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LINEARLY COMPACT MODULES OVER HNP RINGS

Dedicated to Professor Hirosi Nagao for his 60th birthday

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Let R be a hereditary noetherian prime ring (an HNP ring for short) and let F be a non-trivial right Gabriel topology on R, i.e., F consists of essential right ideals of R (see §1 of [9]). Then R is a topological ring with elements of F as a fundamental system of neighborhoods of 0. Let M be a topological right R-module with a fundamental system of neighborhoods of 0 consisting of submodules. Then M is called *F-linearly compact* (*F-l.c.* for short) if
(i) it is Hausdorff,
(ii) if every finite subset of the set of congruences $x \equiv m_\alpha \pmod{N_\alpha}$, where N_α are closed submodules of M, has a solution in M, then the entire set of the congruences has a solution in M.

This paper is concerned with *F-l.c.* modules over HNP rings in the case F is special. Let A be a maximal invertible ideal of R and let F_A be the right Gabriel topology consisting of all right ideals containing some power of A. Then we give, in §2, a complete algebraic structure of F_A-*l.c.* modules by using Kaplansky's duality theorem and basic submodules. From this result we get: "F_A-*l.c.* modules"$\Rightarrow$"F_A^ω-pure injective modules". This implication is not necessary to hold for any right Gabriel topology as it is shown in §3. It is established that there is a duality between F_A-*l.c.* modules and left $\hat{R}_A$-modules, where $\hat{R}_A$ is the completion of R with respect to A (see Theorem 2.6). Main results in this paper were announced without proofs in [11].

Concerning our terminologies and notations we refer to [8] and [9].

1. Throughout this paper, R denotes an HNP ring with quotient ring Q and $K=Q/R\neq 0$. Let F be any non-trivial right Gabriel topology on R; "*trivial*" means that either all modules are F-torsion-free or all modules are F-torsion. Then F consists of essential right ideals of R (see [9, p. 96]). Let I be any essential right ideal of R. Define $(R: I)_l=\{q\in Q \mid qI\subseteq R\}$. Similarly $(R: J)_r=\{q\in Q \mid Jq\subseteq R\}$ for any essential left ideal J of R. An ideal X of R is called *invertible* if $(R: X)_l X=R=X(R: X)_r$. In this case we have $(R: X)_l =(R: X)_r$, denoted by X^{-1}. For any right Gabriel topology F, put $Q_F=\cup (R: I)_l$ $(I\in F)$, *the ring of quotients of* R *with respect to* F. The family F_l of left ideals J of R such that $Q_F J=Q_F$ is a left Gabriel topology on R, which is called *the left Gabriel topology corresponding to F*. It is clear that $Q_F=Q_{F_l}=\cup(R: J)_r$ $(J\in F_l)$. Define $\hat{R}_F=\varprojlim R/I$ $(I\in F)$, the inverse limit of the modules R/I, and $\hat{R}_{F_l}=\varprojlim R/J$ $(J\in F_l)$. Then both $\hat{R}_F$ and $\hat{R}_{F_l}$ are rings (see [16, §4]). Let M be an F-torsion module. Then it is an $\hat{R}_F$-module as follows; for any $m\in M$ and $\hat{r}=([r_I+I])\in\hat{R}_F$, we define $m\hat{r}=mr_L$, where L is any element in F contained in $O(m)=\{r\in R \mid mr=0\}$. Similarly, an F_l-rotsion left module is an $\hat{R}_{F_l}$-module. In [7], we studied *F-l.c.* modules over a Dedekind prime ring. All results in [7, §2] are carried over *F-l.c.* modules over any HNP rings without any changes of the proofs. Here we pick up some of them which are frequently used in §2. Let η: $R\rightarrow\hat{R}_F$ be the canonical map and $\hat{F}=\{\hat{L}$: right ideals of $\hat{R}_F \mid \hat{L}\supset\eta(I)\hat{R}_F$ for some $I\in F\}$. Then $\hat{R}_F$ is a topological ring with elements of $\hat{F}$ as a fundamental system of neighborboods of 0. For any $\hat{R}_F$-module, we can define the concept of *$\hat{F}$-l.c. modules*.

(1.1) *A module is an F-l.c. module if and only if it is an $\hat{R}_F$-module and is an $\hat{F}$-l.c. module* (see Proposition 2.10 of [7]).

