Ballico, E. and Keem, C.
Osaka J. Math.
39 (2002), 283–292

CLIFFORD INDEX OF SMOOTH ALGEBRAIC CURVES OF ODD GONALITY WITH BIG $W^r_d(C)$*

Dedicated to Professor Sang Moon Kim on the occassion of his retirement.

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(Received July 19, 2000)

0. Introduction

Let C be a smooth projective irreducible algebraic curve over the field of complex numbers $\mathbb{C}$ or a compact Riemann surface of genus g. Let $J(C)$ be the Jacobian variety of the curve C, which is a g-dimensional abelian variety parameterizing all the line bundles of given degree d on C. We denote by $W^r_d(C)$ a subvariety of the Jacobian variety $J(C)$ consisting of line bundles of degree d with $r+1$ or more independent global sections.

If $d > g+r-2$, one can compute the dimension of $W^r_d(C)$ by using the Riemann-Roch formula, and this dimension is independent of C. If $d \leq g+r-2$, the dimension of $W^r_d(C)$ is known to be greater than or equal to the Brill-Noether number $\rho(d,g,r) := g-(r+1)(g-d+r)$ for any curve C, and is equal to $\rho(d,g,r)$ for general curve C by theorems of Kleiman-Laksov [13] and Griffiths-Harris [7]. On the other hand, the maximal possible dimension of $W^r_d(C)$ for this range of d, g and r is $d-2r$ and the maximum is attained if and only if C is hyperelliptic by a well known theorem of H. Martens [16].

From a result of M. Coppens, G. Martens and C. Keem [4, Corollary 3.3.2], it is known that for curves of odd gonality — i.e. curves for which the minimal number of sheets of a covering over $\mathbb{P}^1$ is odd — the theorem of H. Martens can be refined significantly.

*The authors are grateful to Professors Akira Ohbuchi and Takao Kato for several comments and suggestions on a previous version of this paper.
† Partially supported by MURST.
‡ Partially supported by Seoam Scholarship Foundation for a visit to the Department of Mathematics of University of Notre Dame where this manuscript was prepared for publication. Also supported in part by KOSEF 981-0101-005-1.

Proposition A (Coppens, Keem and G. Martens). *Let C be a smooth algebraic curve of odd gonality. Then*

$$\dim W^r_d(C) \leq d - 3r$$

for $d \leq g - 1$.

Furthermore, by a recent progress made by G. Martens [14] as well as a result of T. Kato and C. Keem [11], it is known that if the dimension of $W^r_d(C)$ for curves C of odd gonality is near to the maximum possible value, then C is of very special type of curves.

Proposition B (G. Martens [14, Theorem 2]). *Let C be a smooth projective irreducible curve of genus g over the complex number field. Assume that the gonality of C is odd. If* $\dim W^r_d(C) = d - 3r$ *for some* $d \leq g - 2$ *and* $r > 0$ *then C is either trigonal, smooth plane sextic, birational to a plane curve of degree* 7 (*in this case only* $g = 13$ *and* $g = 14$ *occur*; *with a simple* $g^4_{12} = g^1_5 + g^2_7$ *or a very ample* $g^4_{12} = g^1_5 + g^2_7$ *respectively*) *or an extremal space curve of degree* 10 *with a very ample* $g^5_{15} = g^3_{10} + g^1_5$.

Proposition C (T. Kato, C. Keem [11, Theorem 1]). *Let C be a smooth irreducible projective curve of genus g over the complex number field. Assume the gonality of C is odd and* $\dim W^r_d(C) = d - 3r - 1$ *for some* $d \leq g - 4$ *and* $r > 0$. *Then C is* 5-*gonal with* $10 \leq g \leq 18$, $g = 20$ *or* 7-*gonal of genus* 21; *furthermore C is a smooth plane sextic* (*resp. octic*) *in case* $\mathrm{gon}(C) = 5$, $g = 10$ (*resp.* $\mathrm{gon}(C) = 7$, $g = 21$).

