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LINEARLY COMPACT MODULES OVER HNP RINGS

Dedicated to Professor Hirosi Nagao for his 60th birthday

HIDETOSHI MARUBAYASHI

(Received May 4, 1984)

Let $R$ be a hereditary noetherian prime ring (an HNP ring for short) and let $F$ be a non-trivial right Gabriel topology on $R$, i.e., $F$ consists of essential right ideals of $R$ (see §1 of [9]). Then $R$ is a topological ring with elements of $F$ as a fundamental system of neighborhoods of 0. Let $M$ be a topological right $R$-module with a fundamental system of neighborhoods of 0 consisting of submodules. Then $M$ is called $F$-linearly compact ($F$-l.c. for short) if

(i) it is Hausdorff,

(ii) if every finite subset of the set of congruences $x \equiv m_a \pmod{N_a}$, where $N_a$ are closed submodules of $M$, has a solution in $M$, then the entire set of the congruences has a solution in $M$.

This paper is concerned with $F$-l.c. modules over HNP rings in the case $F$ is special. Let $A$ be a maximal invertible ideal of $R$ and let $F_A$ be the right Gabriel topology consisting of all right ideals containing some power of $A$. Then we give, in §2, a complete algebraic structure of $F_A$-l.c. modules by using Kaplansky's duality theorem and basic submodules. From this result we get: "$F_A$-l.c. modules$\Rightarrow$"$F_A$-pure injective modules". This implication is not necessary to hold for any right Gabriel topology as it is shown in §3. It is established that there is a duality between $F_A$-l.c. modules and left $\hat{R}_A$-modules, where $\hat{R}_A$ is the completion of $R$ with respect to $A$ (see Theorem 2.6). Main results in this paper were announced without proofs in [11].

Concerning our terminologies and notations we refer to [8] and [9].
left ideals \( J \) of \( R \) such that \( Q_f J = Q_f \) is a left Gabriel topology on \( R \), which is called the left Gabriel topology corresponding to \( F \). It is clear that \( Q_f = Q_{fJ} = \bigcup (R : J), (J \in F) \). Define \( \hat{R_f} = \lim R/I (I \in F) \), the inverse limit of the modules \( R/I \), and \( \hat{R_{fI}} = \lim R/J (J \in F) \). Then both \( \hat{R_f} \) and \( \hat{R_{fI}} \) are rings (see [16, §4]).

Let \( M \) be an \( F \)-torsion module. Then it is an \( \hat{R_f} \)-module as follows; for any \( \bar{m} \in \hat{M} \) and \( \bar{r} = (r_1 + I) \in \hat{R_f} \), we define \( m \bar{r} = mr \), where \( L \) is any element in \( F \) contained in \( Q(m) = \{ r \in R | mr = 0 \} \). Similarly, an \( F \)-torsion left module is an \( \hat{R_{fI}} \)-module. In [7], we studied \( F \)-l.c. modules over a Dedekind prime ring. All results in [7, §2] are carried over \( F \)-l.c. modules over any HNP rings without any changes of the proofs. Here we pick up some of them which are frequently used in §2. Let \( \eta \colon R \to \hat{R_f} \) be the canonical map and \( \hat{F} = \{ \hat{L} \colon \text{right ideals of } \hat{R_f} \} \). Then \( \hat{R_f} \) is a topological ring with elements of \( \hat{F} \) as a fundamental system of neighborhoods of 0. For any \( \hat{R_f} \)-module, we can define the concept of \( \hat{F} \)-l.c. modules.

\( (1.1) \) A module is an \( F \)-l.c. module if and only if it is an \( \hat{R_f} \)-module and is an \( \hat{F} \)-l.c. module (see Proposition 2.10 of [7]).

Let \( M \) be an \( F \)-l.c. module. Then \( M^* \) means the left module of all continuous homomorphisms from \( M \) into \( K_f (=Q_f / R) \), where \( K_f \) is equipped with the discrete topology. It is evident that an element \( f \in \text{Hom}_k(M, K_f) \) is continuous if and only if \( \text{Ker } f \) is open. Let \( G \) be a left \( \hat{R_{fI}} \)-module. Then we denote by \( G^* \) the right module \( \text{Hom}_k(G, K_f) \) and define its finite topology by taking the submodules \( \text{Ann}(N) = \{ f \in G^* | (N) f = 0 \} \) as a fundamental system of neighborhoods of zero, where \( N \) runs over all finitely generated \( \hat{R_{fI}} \)-submodules of \( G \).

