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## CLIFFORD INDEX OF SMOOTH ALGEBRAIC CURVES OF ODD GONALITY WITH BIG $W'_d(C)^*$

Dedicated to Professor Sang Moon Kim on the occasion of his retirement.

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### 0. Introduction

Let  $C$  be a smooth projective irreducible algebraic curve over the field of complex numbers  $\mathbb{C}$  or a compact Riemann surface of genus  $g$ . Let  $J(C)$  be the Jacobian variety of the curve  $C$ , which is a  $g$ -dimensional abelian variety parameterizing all the line bundles of given degree  $d$  on  $C$ . We denote by  $W'_d(C)$  a subvariety of the Jacobian variety  $J(C)$  consisting of line bundles of degree  $d$  with  $r+1$  or more independent global sections.

If  $d > g+r-2$ , one can compute the dimension of  $W'_d(C)$  by using the Riemann-Roch formula, and this dimension is independent of  $C$ . If  $d \leq g+r-2$ , the dimension of  $W'_d(C)$  is known to be greater than or equal to the Brill-Noether number  $\rho(d, g, r) := g - (r+1)(g-d+r)$  for any curve  $C$ , and is equal to  $\rho(d, g, r)$  for general curve  $C$  by theorems of Kleiman-Laksov [13] and Griffiths-Harris [7]. On the other hand, the maximal possible dimension of  $W'_d(C)$  for this range of  $d$ ,  $g$  and  $r$  is  $d-2r$  and the maximum is attained if and only if  $C$  is hyperelliptic by a well known theorem of H. Martens [16].

From a result of M. Coppens, G. Martens and C. Keem [4, Corollary 3.3.2], it is known that for curves of odd gonality — i.e. curves for which the minimal number of sheets of a covering over  $\mathbb{P}^1$  is odd — the theorem of H. Martens can be refined significantly.

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**Proposition A** (Coppens, Keem and G. Martens). *Let  $C$  be a smooth algebraic curve of odd gonality. Then*

$$\dim W_d^r(C) \leq d - 3r$$

for  $d \leq g - 1$ .

Furthermore, by a recent progress made by G. Martens [14] as well as a result of T. Kato and C. Keem [11], it is known that if the dimension of  $W_d^r(C)$  for curves  $C$  of odd gonality is near to the maximum possible value, then  $C$  is of very special type of curves.

**Proposition B** (G. Martens [14, Theorem 2]). *Let  $C$  be a smooth projective irreducible curve of genus  $g$  over the complex number field. Assume that the gonality of  $C$  is odd. If  $\dim W_d^r(C) = d - 3r$  for some  $d \leq g - 2$  and  $r > 0$  then  $C$  is either trigonal, smooth plane sextic, birational to a plane curve of degree 7 (in this case only  $g = 13$  and  $g = 14$  occur; with a simple  $g_{12}^4 = g_5^1 + g_7^2$  or a very ample  $g_{12}^4 = g_5^1 + g_7^2$  respectively) or an extremal space curve of degree 10 with a very ample  $g_{15}^5 = g_{10}^3 + g_5^1$ .*

**Proposition C** (T. Kato, C. Keem [11, Theorem 1]). *Let  $C$  be a smooth irreducible projective curve of genus  $g$  over the complex number field. Assume the gonality of  $C$  is odd and  $\dim W_d^r(C) = d - 3r - 1$  for some  $d \leq g - 4$  and  $r > 0$ . Then  $C$  is 5-gonal with  $10 \leq g \leq 18$ ,  $g = 20$  or 7-gonal of genus 21; furthermore  $C$  is a smooth plane sextic (resp. octic) in case  $\text{gon}(C) = 5$ ,  $g = 10$  (resp.  $\text{gon}(C) = 7$ ,  $g = 21$ ).*