The purpose of this paper is to chase a further generalization of the above results of G. Martens and Kato–Keem. We use standard notation for divisors, linear series, invertible sheaves and line bundles on algebraic curves following [3]. As usual, g^r_d is an r-dimensional linear series of degree d on C, which may be possibly incomplete. If D is a divisor on C, we write $|D|$ for the associated complete linear series on C. By K_C or K we denote a canonical divisor on C. If L is a line bundle (or an invertible sheaf) we sometimes abbreviate the notation $H^i(C, L)$ (resp. $\dim H^i(C, L)$) by $H^i(L)$ (resp. $h^i(L)$) for simplicity when no confusion is likely to occur. Also, for a divisor D on C we write $H^i(D), h^i(D)$ instead of $H^i(C, \mathcal{O}_C(D))$, $\dim H^i(C, \mathcal{O}_C(D))$. A base-point-free g^r_d on C defines a morphism $f : C \to \mathbb{P}^r$ onto a non-degenerate irreducible (possibly singular) curve in $\mathbb{P}^r$. If f is birational onto its image $f(C)$ the given g^r_d is called simple or birationally very ample. In case the given g^r_d is not simple, let C' be the normalization of $f(C)$. Then there is a morphism (a non-trivial covering map) $C \to C'$ and we use the same notation f for this covering map of some degree k induced by the original morphism $f : C \to \mathbb{P}^r$. The gonality of C which is the minimal sheet number of a covering over $\mathbb{P}^1$ is denoted by $\mathrm{gon}(C)$. We also recall that given a line bundle $L \in \mathrm{Pic}(C)$, the Clifford index $\mathrm{Cliff}(L)$ of L is defined by

$\text{Cliff}(L) := \deg L - 2(h^0(L) - 1)$, and the Clifford index $\text{Cliff}(C)$ of C is defined by

$$\text{Cliff}(C) := \min\{\, \text{Cliff}(L) : L \in \text{Pic}(C) \text{ with } h^0(L) \geq 2 \text{ and } h^1(L) \geq 2 \,\}.$$

We say that a line bundle L contributes to the Clifford index of C if $h^0(L) \geq 2$ and $h^1(L) \geq 2$. As is well-known, the Clifford index of a smooth algebraic curve is a measurement how special a curve is in the sense of moduli. Specifically, if $k = \text{gon}(C)$ then $\text{Cliff}(C) \leq k - 2$ for any curve C and $\text{Cliff}(C) = k - 2$ for a general k-gonal curve; cf. [12] for more details. The result of this paper is the following theorem.

Theorem 1. *Let $e \geq 0$ be a fixed integer and let C be a smooth algebraic curve of genus $g \geq 4e+7$. Suppose that the gonality $\text{gon}(C)$ of the curve C is an odd integer. Assume that*

$$d - 3r - e \leq \dim W^r_d(C)$$

for some $d, r \geq 1$ such that $d \leq g - e - 3$. Then

$$\text{Cliff}(C) \leq 2(e + 1).$$

In proving our result, we use standard techniques in the theory of linear series on curves such as the Castelnuovo-Severi inequality, excess linear series argument as well as the Accola-Griffiths-Harris theorem.

1. Proof of Theorem 1

A proof of Theorem 1 requires several preparatory results and we begin with the following theorem due to Matelski [15]; see also [9, Corollary 1].

Lemma 2. *Let C be a smooth algebraic curve of genus $g \geq 4j + 3$, $j \geq 0$. If $\dim W^1_d(C) = d-2-j$ for some d such that $j+2 \leq d \leq g-1-j$, then $\dim W^1_{2j+2}(C) \geq j$.*

For positive integers d, r, let $m = [(d-1)/(r-1)]$, $\varepsilon = d-m(r-1)-1$, $\varepsilon_1 = d-m_1 r-1$. We set

$$\pi(d, r) = \frac{m(m-1)}{2}(r-1) + m\varepsilon.$$

Lemma 3 (Castelnuovo's bound). *Assume C admits a base-point-free and simple linear series g^r_d. Then $g \leq \pi(d, r)$.*

Lemma 4 ([1, §7]). *If C admits infinite number of base-point-free simple linear series g^r_d's, then $g \leq \pi(d + 1, r + 1)$.*

Lemma 5 (Excess linear series [3, VII Exercise C, page 329]). *On any curve C,*

$$\dim W^r_{d-1}(C) \geq \dim W^r_d(C) - r - 1.$$

The following is a special case of the so-called Castelnuovo-Severi inequality.

Lemma 6 (Castelnuovo-Severi bound [2, Theorem 3.5]). *Assume there exist two curves C_1 and C_2 of genus g_1 and g_2, respectively, so that C is a k_i-sheeted covering of C_i $(i = 1,\ 2)$. If k_1 and k_2 are coprime, then*

$$g \leq (k_1 - 1)(k_2 - 1) + k_1 g_1 + k_2 g_2.$$

Lemma 7 (Extension of H. Martens' theorem [10]). *Let d and r be positive integers such that $d \leq g + r - 4$, $r \geq 1$. If*

$$\dim W^r_d(C) \geq d - 2r - 2 \geq 0$$

then C is either hyperelliptic, trigonal, bi-elliptic, tetragonal, a smooth plane sextic or a double covering of a curve of genus 2.