Let M be an *F-l.c.* module. Then M^* means the left module of all continuous homomorphisms from M into K_F $(=Q_F/R)$, where K_F is equipped with the discrete topology. It is evident that an element $f\in\mathrm{Hom}_R(M, K_F)$ is continuous if and only if $\mathrm{Ker}\, f$ is open. Let G be a left $\hat{R}_{F_l}$-module. Then we denote by $G^\#$ the right module $\mathrm{Hom}_{\hat{R}_{F_l}}(G, K_F)$ and define its finite topology by taking the submodules $\mathrm{Ann}(N)=\{f\in G^\# \mid (N)f=0\}$ as a fundamental system of neighborhoods of zero, where N runs over all finitely generated $\hat{R}_{F_l}$-submodules of G.

(1.2) (Kaplansky's duality theorem) *Let M be an F-l.c. module. Then M^* is a left $\hat{R}_{F_l}$-module and M is isomorphic to $M^{*\#}$ as topological modules, where $M^{*\#}$ is equipped with the finite topology induced by M^* as the above* (see Lemma 2.11 and Theorem 2.12 of [7]).

2. Let A be a maximal invertible ideal of R and let $F_A=\{I$: right ideal of $R \mid I\supseteqq A^n$ for some $n>0\}$, a right Gabriel topology. Then $F_{A_l}=\{J$; left ideal of $R \mid J\supseteqq A^m$ for some $m>0\}$. We denote the inverse limit of the modules R/A^n $(n=1, 2, \cdots)$ by $\hat{R}$. Then $\hat{R}_{F_A}=\hat{R}=\hat{R}_{F_{A_l}}$ and it is an HNP ring with the Jacobson radical $\hat{A}=A\hat{R}=\hat{R}A$ and with quotient ring $\hat{Q}=Q\otimes_R\hat{R}$ (see Lemma 1.2 and Theorem 1.1 of [8]). F_A-*l.c.* modules and F_A-torsion modules are said to be *A-l.c. modules* and *A-primary modules*, respectively. We note that $K_{F_A}=\hat{Q}/\hat{R}$, because $K_{F_A}=\cup A^{-n}/R=(\cup A^{-n}/R)\otimes_R\hat{R}\cong\cup\hat{A}^{-n}/\hat{R}=\hat{Q}/\hat{R}$.

In this section, we shall give a complete algebraic structure of *A-l.c.* modules. We can see from (1.1) that a module is *A-l.c.* if and only if it is an $\hat{R}$-module and an $\hat{A}$-*l.c.* module. If A is a maximal ideal of R, then $\hat{R}$ is a Dedekind prime ring with unique maximal ideal $\hat{A}$. Thus, in this case, the algebraic structure of A-*l.c.* modules has been characterized in Theorem 3.4 of [7]. If A is not maximal ideal, then $A=M_1\cap\cdots\cap M_p$, where $M_1, \cdots, M_p$ are all maximal idempotent ideals of R and is a cycle, i.e., $O_r(M_1)=O_l(M_2), \cdots, O_r(M_p)=O_l(M_1)$, where $O_r(M_1)=\{q\in Q\mid M_1q\subseteqq M_1\}$ and $O_l(M_2)=\{q\in Q\mid qM_2\subseteqq M_2\}$. Furthermore, we have the following (see Theorem 1.1 of [8] and Lemma 4 of [10]):

(a) $\hat{R}=\overbrace{(e_1\hat{R}\oplus\cdots\oplus e_1\hat{R})}^{k_1}\oplus\cdots\oplus\overbrace{(e_p\hat{R}\oplus\cdots\oplus e_p\hat{R})}^{k_p}$, *where each* $e_i\hat{R}$ *is a uniform right ideal of* $\hat{R}$, e_i *is idempotent in* $\hat{R}$, $e_i\hat{R}/e_i\hat{A}$ *is a simple module annihilated by* M_i *and* k_i *is the Goldie dimension of* R/M_i.
(b) $\hat{A}=\hat{M}_1\cap\cdots\cap\hat{M}_p$, *where* $\hat{M}_1, \cdots, \hat{M}_p$ *are all maximal idempotent ideals of* $\hat{R}$ *and is a cycle, and* $\hat{M}_i=M_i\hat{R}=\hat{R}M_i$ *for each* i $(1\leqq i\leqq p)$.