\( (1.2) \) (Kaplansky's duality theorem) Let \( M \) be an \( F \)-l.c. module. Then \( M^* \) is a left \( \hat{R_f} \)-module and \( M \) is isomorphic to \( M^{**} \) as topological modules, where \( M^{**} \) is equipped with the finite topology induced by \( M^* \) as the above (see Lemma 2.11 and Theorem 2.12 of [7]).

2. Let \( A \) be a maximal invertible ideal of \( R \) and let \( F_A = \{ I \colon \text{right ideal of } R/J \supseteq A^n \text{ for some } n > 0 \} \), a right Gabriel topology. Then \( F_{A_i} = \{ J \colon \text{left ideal of } R/J \supseteq A^n \text{ for some } m > 0 \} \). We denote the inverse limit of the modules \( R/J \supseteq A^n \) (\( n = 1, 2, \ldots \) ) by \( \hat{R} \). Then \( \hat{F}_{A} = \hat{R} = \hat{R}_{F_A} \) and it is an HNP ring with the Jacobson radical \( \hat{A} = A \hat{R} = \hat{R}A \) and with quotient ring \( \hat{Q} = \hat{Q} \otimes_R \hat{R} \) (see Lemma 1.2 and Theorem 1.1 of [8]). \( F_A \)-l.c. modules and \( F_A \)-torsion modules are said to be \( A \)-l.c. modules and \( A \)-primary modules, respectively. We note that \( K_{F_A} = \hat{Q} / \hat{R} \), because \( K_{F_A} = \bigcup A^{-n} / R = (\bigcup A^{-n} / R) \otimes_R \hat{R} = \bigcup A^{-n} / \hat{R} = \hat{Q} / \hat{R} \).

In this section, we shall give a complete algebraic structure of \( A \)-l.c. modules. We can see from (1.1) that a module is \( A \)-l.c. if and only if it is an \( \hat{R} \)-module and an \( \hat{A} \)-l.c. module. If \( A \) is a maximal ideal of \( R \), then \( \hat{R} \) is a Dede-
kind prime ring with unique maximal ideal \( \hat{A} \). Thus, in this case, the algebraic structure of \( A\text{-l.c.} \) modules has been characterized in Theorem 3.4 of [7]. If \( A \) is not maximal ideal, then \( A = M_1 \cap \cdots \cap M_p \), where \( M_1, \ldots, M_p \) are all maximal idempotent ideals of \( R \) and is a cycle, i.e., \( O_i(M_1) = O_i(M_2) = \cdots = O_i(M_p) \), where \( O_i(M_1) = \{ q \in Q \mid qM_1 \subseteq M_1 \} \) and \( O_i(M_2) = \{ q \in Q \mid qM_2 \subseteq M_2 \} \). Furthermore, we have the following (see Theorem 1.1 of [8] and Lemma 4 of [10]):

(a) \( \hat{R} = (e_1 \hat{R} \oplus e_2 \hat{R} \oplus \cdots \oplus e_p \hat{R}) \oplus \cdots \oplus (e_1 \hat{R} \oplus e_2 \hat{R} \oplus \cdots \oplus e_p \hat{R}) \), where each \( e_i \hat{R} \) is a uniform right ideal of \( \hat{R} \), \( e_i \) is idempotent in \( \hat{R} \), \( e_i \hat{R}/e_i \hat{A} \) is a simple module annihilated by \( M_i \) and \( k_i \) is the Goldie dimension of \( R/M_i \).

(b) \( \hat{A} = \hat{M}_1 \cap \cdots \cap \hat{M}_p \), where \( \hat{M}_1, \ldots, \hat{M}_p \) are all maximal idempotent ideals of \( \hat{R} \) and is a cycle, and \( \hat{M}_i = M_i \hat{R} = R \hat{M}_i \) for each \( i (1 \leq i \leq p) \).

**Lemma 2.1.** Under the same notations as in (a) and (b), we have the following:

(1) \( (e_i \hat{A}^{-1} + \cdots + e_i \hat{A}^{-1}) + \hat{R} = O_i(M_{i+1}) = O_i(\hat{M}_i) \) \( (1 \leq i \leq p \) and \( p + 1 = 1) \).

(2) \( \hat{R} e_i \hat{A} e_i \) is left \( M_i \)-primary, i.e., each element of \( \hat{R} e_i \hat{A} e_i \) is annihilated by \( M_i \).