The purpose of this paper is to chase a further generalization of the above results of G. Martens and Kato–Keem. We use standard notation for divisors, linear series, invertible sheaves and line bundles on algebraic curves following [3]. As usual,  $g_d^r$  is an  $r$ -dimensional linear series of degree  $d$  on  $C$ , which may be possibly incomplete. If  $D$  is a divisor on  $C$ , we write  $|D|$  for the associated complete linear series on  $C$ . By  $K_C$  or  $K$  we denote a canonical divisor on  $C$ . If  $L$  is a line bundle (or an invertible sheaf) we sometimes abbreviate the notation  $H^i(C, L)$  (resp.  $\dim H^i(C, L)$ ) by  $H^i(L)$  (resp.  $h^i(L)$ ) for simplicity when no confusion is likely to occur. Also, for a divisor  $D$  on  $C$  we write  $H^i(D)$ ,  $h^i(D)$  instead of  $H^i(C, \mathcal{O}_C(D))$ ,  $\dim H^i(C, \mathcal{O}_C(D))$ . A base-point-free  $g_d^r$  on  $C$  defines a morphism  $f : C \rightarrow \mathbb{P}^r$  onto a non-degenerate irreducible (possibly singular) curve in  $\mathbb{P}^r$ . If  $f$  is birational onto its image  $f(C)$  the given  $g_d^r$  is called simple or birationally very ample. In case the given  $g_d^r$  is not simple, let  $C'$  be the normalization of  $f(C)$ . Then there is a morphism (a non-trivial covering map)  $C \rightarrow C'$  and we use the same notation  $f$  for this covering map of some degree  $k$  induced by the original morphism  $f : C \rightarrow \mathbb{P}^r$ . The gonality of  $C$  which is the minimal sheet number of a covering over  $\mathbb{P}^1$  is denoted by  $\text{gon}(C)$ . We also recall that given a line bundle  $L \in \text{Pic}(C)$ , the Clifford index  $\text{Cliff}(L)$  of  $L$  is defined by

$\text{Cliff}(L) := \deg L - 2(h^0(L) - 1)$ , and the Clifford index  $\text{Cliff}(C)$  of  $C$  is defined by

$$\text{Cliff}(C) := \min \{ \text{Cliff}(L) : L \in \text{Pic}(C) \text{ with } h^0(L) \geq 2 \text{ and } h^1(L) \geq 2 \}.$$

We say that a line bundle  $L$  contributes to the Clifford index of  $C$  if  $h^0(L) \geq 2$  and  $h^1(L) \geq 2$ . As is well-known, the Clifford index of a smooth algebraic curve is a measurement how special a curve is in the sense of moduli. Specifically, if  $k = \text{gon}(C)$  then  $\text{Cliff}(C) \leq k - 2$  for any curve  $C$  and  $\text{Cliff}(C) = k - 2$  for a general  $k$ -gonal curve; cf. [12] for more details. The result of this paper is the following theorem.

**Theorem 1.** *Let  $e \geq 0$  be a fixed integer and let  $C$  be a smooth algebraic curve of genus  $g \geq 4e+7$ . Suppose that the gonality  $\text{gon}(C)$  of the curve  $C$  is an odd integer. Assume that*

$$d - 3r - e \leq \dim W_d^r(C)$$

for some  $d, r \geq 1$  such that  $d \leq g - e - 3$ . Then

$$\text{Cliff}(C) \leq 2(e + 1).$$

In proving our result, we use standard techniques in the theory of linear series on curves such as the Castelnuovo-Severi inequality, excess linear series argument as well as the Accola-Griffiths-Harris theorem.

### 1. Proof of Theorem 1

A proof of Theorem 1 requires several preparatory results and we begin with the following theorem due to Matelski [15]; see also [9, Corollary 1].

