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ON MARKOV PROCESS GENERATED BY PSEUDODIFFERENTIAL OPERATOR OF VARIABLE ORDER

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1. Introduction

The relationship between diffusion processes and differential equations has been studied with great details. However the relationship between pure jump type Markov processes and evolution equations has not been studied in full details.

Recently, N. Jacob and H. Leopld [2] have shown that there exists a Feller semigroup generated by the pseudodifferential operator whose symbol is the function $-\langle \xi \rangle^{\sigma(x)} = -(\sqrt{1+|\xi|^2})^{\sigma(x)}$, which we denote $-\langle D_x \rangle^{\sigma(X)}$. Here the order function $\sigma(x)$ is the sum of some function in Schwartz class and some constant and satisfies $0 < \inf \sigma \leq \sup \sigma \leq 2$. For this purpose they have introduced a suitable function space. This space is a kind of Sobolev space of variable order. Here $\langle \xi \rangle^{\sigma(x)}$ have been used as a weight function. They have shown that the restriction of $\langle D_x \rangle^{\sigma(X)}$ to this Sobolev space satisfies the conditions in Hille-Yosida-Ray theorem. But they do not give conditions for that a general pseudodifferential operator generates a Feller semigroup. In this paper we give one answer to this problem.

We want to inverstigate the relationship between Markov processes and evolution equations with respect to pseudodifferential operators by developing their ideas. In order to complete our theory we should restrict functions which we treat to $H^{-\infty}$ instead of $\mathcal{S}'(\boldsymbol{R}^d)$ at first of all. And considering the function $\langle \xi \rangle$ $(\xi \in \boldsymbol{R}^d)$ as standard weight function, we define the Sobolev space $H^{\sigma(\cdot)}(\boldsymbol{R}^d)$ with variable order by the same way as in the definition of such spaces with constant order, where the order function σ is in $\mathcal{B}^\infty(\boldsymbol{R}^d)$. This definition which is slightly different from one in [2] allows us to show that if σ_1 and σ_2 are in $\mathcal{B}^\infty(\boldsymbol{R}^d)$ and satisfy $\sigma_1(x) \leq \sigma_2(x)$ for any $x \in \boldsymbol{R}^d$, then $H^{\sigma_2(\cdot)}(\boldsymbol{R}^d) \subset H^{\sigma_1(\cdot)}(\boldsymbol{R}^d)$, and if P is an elliptic pseudodifferential operator in the class $\mathcal{S}^{\sigma}_{\rho,\delta}(0 \leq \delta < \rho \leq 1)$, then the space $\{u \in H^{-\infty}; Pu \in L^2(\boldsymbol{R}^d)\}$ coincides with $H^{\sigma(\cdot)}(\boldsymbol{R}^d)$. Combining the ideas in N. Jacob and H. Leopld [2] with these facts, we obtain that there exists a Feller semigroup $\{T_t\}_{t\geq 0}$ generated by a storngly elliptic pseudodifferential operator P with suitable variable order. Using the method proposed in [6], we see that, for $u_0 \in C_0^\infty(\boldsymbol{R}^d)$, $u = T_t u_0$ is a unique solution to the initial-value problem $\partial_t u - Pu = 0$, $u(0) = u_0$ and also we can construct its fundamental solution which is a pseudodifferential operator with a smooth kernel function. Therefore, we see that this semigroup has a transition density. Moreover, combining these results with the method proposed in [6], we obtain the local Hölder conditions for sample paths of Markov process to which such Feller semigroup corresponds.

The organization of this paper is as follows. Section 2 is devoted to preparing basic results of pseudodifferential operators and Sobolev spaces of variable order of differentiation. In section 3 we prove that there exists a unique Feller semigroup generated by a strongly elliptic pseudodifferential operator in $\$^{\sigma}_{\rho,\delta}$ $(0 < \inf \sigma(x) \leq \sup \sigma(x) \leq 2)$ and that this Feller semigroup has a transition density. And we mention local Hölder conditions for sample paths of the Markov processes. In appendix we mention some properties of Sobolev spaces of variable order.

