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<th><strong>Title</strong></th>
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Throughout this paper, we assume that $R$ is an associative ring with identity and $\{M_\alpha\}_I$ is an infinite set of completely indecomposable right $R$-modules. We put $M=\sum_I \oplus M_\alpha$ and $\bar{M}=M/J(M)$, where $J(M)=\sum_I \oplus J(M_\alpha)$ denotes the Jacobson radical of $M$.

If each $M_\alpha$ is a cyclic hollow module, then $M$ is completely reducible. In this case, $M$ is said to have the lifting property of simple modules modulo the radical if every simple submodule of $M$ is induced from a direct summand of $M$ ([3]). On the other hand, for the family $\mathcal{H}$ of all maximal submodules of $M$, $M$ is said to have the lifting property of modules for $\mathcal{H}$ if every member $A$ in $\mathcal{H}$ is co-essentially lifted to a direct summand of $M$, that is, there exists a decomposition $M=A^* \oplus A^{**}$ such that $A^* \subseteq A$ and $A \cap A^{**}$ is small in $M$ ([5]). These two concepts are both dual to 'extending property of simple modules' mentioned in [4]. Therefore, we must observe whether these two lifting properties coincide or not. In this paper, we study this problem and show the following result: $M$ has the lifting property of modules for $\mathcal{H}$ if and only if it has the lifting property of simple modules modulo the radical and satisfies the following condition: For any $\{M_\alpha\}_I \subseteq \{M_\alpha\}_I$ and epimorphisms $\{f_i: M_\alpha \to M_{\alpha_i+1}\}_{i=1}^n$, there exist $n$ (depending on the sets) and epimorphism $g: M_{\alpha_{n+1}} \to M_{\alpha_n}$ such that $g=f_n^{-1}$, where $g$ and $f_n$ are the induced isomorphisms: $M_{\alpha_{n+1}} \to M_{\alpha_n}$ and $M_{\alpha_n} \to M_{\alpha_{n+1}}$, respectively (Theorem 10).

NOTATION. By $P(M)$ we denote the set of all submodules $X$ of $M$ such that $X \cap M_\alpha=M_\alpha$ for all $\alpha \in I$ and $X=\sum_I \oplus (X \cap M_\alpha)$.

We first show

Theorem 1. The following conditions are equivalent:

1) For any pair $\alpha, \beta \in I$, every epimorphism from $M_\alpha$ to $M_\beta$ is an isomorphism.

2) Let $\{A_\beta\}_I$ be a family of indecomposable direct summands of $M$. If $A_{\beta_1}+\cdots+A_{\beta_n}+X\cong A_{\beta_{n+1}}$ for any $X \in P(M)$ and any finite subset $\{\beta_1, \cdots, \beta_{n+1}\}$
If \( J \subseteq I \), then \( \sum J A_\beta \) is a direct sum (and a locally direct summand of \( M \)).

Proof. 1) \( \Rightarrow \) 2). Let \( \beta_1, \ldots, \beta_{n+1} \in J \) and assume that \( A = \sum \bigoplus \oplus M_{\gamma} \). We may show \( A = \sum_{\beta} A_\beta \oplus M\). We see from [1] and [6] that every indecomposable direct summand of \( M \) satisfies the exchange property and hence we have a subset \( I' = \{ \alpha_1, \ldots, \alpha_n \} \subseteq I \) satisfying \( M = A \oplus \bigoplus \oplus M_{\gamma} \). We get either \( M = A \oplus \bigoplus \oplus M_{\gamma} \) for some \( \nu \in I - I' \) or \( M = \sum_{J \subseteq I'} \bigoplus A_{\beta} \oplus \bigoplus \oplus \bigoplus M_{\gamma} \) for some \( i \).

In the former case, \( A = \bigoplus M_{\gamma} \) as desired. In the latter case, \( M = \bigoplus A_{\beta} \oplus \bigoplus \bigoplus M_{\gamma} \) for some \( \alpha \subseteq \{ \alpha_1, \ldots, \alpha_n \} \). For each \( \gamma \in I - I' \), \( \pi_{\gamma} \) denotes the projection: \( M = \bigoplus \bigoplus M_{\gamma} \). If \( \pi_{\gamma}(A_{\beta+1}) = M_{\gamma} \) for all \( \gamma \in I - I' \) then \( X = \bigoplus \bigoplus \bigoplus \pi_{\gamma}(A_{\beta+1}) = \bigoplus \bigoplus M_{\gamma} \). Therefore, \( \pi_{\gamma}(A_{\beta+1}) = M_{\gamma} \) for some \( \gamma \in I - I' \). Since \( M = \bigoplus A_{\beta+1} \oplus \bigoplus \bigoplus \bigoplus M_{\gamma} \) is an isomorphism by the assumption. Hence it follows that \( M = \bigoplus A_{\beta} \oplus \bigoplus \bigoplus M_{\gamma} \). Since \( M = \bigoplus A_{\beta+1} \oplus \bigoplus \bigoplus M_{\gamma} \), where \( K = \{ I - I' \} - \{ \gamma \} \).