**Lemma 2.** *Let  $C$  be a smooth algebraic curve of genus  $g \geq 4j + 3$ ,  $j \geq 0$ . If  $\dim W_d^1(C) = d - 2 - j$  for some  $d$  such that  $j + 2 \leq d \leq g - 1 - j$ , then  $\dim W_{2j+2}^1(C) \geq j$ .*

For positive integers  $d, r$ , let  $m = [(d-1)/(r-1)]$ ,  $\varepsilon = d - m(r-1) - 1$ ,  $\varepsilon_1 = d - m_1r - 1$ . We set

$$\pi(d, r) = \frac{m(m-1)}{2}(r-1) + m\varepsilon.$$

**Lemma 3** (Castelnuovo's bound). *Assume  $C$  admits a base-point-free and simple linear series  $g_d^r$ . Then  $g \leq \pi(d, r)$ .*

**Lemma 4** ([1, §7]). *If  $C$  admits infinite number of base-point-free simple linear series  $g_d^r$ 's, then  $g \leq \pi(d+1, r+1)$ .*

**Lemma 5** (Excess linear series [3, VII Exercise C, page 329]). *On any curve  $C$ ,*

$$\dim W_{d-1}^r(C) \geq \dim W_d^r(C) - r - 1.$$

The following is a special case of the so-called Castelnuovo-Severi inequality.

**Lemma 6** (Castelnuovo-Severi bound [2, Theorem 3.5]). *Assume there exist two curves  $C_1$  and  $C_2$  of genus  $g_1$  and  $g_2$ , respectively, so that  $C$  is a  $k_i$ -sheeted covering of  $C_i$  ( $i = 1, 2$ ). If  $k_1$  and  $k_2$  are coprime, then*

$$g \leq (k_1 - 1)(k_2 - 1) + k_1 g_1 + k_2 g_2.$$

**Lemma 7** (Extension of H. Martens' theorem [10]). *Let  $d$  and  $r$  be positive integers such that  $d \leq g + r - 4$ ,  $r \geq 1$ . If*

$$\dim W_d^r(C) \geq d - 2r - 2 \geq 0$$

*then  $C$  is either hyperelliptic, trigonal, bi-elliptic, tetragonal, a smooth plane sextic or a double covering of a curve of genus 2.*

We also need the following result due to M. Coppens and G. Martens which may be considered as a “Clifford’s theorem” for curves of odd gonality.

**Lemma 8** (M. Coppens, G. Martens [5]). *Let  $D$  be an effective divisor on a curve  $C$  of genus  $g$  and of odd gonality such that  $\deg D < g$ . Then  $\dim |D| \leq (1/3)\deg D$ .*

Proof of Theorem 1. For  $e = 0$ , the result holds by Proposition B if  $C$  does not belong to the following special classes of curves described in Proposition B;

- (i) a 5-gonal curve of genus  $g = 14$  with a very ample  $g_{12}^4 = g_5^1 + g_7^2$
- (ii) a 5-gonal curve of genus  $g = 13$  with a simple  $g_{12}^4 = g_5^1 + g_7^2$
- (iii) a 5-gonal extremal space curve of degree 10 and genus  $g = 16$  with a very ample  $g_{15}^5 = g_5^1 + g_{10}^3$ .

We first argue that these curves do not satisfy  $\dim W_d^r(C) = d - 3r$  for any  $d \leq g - 3$  and  $r > 0$ . If  $\dim W_d^r(C) = d - 3r$  for some  $d \leq g - 3$  with  $r = 1$  or  $r = 2$ , then  $C$  must be a curve of gonality  $\text{gon}(C) \leq 4$  by Lemma 7. Therefore we now assume that  $\dim W_d^r(C) = d - 3r$  for some  $d \leq g - 3$  with  $r \geq 3$ .

CASE (i): If  $C$  is a 5-gonal curve of genus  $g = 14$  with a very ample  $g_{12}^4 = g_5^1 + g_7^2$ ,  $W_d^r(C) = \emptyset$  for any  $r \geq 3$  and  $d \leq 9$  by Lemma 3 (Castelnuovo genus bound). Since  $g = 14$  and  $d \leq g - 3$ , we have  $r \leq 3$  by Lemma 8. Furthermore, it is easy to see that  $\dim W_{10}^3(C) \leq 0$ . Suppose otherwise. Then there exist infinitely many  $g_{10}^3 \in W_{10}^3(C)$  which must be base-point-free and simple. Therefore one can apply