2. Sobolev spaces of variable order

First of all we give a definition of pseudodifferential operator of variable order. Let δ and ρ be real numbers with $0 \leq \delta < \rho \leq 1$, σ be a real-valued function in $\mathcal{B}^{\infty}(\boldsymbol{R}^d)$, the set of all C^{∞} functions whose derivatives of each order are bounded. We say that a function $p(x,\xi) \in \mathcal{B}^{\infty}(\boldsymbol{R}^d_x \times \boldsymbol{R}^d_\xi)$ is a member of $S^{\sigma}_{\rho,\delta}$ if and only if for any multi-indices α and β there exists some positive constant $C_{\alpha,\beta}$ such that

$$|p^{(\alpha)}_{(\beta)}(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{\sigma(x)-\rho|\alpha|+\delta|\beta|}.$$

For $u \in \mathcal{S}(\boldsymbol{R}^d)$ (the set of all rapidly decreasing functions) and $p \in S^{\sigma}_{\rho,\delta}$ we define a function $Pu \in \mathcal{S}(\boldsymbol{R}^d)$ by

$$Pu(x) = \int_{\boldsymbol{R}^d} e^{ix\xi} p(x,\xi)\hat{u}(\xi) đ\xi,$$

where $\hat{u}(\xi) = \int_{\boldsymbol{R}^d} e^{-ix\xi}u(x)dx$ (the Fourier transformation of u) and $đ\xi = (2\pi)^{-n}d\xi$. We easily see that P is an operator from $\mathcal{S}(\boldsymbol{R}^d)$ to $\mathcal{S}(\boldsymbol{R}^d)$. Now we call that P is a pseudodifferential operator with symbol $p(x,\xi)$, and we denote $P = p(X,D_x)$ and $\sigma(P) = p(x,\xi)$. The set of all pseudodifferential operators with symbol p of class $S^{\sigma}_{\rho,\delta}$ is denoted by $\$^{\sigma}_{\rho,\delta}$, that is,

$$\$^{\sigma}_{\rho,\delta} = \{p(X,D_x); p \in S^{\sigma}_{\rho,\delta}\}.$$

It is clear that $S^{\sigma_1}_{\rho,\delta} \subset S^{\sigma_2}_{\rho,\delta}$ for $\sigma_1(x) \leq \sigma_2(x)$, and especially $S^{\underline{\sigma}}_{\rho,\delta} \subset S^{\sigma}_{\rho,\delta} \subset S^{\overline{\sigma}}_{\rho,\delta}$, where $\overline{\sigma} = \sup_{x\in\boldsymbol{R}^d}\sigma(x)$ and $\underline{\sigma} = \inf_{x\in\boldsymbol{R}^d}\sigma(x)$. For $p \in S^{\sigma}_{\rho,\delta}$ we define the semi-norms $|p|^{(\sigma)}_{\ell}$ $(\ell = 1,2,\cdots)$ by

$$|p|^{(\sigma)}_{\ell} = \max_{|\alpha+\beta|\leq\ell} \sup_{(x,\xi)\in\boldsymbol{R}^d\times\boldsymbol{R}^d} \{|p^{(\alpha)}_{(\beta)}(x,\xi)|\langle\xi\rangle^{-(\sigma(x)-\rho|\alpha|+\delta|\beta|)}\}.$$

The following facts have been obtained in [6].

Theorem 2.1. *Assume that* $0 \leq \delta < \rho \leq 1$.

(1) *Let* $\sigma_j (j = 1,2)$ *be functions of* $\mathcal{B}(\boldsymbol{R}^d)$ *and* P_j*'s are pseudodifferential operators with symbols* $p_j(x,\xi) \in S^{\sigma_j}_{\rho,\delta} (j = 1,2)$. *Then* $P = P_1 \cdot P_2$ *belongs to* $\mathcal{S}^{\sigma_1+\sigma_2}_{\rho,\delta}$.

(2) *For* $P = p(x,D_x) \in \mathcal{S}^{\sigma}_{\rho,\delta}$ *the formally adjoint operator* P^*, *defined by* $(Pu,v) = (u,P^*v)$ *for* $u,v \in \mathcal{S}(\boldsymbol{R}^d)$, *belongs to* $\mathcal{S}^{\sigma}_{\rho,\delta}$.

