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CLIFFORD INDEX OF SMOOTH ALGEBRAIC CURVES
OF ODD GONALITY WITH BIG $W^r_d(C)^*$

Dedicated to Professor Sang Moon Kim on the occasion of his retirement.

EDOARDO BALLICO$^\dagger$ and CHANGHO KEEM$^\ddagger$

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0. Introduction

Let $C$ be a smooth projective irreducible algebraic curve over the field of complex numbers $\mathbb{C}$ or a compact Riemann surface of genus $g$. Let $J(C)$ be the Jacobian variety of the curve $C$, which is a $g$-dimensional abelian variety parameterizing all the line bundles of given degree $d$ on $C$. We denote by $W^r_d(C)$ a subvariety of the Jacobian variety $J(C)$ consisting of line bundles of degree $d$ with $r+1$ or more independent global sections.

If $d > g+r-2$, one can compute the dimension of $W^r_d(C)$ by using the Riemann-Roch formula, and this dimension is independent of $C$. If $d \leq g+r-2$, the dimension of $W^r_d(C)$ is known to be greater than or equal to the Brill-Noether number $\rho(d, g, r) := g - (r + 1)(g - d + r)$ for any curve $C$, and is equal to $\rho(d, g, r)$ for general curve $C$ by theorems of Kleiman-Laksov [13] and Griffiths-Harris [7]. On the other hand, the maximal possible dimension of $W^r_d(C)$ for this range of $d$, $g$ and $r$ is $d - 2r$ and the maximum is attained if and only if $C$ is hyperelliptic by a well known theorem of H. Martens [16].

From a result of M. Coppens, G. Martens and C. Keem [4, Corollary 3.3.2], it is known that for curves of odd gonality — i.e. curves for which the minimal number of sheets of a covering over $\mathbb{P}^1$ is odd — the theorem of H. Martens can be refined significantly.

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Proposition A (Coppens, Keem and G. Martens). Let $C$ be a smooth algebraic curve of odd gonality. Then
\[ \dim W_d^r(C) \leq d - 3r \]
for $d \leq g - 1$.

Furthermore, by a recent progress made by G. Martens [14] as well as a result of T. Kato and C. Keem [11], it is known that if the dimension of $W_d^r(C)$ for curves $C$ of odd gonality is near to the maximum possible value, then $C$ is of very special type of curves.

Proposition B (G. Martens [14, Theorem 2]). Let $C$ be a smooth projective irreducible curve of genus $g$ over the complex number field. Assume that the gonality of $C$ is odd. If $\dim W_d^r(C) = d - 3r$ for some $d \leq g - 2$ and $r > 0$ then $C$ is either trigonal, smooth plane sextic, birational to a plane curve of degree 7 (in this case only $g = 13$ and $g = 14$ occur; with a simple $g_{12}^4 = g_3^1 + g_3^2$ or a very ample $g_{12}^4 = g_3^1 + g_3^2$ respectively) or an extremal space curve of degree 10 with a very ample $g_5^3 = g_3^1 + g_3^2$.

Proposition C (T. Kato, C. Keem [11, Theorem 1]). Let $C$ be a smooth irreducible projective curve of genus $g$ over the complex number field. Assume the gonality of $C$ is odd and $\dim W_d^r(C) = d - 3r - 1$ for some $d \leq g - 4$ and $r > 0$. Then $C$ is 5-gonal with $10 \leq g \leq 18$, $g = 20$ or 7-gonal of genus 21; furthermore $C$ is a smooth plane sextic (resp. octic) in case $\text{gon}(C) = 5$, $g = 10$ (resp. $\text{gon}(C) = 7$, $g = 21$).