Lemma 2.1. *Under the same notations as in* (a) *and* (b), *we have the following*
(1) $(e_i\hat{A}^{-1}+\cdots+e_i\hat{A}^{-1})+\hat{R}=O_l(\hat{M}_{i+1})=O_r(\hat{M}_i)$ $(1\leqq i\leqq p$ *and* $p+1=1)$.
(2) $\hat{R}e_i/\hat{A}e_i$ *is left* M_i*-primary, i.e., each element of* $\hat{R}e_i/\hat{A}e_i$ *is annihilated by* M_i.

Proof. Firstly we note that $\hat{A}^{-1}=(e_1\hat{A}^{-1}\oplus\cdots\oplus e_1\hat{A}^{-1})\oplus\cdots\oplus(e_p\hat{A}^{-1}\oplus\cdots\oplus e_p\hat{A}^{-1})$ and $\hat{A}^{-1}=O_l(\hat{M}_1)+\cdots+O_l(\hat{M}_p)$, because $O_l(\hat{M}_i)=(\hat{R}\colon \hat{M}_i)_l$. Thus we have
(c) $\hat{A}^{-1}/\hat{R}=(e_1\hat{A}^{-1}+\cdots+e_1\hat{A}^{-1}+\hat{R})/\hat{R}\oplus\cdots\oplus(e_p\hat{A}^{-1}+\cdots+e_p\hat{A}^{-1}+\hat{R})/\hat{R}$, and
(d) $\hat{A}^{-1}/\hat{R}=O_l(\hat{M}_1)/\hat{R}\oplus\cdots\oplus O_l(\hat{M}_p)/\hat{R}$.
It is clear that $O_l(\hat{M}_i)/\hat{R}$ is M_i-primary. Since $e_i\hat{Q}/e_i\hat{A}$ is a uniform and injective $\hat{R}$-module, it is a uniform and injective R-module by Lemma 2.4 of [8]. Thus we have $e_i\hat{A}^{-1}/e_i\hat{R}$ is M_{i+1}-primary by periodicity theorem and (a) (see Theorem 22 of [4]). It follows that $(e_i\hat{A}^{-1})\hat{M}_{i+1}\subseteqq e_i\hat{R}\subseteqq\hat{R}$ and $e_i\hat{A}^{-1}\subseteqq O_l(\hat{M}_{i+1})$. Thus (1) follows from (c) and (d).

(2) Since $O_r(\hat{M}_i)=O_l(\hat{M}_{i+1})$, we have $\hat{M}_i(e_i\hat{A}^{-1})\subseteqq\hat{R}$ by (1) and hence $\hat{M}_ie_i\subseteqq\hat{A}e_i$. This implies that $\hat{R}e_i/\hat{A}e_i$ is M_i-primary as left modules.

Let M be an $\hat{R}$-module. Then write $M^{\#}=\mathrm{Hom}_{\hat{R}}(M, K_{F_A})$.

Lemma 2.2. *Under the same notations as in* (a) *and* (b), *we have*
(1) *for any positive integer* n *and any* i $(1\leqq i\leqq p)$, $(e_i\hat{R}/e_i\hat{A}^n)^{\#}=\hat{R}e_j/\hat{A}^ne_j$ *for some* j $(1\leqq j\leqq p)$.
(2) $(e_i\hat{R})^{\#}=\hat{Q}e_i/\hat{R}e_i=E(\hat{R}e_{i-1}/\hat{A}e_{i-1})$, *the injective hull of* $\hat{R}e_{i-1}/\hat{A}e_{i-1}$, *where* $1\leqq i\leqq p$ *and* $i-1=p$ *if* $i=1$.
(3) $(e_i\hat{Q})^{\#}=\hat{Q}e_i$ *for each* i $(1\leqq i\leqq p)$.
(4) $(e_i\hat{Q}/e_i\hat{R})^{\#}=\hat{R}e_i$ *for each* i $(1\leqq i\leqq p)$.
These modules are all A*-l.c. modules.*

Proof. (1) Clearly $(e_i\hat{R}/e_i\hat{A}^n)^\# = \hat{A}^{-n}e_i + \hat{R}/\hat{R} = \hat{A}^{-n}e_i/\hat{R}e_i$ by left multiplications of elements in $\hat{A}^{-n}e_i$. $\hat{A}^{-n}e_i/\hat{R}e_i$ is a uniserial module of length n with composition factor modules $\hat{A}^{-k}e_i/\hat{A}^{-(k-1)}e_i$ ($1 \leqq k \leqq n$ and $\hat{A}^{-0} = \hat{R}$). There is j ($1 \leqq j \leqq p$) such that $\hat{A}^{-n}e_i/\hat{A}^{-(n-1)}e_i \cong \hat{R}e_j/\hat{A}e_j$ and then $\hat{R}e_j/\hat{A}^n e_j \cong \hat{A}^{-n}e_i/\hat{R}e_i$ by the periodicity theorem.