Let $P = p(X,D_x) \in \mathcal{S}^{\sigma}_{\rho,\delta}$. We say that P is elliptic if there exist $c_0 > 0$ and $M > 0$ such that

$$|p(x,\xi)| \geq c_0 \langle \xi \rangle^{\sigma(x)} \quad (|\xi| \geq M).$$

Let $P \in \mathcal{S}^{\sigma}_{\rho,\delta}$. We say that $Q \in \mathcal{S}^{\infty}_{\rho,\delta} = \bigcup_{m \in \boldsymbol{R}} \mathcal{S}^m_{\rho,\delta}$ is a left (resp. right) parametrix of P if there exists R_L (resp. R_R) $\in \mathcal{S}^{-\infty} = \bigcap_{m \in \boldsymbol{R}} \mathcal{S}^m_{\rho,\delta}$ such that

$$QP = I + R_L \quad (\text{resp.} \quad PQ = I + R_R).$$

We say simply that Q is a parametrix for P if Q is simultaneously a left and right parametrix of P.

The following important result on elliptic operators also holds for pseudodifferential operators of variable order.

Theorem 2.2. *If* $P \in \mathcal{S}^{\sigma}_{\rho,\delta}$ *is elliptic, then there exists a parametrix of* P *in* $\mathcal{S}^{-\sigma}_{\rho,\delta}$.

Proof. We can construct a parametrix in the same way as in the case of constant order (see the proof of Theorem 5.4 of Chapter 2 of [4]). If we check precisely, we see that the symbol of the parametrix is really a member of $S^{-\sigma}_{\rho,\delta}$. □

Now we give a definition of the Sobolev spaces of order σ, a real-valued function in $\mathcal{B}^{\infty}(\boldsymbol{R}^d)$, which is wider than the class treated in former literatures. Let us denote

$$H^{\sigma(\cdot)}(\boldsymbol{R}^d) = \{u \in H^{-\infty}(\boldsymbol{R}^d); \langle D_x \rangle^{\sigma(X)} u \in L^2(\boldsymbol{R}^d)\}. \tag{2.1}$$

REMARK. We still do not give any topologies in $H^{\sigma(\cdot)}(\boldsymbol{R}^d)$. It is introduced after the proof of Theorem 2.4.

The following fact is fundamental in our theory.

Lemma 2.3. *Let* u *be a member of* $H^{\sigma(\cdot)}(\boldsymbol{R}^d)$. *Then, for any* $P \in \mathcal{S}^{\sigma}_{\rho,\delta}$, Pu *belongs to* $L^2(\boldsymbol{R}^d)$.

Proof. Let us denote a parametrix of $\langle D_x\rangle^{\sigma(X)}$ by Q_σ. Then there exists some $R_L \in \mathcal{S}^{-\infty}$ such that $Q_\sigma\langle D_x\rangle^{\sigma(X)} = I + R_L$. Now we write

$$(2.2)\qquad Pu = P(Q_\sigma\langle D_x\rangle^{\sigma(X)} - R_L)u = PQ_\sigma(\langle D_x\rangle^{\sigma(X)}u) - PR_Lu.$$

Theorem 2.1 implies $PQ_\sigma \in \mathcal{S}^0_{\rho,\delta}$. Thus L^2 boundedness theorem implies the conclusion since $u \in H^{-\infty}(\boldsymbol{R}^d)$ and $\langle D_x\rangle^{\sigma(X)}u \in L^2(\boldsymbol{R}^d)$. □

Hereafter we use above notation Q_σ without specification.

Theorem 2.4. *Let σ_1 and σ_2 be functions in $\mathcal{B}^\infty(\boldsymbol{R}^d)$ with $\sigma_1(x) \geq \sigma_2(x)$ for any $x \in \boldsymbol{R}^d$. Then we have $H^{\sigma_1(\cdot)}(\boldsymbol{R}^d) \subset H^{\sigma_2(\cdot)}(\boldsymbol{R}^d)$.*

Proof. First we prove $H^{\sigma(\cdot)}(\boldsymbol{R}^d) \subset L^2(\boldsymbol{R}^d)$ when $\sigma \geq 0$. In fact, taking care that $Q_\sigma \in \mathcal{S}^{-\underline{\sigma}}_{\rho,\delta} \subset \mathcal{S}^0_{\rho,\delta}$ we have by L^2 boundedness theorem that $Q_\sigma\langle D_x\rangle^{\sigma(X)}u$ belongs to $L^2(\boldsymbol{R}^d)$ for any $u \in H^{\sigma(\cdot)}(\boldsymbol{R}^d)$. This means that $u = Q_\sigma\langle D_x\rangle^{\sigma(X)}u - R_Lu$ is in $L^2(\boldsymbol{R}^d)$ for any $u \in H^{\sigma(\cdot)}(\boldsymbol{R}^d)$.