The purpose of this paper is to chase a further generalization of the above results of G. Martens and Kato–Keem. We use standard notation for divisors, linear series, invertible sheaves and line bundles on algebraic curves following [3]. As usual, $g_d^r$ is an $r$-dimensional linear series of degree $d$ on $C$, which may be possibly incomplete. If $D$ is a divisor on $C$, we write $|D|$ for the associated complete linear series on $C$. By $K_C$ or $K$ we denote a canonical divisor on $C$. If $L$ is a line bundle (or an invertible sheaf) we sometimes abbreviate the notation $H^i(C, L)$ (resp. $\dim H^i(C, L)$) by $H^i(L)$ (resp. $\dim H^i(L)$) for simplicity when no confusion is likely to occur. Also, for a divisor $D$ on $C$ we write $H^i(D)$, $\dim H^i(D)$ instead of $H^i(C, \mathcal{O}_C(D))$, $\dim H^i(C, \mathcal{O}_C(D))$. A base-point-free $g_d^r$ on $C$ defines a morphism $f : C \to \mathbb{P}^r$ onto a non-degenerate irreducible (possibly singular) curve in $\mathbb{P}^r$. If $f$ is birational onto its image $f(C)$ the given $g_d^r$ is called simple or birationally very ample. In case the given $g_d^r$ is not simple, let $C'$ be the normalization of $f(C)$. Then there is a morphism (a non-trivial covering map) $C \to C'$ and we use the same notation $f$ for this covering map of some degree $k$ induced by the original morphism $f : C \to \mathbb{P}^r$. The gonality of $C$ which is the minimal sheet number of a covering over $\mathbb{P}^1$ is denoted by $\text{gon}(C)$. We also recall that given a line bundle $L \in \text{Pic}(C)$, the Clifford index $\text{Cliff}(L)$ of $L$ is defined by
Clifford index of curves of odd gonality

Cliff(\text{L}) := \deg L - 2(h^0(\text{L}) - 1), and the Clifford index Cliff(\text{C}) of \text{C} is defined by

Cliff(\text{C}) := \min \{ \text{Cliff}(\text{L}) : \text{L} \in \text{Pic}(\text{C}) \text{ with } h^0(\text{L}) \geq 2 \text{ and } h^1(\text{L}) \geq 2 \}.

We say that a line bundle \text{L} contributes to the Clifford index of \text{C} if \text{h}^0(\text{L}) \geq 2 \text{ and } \text{h}^1(\text{L}) \geq 2. As is well-known, the Clifford index of a smooth algebraic curve is a measurement how special a curve is in the sense of moduli. Specifically, if \text{k} = \text{gon}(\text{C}) then Cliff(\text{C}) \leq \text{k} - 2 \text{ for any curve } \text{C} \text{ and } \text{Cliff}(\text{C}) = \text{k} - 2 \text{ for a general } \text{k}-\text{gonal curve}; cf. [12] for more details. The result of this paper is the following theorem.

\textbf{Theorem 1.} Let \text{e} \geq 0 be a fixed integer and let \text{C} be a smooth algebraic curve of genus \text{g} \geq 4\text{e} + 7. Suppose that the gonality \text{gon}(\text{C}) of the curve \text{C} is an odd integer. Assume that

\[ d - 3r - e \leq \dim W^r_d(\text{C}) \]

for some \text{d}, \text{r} \geq 1 such that \text{d} \leq \text{g} - \text{e} - 3. Then

\[ \text{Cliff}(\text{C}) \leq 2(\text{e} + 1). \]

In proving our result, we use standard techniques in the theory of linear series on curves such as the Castelnuovo-Severi inequality, excess linear series argument as well as the Accola-Griffiths-Harris theorem.

\textbf{1. Proof of Theorem 1}

A proof of Theorem 1 requires several preparatory results and we begin with the following theorem due to Matelski [15]; see also [9, Corollary 1].

\textbf{Lemma 2.} Let \text{C} be a smooth algebraic curve of genus \text{g} \geq 4\text{j} + 3, \text{j} \geq 0. If \dim W^1_{2j} = d - 2j \text{ for some } d \text{ such that } j + 2 \leq d \leq g - 1 - j, \text{ then } \dim W^1_{2j + 2}(\text{C}) \geq j.