2) \( \Rightarrow \) 1). Let \( \alpha, \beta \in I \) and consider an epimorphism \( f: M_a \to M_\beta \). Putting \( M_a' = \{ x + f(x) \mid x \in M_a \} \), we see that \( M_a = M_a' \oplus M \) and \( M_a + X \cong M_a \) for any \( X \in \text{P}(M) \); whence, by 2), \( \ker f = M_a' \cap M_a = 0 \) and hence \( f \) is an isomorphism.

Theorem 2. Assume that each \( X \in \text{P}(M) \) is small in \( M \) or each \( M_\alpha \) is cyclic hollow. Then the following condition is equivalent to each of conditions 1) and 2) in Theorem 1.

(K) If \( M = \bigoplus A_\beta \) is an irredundant sum and each \( A_\beta \) is an indecomposable direct summand, then this sum is a direct sum.

Proof. (K) \( \Rightarrow \) 1) is shown by the same proof as in 2) \( \Rightarrow \) 1) in Theorem 1. Now, assume that 2) holds and let \( M = \bigoplus A_\beta \) be an irredundant sum and each \( A_\beta \) an indecomposable direct summand. First, if each \( X \in \text{P}(M) \) is small in \( M \), then we see that \( A_{\beta_1} + \cdots + A_{\beta_n} \oplus X \cong A_{\beta+1} \) for any \( X \in \text{P}(M) \) and any finite subset \( \{ \beta_1, \cdots, \beta_{n+1} \} \subseteq J \). Hence the sum \( M = \bigoplus A_\beta \) is a direct sum by 2).

Next, consider the case when each \( M_\beta \) is cyclic hollow. Assume that there exist a subset \( \{ \beta_1, \cdots, \beta_n \} \subseteq J \) and \( X \in \text{P}(M) \) such that \( A_{\beta_1} + \cdots + A_{\beta_n} + X \cong A_{\beta_1} + Y \). Then we can take a finite subset \( F \subseteq I \) and \( Y \subseteq \bigoplus M_\alpha \) such that \( A_{\beta_1} + \cdots + A_{\beta_n} + Y \cong A_{\beta_1} + Y \). Since \( Y \) is small in \( M \), this implies that \( M = \bigoplus A_\beta \), a contradiction. Therefore, such \( \{ \beta_1, \cdots, \beta_n \} \) and \( X \) do not exist; whence the sum \( M = \bigoplus A_\beta \) is a direct sum by 2).

Theorem 3. The following conditions are equivalent:

1) For any irredundant sum \( \sum J A_\beta \) of direct summands of \( M \) with the pro-
property that \( A_{\beta_1} + \cdots + A_{\beta_n} + X \cong A_{\beta_{n+1}} \) for any \( X \in P(M) \) and any finite subset \( \{\beta_1, \cdots, \beta_n\} \subseteq J \), the sum \( \sum_{\beta \in J} A_{\beta} \) is a direct sum and moreover a direct summand of \( M \).

2) \( \{M_\alpha\} \) is a locally semi-\( T \)-nilpotent set and 2) in Theorem 1 holds.

Proof. 1) \( \Rightarrow \) 2). We may only show the first condition. Let \( \{M_\alpha\}_{\alpha \in I} \subseteq \{M_\alpha\} \) and \( \{f_i: M_\alpha \to M_{\alpha + i}\}_{i = 1}^\infty \) be a set of non-isomorphisms. Then each \( f_i \) is not an epimorphism by Theorem 1. Consider \( M_{\alpha_i} = \{x + f_i(x) | x \in M_\alpha\} \), \( i = 1, 2, \cdots \). Then, as is easily seen, \( \{M_\alpha\}_{\alpha \in I} \) is a set of indecomposable direct summands of \( M \) and satisfies the condition: \( M_{\beta_1} + \cdots + M_{\beta_n} + X \cong M_{\beta_{n+1}} \) for any \( X \in P(M) \) and \( \{\beta_1, \cdots, \beta_{n+1}\} \subseteq \{\alpha_i\}_{i = 1}^\infty \). Hence we get \( M' = \sum_{\alpha} M_{\alpha} \cong (\oplus M_{\alpha}) \). We put \( N = \sum_{\alpha} M_{\alpha} = M' \oplus T \). Assume that \( T \) is not indecomposable and non-zero. Then, by the Krull-Remak-Schmidt Azumaya's theorem, we see \( M' \cap (M_\alpha \oplus M_{\alpha}) = 0 \) for some \( n \neq m \). But we can verify that this is impossible. As a result, \( T \) is indecomposable or zero, from which we get \( N = M' \) or \( N = M' \oplus M_{\alpha} \) for some \( \alpha \). In either case, we see that for every \( x \in M_\alpha \), there exists \( m \) such that \( f_m f_{m-1} \cdots f_i(x) = 0 \). 2) \( \Rightarrow \) 1) is clear from Theorem 1 and [2, Theorem 3.2.5].