Next we show $H^{\sigma_1(\cdot)}(\boldsymbol{R}^d) \subset H^{\sigma_2(\cdot)}(\boldsymbol{R}^d)$ when $\sigma_1(x) \geq \sigma_2(x)$ for each $x \in \boldsymbol{R}^d$. By Theorem 2.1 we have that

$$\langle D_x\rangle^{\sigma_1(X)-\sigma_2(X)}\langle D_x\rangle^{\sigma_2(X)} \in \mathcal{S}^{\sigma_1}_{\rho,\delta}.$$

Hence, for any $u \in H^{\sigma_1(\cdot)}(\boldsymbol{R}^d)$, Lemma 2.3 implies

$$\langle D_x\rangle^{\sigma_1(X)-\sigma_2(X)}\langle D_x\rangle^{\sigma_2(X)}u \in L^2(\boldsymbol{R}^d),$$

that is, $\langle D_x\rangle^{\sigma_2(X)}u \in H^{\sigma_1(\cdot)-\sigma_2(\cdot)}(\boldsymbol{R}^d)$. Thus, since $\sigma_1 - \sigma_2 \geq 0$, we have that $\langle D_x\rangle^{\sigma_2(X)}u$ belongs to $L^2(\boldsymbol{R}^d)$. □

Clearly Theorem 2.4 implies $H^{\sigma(\cdot)}(\boldsymbol{R}^d) \subset H^{\underline{\sigma}}(\boldsymbol{R}^d)$. Taking account of this fact we introduce a topology in $H^{\sigma(\cdot)}(\boldsymbol{R}^d)$. We can prove the following theorem in the same way as in the proof of Theorem 3 of [5].

Theorem 2.5. *$H^{\sigma(\cdot)}(\boldsymbol{R}^d)$ is a Hilbert space with the inner product*

$$(2.3)\qquad (u,v)_\sigma = \int_{\boldsymbol{R}^d}(\langle D_x\rangle^{\sigma(X)}u)(x)\cdot\overline{(\langle D_x\rangle^{\sigma(X)}v)(x)}dx + \int_{\boldsymbol{R}^d}(\langle D_x\rangle^{\underline{\sigma}}u)(x)\cdot\overline{(\langle D_x\rangle^{\underline{\sigma}}v)(x)}dx.$$

Moreover $\mathcal{S}(\boldsymbol{R}^d)$ is dense in $H^{\sigma(\cdot)}(\boldsymbol{R}^d)$.

REMARK 1. In (2.3) we add the second term for the sake of having equivalence between $\| u \|_\sigma = 0$ and $u = 0$ (as usually we denote $\| u \|_\sigma = \sqrt{(u,u)_\sigma}$). Of course the order of this term does not need to be $\underline{\sigma}$. Another norm $\| \langle D_x \rangle^{\sigma(X)} u \|_0 + \| u \|_s$ is equivalent to $\| u \|_\sigma$ for any constant s with $s \leq \underline{\sigma}$.

The second term of (2.3) is also helpful to prove the completeness of the space in the proof of Theorem 2.5.

REMARK 2. Our argument above implies that

$$H^{\sigma(\cdot)}(\boldsymbol{R}^d) = \{u \in H^{-\infty}(\boldsymbol{R}^d); \| u \|_\sigma < \infty\},$$

which is the definition of $H^{\sigma(\cdot)}(\boldsymbol{R}^d)$ adapted in former literatures (see, for example, [7]). Our definition seems to be more natural.

Lemma 2.3 implies the following fact.