**Proof.** Firstly we note that \( \hat{A}^{-1} = (e_1 \hat{A}^{-1} \oplus \cdots \oplus e_p \hat{A}^{-1}) \oplus \cdots \oplus (e_1 \hat{A}^{-1} \oplus \cdots \oplus e_p \hat{A}^{-1}) \) and \( \hat{A}^{-1} = O_i(\hat{M}_1) + \cdots + O_i(\hat{M}_p) \), because \( O_i(\hat{M}_i) = (\hat{R} : \hat{M}_i) \). Thus we have

(c) \( \hat{A}^{-1}/\hat{R} = (e_1 \hat{A}^{-1} + \cdots + e_i \hat{A}^{-1} + \hat{R})/\hat{R} \oplus \cdots \oplus (e_p \hat{A}^{-1} + \cdots + e_p \hat{A}^{-1} + \hat{R})/\hat{R} \),

(d) \( \hat{A}^{-1}/\hat{R} = O_i(\hat{M}_1)/\hat{R} \oplus \cdots \oplus O_i(\hat{M}_p)/\hat{R} \).

It is clear that \( O_i(\hat{M}_i)/\hat{R} \) is \( M_i \)-primary. Since \( e_i \hat{Q}/e_i \hat{A} \) is a uniform and injective \( \hat{R} \)-module, it is a uniform and injective \( R \)-module by Lemma 2.4 of [8]. Thus we have \( e_i \hat{A}^{-1} e_i \hat{R} \) is \( M_{i+1} \)-primary by periodicity theorem and (a) (see Theorem 22 of [4]). It follows that \( (e_i \hat{A}^{-1}) \hat{M}_{i+1} \subseteq e_i \hat{R} \subseteq \hat{R} \) and \( e_i \hat{A}^{-1} \subseteq O_i(\hat{M}_{i+1}) \). Thus (1) follows from (c) and (d).

(2) Since \( O_i(\hat{M}_i) = O_i(\hat{M}_{i+1}) \), we have \( \hat{M}_i (e_i \hat{A}^{-1}) \subseteq \hat{R} \) by (1) and hence \( \hat{M}_i e_i \subseteq \hat{A} e_i \). This implies that \( \hat{R} e_i \hat{A} e_i \) is \( M_i \)-primary as left modules.

Let \( M \) be an \( \hat{R} \)-module. Then write \( M^* = \text{Hom}_R(M, K_{F_A}) \).

**Lemma 2.2.** Under the same notations as in (a) and (b), we have

(1) for any positive integer \( n \) and any \( i \) \( (1 \leq i \leq p) \), \( (e_i \hat{R}/e_i \hat{A}^n)^* = \hat{R} e_i \hat{A}^n e_i \) for some \( j \) \( (1 \leq j \leq p) \).

(2) \( (e_i \hat{R})^* = \hat{Q}_e_i \hat{R} e_i = E(\hat{R} e_{i-1} \hat{A} e_{i-1}) \), the injective hull of \( \hat{R} e_{i-1} \hat{A} e_{i-1} \), where \( 1 \leq i \leq p \) and \( i - 1 = p \) if \( i = 1 \).

(3) \( (e_i \hat{Q})^* = \hat{Q} e_i \) for each \( i \) \( (1 \leq i \leq p) \).

(4) \( (e_i \hat{Q}/e_i \hat{R})^* = \hat{R} e_i \) for each \( i \) \( (1 \leq i \leq p) \).

These modules are all \( A\text{-l.c.} \) modules.
Proof. (1) Clearly \( (e_i \hat{R} | e_i \hat{A}^*) = \hat{A}^{-*} e_i + \hat{R} = \hat{A}^{-*} e_i | \hat{R} e_i \) by left multiplications of elements in \( \hat{A}^{-*} e_i \). \( \hat{A}^{-*} e_i | \hat{R} e_i \) is a uniserial module of length \( n \) with composition factor modules \( \hat{A}^{-k} e_i | \hat{A}^{-k*} e_i \) \( (1 \leq k \leq n \) and \( \hat{A}^{-0*} = \hat{R} e_i \). There is \( j (1 \leq j \leq p) \) such that \( \hat{A}^{-*} e_i | \hat{A}^{-*} e_i \approx \hat{R} e_j | \hat{A} e_j \) and then \( \hat{R} e_j | \hat{A} e_j \approx \hat{A}^{-*} e_i | \hat{R} e_i \) by the periodicity theorem.