We also need the following result due to M. Coppens and G. Martens which may be considered as a "Clifford's theorem" for curves of odd gonality.

Lemma 8 (M. Coppens, G. Martens [5]). *Let D be an effective divisor on a curve C of genus g and of odd gonality such that $\deg D < g$. Then $\dim |D| \leq (1/3)\deg D$.*

Proof of Theorem 1. For $e = 0$, the result holds by Proposition B if C does not belong to the following special classes of curves described in Proposition B;
(i) a 5-gonal curve of genus $g = 14$ with a very ample $g^4_{12} = g^1_5 + g^2_7$
(ii) a 5-gonal curve of genus $g = 13$ with a simple $g^4_{12} = g^1_5 + g^2_7$
(iii) a 5-gonal extremal space curve of degree 10 and genus $g = 16$ with a very ample $g^5_{15} = g^1_5 + g^3_{10}$.
We first argue that these curves do not satisfy $\dim W^r_d(C) = d - 3r$ for any $d \leq g - 3$ and $r > 0$. If $\dim W^r_d(C) = d - 3r$ for some $d \leq g - 3$ with $r = 1$ or $r = 2$, then C must be a curve of gonality $\mathrm{gon}(C) \leq 4$ by Lemma 7. Therefore we now assume that $\dim W^r_d(C) = d - 3r$ for some $d \leq g - 3$ with $r \geq 3$.

CASE (i): If C is a 5-gonal curve of genus $g = 14$ with a very ample $g^4_{12} = g^1_5 + g^2_7$, $W^r_d(C) = \emptyset$ for any $r \geq 3$ and $d \leq 9$ by Lemma 3 (Castelnuovo genus bound). Since $g = 14$ and $d \leq g - 3$, we have $r \leq 3$ by Lemma 8. Furthermore, it is easy to see that $\dim W^3_{10}(C) \leq 0$. Suppose otherwise. Then there exist infinitely many $g^3_{10} \in W^3_{10}(C)$ which must be base-point-free and simple. Therefore one can apply

Lemma 4 to get the contradiction $g \leq 12$. Finally, suppose that $\dim W^3_{11}(C) = 2$. Since we already have $\dim W^3_{10}(C) \leq 0$, it is clear that a general $\mathcal{L} \in W^3_{11}(C)$ is base-point-free and hence birationally very ample. For a general $\mathcal{L} = g^3_{11} \in W^3_{11}(C)$, we consider $h^0(C, K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g^1_5))$. If $h^0(C, K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g^1_5)) \geq 4$, then $|K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g^1_5)| = g^3_{10}$ for a general $\mathcal{L} \in W^3_{11}(C)$, and hence $\dim W^3_{10}(C) = 2$, contrary to $\dim W^3_{10}(C) \leq 0$. Therefore we must have $h^0(C, K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g^1_5)) \leq 3$ for a general $\mathcal{L} \in W^3_{11}(C)$. Then, by the base-point-free pencil trick, applied to the natural map

$$H^0(C, \mathcal{L}) \oplus H^0(C, \mathcal{L}) \longrightarrow H^0(C, \mathcal{L} \otimes \mathcal{O}_C(g^1_5)),$$

one concludes that $h^0(C, \mathcal{L} \otimes \mathcal{O}(-g^1_5)) \geq 2$, for a a general $\mathcal{L} \in W^3_{11}(C)$, which in turn implies $\dim W^1_6(C) = 2$. Then by Lemma 7, we have $\text{gon}(C) \leq 4$, which is a contradiction.

CASE (ii): If C is a 5-gonal curve of genus $g = 13$, exactly the same argument as in the Case (i) is still valid for this case to show that $\dim W^r_d(C) \lneq d - 3r$ for any $d \leq g - 3$ and $r > 0$.