Theorem 2.6. *Let $P \in \mathcal{S}^{\sigma}_{\rho,\delta}$ be elliptic. Then $H^{\sigma(\cdot)}(\boldsymbol{R}^d) = \{u \in H^{-\infty}(\boldsymbol{R}^d); Pu \in L^2(\boldsymbol{R}^d)\}$ as a set. Moreover the norm $\| u \|_\sigma$ is equivalent to another norm $\| u \|_{\sigma,P} := (\| Pu \|_0^2 + \| u \|_{\underline{\sigma}}^2)^{1/2}$.*

Proof. Lemma 2.3 shows "$\subset$", and (2.2) gives the inequality

$$\| Pu \|_0 \leq C(\| \langle D_x \rangle^{\sigma(X)} u \|_0 + \| u \|_{\underline{\sigma}}) \tag{2.4}$$

(for some constant C), which implies $\| u \|_{\sigma,P} \leq (C+1) \| u \|_\sigma$. On the other hand, since P is elliptic, we can replace $\langle D_x \rangle^{\sigma(X)}$ and P with each other in the argument of the proof of Lemma 2.3. Then

$$\| \langle D_x \rangle^{\sigma(X)} u \|_0 \leq C(\| Pu \|_0 + \| u \|_{-s})$$

for some sufficiently large s. (Since $u \in H^{-\infty}(\boldsymbol{R}^d)$, there exist such an s.) Thus we have the converse inclusion, and from this, we easily have the converse inequality to (2.4). □

Now we generalize the boundedness theorem of pseudodifferential operators to those of variable orders. The next theorem gives some key estimates in this paper.

Theorem 2.7. *Let σ and τ be function in $\mathcal{B}^\infty(\boldsymbol{R}^d)$. Suppose that $P \in \mathcal{S}^{\sigma}_{\rho,\delta}$. Then there exist some constant $C > 0$ independent of P and some positive integer ℓ depending only on σ, τ, ρ, δ, and n such that*

$$\| Pu \|_\tau \leq C |p|_\ell^{(\sigma)} \| u \|_{\sigma+\tau}$$

for $u \in H^{\sigma(\cdot)+\tau(\cdot)}(\boldsymbol{R}^d)$.

Proof. Note the equality

$$\langle D_x\rangle^{\tau(X)}Pu = \langle D_x\rangle^{\tau(X)}P(Q_{\sigma+\tau}\langle D_x\rangle^{\sigma(X)+\tau(X)} - R_L)u = \langle D_x\rangle^{\tau(X)}PQ_{\sigma+\tau}\langle D_x\rangle^{\sigma(X)+\tau(X)}u - \langle D_x\rangle^{\tau(X)}PR_L u. \tag{2.5}$$

Since $\langle D_x\rangle^{\tau(X)}PQ_{\sigma+\tau} \in \$^0_{\rho,\delta}$, we obtain, by L^2 boundedness theorem, that

$$\| Pu \|_\tau \leq C(|\sigma(\langle D_x\rangle^{\tau(X)}PQ_{\sigma+\tau})|^{(\sigma)}_{\ell_0} + |\sigma(\langle D_x\rangle^{\tau(X)}PR_L\langle D_x\rangle^{-\underline{(\sigma+\tau)}}|^{(\sigma)}_{\ell_0}) \| u \|_{\sigma+\tau} \,. \tag{2.6}$$

By Theorem 2.5 of Chapter 7 of [4] we have

$$\sigma(\langle D_x\rangle^{\tau(X)}PQ_{\sigma+\tau})(x,\xi) = p_0(x,\xi) + p_1(x,\xi) + \cdots + p_{N-1}(x,\xi) + r_N(x,\xi), \tag{2.7}$$

where each p_j are linear combinations of the products of derivatives of $\langle\xi\rangle^{\tau(x)}$, $p(x,\xi)$, and $\sigma(Q_{\sigma+\tau})(x,\xi)$ of order at most j. Thus we have

$$|p_j|^{(-j)}_{\ell_0} \leq C|p|^{(\sigma)}_{\ell}, \tag{2.8}$$

where C is a constant independent of P, and ℓ is an integer depending only on ℓ_0, σ, τ, ρ, δ, and n. On the other hand r_N of (2.7) is represented as follows :

$$r_N(x,\xi) = Os - \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} e^{-iy_1\eta_1 - iy_2\eta_2}\left(\int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!}\frac{d^N}{d\theta^N}\{\langle\xi+\theta\eta_1\rangle^{\tau(x)} p(x+y_1,\xi+\theta\eta_2)\sigma(Q_{\sigma+\tau})(x+y_1+y_2,\xi)\}d\theta\right)dy_1 đ\eta_1 dy_2 đ\eta_2.$$