For positive integers \text{d}, \text{r}, let \text{m} = [(\text{d} - 1)/(\text{r} - 1)], \text{e} = \text{d} - \text{m}(\text{r} - 1) - 1, \text{e}_1 = \text{d} - \text{m}_1\text{r} - 1. We set

\[ \pi(d, r) = \frac{m(m - 1)}{2}(r - 1) + m\text{e}. \]

\textbf{Lemma 3 (Castelnuovo’s bound).} Assume \text{C} admits a base-point-free and simple linear series \text{g}^r_d. Then \text{g} \leq \pi(d, r).

\textbf{Lemma 4 ([1, §7]).} If \text{C} admits infinite number of base-point-free simple linear series \text{g}^r_d’s, then \text{g} \leq \pi(d + 1, r + 1).
Lemma 5 (Excess linear series [3, VII Exercise C, page 329]). On any curve $C$, 
\[ \dim W^r_d(C) \geq \dim W^r_d(C) - r - 1. \]

The following is a special case of the so-called Castelnuovo-Severi inequality.

Lemma 6 (Castelnuovo-Severi bound [2, Theorem 3.5]). Assume there exist two 
curves $C_1$ and $C_2$ of genus $g_1$ and $g_2$, respectively, so that $C$ is a $k_i$-sheeted covering 
of $C_i$ $(i = 1, 2)$. If $k_1$ and $k_2$ are coprime, then 
\[ g \leq (k_1 - 1)(k_2 - 1) + k_1 g_1 + k_2 g_2. \]

Lemma 7 (Extension of H. Martens’ theorem [10]). Let $d$ and $r$ be positive integers such that $d \leq g + r - 4$, $r \geq 1$. If 
\[ \dim W^r_d(C) \geq d - 2r - 2 \geq 0 \]
then $C$ is either hyperelliptic, trigonal, bi-elliptic, tetragonal, a smooth plane sextic or 
a double covering of a curve of genus 2.

We also need the following result due to M. Coppens and G. Martens which may 
be considered as a “Clifford’s theorem” for curves of odd gonality.

Lemma 8 (M. Coppens, G. Martens [5]). Let $D$ be an effective divisor on a 
curve $C$ of genus $g$ and of odd gonality such that $\deg D < g$. Then $\dim |D| \leq (1/3) \deg D$.

Proof of Theorem 1. For $e = 0$, the result holds by Proposition B if $C$ does not 
belong to the following special classes of curves described in Proposition B:
(i) a 5-gonal curve of genus $g = 14$ with a very ample $g^4_{12} = g^4_5 + g^2_7$
(ii) a 5-gonal curve of genus $g = 13$ with a simple $g^4_{12} = g^4_5 + g^2_7$
(iii) a 5-gonal extremal space curve of degree 10 and genus $g = 16$ with a very ample 
$g^5_{15} = g^3_7 + g^2_{10}$.

We first argue that these curves do not satisfy $\dim W^r_d(C) = d - 3r$ for any $d \leq g - 3$ 
and $r > 0$. If $\dim W^r_d(C) = d - 3r$ for some $d \leq g - 3$ with $r = 1$ or $r = 2$, then $C$ 
must be a curve of gonality $\text{gon}(C) \leq 4$ by Lemma 7. Therefore we now assume that 
$\dim W^r_d(C) = d - 3r$ for some $d \leq g - 3$ with $r \geq 3$.