CASE (iii): Let C be a 5-gonal extremal space curve of degree 10 and genus $g = 16$. Note that C is a complete intersection of a quintic and a quadric in $\mathbb{P}^3$. For $d \leq 9$ and $r \geq 3$, $W^r_d(C) = \emptyset$ by Lemma 3. For the case $(d, r) = (10, 3)$, we apply the same argument as in the case (i) above to show that $\dim W^3_{10}(C) \leq 0$. For the case $(d, r) = (11, 3)$, suppose that $\dim W^3_{11}(C) = 2$. Since we already have $\dim W^3_{10}(C) \leq 0$, a general g^3_{11} must be base-point-free and simple. Then by Lemma 4 we get a contradiction $g \leq 15$. Let $(d, r) = (12, 3)$ and assume that $\dim W^3_{12}(C) = 3$. For a general $\mathcal{L} = g^3_{12} \in W^3_{12}(C)$, we again consider $h^0(C, K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g^1_5))$. If $h^0(C, K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g^1_5)) \geq 5$, then $|K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g^1_5)| = g^4_{13}$ for a general $\mathcal{L} \in W^3_{12}(C)$, and hence $\dim W^4_{13}(C) \geq 3$, a contradiction to Proposition A. Therefore we must have $h^0(C, K\mathcal{L}^{-1} \otimes \mathcal{O}_C(-g^1_5)) \leq 4$ for a general $\mathcal{L} \in W^3_{12}(C)$. By applying the base-point-free pencil trick to the natural map

$$H^0(C, \mathcal{L}) \oplus H^0(C, \mathcal{L}) \longrightarrow H^0(C, \mathcal{L} \otimes \mathcal{O}_C(g^1_5)),$$

one concludes that $h^0(C, \mathcal{L} \otimes \mathcal{O}(-g^1_5)) \geq 2$, for a a general $\mathcal{L} \in W^3_{12}(C)$, which in turn implies $\dim W^1_7(C) \geq 3$. Then by Lemma 7, we have $\text{gon}(C) \leq 4$, which is a contradiction. Let $(d, r) = (12, 4)$ and assume that $\dim W^4_{12}(C) = 0$. If g^4_{12} is not simple, then C is either trigonal or a double cover of a curve of genus $h \leq 2$, a contradiction. If g^4_{12} is simple, then $g \leq 15$ by Lemma 3, again a contradiction. For the case $(d, r) = (13, 3)$, we can use an argument almost parallel to the case $(d, r) = (12, 3)$ to show that $\dim W^3_{13}(C) \lneq 4$. Finally let $(d, r) = (13, 4)$ and assume that $\dim W^4_{13}(C) = 1$. Since we already know $W^4_{12}(C) = \emptyset$, every $g^4_{13} \in W^4_{13}(C)$ is base-point-free and hence simple. Therefore one applies Lemma 4 to get the contradiction $g \leq 15$. In all, we conclude that our theorem holds for $e = 0$.

For $e = 1$, the theorem is valid by Proposition C. Hence from now on, we may assume that $e \geq 2$ and $\text{gon}(C) \geq 7$; note that if $g \geq 4e + 7$, the curves C in

Proposition B and Proposition C have $\text{gon}(C) \leq 5$. By induction, we assume that $\dim W^r_d(C) = d - 3r - e$ for some $d \leq g - e - 3$ and $r \geq 1$.

Let Z be an irreducible component of $W^r_d(C)$ of dimension $d-3r-e$ and let $g^r_d(z)$ be the linear series associated to an element $z \in Z$. By the fact that no component of $W^r_d(C)$ is properly contained in a component of $W^{r+1}_d(C)$, we may assume that $g^r_d(z)$ is complete for a general $z \in Z$; cf. [3, Lemma 3.5–page 182]. By shrinking if necessary, one may further assume that $g^r_d(z)$ is base-point-free for a general $z \in Z$. We first treat the case $r = 1$, which is relatively easy.

CLAIM 1. If $r = 1$, then $\text{Cliff}(C) \leq 2(e+1)$.

For $r = 1$, we set $\dim W^1_d(C) = d - 2 - j = d - 3 - e \geq 0$; $j = e + 1$. Therefore we have $j + 2 \leq e + 3 \leq d \leq g - 1 - j$, where the last inequality comes from our assumption $d \leq g - e - 3$. Hence Lemma 2 applies to get the inequality

$$\dim W^1_{2(e+1)+2}(C) = \dim W^1_{2e+4}(C) \geq e + 1.$$

By Lemma 5, one has $\dim W^1_{2e+3}(C) \geq e - 1 \geq 0$ and it follows that

$$\text{Cliff}(C) \leq (2e+3) - 2 = 2e + 1 \leq 2e + 2,$$

as wanted; note that $g^1_{2e+3} \in W^1_{2e+3}(C)$ contributes to the Clifford index of C by the genus assumption $g \geq 4e+7$. Therefore, for the rest of the proof, we may assume that $r \geq 2$ and that

$$\dim W^1_n(C) \leq n - 4 - e \tag{1}$$

for any $n \leq g - e - 3$.