Lemma 4 to get the contradiction  $g \leq 12$ . Finally, suppose that  $\dim W_{11}^3(C) = 2$ . Since we already have  $\dim W_{10}^3(C) \leq 0$ , it is clear that a general  $\mathcal{L} \in W_{11}^3(C)$  is base-point-free and hence birationally very ample. For a general  $\mathcal{L} = g_{11}^3 \in W_{11}^3(C)$ , we consider  $h^0(C, K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g_5^1))$ . If  $h^0(C, K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g_5^1)) \geq 4$ , then  $|K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g_5^1)| = g_{10}^3$  for a general  $\mathcal{L} \in W_{11}^3(C)$ , and hence  $\dim W_{10}^3(C) = 2$ , contrary to  $\dim W_{10}^3(C) \leq 0$ . Therefore we must have  $h^0(C, K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g_5^1)) \leq 3$  for a general  $\mathcal{L} \in W_{11}^3(C)$ . Then, by the base-point-free pencil trick, applied to the natural map

$$H^0(C, \mathcal{L}) \oplus H^0(C, \mathcal{L}) \longrightarrow H^0(C, \mathcal{L} \otimes \mathcal{O}_C(g_5^1)),$$

one concludes that  $h^0(C, \mathcal{L} \otimes \mathcal{O}_C(-g_5^1)) \geq 2$ , for a general  $\mathcal{L} \in W_{11}^3(C)$ , which in turn implies  $\dim W_6^1(C) = 2$ . Then by Lemma 7, we have  $\text{gon}(C) \leq 4$ , which is a contradiction.

CASE (ii): If  $C$  is a 5-gonal curve of genus  $g = 13$ , exactly the same argument as in the Case (i) is still valid for this case to show that  $\dim W_d^r(C) \leq d - 3r$  for any  $d \leq g - 3$  and  $r > 0$ .

CASE (iii): Let  $C$  be a 5-gonal extremal space curve of degree 10 and genus  $g = 16$ . Note that  $C$  is a complete intersection of a quintic and a quadric in  $\mathbb{P}^3$ . For  $d \leq 9$  and  $r \geq 3$ ,  $W_d^r(C) = \emptyset$  by Lemma 3. For the case  $(d, r) = (10, 3)$ , we apply the same argument as in the case (i) above to show that  $\dim W_{10}^3(C) \leq 0$ . For the case  $(d, r) = (11, 3)$ , suppose that  $\dim W_{11}^3(C) = 2$ . Since we already have  $\dim W_{10}^3(C) \leq 0$ , a general  $g_{11}^3$  must be base-point-free and simple. Then by Lemma 4 we get a contradiction  $g \leq 15$ . Let  $(d, r) = (12, 3)$  and assume that  $\dim W_{12}^3(C) = 3$ . For a general  $\mathcal{L} = g_{12}^3 \in W_{12}^3(C)$ , we again consider  $h^0(C, K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g_5^1))$ . If  $h^0(C, K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g_5^1)) \geq 5$ , then  $|K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g_5^1)| = g_{13}^4$  for a general  $\mathcal{L} \in W_{12}^3(C)$ , and hence  $\dim W_{13}^4(C) \geq 3$ , a contradiction to Proposition A. Therefore we must have  $h^0(C, K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g_5^1)) \leq 4$  for a general  $\mathcal{L} \in W_{12}^3(C)$ . By applying the base-point-free pencil trick to the natural map

$$H^0(C, \mathcal{L}) \oplus H^0(C, \mathcal{L}) \longrightarrow H^0(C, \mathcal{L} \otimes \mathcal{O}_C(g_5^1)),$$