**Definition ([5]).** Let \( \{A_1, \cdots, A_n\} \) be a family of submodules of \( M \). We say that the family is **co-independent** if the canonical map: \( M \to \bigoplus_{i=1}^n (M/A_i) \) is an epimorphism.

**Theorem 4.** The following conditions are equivalent:

1) For any \( \alpha \in I \), every epimorphism from \( \sum_{I \setminus \{\alpha\}}^1 \oplus M_{\beta} \) to \( M_{\alpha} \) splits.

2) If \( \{A_1, \cdots, A_n\} \) is a co-independent family of direct summands of \( M \) such that \( M/A_i \) is indecomposable, then \( \bigcap_{i=1}^n A_i \) is a direct summand of \( M \).

Proof. By [1] and [6], we see that every indecomposable direct summand of \( M \) is isomorphic to some member in \( \{M_\alpha\} \) and hence satisfies the finite exchange property.

2) \( \Rightarrow \) 1). Let \( \alpha \in I \) and \( f: T = \sum_{I \setminus \{\alpha\}}^1 \oplus M_{\beta} \to M_{\alpha} \) be an epimorphism. Putting \( N = \{x + f(x) | x \in T\} \), we see that \( M = N + T \), whence \( \{N, T\} \) is co-independent. Thus \( \ker f = T \cap N \subseteq \oplus M \).

1) \( \Rightarrow \) 2). We show this by induction. So, let \( \{A_1, \cdots, A_n, A\} \) be a co-independent family of direct summands of \( M \) such that each \( M/A_i \) and \( M/A \) are indecomposable, and assume \( B = \bigcap_{i=1}^n A_i \subseteq \oplus M \). Setting

\[ M = A \oplus A^* \]
we see, by the above remark, that either

\[ M = B \oplus X \oplus A^* \]

for some \( X \subseteq B^* \) or

\[ M = B' \oplus A^* \oplus B^* \]

for some \( B' \subseteq B \).

We first assume the former case, and let \( \pi_A : M = A \oplus A^* \to A \) and \( \pi_{A^*} : M = A \oplus A^* \to A^* \) be the projections. Since \( M = A + B \) and \( B \oplus A^* \uplus M \) we see \( \pi_A(B) = A^* \) and \( B \subseteq \pi_A(B) \oplus \pi_{A^*}(B) = B \oplus A^* \uplus M \); so \( \pi_A(B) \uplus M \). Since \( B \cap A = 0 \), the mapping \( f : \pi_A(B) \to A^* \) given by \( \pi_A(b) \to \pi_{A^*}(b) \) is well defined and an epimorphism. As a result, \( B \cap A = \ker f \uplus M \) by the condition 1).

Next consider the latter case:

\[ M = B' \oplus A^* \oplus B^* \]

where \( B' \subseteq B \). Since \( B^* = M/B = M/A_1 \oplus \cdots \oplus M/A_n \), \( B^* \) has the exchange property (cf. [1], [6]) and so does \( A^* \oplus B^* \). Therefore

\[ M = B' \oplus A^* \oplus B^* \]

for some \( A' \subseteq A \). Consider the projections:

\[ \pi_{A^*} : M \to A^*, \pi_{B^*} : M \to B^* \]

with respect to \( M = A' \oplus A^* \oplus B^* \), and

\[ \tau_A : M \to A^*, \tau_{B^*} : M \to B^* \]

with respect to \( M = B' \oplus A^* \oplus B^* \).

Here the mapping \( f : B^* \to A^* \) given by \( \pi_{B^*}(a) \to \pi_{A^*}(a) \) for \( a \in A \) and \( g : A^* \to B^* \)
given by \( \tau_{A^*}(b) \to \tau_{B^*}(b) \) for \( b \in B \) are well defined. Put

\[ X = \{ \pi_{B^*}(a) + \pi_{A^*}(a) \mid a \in A \} , \]

\[ Y = \{ \tau_{A^*}(b) + \tau_{B^*}(b) \mid b \in B \} . \]

Then \( A = A' \oplus X, B = B' \oplus Y, X \oplus A^* = Y \oplus B^* = A^* \oplus B^* \) and

\[ M = A' \oplus X \oplus A^* \]

\[ = B' \oplus Y \oplus B^* . \]

If \( X \oplus A^* = X \oplus T \) for some \( T \subseteq B^* \), then \( B = \{ \delta(b) + \delta'(b) \mid b \in B \} \) where \( \delta \) and
\( \delta' \) are the projections: \( M \to A' \oplus X \) and \( M \to T \), respectively with respect to \( M = A' \oplus X \oplus T \). Noting \( M = A + B \) and \( B \cap T = 0 \), we see \( \delta(B) = A \) and \( \delta'(B) = T \), and further the mapping \( \phi: A \to T \) given by \( \delta(b) \to \delta'(b) \) is well defined and an epimorphism. Consequently \( A \cap B = \ker \phi \oplus M \).