CASE (i): If $C$ is a 5-gonal curve of genus $g = 14$ with a very ample $g^4_{12} = g^4_5 + g^2_7$, $W^r_d(C) = \emptyset$ for any $r \geq 3$ and $d \leq 9$ by Lemma 3 (Castelnuovo genus 
bound). Since $g = 14$ and $d \leq g - 3$, we have $r \leq 3$ by Lemma 8. Furthermore, it is 
easy to see that $\dim W^r_d(C) = 0$. Suppose otherwise. Then there exist infinitely many $g^3_{10} \in W^3_{10}(C)$ which must be base-point-free and simple. Therefore one can apply
Lemma 4 to get the contradiction $g \leq 12$. Finally, suppose that $\dim W^3_{10}(C) \leq 0$. Since we already have $\dim W^3_{11}(C) \leq 0$, it is clear that a general $L \in W^3_{11}(C)$ is base-point-free and hence birationally very ample. For a general $L = g^3_{11} \in W^3_{11}(C)$, we consider $h^0(C, KL^{-1} \otimes O_C(-g^1_3))$. If $h^0(C, KL^{-1} \otimes O_C(-g^1_3)) \geq 4$, then $|KL^{-1} \otimes O_C(-g^1_3)| = g^3_{10}$ for a general $L \in W^3_{11}(C)$, and hence $\dim W^3_{10}(C) = 2$, contrary to $\dim W^3_{10}(C) \leq 0$. Therefore we must have $h^0(C, KL^{-1} \otimes O_C(-g^1_3)) \leq 3$ for a general $L \in W^3_{11}(C)$. Then, by the base-point-free pencil trick, applied to the natural map

$$H^0(C, L) \otimes H^0(C, L) \to H^0(C, L \otimes O_C(g^1_3)),$$

one concludes that $h^0(C, L \otimes O(-g^1_3)) \geq 2$, for a a general $L \in W^3_{11}(C)$, which in turn implies $\dim W^3_{3}(C) = 2$. Then by Lemma 7, we have $\text{gon}(C) \leq 4$, which is a contradiction.

CASE (ii): If $C$ is a 5-gonal curve of genus $g = 13$, exactly the same argument as in the Case (i) is still valid for this case to show that $\dim W^r_{d}(C) \leq d - 3r$ for any $d \leq g - 3$ and $r > 0$.

CASE (iii): Let $C$ be a 5-gonal extremal space curve of degree 10 and genus $g = 16$. Note that $C$ is a complete intersection of a quintic and a quadric in $\mathbb{P}^3$. For $d \leq 9$ and $r \geq 3$, $W^r_d(C) = \emptyset$ by Lemma 3. For the case $(d, r) = (10, 3)$, we apply the same argument as in the case (i) above to show that $\dim W^3_{10}(C) \leq 0$. For the case $(d, r) = (11, 3)$, suppose that $\dim W^3_{11}(C) = 2$. Since we already have $\dim W^3_{10}(C) \leq 0$, a general $g^3_{11}$ must be base-point-free and simple. Then by Lemma 4 we get a contradiction $g \leq 15$. Let $(d, r) = (12, 3)$ and assume that $\dim W^3_{12}(C) = 3$. For a general $L = g^3_{12} \in W^3_{12}(C)$, we again consider $h^0(C, KL^{-1} \otimes O_C(-g^1_3))$. If $h^0(C, KL^{-1} \otimes O_C(-g^1_3)) \geq 5$, then $|KL^{-1} \otimes O_C(-g^1_3)| = g^3_{13}$ for a general $L \in W^3_{12}(C)$, and hence $\dim W^3_{13}(C) \geq 3$, a contradiction to Proposition A. Therefore we must have $h^0(C, KL^{-1} \otimes O_C(-g^1_3)) \leq 4$ for a general $L \in W^3_{12}(C)$. By applying the base-point-free pencil trick to the natural map

$$H^0(C, L) \otimes H^0(C, L) \to H^0(C, L \otimes O_C(g^1_3)),$$

one concludes that $h^0(C, L \otimes O(-g^1_3)) \geq 2$, for a a general $L \in W^3_{12}(C)$, which in turn implies $\dim W^3_{4}(C) \geq 3$. Then by Lemma 7, we have $\text{gon}(C) \leq 4$, which is a contradiction. Let $(d, r) = (12, 4)$ and assume that $\dim W^3_{13}(C) = 0$. If $g^4_{12}$ is not simple, then $C$ is either trigonal or a double cover of a curve of genus $h \leq 2$, a contradiction.