Thus by Theorem 2.4 of Chapter 7 of [4] we obtain

$$|r_N|^{(\overline{\sigma}+\overline{\tau}-\underline{(\sigma+\tau)}-N)}_{\ell_0} \leq C|p|^{(\overline{\sigma})}_{\ell}, \tag{2.9}$$

where C is a constant independent of P, and ℓ is an integer depending only on ℓ_0, σ, τ, ρ, δ, and n. Since $|p|^{(\overline{\sigma})}_{\ell} \leq |p|^{(\sigma)}_{\ell}$, we have, by (2.7), (2.8), and (2.9) that

$$|\sigma(\langle D_x\rangle^{\tau(X)}PQ_{\sigma+\tau})|^{(\sigma)}_{\ell_0} \leq C_0|p|^{(\sigma)}_{\ell}. \tag{2.10}$$

Similarly we have

$$|\sigma(\langle D_x\rangle^{\tau(X)}PR_L\langle D_x\rangle^{-\underline{(\sigma+\tau)}})|^{(\sigma)}_{\ell_0} \leq C_1|p|^{(\sigma)}_{\ell}. \tag{2.11}$$

Thus we obtain the conclusion by (2.6), (2.10), and (2.11). □

3. Feller semigroup

Let $C_\infty(\boldsymbol{R}^d)$ be the completion of $\mathcal{S}(\boldsymbol{R}^d)$ by the L^∞-norm.

We call a family $\{T_t\}_{t\geq 0}$ of linear operators from $C_\infty(\boldsymbol{R}^d)$ to itself a Feller semigroup if the following conditions are fullfilled.

i) $\{T_t\}_{t\geq 0}$ is a semigroup, that is, we have $T_{s+t}=T_sT_t$ for all $s,t\geq 0$ and $T_0=id$.

ii) $\{T_t\}_{t\geq 0}$ is strongly continuous, that is, $\lim_{t\to 0}\| T_tu-u\|_{L^\infty}=0$ for all $u\in C_\infty(\boldsymbol{R}^d)$.

iii) Each of the operator T_t is a positivity preserving contraction, that is, for $u\in C_\infty(\boldsymbol{R}^d)$ with $0\leq u\leq 1$ we have $0\leq T_tu\leq 1$.

The following well-known theorem is often called Hille-Yosida-Ray theorem.

Theorem 3.1. *Let $\tilde{P}$ be a linear operator on $C_\infty(\boldsymbol{R}^d)$.*

In order that $-\tilde{P}$ extends to a generator of a Feller semigroup, it is necessary and sufficient that the following conditions are fullfilled.

i) *The domain $\mathcal{D}(\tilde{P})$ of $\tilde{P}$ is dense in $C_\infty(\boldsymbol{R}^d)$.*

ii) *$-\tilde{P}$ satisfies the positive maximum principle on $\mathcal{D}(\tilde{P})$, that is, if $u\in\mathcal{D}(\tilde{P})$ and $x_0\in\boldsymbol{R}^d$ such that $u(x_0)=\sup_{x\in\boldsymbol{R}^d}u(x)\geq 0$, then it follows that $-\tilde{P}u(x_0)\leq 0$ holds.*

iii) *For some $\lambda\geq 0$ the operator $\tilde{P}+\lambda$ maps $\mathcal{D}(\tilde{P})$ onto a dense subspace of $C_\infty(\boldsymbol{R}^d)$.*

Ph. Courrège [1] gives a sufficient condition to that a pseudodifferential operator satisfies the positive maximum principle. In order to state this theorem we should give the definition of negative definite function.

DEFINITION. A function $f:\boldsymbol{R}^d\to\boldsymbol{C}$ is called a negative definite function if for all $m\in\boldsymbol{N}$ and points $x^1,\cdots,x^m\in\boldsymbol{R}^d$ the matrix $(f(x^i)+\overline{f(x^j)}-f(x^i-x^j))_{i,j=1,\cdots,m}$ is non-negative Hermitian.

Theorem 3.2. *Let $p(x,\xi)$ be a continuous function on $\boldsymbol{R}^d_x\times\boldsymbol{R}^d_\xi$ such that for each $x\in\boldsymbol{R}^n$ the function $\xi\mapsto p(x,\xi)$ is negative definite. Then the pseudodifferential operator $-p(x,D_x)$ satisfies the positive maximum principle on $C_0^\infty(\boldsymbol{R}^d)$.*

In [2] Jacob and Leopold apply these two theorems to the operator $\langle D_x\rangle^{\sigma(X)}$, and obtain that this operator generates a Feller semigroup. They use the theory of pseudodifferential operator of order varying weight function and Sobolev spaces related to this weight function. We replace this part of their idea by ours which is mentioned in section 2, and we have the following theorem.