If the case: \( X \oplus A^* = X \oplus T \) for some \( T \subseteq B^* \) does not occur, we must have \( A^* \oplus B^* = X \oplus Y \), so

\[
M = A' \oplus X \oplus Y = B' \oplus Y \oplus X.
\]

Then let \( \eta_A: M \to A' \) and \( \eta_X: M \to X \) be the projections with respect to \( M = A' \oplus X \oplus Y \). Putting \( Z = \{ \eta_A(b') + \eta_X(b') \mid b' \in B' \} \), we get \( Z \subset A = A' \oplus X \) and \( A \cap B = Z \subset \ker \phi \). The proof is now completed.

**Remark.** a) Under the assumptions that each \( M_a \) is cyclic hollow and \( J(M) \) is small in \( M' \) the equivalence of 1) in Theorem 1 and \( (K) \) in Theorem 2 was shown in [3]. Theorem 2 says that this second assumption is superfluous. b) In the case when each \( M_a \) is cyclic hollow, the condition 1) in Theorem 1 and 1) in Theorem 4 are clearly equivalent and hence all conditions in Theorems 1, 2 and 4 are equivalent. We also know from [3] that the following condition is also an equivalent condition: If \( \{ A_a \} \) is a family of direct summands of \( M \) such that \( \{ \bar{A}_a \} \) is independent in \( \bar{M} = M / J(M) \), then the sum \( \sum \bar{A}_a \) is a direct sum and a locally direct summand.

**Theorem 5.** The following conditions are equivalent:

1) For any independent family \( \{ A_\beta \} \) of indecomposable direct summands of \( M \), \( \sum_\beta A_\beta \) is a locally direct summand.

2) For any \( \alpha \in I \) and any monomorphism \( f: M_\alpha \to \sum_\beta \alpha A_\beta \), \( f(M_\alpha) \) is a direct summand of \( \sum_\beta \alpha A_\beta \).

**Proof.** The proof is done as in the proof of [4, Theorem 13].

1) => 2). Let \( \alpha \in I \) and consider a monomorphism \( f: M_\alpha \to T = \sum_\beta \alpha M_\beta \).

Put \( M'_\alpha = \{ x + f(x) \mid x \in M_\alpha \} \). Then \( M'_\alpha \cap T = 0 \) and \( M'_\alpha \oplus T = M_\alpha \oplus T \); whence \( M'_\alpha = M_\alpha \) and \( M'_\alpha \) is a direct summand of \( M_\alpha \oplus M_\beta \). Further \( M'_\alpha \cap M_\alpha = 0 \) and hence it follows from 1) that \( M'_\alpha \oplus M_\alpha = M_\alpha \oplus \text{Im } f \oplus M \); so \( \text{Im } f \oplus T \).

2) => 1). We may show the following: If \( \{ A_1, \ldots, A_n \} \) is an independent set of indecomposable direct summands of \( M \), \( A_1 \oplus \cdots \oplus A_n \) is also a direct summand of \( M \).

If \( n = 1 \), this is clear. Assume \( n > 1 \) and \( A = A_1 \oplus \cdots \oplus A_{n-1} \oplus M \). Since each member of \( \{ A_1, \ldots, A_{n-1} \} \) is isomorphic to some member in \( \{ M_a \} \) (cf. [1]), \( A \) has the exchange property (cf. [6]), so
for some subset \( J \subseteq I \). Since \( A_n \) has the exchange property,

\[
M = A_1 \oplus \cdots \oplus A_{k-1} \oplus A_{k+1} \oplus \cdots \oplus A_{n-1} \oplus A_n \oplus \sum_j \oplus M_j \cdots \quad (*)
\]

for some \( k \) or

\[
M = A \oplus A_n \oplus \sum_j \oplus M_j
\]

for some \( \sigma \in J \). In the latter case the proof is completed. In the former case, \( A_\lambda \supseteq \sum_j \oplus M_j \) for some \( \lambda \in I - J \) and \( f=\pi|A_k: A_k \to \sum_j \oplus M_j \) is a monomorphism, where \( \pi \) denotes the projection: \( M \to \sum_j \oplus M_j \) with respect to \((*)\). By 1), \( f(A_\lambda) \oplus M \) and hence we see that \( A \oplus A_n \oplus M \).

**Theorem 6.** Assume that each \( M_\alpha \) is uniform. Then the following conditions are equivalent:

1) For any pair \( \alpha, \beta \in I \), every monomorphism from \( M_\alpha \) to \( M_\beta \) is an isomorphism.

2) For any \( \alpha \in I \) and any monomorphism \( f \) from \( M_\alpha \) to \( \sum_{j \in (\alpha)} \oplus M_\beta \), the image \( f(M_\alpha) \) is a direct summand.