CLAIM 2. If $r \geq 2$, then $g^r_d(z)$ is simple for a general $z \in Z$.

Assume $g^r_d(z)$ is compounded for a general $z \in Z$. Then $g^r_d(z)$ induces an n-sheeted covering map $\pi : C \to C'$ onto a smooth curve C' of genus g' with $n \mid d$ and $n \geq 2$. Then $g^r_d(z)$ is the pull back of a base-point-free complete series $g^r_{d/n}$ on C' with respect to π; i.e. $g^r_d(z) = \pi^*(g^r_{d/n})$.

Let $g' = 0$. Then $(d/n) - r = g' = 0$ and $Z \subset r \cdot W^1_n(C)$. Hence one has

$$d - 3r - e \leq \dim W^1_n(C) \leq n - 4 - e,$$

where the second inequality follows from (1). Therefore $(n-3)(r-1) \leq -1$ and hence it follows that $n = 2$; but this is a contradiction since C is non-hyperelliptic.

Next, we assume $g' > 0$. By de Franchis' theorem, we may assume that the map

$W^r_{d/n}(C') \xrightarrow{\pi^*} Z$ is finite dominant map. Hence,

$$0 \leq d - 3r - e = \dim Z \leq \dim W^r_{d/n}(C').$$

Assume $g^r_{d/n}$ is special. Then $\dim W^r_{d/n}(C') \leq (d/n) - 2r$ by H. Martens' theorem [16]. Hence, we have $0 \leq d - 3r - e = \dim Z \leq (d/n) - 2r$. Therefore it follows that $(n-1)d \leq n(r+e)$ and $d \geq 3r + e$. Hence we have

$$\mathrm{Cliff}(C) \leq d - 2r \leq \frac{n}{n-1}(r+e) - 2r$$

and a simple computation leads to $\mathrm{Cliff}(C) \leq 2e + 2$ as wanted.
Assume $g^r_{d/n}$ is non-special. Again by de Franchis' theorem, the map $J(C') = W^r_{d/n}(C') \xrightarrow{\pi^*} Z$ is a finite dominant map and

$$\dim W^r_{d/n}(C') = \dim \mathrm{Jac}(C') = g' = \frac{d}{n} - r = \dim Z = d - 3r - e. \tag{2}$$

We shall treat the cases $n = 2$ and $n \geq 3$ separately.
$n = 2$: Since $\mathrm{gon}(C) = k$ is odd, the morphism $C \longrightarrow \mathbb{P}^1$ induced by a g^1_k does not factor through π. Hence, Lemma 6 (Castelnuovo-Severi bound) gives $g \leq k - 1 + 2g'$. Since $k \leq 2 \cdot \mathrm{gon}(C') \leq 2 \cdot (g'+3)/2$, we get $g \leq 3g' + 2$. Note that the equality (2) for $n = 2$ implies $d = 4r + 2e$ and $g' = r + e$. Therefore from the assumption $d \leq g - e - 3$, we have $d + e + 3 \leq g \leq 3g' + 2 \Rightarrow 4r + 2e + e + 3 \leq 3g' + 2 \Rightarrow g' \leq e - 1$. Hence $g \leq 3(e-1) + 2$, a contradiction to $g \geq 4e + 7$.
$n \geq 3$: We remark that $\pi^*(W^1_{d/n-r+1}(C')) \subset W^1_{d-n(r-1)}(C)$. Hence by the equality (2), we have

$$\begin{aligned} \dim \pi^*(W^1_{d/n-r+1}(C')) = \dim W^1_{d/n-r+1}(C') = \dim J(C') = d - 3r - e \\ \leq \dim W^1_{d-n(r-1)}(C). \end{aligned} \tag{3}$$

Since $d - 3r - e \geq d - n(r-1) - 3 - e$ for $n \geq 3$ and $d - n(r-1) \leq g - e - 3$, the above inequality (3) is contradictory to our assumption (1). And this finishes the proof of Claim 2.