one concludes that  $h^0(C, \mathcal{L} \otimes \mathcal{O}_C(-g_5^1)) \geq 2$ , for a general  $\mathcal{L} \in W_{12}^3(C)$ , which in turn implies  $\dim W_7^1(C) \geq 3$ . Then by Lemma 7, we have  $\text{gon}(C) \leq 4$ , which is a contradiction. Let  $(d, r) = (12, 4)$  and assume that  $\dim W_{12}^4(C) = 0$ . If  $g_{12}^4$  is not simple, then  $C$  is either trigonal or a double cover of a curve of genus  $h \leq 2$ , a contradiction. If  $g_{12}^4$  is simple, then  $g \leq 15$  by Lemma 3, again a contradiction. For the case  $(d, r) = (13, 3)$ , we can use an argument almost parallel to the case  $(d, r) = (12, 3)$  to show that  $\dim W_{13}^3(C) \leq 4$ . Finally let  $(d, r) = (13, 4)$  and assume that  $\dim W_{13}^4(C) = 1$ . Since we already know  $W_{12}^4(C) = \emptyset$ , every  $g_{13}^4 \in W_{13}^4(C)$  is base-point-free and hence simple. Therefore one applies Lemma 4 to get the contradiction  $g \leq 15$ . In all, we conclude that our theorem holds for  $e = 0$ .

For  $e = 1$ , the theorem is valid by Proposition C. Hence from now on, we may assume that  $e \geq 2$  and  $\text{gon}(C) \geq 7$ ; note that if  $g \geq 4e + 7$ , the curves  $C$  in

Proposition B and Proposition C have  $\text{gon}(C) \leq 5$ . By induction, we assume that  $\dim W_d^r(C) = d - 3r - e$  for some  $d \leq g - e - 3$  and  $r \geq 1$ .

Let  $Z$  be an irreducible component of  $W_d^r(C)$  of dimension  $d - 3r - e$  and let  $g_d^r(z)$  be the linear series associated to an element  $z \in Z$ . By the fact that no component of  $W_d^r(C)$  is properly contained in a component of  $W_d^{r+1}(C)$ , we may assume that  $g_d^r(z)$  is complete for a general  $z \in Z$ ; cf. [3, Lemma 3.5–page 182]. By shrinking if necessary, one may further assume that  $g_d^r(z)$  is base-point-free for a general  $z \in Z$ . We first treat the case  $r = 1$ , which is relatively easy.

CLAIM 1. If  $r = 1$ , then  $\text{Cliff}(C) \leq 2(e + 1)$ .

For  $r = 1$ , we set  $\dim W_d^1(C) = d - 2 - j = d - 3 - e \geq 0$ ;  $j = e + 1$ . Therefore we have  $j + 2 \leq e + 3 \leq d \leq g - 1 - j$ , where the last inequality comes from our assumption  $d \leq g - e - 3$ . Hence Lemma 2 applies to get the inequality

$$\dim W_{2(e+1)+2}^1(C) = \dim W_{2e+4}^1(C) \geq e + 1.$$

By Lemma 5, one has  $\dim W_{2e+3}^1(C) \geq e - 1 \geq 0$  and it follows that

$$\text{Cliff}(C) \leq (2e + 3) - 2 = 2e + 1 \leq 2e + 2,$$

as wanted; note that  $g_{2e+3}^1 \in W_{2e+3}^1(C)$  contributes to the Clifford index of  $C$  by the genus assumption  $g \geq 4e + 7$ . Therefore, for the rest of the proof, we may assume that  $r \geq 2$  and that

$$(1) \quad \dim W_n^1(C) \leq n - 4 - e$$

for any  $n \leq g - e - 3$ .

CLAIM 2. If  $r \geq 2$ , then  $g_d^r(z)$  is simple for a general  $z \in Z$ .

Assume  $g_d^r(z)$  is compounded for a general  $z \in Z$ . Then  $g_d^r(z)$  induces an  $n$ -sheeted covering map  $\pi : C \rightarrow C'$  onto a smooth curve  $C'$  of genus  $g'$  with  $n \mid d$  and  $n \geq 2$ . Then  $g_d^r(z)$  is the pull back of a base-point-free complete series  $g_{d/n}^r$  on  $C'$  with respect to  $\pi$ ; i.e.  $g_d^r(z) = \pi^*(g_{d/n}^r)$ .

Let  $g' = 0$ . Then  $(d/n) - r = g' = 0$  and  $Z \subset r \cdot W_n^1(C)$ . Hence one has

$$d - 3r - e \leq \dim W_n^1(C) \leq n - 4 - e,$$

where the second inequality follows from (1). Therefore  $(n - 3)(r - 1) \leq -1$  and hence it follows that  $n = 2$ ; but this is a contradiction since  $C$  is non-hyperelliptic.