Proof. 2) \( \Rightarrow \) 1) is clear. Assume 1). Let \( \alpha \in I \) and consider a monomorphism \( f: M_\alpha \to \sum_{j \in (\alpha)} \oplus M_\beta \). Put \( T=f(M_\alpha) \). Since each \( M_\gamma \) is uniform, we can take \( \beta \in I - \{\alpha\} \) such that \( T \cap \sum_{j \in (\beta)} \oplus M_\gamma = 0 \). Let \( \pi \) be the projection: \( M=\sum_{j \in (\beta)} \oplus M_\alpha \to M_\beta \). Then \( g=\pi|T: T \to M_\beta \) is a monomorphism and hence \( gf: M_\alpha \to M_\beta \) is a monomorphism. Therefore \( g \) is an isomorphism by 1) and it follows that \( M=T \oplus \sum_{j \in (\beta)} \oplus M_\gamma \).

**Remark.** Under the assumption that each \( M_\alpha \) is uniform, all conditions in Theorems 5 and 6 are equivalent (cf. [4, Theorem 13]).

**Definition.** Let \( \mathcal{A} \) be a family of submodules of \( M \). \( M \) is said to have the lifting property of modules for \( \mathcal{A} \) if, for any \( A \) in \( \mathcal{A} \), there exists a decomposition \( M=A^* \oplus A^{**} \) such that \( A^* \subseteq A \) and \( A \cap A^{**} \) is small in \( M \) (see [5]).

**Notation.** By \( \mathcal{H}(M) \), we denote the set of all submodules \( A \) of \( M \) such that \( M/A \) is a cyclic hollow module and define \( \mathcal{H}^*(M) = \{ A \in \mathcal{H}(M) | A \) contains almost all \( M_\alpha \) but finit \}.

**Theorem 7.** Assume that each \( M_\alpha \) is cyclic hollow. Then the following conditions are equivalent:

1) \( M \) has the lifting property of modules for \( \mathcal{H}^*(M) \).
2) For any pair $\alpha, \beta \in I$, any $X \subseteq M_\beta$ and any epimorphism $f: M_\alpha \to M_\beta/X$, there exists either $g: M_\alpha \to M_\beta$ or $h: M_\beta \to M_\alpha$ such that

\[
M_\alpha \xrightarrow{g} M_\beta \quad \text{or} \quad M_\beta \xleftarrow{h} M_\alpha
\]

is commutative, where $\phi$ is the canonical map.

Proof. 1) $\Rightarrow$ 2). Let $\alpha, \beta \in I$ and consider submodules $X_\alpha \subseteq M_\alpha$ and $X_\beta \subseteq M_\beta$. Put $\overline{M} = M/(X_\alpha \oplus X_\beta \oplus \sum_{f \in (I_{-\alpha, \beta})} M_f)$ and let $f: \overline{M} \to \overline{M}$ be an isomorphism. If we put $A = \{x \in M_\alpha \oplus M_\beta \mid x \in \{y + f(y) \mid y \in M_\alpha\}\}$, then $M/A \cong \overline{M}$ and hence $A \oplus \sum_{f \in (I_{-\alpha, \beta})} M_f \in \mathcal{A}(M)$. So, by 1), there exists a decomposition $M = A^* \oplus A^{**}$ such that $A^* \subseteq A$ and $A \cap A^{**}$ is small in $M$. Since $M/A = A^{**}/(A \cap A^{**})$ is cyclic hollow, $A^{**}$ is also cyclic hollow. Hence $A^{**}$ can be exchanged by some member in $\{M_\alpha\}_I$. Since $\overline{M} = A^*$, $A^{**}$ must be in fact exchanged by $M_\alpha$ or $M_\beta$; whence we get either $M = A^* \oplus M_\alpha$ or $M = A^* \oplus M_\beta$. In the former case, let $\pi: M = A^* \oplus M_\beta \to M_\beta$ be the projection. Then the diagram

\[
\begin{array}{ccc}
\overline{M} & \xrightarrow{f} & \overline{M} \\
\varphi_\alpha & \uparrow & \varphi_\beta \\
M_\alpha & \xrightarrow{f'} & M_\alpha
\end{array}
\]

is commutative, where $f' = -\pi|M_\alpha$ and $\varphi_\alpha$ and $\varphi_\beta$ are the canonical maps. In the latter case, we can obtain the desired epimorphism: $M \to M_\alpha$ by considering the projection: $M = A^* \oplus M_\alpha \to M_\alpha$.