(2) The first isomorphism is also obtained by left multiplication of elements in \( \hat{Q} e_i \). The second isomorphism follows from the periodicity theorem.

(3) Let \( x = xe_i \) be any element of \( \hat{Q} e_i \). Then a mapping \( \lambda_x : e_i \hat{Q} \rightarrow K_{F_A} \) given by \( \lambda_x(y) = [xy + \hat{R}] \) \( (y \in e_i \hat{Q}) \) is a homomorphism. Assume that \( \lambda_x = 0 \) and \( x \neq 0 \). Then \( x \hat{Q} = xe_i \hat{Q} \subseteq \hat{R} \), that is, \( x \in \hat{R} \). Hence \( \hat{R} x \hat{Q} = \hat{Q} \), a contradiction. Hence we may assume that \( \hat{Q} e_i \subseteq (e_i \hat{Q})^* \). Conversely, let \( f \) be any non zero element in \( (e_i \hat{Q})^* \) and let \( f(e_i) = [q + \hat{R}] \), where \( q = ge_i \in \hat{Q} \). Since \( (f - \lambda_q)(e_i \hat{R}) = 0 \), \( f - \lambda_q \) induces an element \( f - \lambda_q \) in \( (e_i \hat{Q})^* \). Since \( \hat{Q} / \hat{R} e_i \hat{Q} \cong (1 - e_i) \hat{Q} / (1 - e_i) \hat{R} \), we may consider that \( f - \lambda_q \in (\hat{Q} / \hat{R} e_i \hat{Q})^* \). By Proposition A.3 of [8], \( \hat{R} e_i \hat{Q} \cong (\hat{Q} / \hat{R} e_i \hat{Q})^* \). Hence \( f - \lambda_q = \lambda_r \) for some \( r \in \hat{R} \) and \( f - \lambda_q = \lambda_r \). So we get that \( (e_i \hat{Q})^* \subseteq \hat{Q} e_i \) and therefore \( (e_i \hat{Q})^* = \hat{Q} e_i \).

(4) The exact sequence \( 0 \rightarrow e_i \hat{R} \rightarrow e_i \hat{Q} \rightarrow e_i \hat{Q} / e_i \hat{R} \rightarrow 0 \) induces the exact sequence \( 0 \rightarrow (e_i \hat{Q} / e_i \hat{R})^* \rightarrow (e_i \hat{Q})^* \rightarrow (e_i \hat{R})^* \rightarrow 0 \), because \( K_{F_A} \) is injective. The assertion follows from (2) and (3). The left modules in (1) and (2) are artinian and \( A \)-primary. So they are \( A \)-l.c. modules in the discrete topology by Lemma 2.1 of [7] (as it has been pointed out in §1, all results in [7, §2] hold in \( F \)-l.c. modules over any HNP rings). \( \hat{R} \) is an \( A \)-l.c. modules by Lemma 2.4 of [7]. Thus it follows that \( \hat{R} e_i \) is also an \( A \)-l.c. module. Finally consider the exact sequence \( 0 \rightarrow \hat{R} e_i \rightarrow \hat{Q} e_i \rightarrow \hat{Q} e_i / \hat{R} e_i \rightarrow 0 \). \( \hat{Q} e_i \) is a topological module by taking as a fundamental system of 0 the submodules \( \{\hat{A}^{-n} e_i | n = 0, \pm 1, \pm 2, \ldots\} \). Hence \( \hat{Q} e_i \) is an \( A \)-l.c. module by Proposition 9 of [20].

Following [9], a submodule \( L \) of a module \( M \) is called \( F^* \)-pure if \( MJ \cap L = LJ \) for any \( J \in F \). Let \( F_0 \) be the right Gabriel topology of all essential right ideals of \( R \). Then “an \( F^* \)-pure submodule” is merely called a pure submodule.

Consider the following condition:

(e) all finitely generated \( F \) and \( F_1 \)-torsion modules are a direct sum of cyclic modules.

This condition is satisfied by any topologies \( F \) and \( F_1 \) on \( R \) if \( R \) has enough invertible ideals and so, especially, if \( R \) has a non zero Jacobson radical (see Corollary 3.4 and Theorems 4.12, 4.13 of [3]). If all \( F \) and \( F_1 \)-torsion modules are of bounded orders, i.e., unfaithful modules, then this condition is satisfied, because every factor ring of an HNP ring is serial (Corollary 3.2 of [1]). Note that [9, Lemma 1.2] is still valid for topologies \( F \) and \( F_1 \) on any HNP ring \( R \) satisfying the condition (e). Furthermore, if \( R \) has a nonzero Jacobson radical, then a submodule \( L \) of a module \( M \) is pure if and only if \( Mc \cap L = Lc \) for any
regular element $c$ in $R$ by Proposition 3 of [19] and the remark to Theorem 3.6 of [15].