(2) The first isomorphism is also obtained by left multiplication of elements in $\hat{Q}e_i$. The second isomorphism follows from the periodicity theorem.

(3) Let $x = xe_i$ be any element of $\hat{Q}e_i$. Then a mapping λ_x: $e_iQ \rightarrow K_{F_A}$ given by $\lambda_x(y) = [xy + \hat{R}]$ ($y \in e_i\hat{Q}$) is a homomorphism. Assume that $\lambda_x = 0$ and $x \neq 0$. Then $x\hat{Q} = xe_i\hat{Q} \subseteqq \hat{R}$, that is, $x \in \hat{R}$. Hence $\hat{R}x\hat{R}\hat{Q} \subseteqq \hat{R}$. But $\hat{R}x\hat{R}$ contains a regular element in $\hat{R}$ and so $\hat{R}x\hat{R}\hat{Q} = \hat{Q}$, a contradiction. Hence we may assume that $\hat{Q}e_i \subseteqq (e_i\hat{Q})^\#$. Conversely, let f be any non zero element in $(e_i\hat{Q})^\#$ and let $f(e_i) = [q + \hat{R}]$, where $q = qe_i \in \hat{Q}$. Since $(f - \lambda_q)(e_i\hat{R}) = 0$, $f - \lambda_q$ induces an element $\overline{f - \lambda_q}$ in $(e_i\hat{Q}/e_i\hat{R})^\#$. Since $\hat{Q}/\hat{R} = e_i\hat{Q}/e_i\hat{R} \oplus (1 - e_i)\hat{Q}/(1 - e_i)\hat{R}$, we may consider that $\overline{f - \lambda_q} \in (\hat{Q}/\hat{R})^\#$. By Proposition A.3 of [8], $\hat{R} \cong (\hat{Q}/\hat{R})^\#$. Hence $\overline{f - \lambda_q} = \lambda_r$ for some $r \in \hat{R}$ and $f - \lambda_q = \lambda_r$. So we get that $(e_i\hat{Q})^\# \subset \hat{Q}e_i$ and therefore $(e_i\hat{Q})^\# = \hat{Q}e_i$.

(4) The exact sequence $0 \rightarrow e_i\hat{R} \rightarrow e_i\hat{Q} \rightarrow e_i\hat{Q}/e_i\hat{R} \rightarrow 0$ induces the exact sequence $0 \rightarrow (e_i\hat{Q}/e_i\hat{R})^\# \rightarrow (e_i\hat{Q})^\# \rightarrow (e_i\hat{R})^\# \rightarrow 0$, because K_{F_A} is injective. The assertion follows from (2) and (3). The left modules in (1) and (2) are artinian and A-primary. So they are A-*l.c.* modules in the discrete topology by Lemma 2.1 of [7] (as it has been pointed out in §1, all results in [7, §2] hold in F-*l.c.* modules over any HNP rings). $\hat{R}$ is an A-*l.c.* modules by Lemma 2.4 of [7]. Thus it follows that $\hat{R}e_i$ is also an A-*l.c.* module. Finally consider the exact sequence $0 \rightarrow \hat{R}e_i \rightarrow \hat{Q}e_i \rightarrow \hat{Q}e_i/\hat{R}e_i \rightarrow 0$. $\hat{Q}e_i$ is a topological module by taking as a fundamental system of 0 the submodules $\{\hat{A}^n e_i \mid n = 0, \pm 1, \pm 2, \cdots\}$. Hence $\hat{Q}e_i$ is an A-*l.c.* module by Proposition 9 of [20].

Following [9], a submodule L of a module M is called F^ω-*pure* if $MJ \cap L = LJ$ for any $J \in F_l$. Let F_o be the right Gabriel topology of all essential right ideals of R. Then "an F_o^ω-pure submodule" is merely called a *pure submodule*.