2) $\Rightarrow$ 1). Let $A \in \mathcal{A}(M)$. Then we can take $F = \{\alpha_1, \ldots, \alpha_n\} \subseteq I$ and submodule $T \subseteq M_{\alpha_1} \oplus \cdots \oplus M_{\alpha_n}$ such that $A = \sum_{i=1}^n M_{\alpha_i} \oplus T$ and $M = A + M_{\alpha_i}$, $i = 1, \ldots, n$. We put $X = \sum_{i=1}^n M_{\alpha_i}$ and $\overline{M} = M/X$. Then

\[
\begin{align*}
\overline{M} &= A \oplus \overline{M}_{\alpha_1} \oplus \cdots \oplus \overline{M}_{\alpha_n} \\
\overline{M}_{\alpha_i} &= M_{\alpha_i}/(A \cap M_{\alpha_i}) \quad \text{(canonically), } i = 1, \ldots, n.
\end{align*}
\]

Let $\pi_i: \overline{M} = A \oplus \overline{M}_{\alpha_i} \to \overline{M}_{\alpha_i}$ be the projection, $i = 1, \ldots, n$. Then $\pi_i(\overline{M}_{\alpha_j}) = \overline{M}_{\alpha_i}$ and $\{x + \pi_i(x) \mid x \in M_{\alpha_j}\} \subseteq A$ for $j \neq i$. Here, using 2), we can take $i_0 \in \{1, \ldots, n\}$ and mappings $\{f_j: M_{\alpha_i} \to M_{\alpha_{i_0}} \mid j \neq i_0\}$ such that

\[
\overline{f_j}(x) = \pi_{i_0}(x)
\]
for all \( x \in M_{a_j} \) and \( j = i_0 \). Putting \( A_j = \{ x + f_j(x) \mid x \in M_{a_j} \} \) and \( T = A_1 \oplus \cdots \oplus A_{i_0-1} \oplus A_{i_0+1} \oplus \cdots \oplus A_s \oplus \sum_{j} \oplus M_b \), we see that \( T \subseteq A \) and \( M = T \oplus M_{a_{i_0}} \).

**Notation.** By \( \mathcal{H}(M) \) we denote the set of all maximal submodules of \( M \) and put \( \mathcal{H}^*(M) = \{ A \in \mathcal{H}(M) \mid A \text{ contains almost all } M \text{ but finite} \} \).

**Theorem 8.** Assume that each \( M_a \) is a cyclic hollow module. Then the following conditions are equivalent:

1) \( M \) has the lifting property of simple modules modulo the radical.
2) \( M \) has the lifting property of modules for \( \mathcal{H}^*(M) \).
3) For any pair \( \alpha, \beta \) in \( I \) such that \( \bar{M}_\alpha = \bar{M}_\beta \) and any isomorphism \( f: \bar{M}_\alpha \rightarrow \bar{M}_\beta \) (where \( \bar{M} = M/J(M) \)) there exists an epimorphism \( g \) of either \( M_{\alpha} \) onto \( M_{\beta} \) or \( M_{\beta} \) onto \( M_{\alpha} \) such that \( g = f \) or \( g = f^{-1} \), where \( g \) is the induced isomorphism.

Proof. 1)\( \Leftrightarrow \)3) is due to Harada ([3]). 2)\( \Leftrightarrow \)3) is shown by the quite same argument as in the proof of Theorem 7.

**Notation.** Let \( \{ M_{a_i} \}_{i=1}^r \subseteq \{ M_{a_i} \}_{i=1}^r \) and let \( \{ f_i: M_{a_i} \rightarrow M_{a_{i+1}} \} \) be a set of epimorphisms. By \( X_i \) we denote the set of all \( x \) in \( M_{a_i} \) such that \( f_i(x) = 0 \) for some \( n \) (depending on \( n \)). Put \( X_i = \sum_{i=1}^n \oplus X_i \) and \( \bar{M} = M/X_i \). Then, as is easily seen, \( f_i \) induces an isomorphism \( \bar{f}_i: \bar{M}_{a_i} \rightarrow \bar{M}_{a_{i+1}} \). Here we shall consider the following condition:

\((*)\) For any such \( \{ M_{a_i} \}_{i=1}^r \), epimorphisms \( \{ f_i: M_{a_i} \rightarrow M_{a_{i+1}} \}_{i=1}^r \) and \( \bar{M} \), there exist \( n \) (depending on the sets) and epimorphism \( g: M_{a_{n+1}} \rightarrow M_{a_n} \) such that \( g \) induces \( \bar{f}_i^{-1} \)

**Theorem 9.** Assume that each \( M_a \) is cyclic hollow. Then the following conditions are equivalent:

1) \( M \) has the lifting property of modules for \( \mathcal{H}(M) \).
2) \( M \) has the lifting property of modules for \( \mathcal{H}^*(M) \) and satisfies the condition \((*)\).