**Lemma 2.3.** Let $R$ be an HNP ring with the Jacobson radical $A$ and $A$ be a maximal invertible ideal of $R$. If a short exact sequence $0 \to L \to M \to N \to 0$ is pure, then $0 \to \text{Hom}_R(N, K) \to \text{Hom}_R(M, K) \to \text{Hom}_R(L, K) \to 0$ is pure as left $R$-modules.

**Proof.** Let $c$ be any regular element of $R$ and let $cf = g\beta$ be any element in $c\text{Hom}_R(M, K) \cap (\text{Hom}_R(N, K))\beta^\ast$, where $f \in \text{Hom}_R(M, K)$ and $g \in \text{Hom}_R(N, K)$. Since $g\beta\alpha(L) = 0$, we have $\alpha(L) \subseteq \text{Ker } g\beta = \text{Ker } cf$. There is a nature number $n$ such that $Rc \supset A^n$. It follows that $0 = Rcf\alpha(L) \supseteq A^n\alpha(L)$. Put $f\alpha(L) = X/R$, where $X$ is a submodule of $Q$ containing $R$. Then $A^nX \subset R$ and so $X \subset (R; A^n), A^n = (R; A^n)$. Thus we have $XA^n \subset R$. This implies that $f\alpha(L)A^n = 0$. Put $\overline{M} = M/\alpha(L)A^n$. Then $L = \alpha(L)/\alpha(L)A^n$ is pure in $\overline{M}$, because $L$ is pure in $M$. It follows from Theorem 3 of [13] and Theorem 1.3 of [14] that $L$ is a direct summand of $\overline{M}$, because $L$ is of bounded order. Thus we have the following sequence;

$$M \xrightarrow{\eta} \overline{M} = L \oplus \overline{M}_1 \xrightarrow{\pi} \overline{M}_1 \xrightarrow{f_1} K,$$

where $\eta$ is a natural homomorphism, $\pi$ is a projection map from $\overline{M}$ to $\overline{M}_1$ ($\overline{M}_1$ is a submodule of $M$) and $f_1$ is the map induced by $f$ (note that $f\alpha(L)A^n = 0$). Put $h = f_1\pi\eta$ and let $x$ be any element of $M$. Write $x = x_1 + x_2$ ($x_1 \in \alpha(L)$ and $x_2 \in \overline{M}_1$). Then $ch(x) = c\tau(x_1 + x_2) = cf(x_1) = cf(x_2)$. Since $x - x_2 \in \alpha(L) + \alpha(L)A^n \subseteq \alpha(L)$ and $cf(\alpha(L)) = 0$, we have $ch(x) = cf(x_2) = cf(x)$. Therefore $ch = cf$. By the construction of $h$, $h(\alpha(L)) = 0$. This entails that $h$ induces a map $k: N \to K$ such that $k\beta = h$. Hence we have $cf = ch = ch\beta \in c(\text{Hom}_R(N, K))\beta^\ast$, as desired.

**Theorem 2.4.** Under the same notations as in (a) and (b), a module is an $A$-l.c. module if and only if it is isomorphic to a direct product of modules of the following types:

$$e_i\hat{R}/e_i\hat{A}^n \text{ (} n = 1, 2, \cdots \text{)}, E(e_i\hat{R}/e_i\hat{A}), \text{ the injective hull of } e_i\hat{R}/e_i\hat{A}, e_i\hat{R} \text{ and } e_i(Q \otimes_R \hat{R}) \text{ (} 1 \leq i \leq p \text{)}.$$