Next, we assume  $g' > 0$ . By de Franchis' theorem, we may assume that the map

$W_{d/n}^r(C') \xrightarrow{\pi^*} Z$  is finite dominant map. Hence,

$$0 \leq d - 3r - e = \dim Z \leq \dim W_{d/n}^r(C').$$

Assume  $g_{d/n}^r$  is special. Then  $\dim W_{d/n}^r(C') \leq (d/n) - 2r$  by H. Martens' theorem [16]. Hence, we have  $0 \leq d - 3r - e = \dim Z \leq (d/n) - 2r$ . Therefore it follows that  $(n-1)d \leq n(r+e)$  and  $d \geq 3r+e$ . Hence we have

$$\text{Cliff}(C) \leq d - 2r \leq \frac{n}{n-1}(r+e) - 2r$$

and a simple computation leads to  $\text{Cliff}(C) \leq 2e+2$  as wanted.

Assume  $g_{d/n}^r$  is non-special. Again by de Franchis' theorem, the map  $J(C') = W_{d/n}^r(C') \xrightarrow{\pi^*} Z$  is a finite dominant map and

$$(2) \quad \dim W_{d/n}^r(C') = \dim \text{Jac}(C') = g' = \frac{d}{n} - r = \dim Z = d - 3r - e.$$

We shall treat the cases  $n = 2$  and  $n \geq 3$  separately.

$n = 2$ : Since  $\text{gon}(C) = k$  is odd, the morphism  $C \rightarrow \mathbb{P}^1$  induced by a  $g_k^1$  does not factor through  $\pi$ . Hence, Lemma 6 (Castelnuovo-Severi bound) gives  $g \leq k - 1 + 2g'$ . Since  $k \leq 2 \cdot \text{gon}(C') \leq 2 \cdot (g'+3)/2$ , we get  $g \leq 3g'+2$ . Note that the equality (2) for  $n = 2$  implies  $d = 4r+2e$  and  $g' = r+e$ . Therefore from the assumption  $d \leq g - e - 3$ , we have  $d + e + 3 \leq g \leq 3g' + 2 \Rightarrow 4r + 2e + e + 3 \leq 3g' + 2 \Rightarrow g' \leq e - 1$ . Hence  $g \leq 3(e-1) + 2$ , a contradiction to  $g \geq 4e+7$ .

$n \geq 3$ : We remark that  $\pi^*(W_{d/n-r+1}^1(C')) \subset W_{d-n(r-1)}^1(C)$ . Hence by the equality (2), we have

$$(3) \quad \begin{aligned} \dim \pi^*(W_{d/n-r+1}^1(C')) &= \dim W_{d/n-r+1}^1(C') = \dim J(C') = d - 3r - e \\ &\leq \dim W_{d-n(r-1)}^1(C). \end{aligned}$$

Since  $d - 3r - e \geq d - n(r-1) - 3 - e$  for  $n \geq 3$  and  $d - n(r-1) \leq g - e - 3$ , the above inequality (3) is contradictory to our assumption (1). And this finishes the proof of Claim 2.

Since  $g_d^r(z)$  is simple for a general  $z \in Z$  if  $r \geq 2$ , we may apply Accola-Griffiths-Harris theorem [8, page 73] to our current situation and we have the following inequality;

$$d - 3r - e \leq \dim W_d^r(C) \leq \dim T_{|D|} W_d^r(C) \leq h^0(2D) - 3r \quad \text{for } D \in g_d^r(z),$$

and it follows that

$$d - e \leq h^0(2D) = 2d + 1 - g + h^1(2D).$$

On the other hand, by the numerical bound  $d \leq g - e - 3$  which we have assumed, we see that  $h^1(2D) \geq g - d - 1 - e \geq 2$  and hence the linear series  $|2D|$  contributes to the Clifford index of  $C$ . Therefore we finally have

$$(4) \quad \text{Cliff}(C) \leq \text{Cliff}(2D) = 2d - 2h^0(2D) + 2 \leq 2d - 2(d - e - 1) = 2(e + 1),$$

and this finishes the proof of the theorem.  $\square$

One may refine the statement in Theorem 1 for small  $e \leq 6$  as follows by looking at our proof more carefully, which Takao Kato has kindly informed the authors through Akira Ohbuchi.