Proof. 1)\( \Rightarrow \)2). The first part is clear. Let \( \{ M_{a_i} \}_{i=1}^r \subseteq \{ M_{a_i} \}_{i=1}^r \) and let \( \{ f_i: M_{a_i} \rightarrow M_{a_{i+1}} \}_{i=1}^r \) be a set of epimorphisms. To verify \((*)\) for these sets we can assume that \( \{ M_{a_i} \}_{i=1}^r \), since \( \sum_{i=1}^r \oplus M_{a_i} \) also has the lifting property of modules for \( \mathcal{H}(\sum_{i=1}^r \oplus M_{a_i}) \). Now, we put \( X_i = \{ x \in M_{a_i} \mid \exists n: f_i f_{i-1} \cdots f_i(x) = 0 \} \), \( X = \sum_{i=1}^r \oplus X_i \), and \( \bar{M} = M/X \). Since each \( M_{a_i} \) is cyclic hollow, we can put \( M_{a_i} = m_i R \) with \( f_i(m_i) = m_{i+1} \) for some \( \{ m_i \}_{i=1}^r \). Putting \( A = \sum_{i=1}^r (m_i + m_{i+1}) R \), we see that \( M = m_i R + A \) and \( m_i R \cap A = X_i, i = 1, 2, \ldots \). Since \( M/A = (m_i R + A)/A = m_i R/(A \cap R), A \) lies in \( \mathcal{H}(M) \). Hence there exists a decomposition \( M = A^* \oplus A^{**} \) such that \( A^* \subseteq A \) and \( A \cap A^{**} \) is small in \( M \). Since \( M/A = \ldots \)
A**/(A ∩ A**), A**/(A ∩ A**) is cyclic hollow and hence so is A**. As a result, we can assume that A** coincides with some member in \( \{ M_{a_i} \}_{i=1}^\infty \) by the Krull-Remak-Schmidt-Azumaya's theorem: say

\[ M = A^* \oplus M_{a_n} \]

with \( A^* \subseteq A \). We express \( m_{n+1} \) as

\[ m_{n+1} = -m_n r_n + (m_n + m_{n+1}) r_n + m_{n+1} r_{n+1} \]

with \( m_{n+1} r_{n+1} \in X_{n+1} \).

Now the mapping \( g: M_{a_{n+1}} \to M_{a_n} \) given by the rule \( m_{n+1} r \mapsto m_n r \) is well defined and an epimorphism. We claim that \( \hat{g} = \hat{f}_n^{-1} \). In fact, it is easy to see that \( m_n r \in X_n \) if and only if \( m_{n+1} r \in X_{n+1} \); whence \( g \) induces an isomorphism \( \hat{g} \) from \( \hat{M}_{a_{n+1}} \) to \( \hat{M}_{a_n} \) and moreover \( \hat{m}_{n+1} = \hat{m}_n r_n = \hat{f}_n(\hat{m}_n r_n) = \hat{f}_n \hat{g}(\hat{m}_{n+1}) \) and hence \( \hat{g} = \hat{f}_n^{-1} \).

2) \( \Rightarrow \) 1). We fix \( \alpha \in I \) and put \( M_{a_0} = M_{a_\alpha} R \). Let \( A \in \mathcal{H}(M) \). To show that \( A \) can be co-essentially lifted to a direct summand of \( M \), we may assume that each \( M_{a_\alpha} \) is not contained in \( A \), namely, \( M = M_{a_\alpha} + A \) for all \( \alpha \in I \). Put \( Y_\alpha = M_{a_\alpha} \cap A \) for all \( \alpha \in I \), \( Y = \sum_{\alpha \in I} Y_\alpha \) and \( \tilde{M} = M/\tilde{Y} \). For any \( \beta \in I - \{ \alpha_0 \} \), we see

\[ \tilde{M} = \tilde{M}_{a_\alpha} \oplus \tilde{A} = \tilde{M}_\beta \oplus \tilde{A} \]

So, there exist \( m_\beta \in M_\beta \) and \( a_\beta \in A \) such that

\[ \tilde{m}_{a_\alpha} = m_\beta + a_\beta \]

Clearly the rule \( \tilde{m}_{a_\alpha} r \mapsto m_\beta r \) defines an isomorphism from \( \tilde{M}_{a_\alpha} \) to \( \tilde{M}_\beta \). Therefore the rule \( \tilde{m}_{a_\alpha} r \mapsto m_\beta r \) define an isomorphism \( \eta_{a_\alpha}^{a_\beta} : \tilde{M}_{a_\alpha} \to \tilde{M}_{a_\beta} \) for any pair \( \beta, \beta' \) in \( I \). Here we shall show that there does not exist the following subset \( \{ \alpha_i \}_{i=1}^\infty \subseteq I - \{ \alpha_0 \} \):

i) there exists a set \( \{ f_i : M_{a_i} \to M_{a_{i+1}} \}_{i=1}^\infty \) of epimorphisms such that each \( f_i \) induces the isomorphism \( \eta_{a_i}^{a_{i+1}} \)

ii) but for all \( i \) there does not exist any epimorphism \( g : M_{a_{i+1}} \to M_{a_i} \) which induces the isomorphism \( \eta_{a_i}^{a_{i+1}} \)^{-1}.