**Proof.** The sufficiency follows from Proposition 1 of [20] and Lemma 2.2. Conversely let $M$ be an $A$-l.c. module. Then $M^\ast$ is a left $\hat{R}$-module by (1.2). So $M^\ast$ has a basic submodule $B$ by Theorem 2.1 of [8]. Then $B$ is a direct sum of modules of types; $\hat{R}e_i/\hat{A}^n e_i$ and $\hat{R}e_i$ ($1 \leq i \leq p$) and $n = 1, 2, \cdots$), and $M^\ast/B$ is a direct sum of modules of types; $E(\hat{R}e_i/\hat{A}e_i)$ and $(Q \otimes_R \hat{R})e_i$ (see Theorem 2.2 of [8]). Then from pure exact sequence $0 \to B \to M^\ast \to M^\ast/B \to 0$, we derive the pure exact sequence $0 \to (M^\ast/B)^\ast \to M^\ast \to B^\ast \to 0$ (as right $\hat{R}$-modules) by Lemma 2.3. By Lemma 2.2, $(M^\ast/B)^\ast$ is a direct product of modules of types; $e_i(Q \otimes_R \hat{R})$ and $e_i\hat{R}$. Here $e_i(Q \otimes_R \hat{R})$ is an injective $\hat{R}$-module.
Since $\hat{R} \cong \text{Hom}_R(K_{F_A}, K_{F_A}) \cong \text{Hom}_R(\hat{Q}/\hat{R}, \hat{Q}/\hat{R})$ (see Lemma 1.5 and Proposition A.3 of [8]), $\hat{R}$ is a pure injective $\hat{R}$-module by Propositions A.5, A.6 of [8] and Theorem 3.5, the remark to Proposition A.5 of [9], i.e., $\hat{R}$ has the injective property relative to the class of pure exact sequences and so is $\epsilon_i \hat{R}$. Hence $(M^*/B)^i$ is also pure injective. This entails that $M^{**} \cong (M^*/B)^i \oplus B^i$, and the assertion follows from (1.2) and Lemma 2.2.

**Lemma 2.5.** Let $M$ be a left $\hat{R}$-module and let $m$ be any non-zero element of $M$. Then there is an element $f$ in $M^i$ such that $(m)f \neq 0$.

**Proof.** $\hat{R}m$ is a finite direct sum of modules of types; $\hat{R}e_i/\hat{A}^i e_i$ and $\hat{R}e_i$ by Theorems 2.1 and 2.2 of [8]. Thus the assertion follows from Lemma 2.2, because $K_{F_A}$ is an injective $\hat{R}$-module.

**Theorem 2.6.** Let $R$ be an HNP ring and let $A$ be a maximal invertible ideal of $R$. Then

1. Let $M$ be any $A$-l.c. module. Then $M^*$ is a left $\hat{R}$-module and $M \cong M^{**}$.

2. Let $M$ be any left $\hat{R}$-module. Then $M^*$ is an $A$-l.c. module in a certain topology and $M \cong M^{**}$ ($M^*$ is equipped with the finite topology).

**Proof.**

1. is clear from (1.2).

2. Let $M$ be any left $\hat{R}$-module. Then $M^i$ is a direct product of modules of types in Theorem 2.4 (this is proved in the same way as in Theorem 2.4 by using basic submodules). Thus $M^i$ is an $A$-l.c. module. Now $M^i$ is equipped with the finite topology (it is not requested that $M^i$ is an $A$-l.c. module in the finite topology). Let $\beta : M \to M^{**}$ be the natural map given by $((m) / \beta) (f) = (m)f$, where $m \in M$ and $f \in M^i$. Note that $(m)\beta \in M^{**}$, because $\ker (m)\beta = \{g \in M^i | (m)g = 0\}$. By Lemma 2.5, $\beta$ is a monomorphism. To prove that $\beta$ is an epimorphism, let $g$ be any element in $M^{**}$. Since $\ker g$ is open in $M^i$, there is a finitely generated left module $N$ of $M$ such that $\ker g \supseteq \text{Ann}(N)$. Write

\[ N \cong \bigoplus_{i=1}^p \bigoplus_{i} \hat{R}e_i / \hat{A}^i e_i \oplus \bigoplus_{i} \hat{R}e_i \text{ (1} \leq k \leq p) \text{,} \]

where $n_{ij} \geq 0$. Thus $N$ is a left $A$-l.c. module by Lemma 2.2. Consider the following commutative diagram;

\[
\begin{array}{ccc}
M^i & \xrightarrow{\delta} & M^i/\text{Ann}(N) \\
\downarrow{\gamma} & & \downarrow{g} \\
N^i & \cong & K_{F_A} \\
\end{array}
\]

where $\gamma$ is a natural map, $g$ is a map induced by $g$ and $\delta ([f + \text{Ann}(N)]) - f|N$, the restriction map of $f$ to $N$ ($f \in M^i$). Let $h$ be any element of $N^i$. Then
there is a natural number \( n \) such that \( \hat{A}^*(N) h=0 \), because \( N \) is finitely generated. This entails that \( \text{Ker } h \supseteq \sum_{i=1}^{l} \bigoplus_{i} \hat{A}^e_i \bigoplus_{i} \hat{A}^e_n \), open in \( N \) (in the topology given in Lemma 2.2). Thus we have \( h \in N^* \) and hence \( N^* = N^3 \).