Consider the following condition:

(e) *all finitely generated F and F_l-torsion modules are a direct sum of cyclic modules.*

This condition is satisfied by any topologies F and F_l on R if R has enough invertible ideals and so, especially, if R has a non zero Jacobson radical (see Corollary 3.4 and Theorems 4.12, 4.13 of [3]). If all F and F_l-torsion modules are of bounded orders, i.e., unfaithful modules, then this condition is satisfied, because every factor ring of an HNP ring is serial (Corollary 3.2 of [1]). Note that [9, Lemma 1.2] is still valid for topologies F and F_l on any HNP ring R satisfying the condition (e). Furthermore, if R has a nonzero Jacobson radical, then a submodule L of a module M is pure if and only if $Mc \cap L = Lc$ for any regular element c in R by Proposition 3 of [19] and the remark to Theorem 3.6 of [15].

Lemma 2.3. *Let R be an HNP ring with the Jacobson radical A and A be a maximal invertible ideal of R. If as hort exact sequence $0\to L\overset{\alpha}{\to}M\overset{\beta}{\to}N\to 0$ is pure, then $0\to \mathrm{Hom}_R(N, K)\overset{\beta^*}{\to}\mathrm{Hom}_R(M, K)\overset{\alpha^*}{\to}\mathrm{Hom}_R(L, K)\to 0$ is pure as left R-modules.*

Proof. Let c be any regular element of R and let $cf=g\beta$ be any element in $c\mathrm{Hom}_R(M, K)\cap(\mathrm{Hom}_R(N, K))\beta^*$, where $f\in\mathrm{Hom}_R(M, K)$ and $g\in\mathrm{Hom}_R(N, K)$. Since $g\beta\alpha(L)=0$, we have $\alpha(L)\subset\mathrm{Ker}\ g\beta=\mathrm{Ker}\ cf$. There is a nature number n such that $Rc\supset A^n$. It follows that $0=Rcf\alpha(L)\supseteqq A^nf\alpha(L)$. Put $f\alpha(L)=X/R$, where X is a submodule of Q containing R. Then $A^nX\subseteqq R$ and so $X\subseteqq(R\colon A^n)_r=A^{-n}=(R\colon A^n)_l$. Thus we have $XA^n\subseteqq R$. This implies that $f\alpha(L)A^n=0$. Put $\bar{M}=M/\alpha(L)A^n$. Then $\bar{L}=\alpha(L)/\alpha(L)A^n$ is pure in $\bar{M}$, because L is pure in M. It follows from Theorem 3 of [13] and Theorem 1.3 of [14] that $\bar{L}$ is a direct summand of $\bar{M}$, because $\bar{L}$ is of bounded order. Thus we have the following sequence;

$$M\overset{\eta}{\to}\bar{M}=\bar{L}\oplus\bar{M}_1\overset{\pi}{\to}\bar{M}_1\overset{f_1}{\to}K\,,$$

where η is a natural homomorphism, π is a projection map from $\bar{M}$ to $\bar{M}_1$ (M_1 is a submodule of M) and f_1 is the map induced by f (note that $f\alpha(L)A^n=0$). Put $h=f_1\pi\eta$ and let x be any element of M. Write $\bar{x}=\bar{x}_1+\bar{x}_2$ ($x_1\in\alpha(L)$ and $x_2\in M_1$). Then $ch(x)=cf_1\pi\eta(x)=cf_1(\bar{x}_2)=cf(x_2)$. Since $x-x_2\in\alpha(L)+\alpha(L)A^n\subseteqq\alpha(L)$ and $cf\ \alpha(L)=0$, we have $ch(x)=cf(x_2)=cf(x)$. Therefore $ch=cf$. By the construction of h, $h(\alpha(L))=0$. This entails that h induces a map $k\colon N\to K$ such that $k\beta=h$. Hence we have $cf=ch=ck\beta\in c(\mathrm{Hom}_R(N, K))\,\beta^*$, as desired.