In fact, assume, on the contrary, that such \( \{ \alpha_i \}_{i=1}^\infty \) exists. Put \( X_i = \{ x \in M_{a_i} | f_i f_{n_i} \cdots f_i(x) = 0 \text{ for some } n \geq i \} \), \( X = \sum_{i=1}^\infty X_i \) and \( \tilde{M} = M/X \). Then clearly \( X_i \subseteq Y_i \) and \( f_i(X_i) = X_{i+1} \) for all \( i \). By \( \hat{f}_i \) we denote the induced isomorphism: \( \hat{M}_{a_i} \to \hat{M}_{a_{i+1}} \). Here using the condition 2) we can take \( k \) and an epimorphism \( g : M_{a_k} \to M_{a_{k+1}} \) such that \( g \) induces \( \hat{f}_k^{-1} \). Then \( \hat{m}_k = \hat{g}(\hat{m}_{k+1}) \) and it follows that \( \hat{m}_k = g(m_{k+1}) \). As a result, \( g \) induces \( (\eta_{a_k}^{a_{k+1}})^{-1} \), a contradiction.
Now, by this fact and Theorem 8, we may consider the following two cases.

*) For any \( \alpha \in I - \{ \alpha_0 \} \) there exists an epimorphism \( f_\alpha: M_\alpha \to M_{\alpha_0} \) such that \( f_\alpha \) induces the isomorphism \( \eta_\alpha^*: \tilde{M}_\alpha \to \tilde{M}_{\alpha_0} \).

**) There exist \( J = \{ \alpha_1, \ldots, \alpha_t \} \subseteq I - \{ \alpha_0 \} \) and sets \( \{ f_i^{t+1}: M_{\alpha_i} \to M_{\alpha_{i+1}} \mid i = 0, \ldots, t-1 \} \) and \( \{ f_\beta^\beta: M_\beta \to M_\beta, \beta \in I - \{ J \cup \{ \alpha_0 \} \} \} \) of epimorphisms such that \( f_i^{t+1} \) and \( f_\beta^\beta \) induce \( \eta_{\alpha_i}^{t+1} \) and \( \eta_\beta^\beta \), respectively. Then

\[
\tilde{m}_{\alpha_{i+1}} = f_i^{t+1}(\tilde{m}_{\alpha_i})
\]

for all \( i = 1, 2, \ldots, t-1 \), and

\[
\tilde{m}_\beta = f_\beta^\beta(\tilde{m}_\alpha)
\]

for all \( \beta \in K = I - \{ J \cup \{ \alpha_0 \} \} \).

In the first case, consider the map \( f = \sum_{I - \{ \alpha_0 \}} f_\alpha^\alpha: \sum_{I - \{ \alpha_0 \}} \oplus M_\alpha \to M_{\alpha_0} \) and put \( A^* = \{ x + f(x) \mid x \in \sum_{I - \{ \alpha_0 \}} \oplus M_\alpha \} \). Then \( M = A^* \oplus M_{\alpha_0} \) and it follows from \( A^* = \sum_{I - \{ \alpha_0 \}} \oplus \tilde{a}_\alpha \) that \( A^* \subseteq A \) as desired. In the second case we put \( M_\alpha = \{ x + f_i^{t+1}(x) \mid x \in m_\alpha R \} \) for \( i = 0, 1, \ldots, t-1 \) and \( T = \{ x + g(x) \mid x \in \sum_{\beta} \oplus m_\beta R \} \) where \( g = \sum_{\beta} f_\beta^\beta \). Then

\[
M = \sum_{t=0}^{t-1} \oplus M_\alpha \oplus T \oplus M_{\alpha_0},
\]

\[
\tilde{M}_\alpha = \tilde{a}_\alpha, R \text{ for } i = 1, \ldots, t-1, \text{ and}
\]

\[
\tilde{T} = (\tilde{a}_\beta - \tilde{a}_\beta)R \text{ for all } \beta \in K.
\]

Hence putting \( A^* = \sum_{t=0}^{t-1} \oplus M_\alpha \oplus T \) we see that \( A^* \subseteq A \) and \( M = A^* \oplus M_{\alpha_0} \). Our proof is now completed.

By a similar proof as in the proof of the above theorem, we can obtain the following result which is mentioned in introduction of this paper.

**Theorem 10.** Assume that each \( M_\alpha \) is cyclic hollow. Then the following conditions are equivalent:

1) \( M \) has the lifting property of modules for \( \mathcal{M}(M) \).

2) \( M \) has the lifting property of modules for \( \mathcal{M}*(M) \) and satisfies the following condition: For any subfamily \( \{ M_\alpha \} \subseteq \{ M_\alpha \} \) and epimorphisms \( \{ f_i: M_\alpha \to M_{\alpha_{i+1}} \} \), there exist \( n \) and epimorphism \( g: M_{\alpha_{i+1}} \to M_{\alpha_n} \) satisfying \( f_n^{-1} = g \) on \( \tilde{M} = M/MJ(M) \) where \( f_n \) and \( g \) are the induced isomorphisms.

**References**


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