If $g^4_{12}$ is simple, then $g \leq 15$ by Lemma 3, again a contradiction. For the case $(d, r) = (13, 3)$, we can use an argument almost parallel to the case $(d, r) = (12, 3)$ to show that $\dim W^3_{13}(C) \leq 4$. Finally let $(d, r) = (13, 4)$ and assume that $\dim W^3_{13}(C) = 1$. Since we already know $W^4_{12}(C) = \emptyset$, every $g^4_{13} \in W^3_{13}(C)$ is base-point-free and hence simple. Therefore one applies Lemma 4 to get the contradiction $g \leq 15$. In all, we conclude that our theorem holds for $e = 0$.

For $e = 1$, the theorem is valid by Proposition C. Hence from now on, we may assume that $e \geq 2$ and $\text{gon}(C) \geq 7$; note that if $g \geq 4e + 7$, the curves $C$ in
Proposition B and Proposition C have \( \text{gon}(C) \leq 5 \). By induction, we assume that 
\( \dim W^1_d(C) = d - 3r - e \) for some \( d \leq g - e - 3 \) and \( r \geq 1 \).

Let \( Z \) be an irreducible component of \( W^1_d(C) \) of dimension \( d - 3r - e \) and let \( g^r_d(z) \) be the linear series associated to an element \( z \in Z \). By the fact that no component of \( W^1_d(C) \) is properly contained in a component of \( W^{r+1}_d(C) \), we may assume that \( g^r_d(z) \) is complete for a general \( z \in Z \); cf. [3, Lemma 3.5–page 182]. By shrinking if necessary, one may further assume that \( g^r_d(z) \) is base-point-free for a general \( z \in Z \). We first treat the case \( r = 1 \), which is relatively easy.

**Claim 1.** If \( r = 1 \), then \( \text{Cliff}(C) \leq 2(e + 1) \).

For \( r = 1 \), we set \( \dim W^1_d(C) = d - 2 - j = d - 3 - e \geq 0 \); \( j = e + 1 \). Therefore we have \( j + 2 \leq e + 3 \leq d' \leq g - 1 - j \), where the last inequality comes from our assumption \( d \leq g - e - 3 \). Hence Lemma 2 applies to get the inequality

\[
\dim W^1_{2e+4}(C) = \dim W^1_{2e+4}(C) \geq e + 1.
\]

By Lemma 5, one has \( \dim W^1_{2e+4}(C) \geq e + 1 \geq 0 \) and it follows that

\[
\text{Cliff}(C) \leq (2e + 3) - 2 = 2e + 1 \leq 2e + 2,
\]

as wanted; note that \( g^1_{2e+3} \in W^1_{2e+3}(C) \) contributes to the Clifford index of \( C \) by the genus assumption \( g \geq 4e + 7 \). Therefore, for the rest of the proof, we may assume that \( r \geq 2 \) and that

\[
(1) \quad \dim W^1_n(C) \leq n - 4 - e
\]

for any \( n \leq g - e - 3 \).

**Claim 2.** If \( r \geq 2 \), then \( g^r_d(z) \) is simple for a general \( z \in Z \).

Assume \( g^r_d(z) \) is compounded for a general \( z \in Z \). Then \( g^r_d(z) \) induces an \( n \)-sheeted covering map \( \pi : C \rightarrow C' \) onto a smooth curve \( C' \) of genus \( g' \) with \( n \mid d \) and \( n \geq 2 \). Then \( g^r_d(z) \) is the pull back of a base-point-free complete series \( g^r_d/n \) on \( C' \) with respect to \( \pi \); i.e. \( g^r_d(z) = \pi^*(g^r_d/n) \).