Theorem 2.4. *Under the same notations as in* (a) *and* (b), *a module is an A-l.c. module if and only if it is isomorphic to a direct product of modules of the following types*:

$e_i\hat{R}/e_i\hat{A}^n$ $(n=1, 2, \cdots)$, *$E(e_i\hat{R}/e_i\hat{A})$, the injective hull of $e_i\hat{R}/e_i\hat{A}$, $e_i\hat{R}$ and $e_i(Q\otimes_R\hat{R})$ $(1\leqq i\leqq p)$.*

Proof. The sufficiency follows from Proposition 1 of [20] and Lemma 2.2. Conversely let M be an A-l.c. module. Then M^* is a left $\hat{R}$-module by (1.2). So M^* has a basic submodule B by Theorem 2.1 of [8]. Then B is a direct sum of modules of types; $\hat{R}e_i/\hat{A}^ne_i$ and $\hat{R}e_i$ $(1\leqq i\leqq p)$ and $n=1, 2, \cdots$), and M^*/B is a direct sum of modules of types; $E(\hat{R}e_i/\hat{A}e_i)$ and $(Q\otimes_R\hat{R})e_i$ (see Theorem 2.2 of [8]). Then from pure exact sequence $0\to B\to M^*\to M^*/B\to 0$, we derive the pure exact sequence $0\to(M^*/B)^\#\to M^{*\#}\to B^\#\to 0$ (as right $\hat{R}$-modules) by Lemma 2.3. By Lemma 2.2, $(M^*/B)^\#$ is a direct product of modules of types; $e_i(Q\otimes_R\hat{R})$ and $e_i\hat{R}$. Here $e_i(Q\otimes_R\hat{R})$ is an injective $\hat{R}$-module.

Since $\hat{R}\cong\mathrm{Hom}_R(K_{F_A}, K_{F_A})\cong\mathrm{Hom}_{\hat{R}}(\hat{Q}/\hat{R}, \hat{Q}/\hat{R})$ (see Lemma 1.5 and Proposition A.3 of [8]), $\hat{R}$ is a pure injective $\hat{R}$-module by Propositions A.5, A.6 of [8] and Theorem 3.5, the remark to Proposition A.5 of [9], i.e., $\hat{R}$ has the injective property relative to the class of pure exact sequences and so is $e_i\hat{R}$. Hence $(M^*/B)^\sharp$ is also pure injective. This entails that $M^{*\sharp}\cong(M^*/B)^\sharp\oplus B^\sharp$, and the assertion follows from (1.2) and Lemma 2.2.

Lemma 2.5. *Let M be a left $\hat{R}$-module and let m be any non zero element of M. Then there is an element f in $M^\sharp$ such that $(m)f\neq 0$.*

Proof. $\hat{R}m$ is a finite direct sum of modules of types; $\hat{R}e_i/\hat{A}^n e_i$ and $\hat{R}e_i$ by Theorems 2.1 and 2.2 of [8]. Thus the assertion follows from Lemma 2.2, because K_{F_A} is an injective $\hat{R}$-module.

Theorem 2.6. *Let R be an HNP ring and let A be a maximal invertible ideal of R. Then*

(1) *Let M be any A-l.c. module. Then M^* is a left $\hat{R}$-module and $M\cong M^{*\sharp}$.*

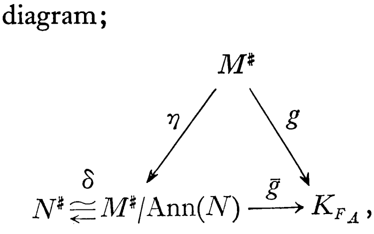
(2) *Let M be any left $\hat{R}$-module. Then $M^\sharp$ is an A-l.c. module in a certain topology and $M\cong M^{\sharp *}$ ($M^\sharp$ is equipped with the finite topology).*

Proof. (1) is clear from (1.2).

(2) Let M be any left $\hat{R}$-module. Then $M^\sharp$ is a direct product of modules of types in Theorem 2.4 (this is proved in the same way as in Theorem 2.4 by using basic submodules). Thus $M^\sharp$ is an A-l.c. module. Now $M^\sharp$ is equipped with the finite topology (it is not requested that $M^\sharp$ is an A-l.c. module in the finite topology). Let β: $M\to M^{\sharp *}$ be the natural map given by $((m)\beta)(f)=(m)f$, where $m\in M$ and $f\in M^\sharp$. Note that $(m)\beta\in M^{\sharp *}$, because $\mathrm{Ker}(m)\beta=\{g\in M^\sharp\,|\,(m)g=0\}$. By Lemma 2.5, β is a monomorphism. To prove that β is an epimorphism, let g be any element in $M^{\sharp *}$. Since $\mathrm{Ker}\, g$ is open in $M^\sharp$, there is a finitely generated left module N of M such that $\mathrm{Ker}\, g\supseteq\mathrm{Ann}(N)$. Write

