^{-\rho/2} < \infty.
\]
Given any sequence $\sigma_n \downarrow 0$, we can choose a subsequence $\sigma'_n$ such that the functions $(\sigma'_n)^{-\rho} p(t, (\sigma'_n)^{1/\nu}, \sigma'_n; V)$ converge pointwise on a dense subset of $\{t > 0\}$ and that

$$
\int_0^\infty dt \mu e^{-\mu t} (\sigma'_n)^{-\rho} p(t, (\sigma'_n)^{1/\nu}, \sigma'_n; V) - \int_0^\infty dt \mu e^{-\mu t} \lim_{n \to \infty} (\sigma'_n)^{-\rho} p(t, (\sigma'_n)^{1/\nu}, \sigma'_n; V).
$$

By uniqueness of the Laplace transform, we deduce, for any $t > 0$,

$$
\lim_{\sigma \to 0} \sigma^{-\rho} p(t, \sigma^{1/\nu}, \sigma; V) = \frac{C_{4}(\alpha, c)\check{C}(x, y)t^{-\rho/2}}{\Gamma(1 - \rho/2)}.
$$

References


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