Since $g^r_d(z)$ is simple for a general $z \in Z$ if $r \geq 2$, we may apply Accola-Griffiths-Harris theorem [8, page 73] to our current situation and we have the following inequality;

$$d - 3r - e \leq \dim W^r_d(C) \leq \dim T_{|D|} W^r_d(C) \leq h^0(2D) - 3r \quad \text{for} \quad D \in g^r_d(z),$$

and it follows that

$$d - e \leq h^0(2D) = 2d + 1 - g + h^1(2D).$$

On the other hand, by the numerical bound $d \leq g - e - 3$ which we have assumed, we see that $h^1(2D) \geq g - d - 1 - e \geq 2$ and hence the linear series $|2D|$ contributes to the Clifford index of C. Therefore we finally have

$$\text{(4)} \qquad \mathrm{Cliff}(C) \leq \mathrm{Cliff}(2D) = 2d - 2h^0(2D) + 2 \leq 2d - 2(d - e - 1) = 2(e + 1),$$

and this finishes the proof of the theorem. □

One may refine the statement in Theorem 1 for small $e \leq 6$ as follows by looking at our proof more carefully, which Takao Kato has kindly informed the authors through Akira Ohbuchi.

Corollary 9. *Let e be a fixed integer with $0 \leq e \leq 6$ and let C be a smooth algebraic curve of genus $g \geq 4e + 7$. Suppose that the gonality $\mathrm{gon}(C)$ of the curve C is an odd integer. Assume that*

$$d - 3r - e \leq \dim W^r_d(C)$$

for some $d, r \geq 1$ such that $d \leq g - e - 3$. Then

$$\mathrm{Cliff}(C) \leq 2(e + 1).$$

Furthermore the equality holds if and only if C is a smooth plane curve of degree $2e + 6$.

Proof. We use the same notations which we used in the proof of Theorem 1. We first remark that everywhere in the course of the proof of Theorem 1, we indeed had $\mathrm{Cliff}(C) \leq 2e + 1$ except for the case $r \geq 2$ and $g^r_d(z)$ is simple for a general $z \in Z$. Therefore, we assume $\mathrm{Cliff}(C) = 2e + 2$ and $g^r_d(z) = |D|$ is simple for a general $z \in Z$ and $r \geq 2$. Hence by the inequality (4), $\mathrm{Cliff}(2D) = \mathrm{Cliff}(C) = 2e + 2$. We now distinguish two cases.
(i) $2d \leq g - 1$: By [5, Theorem C] which provides an upper bound of the degree of a complete linear series $\mathcal{D}$ such that $\mathrm{Cliff}(C) = \mathrm{Cliff}(\mathcal{D})$, we have $2d \leq 4e + 8$. On the other hand

$$2e + 2 = \mathrm{Cliff}(C) \leq \mathrm{Cliff}(D) = d - 2r \leq 2e + 4 - 2r,$$

and it follows that $r \leq 1$, contrary to our assumption $r \geq 2$.
(ii) $2d \geq g - 1$: Note that $|K - 2D| = g^{g-d-2-e}_{2g-2-2d}$ since $\mathrm{Cliff}(K - 2D) = \mathrm{Cliff}(2D)$. We again apply [5, Theorem C] to the linear series $|K - 2D|$; $d' = \deg |K - 2D| = 2g - 2 - 2d \leq 4e + 8$ and hence

$$r' = \dim |K - 2D| \leq e + 3.$$

We now briefly recall the so-called Clifford dimension of a smooth algebraic curve C, denoted by Cliffdim(C), which is defined to be the minimum possible dimension $r(\mathcal{D})$ of a complete linear series $\mathcal{D}$ such that Cliff(C) = Cliff($\mathcal{D}$) and $\mathcal{D}$ contributes to the Clifford index of C; cf. [6, page 174]. By $r' \leq e+3$ and by our numerical hypothesis $e \leq 6$, we have

$$\text{Cliffdim}(C) \leq r' \leq e+3 \leq 9,$$

which in turn implies Cliffdim(C) = 1 or 2 by the last statement in [6, page 203], which asserts in particular that for $3 \leq r \leq 9$ a curve of Clifford dimension r is of even gonality. The case Cliff dim(C) = 1 cannot occur; if then gon(C) = $2e+4$ and C is of even gonality. Therefore Cliffdim(C) = 2 and by a simple fact that a complete linear series $\mathcal{D}$ with dim($\mathcal{D}$) = Cliffdim(C) ≥ 2 is very ample [6, Lemma 1.1, page 177], we deduce that C is a smooth plane curve of degree $2e+6$. □

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