It follows from (1.2) that \( \alpha: N^* = N^3 \). So, for the element \( g\delta^{-1} \in N^3 \) there is an element \( n \in N \) such that \( (n)\alpha = g\delta^{-1} \), i.e., \( (n)\alpha = g \). Now let \( x \) be any element in \( M^* \). Then we have \( g(x) = g\eta(x) = ((n)\alpha) \delta\eta(x) = (n)\{(\delta\eta(x)) = (n)\{\delta \{x + \text{Ann}(N)]) = (n)x = ((n)\beta) (x) \). Hence \( g = (n)\beta \), as desired.

3. In this section, we study relationships between \( F \)-l.c. modules and \( F^w \)-pure injective modules in case \( F \) is special. A module \( G \) is \( F^w \)-pure injective if it has the injective property relative to the class of \( F^w \)-pure exact sequences. Let \( A \) be a maximal invertible ideal of \( R \). Then \( F^w_A \)-pure injective modules are just called \( A \)-pure injective modules. The cancellation set of \( A \), \( C(A) \), is defined to be \( \{c \in R | cx \in A \Rightarrow x \in A \} = \{c \in R | cx \in A \Rightarrow x \in A \} \). By [6], \( R \) satisfies the Ore condition with respect to \( C(A) \) and the local ring \( R_A \) of \( A \) is an HNP ring with Jacobson radical \( AR_A = RA \). Note that a module \( T \) is \( A \)-primary if and only if it is an \( R_A \)-module and torsion as \( R_A \)-modules (see the proof of Lemma 2.4 of [8]). Since \( R_A \) is \( R \)-flat and the inclusion map: \( R \to R_A \) is an epimorphism, we have the following

**Lemma 3.1.** (1) An exact sequence \( 0 \to L \to M \to N \to 0 \) is \( A \)-pure, then the induced sequence \( 0 \to L \otimes_R R_A \to M \otimes_R R_A \to N \otimes_R R_A \to 0 \) is exact and is pure as \( R_A \)-modules.

(2) If an exact sequence \( 0 \to L \to M \to N \to 0 \) of \( R_A \)-modules is pure as \( R_A \)-modules, then it is \( A \)-pure.

Proof. Use (3) in Lemma 1.2 of [9].

**Lemma 3.2.** Let \( F \) be a right Gabriel topology on \( R \) satisfying the condition (e) and let \( G \) be any \( F^w \)-pure injective module. Then \( G = D \oplus H \), where \( D \) is an injective module, and \( H \) is \( F \)-reduced, \( F^w \)-pure injective and \( F^w \)-complete. In particular, \( H \) is an \( R_{f_1} \)-module.

Proof. The proof of Theorem 3.2 of [9] may be used unaltered to yield this lemma.

**Proposition 3.3.** Let \( G \) be a reduced module, i.e., \( G \) has no non zero injective submodules. Then

(1) \( G \) is \( A \)-pure injective if and only if \( G \) is an \( R_A \)-module and is pure injective as \( R_A \)-modules.

(2) \( G \) is \( A \)-pure injective if and only if \( G = \bigoplus G\bigoplus G \).

Proof. (1) It is clear from Lemmas 3.1 and 3.2. (2) follows from Theorems 3.2.4 and 3.3.3 of [18] and (1), because \( R_A \) is a bounded HNP ring.
From Theorem 2.4 and Proposition 3.3, we have

**Corollary 3.4.** A-l.c. modules are A-pure injective modules.

In general, it is not necessary to hold that \((F\text{-l.c. modules}) \Rightarrow (F^\omega\text{-pure injective modules})\). We will end up this paper with giving a counter example. To do this, let \(B\) be an idempotent ideal of \(R\). Then write

\[
F_1 = \{ I \mid IO_i(B) = O_i(B), I: \text{right ideal of } R \}.
\]

\[
F_2 = \{ I \mid IO_i(B) = O_i(B), I: \text{right ideal of } R \}.
\]

Then \(F_1 = \{ J \mid O_i(B)J = O_i(B), J: \text{left ideal of } R \}\), and \(F_2 = \{ J \mid O_i(B)J = O_i(B), J: \text{left ideal of } R \}\). Since \(BO_i(B) = B, O_i(B)B = O_i(B)\), \(BO_i(B) = O_i(B)\) and \(O_i(B)B = B\), we have \(F_1 = \{ J \mid J \cong B \}\), \(F_2 = \{ I \mid I \cong B \}\), \(F_1 \not\cong B\) and \(F_2 \not\cong B\).