Let \( g' = 0 \). Then \( (d/n) - r = g' = 0 \) and \( Z \subset r \cdot W^1_n(C) \). Hence one has

\[
d - 3r - e \leq \dim W^1_n(C) \leq n - 4 - e,
\]

where the second inequality follows from (1). Therefore \( (n - 3)(r - 1) \leq -1 \) and hence it follows that \( n = 2 \); but this is a contradiction since \( C \) is non-hyperelliptic.

Next, we assume \( g' > 0 \). By de Franchis’ theorem, we may assume that the map
$W_{d/n}(C') \xrightarrow{π^*} Z$ is a finite dominant map. Hence,

$$0 \leq d - 3r - e = \dim Z \leq \dim W_{d/n}(C').$$

Assume $g_{d/n}^r$ is special. Then $\dim W_{d/n}(C') \leq (d/n) - 2r$ by H. Martens’ theorem [16]. Hence, we have $0 \leq d - 3r - e = \dim Z \leq (d/n) - 2r$. Therefore it follows that $(n - 1)d \leq n(r + e)$ and $d \geq 3r + e$. Hence we have

$$\text{Cliff}(C) \leq d - 2r \leq \frac{n}{n - 1}(r + e) - 2r$$

and a simple computation leads to $\text{Cliff}(C) \leq 2e + 2$ as wanted.

Assume $g_{d/n}^r$ is non-special. Again by de Franchis’ theorem, the map $J(C') = W_{d/n}(C') \xrightarrow{π^*} Z$ is a finite dominant map and

$$\dim W_{d/n}(C') = \dim \text{Jac}(C') = g' = \frac{d}{n} - r = \dim Z = d - 3r - e.$$ (2)

We shall treat the cases $n = 2$ and $n \geq 3$ separately.

$n = 2$: Since $\text{gon}(C) = k$ is odd, the morphism $C \to \mathbb{P}^1$ induced by a $g^1_k$ does not factor through $π$. Hence, Lemma 6 (Castelnuovo-Severi bound) gives $g \leq k = 1 + 2g'$. Since $k \leq 2 \cdot \text{gon}(C') \leq 2 \cdot (g' + 3)/2$, we get $g \leq 3g' + 2$. Note that the equality (2) for $n = 2$ implies $d = 4r + 2e$ and $g' = r + e$. Therefore from the assumption $d \leq g - e - 3$, we have $d + e + 3 \leq g \leq 3g' + 2 \Rightarrow 4r + 2e + e + 3 \leq 3g' + 2 \Rightarrow g' \leq e - 1$. Hence $g \leq 3(e - 1) + 2$, a contradiction to $g \geq 4e + 7$.

$n \geq 3$: We remark that $π^*(W_{d/n-r+1}(C')) \subset W_{d-n(r-1)}^1(C)$. Hence by the equality (2), we have

$$\dim π^*(W_{d/n-r+1}(C')) = \dim W_{d/n-r+1}(C') = \dim J(C') = d - 3r - e$$ (3)

$$\leq \dim W_{d-n(r-1)}^1(C).$$

Since $d - 3r - e \geq d - n(r - 1) - 3 - e$ for $n \geq 3$ and $d - n(r - 1) \leq g - e - 3$, the above inequality (3) is contradictory to our assumption (1). And this finishes the proof of Claim 2.

Since $g_{d}^r(z)$ is simple for a general $z \in Z$ if $r \geq 2$, we may apply Accola-Griffiths-Harris theorem [8, page 73] to our current situation and we have the following inequality;

$$d - 3r - e \leq \dim W_{d/n}^0(C) \leq \dim T_D W_{d/n}^{r}(C) \leq h^0(2D) - 3r$$ for $D \in g_{d}^r(z),$

and it follows that

$$d - e \leq h^0(2D) = 2d + 1 - g + h^1(2D),$$
On the other hand, by the numerical bound \( d \leq g - e - 3 \) which we have assumed, we see that \( h^1(2D) \geq g - d - 1 - e \geq 2 \) and hence the linear series \(|2D|\) contributes to the Clifford index of \( C \). Therefore we finally have

\[
(4) \quad \text{Cliff}(C) \leq \text{Cliff}(2D) = 2d - 2h^0(2D) + 2 \leq 2d - 2(d - e - 1) = 2(e + 1),
\]

and this finishes the proof of the theorem.