$$(*)\qquad N\cong\sum_{i=1}^{p}\sum_j\oplus\hat{R}e_i/\hat{A}^{n_{ij}}e_i\oplus\sum\oplus\hat{R}e_k\quad(1\leqq k\leqq p),$$

where $n_{ij}\geqq 0$. Thus N is a left A-l.c. module by Lemma 2.2. Consider the following commutative diagram;



where η is a natural map, $\bar{g}$ is a map induced by g and $\delta\,([f+\mathrm{Ann}(N)])=f\,|\,N$, the restriction map of f to N ($f\in M^\sharp$). Let h be any element of $N^\sharp$. Then there is a natural number n such that $\hat{A}^n(N)\, h=0$, because N is finitely generated. This entails that Ker $h \supseteqq \sum_{i=1}^{p} \sum_j \oplus \hat{A}^n e_i / \hat{A}^{n_{ij}} e_i \oplus \sum \oplus \hat{A}^n e_k$, open in N (in the topology given in Lemma 2.2). Thus we have $h \in N^*$ and hence $N^* = N^\sharp$. It follows from (1.2) that $\alpha\colon N \cong N^{\sharp\sharp}$. So, for the element $\bar{g}\delta^{-1} \in N^{\sharp\sharp}$ there is an element $n \in N$ such that $(n)\alpha = \bar{g}\delta^{-1}$, i.e., $((n)\,\alpha)\,\delta = \bar{g}$. Now let x be any element in $M^\sharp$. Then we have $g(x) = \bar{g}\eta(x) = ((n)\,\alpha)\,\delta\eta(x) = (n)\{\delta\eta(x)\} = (n)\ \{\delta\,[x + \mathrm{Ann}(N)]\} = (n)x = ((n)\,\beta)\ (x)$. Hence $g = (n)\,\beta$, as desired.

3. In this section, we study relationships between F-*l.c.* modules and F^ω-pure injective modules in case F is special. A module G is *F^ω-pure injective* if it has the injective property relative to the class of F^ω-pure exact sequences. Let A be a maximal invertible ideal of R. Then F_A^ω-pure injective modules are just called *A-pure injective modules.* The cancellation set of A, $C(A)$, is defined to be $\{c \in R \mid cx \in A \Rightarrow x \in A\} = \{c \in R \mid cx \in A \Rightarrow x \in A\}$. By [6], R satisfies the Ore condition with respect to $C(A)$ and the local ring R_A of R at A is an HNP ring with Jacobson radical $AR_A = R_A A$. Note that a module T is A-primary if and only if it is an R_A-module and torsion as R_A-modules (see the proof of Lemma 2.4 of [8]). Since R_A is R-flat and the inclusion map: $R \to R_A$ is an epimorphism, we have the following

Lemma 3.1. (1) *An exact sequence $0 \to L \to M \to N \to 0$ is A-pure, then the induced sequence $0 \to L \otimes_R R_A \to M \otimes_R R_A \to N \otimes_R R_A \to 0$ is exact and is pure as R_A-modules.*

(2) *if an exact sequence $0 \to L \to M \to N \to 0$ of R_A-modules is pure as R_A-modules, then it is A-pure.*

Proof. Use (3) in Lemma 1.2 of [9].

Lemma 3.2. *Let F be a right Gabriel topology on R satisfying the condition* (e) *and let G be any F^ω-pure injective module. Then $G = D \oplus H$, where D is an injective module, and H is F-reduced, F^ω-pure injective and F_l^ω-complete. In particular, H is an $\hat{R}_{F_l}$-module.*

Proof. The proof of Theorem 3.2 of [9] may be used unaltered to yield this lemma.

Proposition 3.3. *Let G be a reduced module, i.e., G has no non zero injective submodules. Then*

(1) *G is A-pure injective if and only if G is an R_A-module and is pure injective as R_A-modules.*

(2) *G is A-pure injective if and only if $G \cong \hat{G} = \varprojlim G/GA^n$.*

Proof. (1) It is clear from Lemmas 3.1 and 3.2. (2) follows from Theorems 3.2.4 and 3.3.3 of [18] and (1), because R_A is a bounded HNP ring.