**Corollary 9.** *Let  $e$  be a fixed integer with  $0 \leq e \leq 6$  and let  $C$  be a smooth algebraic curve of genus  $g \geq 4e + 7$ . Suppose that the gonality  $\text{gon}(C)$  of the curve  $C$  is an odd integer. Assume that*

$$d - 3r - e \leq \dim W_d^r(C)$$

for some  $d, r \geq 1$  such that  $d \leq g - e - 3$ . Then

$$\text{Cliff}(C) \leq 2(e + 1).$$

Furthermore the equality holds if and only if  $C$  is a smooth plane curve of degree  $2e + 6$ .

**Proof.** We use the same notations which we used in the proof of Theorem 1. We first remark that everywhere in the course of the proof of Theorem 1, we indeed had  $\text{Cliff}(C) \leq 2e + 1$  except for the case  $r \geq 2$  and  $g_d^r(z)$  is simple for a general  $z \in Z$ . Therefore, we assume  $\text{Cliff}(C) = 2e + 2$  and  $g_d^r(z) = |D|$  is simple for a general  $z \in Z$  and  $r \geq 2$ . Hence by the inequality (4),  $\text{Cliff}(2D) = \text{Cliff}(C) = 2e + 2$ . We now distinguish two cases.

(i)  $2d \leq g - 1$ : By [5, Theorem C] which provides an upper bound of the degree of a complete linear series  $\mathcal{D}$  such that  $\text{Cliff}(C) = \text{Cliff}(\mathcal{D})$ , we have  $2d \leq 4e + 8$ . On the other hand

$$2e + 2 = \text{Cliff}(C) \leq \text{Cliff}(D) = d - 2r \leq 2e + 4 - 2r,$$

and it follows that  $r \leq 1$ , contrary to our assumption  $r \geq 2$ .

(ii)  $2d \geq g - 1$ : Note that  $|K - 2D| = g_{2g-2-2d}^{g-d-2-e}$  since  $\text{Cliff}(K - 2D) = \text{Cliff}(2D)$ . We again apply [5, Theorem C] to the linear series  $|K - 2D|$ ;  $d' = \deg |K - 2D| = 2g - 2 - 2d \leq 4e + 8$  and hence

$$r' = \dim |K - 2D| \leq e + 3.$$

We now briefly recall the so-called Clifford dimension of a smooth algebraic curve  $C$ , denoted by  $\text{Cliffdim}(C)$ , which is defined to be the minimum possible dimension  $r(\mathcal{D})$  of a complete linear series  $\mathcal{D}$  such that  $\text{Cliff}(C) = \text{Cliff}(\mathcal{D})$  and  $\mathcal{D}$  contributes to the Clifford index of  $C$ ; cf. [6, page 174]. By  $r' \leq e+3$  and by our numerical hypothesis  $e \leq 6$ , we have

$$\text{Cliffdim}(C) \leq r' \leq e+3 \leq 9,$$

which in turn implies  $\text{Cliffdim}(C) = 1$  or  $2$  by the last statement in [6, page 203], which asserts in particular that for  $3 \leq r \leq 9$  a curve of Clifford dimension  $r$  is of even gonality. The case  $\text{Cliffdim}(C) = 1$  cannot occur; if then  $\text{gon}(C) = 2e+4$  and  $C$  is of even gonality. Therefore  $\text{Cliffdim}(C) = 2$  and by a simple fact that a complete linear series  $\mathcal{D}$  with  $\dim(\mathcal{D}) = \text{Cliffdim}(C) \geq 2$  is very ample [6, Lemma 1.1, page 177], we deduce that  $C$  is a smooth plane curve of degree  $2e+6$ .  $\square$

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