**Proposition 3.5.** Under the same notations as in (f), let \(G\) be any module. Then

1. \(G\) is an \(F_1\)-l.c. module if and only if it is a direct product of modules of types \((R: J)_{n}/R\), where \(J\) is a left ideal of \(R\) containing \(B\).
2. \(G\) is an \(F_2\)-l.c. module, then it is a direct sum of modules of types \(R/J(I \cong B)\).

**Proof.** (1) The sufficiency is evident from Proposition 1 of [20], Proposition A.1 of [8] and Lemma 2.1 of [7]. Let \(G\) be an \(F_1\)-l.c. module. Then \(G^*\) is a left \(\hat{R}_{F_1}(=R/B)\)-module by (1.2). Since \(R/B\) is a serial ring, \(G^*\) is a direct sum of cyclic modules (see Theorem 1.2 and Corollary 3.2 of [1]). Write \(G^* = \sum \oplus R/J_i(I \cong B)\) and then \(G \cong G^{**} = \prod (R: J_i)_{n}/R\) by (1.2).

(2) is clear, because any \(F_2\)-l.c. module is an \(\hat{R}_{F_2}(=R/B)\)-module.

Let \(C\) be an idempotent ideal of \(R\) such that

\[
O_i(B) = O_i(C).
\]

Then \(F_1 = \{ I \mid I \cong C, I: \text{right ideal of } R \}\). Note that there exists an idempotent ideal \(C\) of \(R\) satisfying the condition (g) for any idempotent ideal \(B\) of \(R\) if \(R\) has enough invertible ideals. In the absence of the condition of having enough invertible ideals, we can easily find a pair of idempotent ideals \(B\) and \(C\) satisfying the condition (g) (see [4]).

**Lemma 3.6.** Under the same notations as in (f) and (g), let \(0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0\) be an exact sequence. Then it is \(F^\omega\)-pure if and only if the induced sequence

\[
0 \rightarrow L/LB \rightarrow M/MB
\]

is splitting exact.

**Proof.** The sufficiency is clear. To prove the necessity, assume that
the sequence is $F_\mathfrak{a}^\ast$-pure. Then $(i)$ is exact and is $F_\mathfrak{a}^\ast$-pure. Since a module is a finitely presented $\overline{R}$ ($=R/B$)-module if and only if it is a finitely generated and $F_\mathfrak{a}$-torsion module, $(i)$ is pure as $R$-modules in the sense of [19] by Proposition 3 of [19] and Lemma 1.2 of [9]. Furthermore, since $\overline{M}=M/MB$ satisfies Singh's conditions (I), (II) and (III), $L$ is $h$-pure in the sense of Singh (see Theorem 1.3 of [14]). Hence $L$ is a direct summand of $\overline{M}$ by Theorem 3 of [13], because $L$ is of bounded order.

**Proposition 3.7.** Under the same notations as in (f) and (g), a reduced module is $F_\mathfrak{a}^\ast$-pure injective if and only if it is an $R/B$-module.

Proof. This is clear from Lemmas 3.2 and 3.6.

**Example 3.8.** Under the same notations as (f) and (g), $R/C$ is an $F_\mathfrak{a}$-l.c. module in the discrete topology. But it is not an $F_\mathfrak{a}^\ast$-pure injective module.

Proof. $R/C$ is an artinian and $F_\mathfrak{a}$-torsion module. So it is an $F_\mathfrak{a}$-l.c. module in the discrete topology by Lemma 2.1 of [7]. Assume that it is $F_\mathfrak{a}^\ast$-pure injective. Then it is an $R/B$-module by Proposition 3.7. This implies that $B\subseteq C$ and so $O_i(C):=O_i(B):=(R:B)=O_i(C)$, thus we have $C=O_i(C)\supseteq O_i(C):=O_i(C)\supseteq \mathfrak{a}$, a contradiction.

**References**


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