One may refine the statement in Theorem 1 for small \( e \leq 6 \) as follows by looking at our proof more carefully, which Takao Kato has kindly informed the authors through Akira Ohbuchi.

**Corollary 9.** Let \( e \) be a fixed integer with \( 0 \leq e \leq 6 \) and let \( C \) be a smooth algebraic curve of genus \( g \geq 4e + 7 \). Suppose that the gonality \( \text{gon}(C) \) of the curve \( C \) is an odd integer. Assume that

\[
d - 3r - e \leq \dim W^r_d(C)
\]

for some \( d, r \geq 1 \) such that \( d \leq g - e - 3 \). Then

\[
\text{Cliff}(C) \leq 2(e + 1).
\]

Furthermore the equality holds if and only if \( C \) is a smooth plane curve of degree \( 2e + 6 \).

Proof. We use the same notations which we used in the proof of Theorem 1. We first remark that everywhere in the course of the proof of Theorem 1, we indeed had \( \text{Cliff}(C) \leq 2e + 1 \) except for the case \( r \geq 2 \) and \( g^1_d(z) \) is simple for a general \( z \in Z \). Therefore, we assume \( \text{Cliff}(C) = 2e + 2 \) and \( g^1_d(z) = |D| \) is simple for a general \( z \in Z \) and \( r \geq 2 \). Hence by the inequality (4), \( \text{Cliff}(2D) = \text{Cliff}(C) = 2e + 2 \). We now distinguish two cases.

(i) \( 2d \leq g - 1 \): By [5, Theorem C] which provides an upper bound of the degree of a complete linear series \( D \) such that \( \text{Cliff}(C) = \text{Cliff}(D) \), we have \( 2d \leq 4e + 8 \). On the other hand

\[
2e + 2 = \text{Cliff}(C) \leq \text{Cliff}(D) = d - 2r \leq 2e + 4 - 2r,
\]

and it follows that \( r \leq 1 \), contrary to our assumption \( r \geq 2 \).

(ii) \( 2d \geq g - 1 \): Note that \( |K - 2D| = g^0_{2g - 2 - 2d} \) since \( \text{Cliff}(K - 2D) = \text{Cliff}(2D) \). We again apply [5, Theorem C] to the linear series \(|K - 2D|\); \( d' = \deg |K - 2D| = 2g - 2 - 2d \leq 4e + 8 \) and hence

\[
r' = \dim |K - 2D| \leq e + 3.
\]
We now briefly recall the so-called Clifford dimension of a smooth algebraic curve $C$, denoted by $\text{Cliffdim}(C)$, which is defined to be the minimum possible dimension $r(D)$ of a complete linear series $D$ such that $\text{Cliff}(C) = \text{Cliff}(D)$ and $D$ contributes to the Clifford index of $C$; cf. [6, page 174]. By $r' \leq e+3$ and by our numerical hypothesis $e \leq 6$, we have

$$\text{Cliffdim}(C) \leq r' \leq e+3 \leq 9,$$

which in turn implies $\text{Cliffdim}(C) = 1$ or $2$ by the last statement in [6, page 203], which asserts in particular that for $3 \leq r \leq 9$ a curve of Clifford dimension $r$ is of even gonality. The case $\text{Cliff dim}(C) = 1$ cannot occur; if then $\text{gon}(C) = 2 + 4$ and is of even gonality. Therefore $\text{Cliffdim}(C) = 2$ and by a simple fact that a complete linear series $D$ with $\dim(D) = \text{Cliffdim}(C) \geq 2$ is very ample [6, Lemma 1.1, page 177], we deduce that $C$ is a smooth plane curve of degree $2e+6$.

References

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