From Theorem 2.4 and Proposition 3.3, we have

Corollary 3.4. *A-l.c. modules are A-pure injective modules.*

In general, it is not necessary to hold that (F-*l.c.* modules)$\Rightarrow$(F^{ω}-pure injective modules). We will end up this paper with giving a counter example. To do this, let B be an idempotent ideal of R. Then write

(f) $F_1 = \{I \mid IO_r(B) = O_r(B),\ I\text{: right ideal of } R\}$.
$F_2 = \{I \mid IO_l(B) = O_l(B),\ I\text{: right ideal of } R\}$.

Then $F_{1l}=\{J \mid O_r(B)J=O_r(B),\ J\text{: left ideal of } R\}$, and $F_{2l}=\{J \mid O_l(B)J=O_l(B)$, J: left ideal of $R\}$. Since $BO_r(B)=B$, $O_r(B)B=O_r(B)$, $BO_l(B)=O_l(B)$ and $O_l(B)B=B$, we have $F_{1l}=\{J \mid J \supseteqq B\}$, $F_2=\{I \mid I \supseteqq B\}$, $F_1 \not\ni B$ and $F_{2l} \not\ni B$.

Proposition 3.5. *Under the same notations as in* (f), *let G be any module. Then*

(1) *G is an F_1-l.c. module if and only if it is a direct product of modules of types $(R: J)_r/R$, where J is a left ideal of R containing B.*

(2) *G is an F_2-l.c. module, then it is a direct sum of modules of types R/I $(I \supseteqq B)$.*

Proof. (1) The sufficiency is evident from Proposition 1 of [20], Proposition A.1 of [8] and Lemma 2.1 of [7]. Let G be an F_1-*l.c.* module. Then G^* is a left $\hat{R}_{F_{1l}}(=R/B)$-module by (1.2). Since R/B is a serial ring, G^* is a direct sum of cyclic modules (see Theorem 1.2 and Corollary 3.2 of [1]). Write $G^*=\sum\oplus R/J_i$ $(J_i \supseteqq B)$ and then $G \cong G^{**}=\Pi(R: J_i)_r/R$ by (1.2).

(2) is clear, because any F_2-*l.c.* module is an $\hat{R}_{F_2}$ $(=R/B)$-module.

Let C be an idempotent ideal of R such that

(g) $O_r(B) = O_l(C)$.

Then $F_1=\{I \mid I \supseteqq C,\ I$; right ideal of $R\}$. Note that there exists an idempotent ideal C of R satisfying the condition (g) for any idempotent ideal B of R if R has enough invertible ideals. In the absense of the condition of having enough invertible ideals, we can easily find a pair of idempotent ideals B and C satisfying the condition (g) (see [4]).

Lemma 3.6. *Under the same notations as in* (f) *and* (g), *let $0\to L\to M\to N\to 0$ be an exact sequence. Then it is F_1^{ω}-pure if and only if the induced sequence*

(ι) $0\to L/LB\to M/MB$

is splitting exact.

Proof. The sufficiency is clear. To prove the necessity, assume that the sequence is F_1^ω-pure. Then (ι) is exact and is F_1^ω-pure. Since a module is a finitely presented $\bar{R}$ ($=R/B$)-module if and only if it is a finitely generated and F_1-torsion module, (ι) is pure as R-modules in the sense of [19] by Proposition 3 of [19] and Lemma 1.2 of [9]. Furthermore, since $\bar{M}=M/MB$ satisfies Singh's conditions (I), (II) and (III), $\bar{L}$ is h-pure in the sense of Singh (see Theorem 1.3 of [14]). Hence $\bar{L}$ is a direct summand of $\bar{M}$ by Theorem 3 of [13], because $\bar{L}$ is of bounded order.

Proposition 3.7. *Under the same notations as in* (f) *and* (g), *a reduced module is F_1^ω-pure injective if and only if it is an R/B-module.*

Proof. This is clear from Lemmas 3.2 and 3.6.

EXAMPLE 3.8. Under the same notations as (f) and (g), R/C is an F_1-*l.c.* module in the discrete topology. But it is not an F_1^ω-pure injective module.

Proof. R/C is an artinian and F_1-torsion module. So it is an F_1-*l.c.* module in the discrete topology by Lemma 2.1 of [7]. Assume that it is F_1^ω-pure injective. Then it is an R/B-module by Proposition 3.7. This implies that $B\subseteqq C$ and so $O_l(C)=O_r(B)=(R\colon B)_r\supset(R\colon C)_r=O_r(C)$. Thus we have $C=O_l(C)C\supset O_r(C)C=O_r(C)\supset R$, a contradiction.

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