Theorem 3.3. *Let σ be a real-valued function in $\mathcal{B}^{\infty}(\boldsymbol{R}^d)$ with $0 < \underline{\sigma} \leq \sigma \leq \overline{\sigma} \leq 2$, and let $p(X, D_x) \in \mathcal{S}^{\sigma}_{\rho,\delta}$ be strongly elliptic. Moreover, suppose that for any $x \in \boldsymbol{R}^d$ the function $\xi \mapsto p(x, \xi)$ is negative definite. Then $-p(X, D_x)$ has a closed extension to $C_{\infty}(\boldsymbol{R}^d)$, which is the generator of a Feller semigroup.*

In order to prove Theorem 3.3, we prepare two lemmas. Let k be a nutural number. It follows from Theorem 2.4 and the Sobolev embedding theorem that there exists some $k_0 \in \boldsymbol{N}$ such that for any $k \geq k_0$

$$C_0^{\infty}(\boldsymbol{R}^d) \subset \mathcal{S}(\boldsymbol{R}^d) \subset H^{k\sigma(\cdot)}(\boldsymbol{R}^d) \subset H^{k\underline{\sigma}}(\boldsymbol{R}^d) \subset C_{\infty}(\boldsymbol{R}^d). \tag{3.1}$$

(Here note that $\underline{\sigma} > 0$.)

Lemma 3.4. *Let p be as in Theorem 3.3 and take k as above. Then $-p(X, D_x)$ satisfies the positive maximum principle on $H^{(k+1)\sigma(\cdot)}(\boldsymbol{R}^d)$.*

This lemma can be proved in the same way as in the proof of Proposition 4.1 of [2].

Lemma 3.5. *$\lambda I + p(X, D_x)$ maps $H^{(k+1)\sigma(\cdot)}(\boldsymbol{R}^d)$ onto $H^{k\sigma(\cdot)}(\boldsymbol{R}^d)$.*

Proof. For λ sufficiently large, Theorem 2.7 shows that $\lambda I + p(X, D_x)$ maps from $H^{(k+1)\sigma(\cdot)}(\boldsymbol{R}^d)$ to $H^{k\sigma(\cdot)}(\boldsymbol{R}^d)$. We should know that this map is surjective.

Now put $p_{\lambda}(x, \xi) = \lambda + p(x, \xi)$ and $q^0_{\lambda}(x, \xi) = (\lambda + p(x, \xi))^{-1}$. By the asymptotic expansion formula, we have

$$p_{\lambda}(X, D_x) q^0_{\lambda}(X, D_x) = I + r_{\lambda}(X, D_x),$$

where

$$r_{\lambda}(x, \xi) = \sum_{k=1}^{N-1} \sum_{|\gamma|=k} p^{(\gamma)}_{\lambda}(x, \xi) q^0_{\lambda(\gamma)}(x, \xi) + Os - \int_{\boldsymbol{R}^d} \int_{\boldsymbol{R}^d} e^{-iy\eta} \left(N \int_0^1 \sum_{|\gamma|=N} \frac{(1-\theta)^{N-1}}{\gamma!} p^{(\gamma)}_{\lambda}(x, \xi + \theta\eta) q^0_{\lambda(\gamma)}(x + y, \xi) d\theta \right) dy d\eta.$$

Put $r_{\lambda,k}(x, \xi) = \sum_{|\gamma|=k} p^{(\gamma)}_{\lambda}(x, \xi) q^0_{\lambda(\gamma)}(x, \xi)$. Then, for each k, $r_{\lambda,k}(x, \xi) \in S^{-k(\rho-\delta)}_{\rho,\delta}$, and by an easy calculus, we obtain, for any ε with $0 < \varepsilon < \min\{1, (\rho - \delta)/\overline{\sigma}\}$,

$$|r^{(\alpha)}_{\lambda,k(\beta)}(x, \xi)| \leq C_{\alpha,\beta} (\lambda + \langle \xi \rangle^{\sigma(x)})^{-k\varepsilon} \langle \xi \rangle^{-\rho|\alpha| + \delta|\beta|}, \tag{3.2}$$

where $C_{\alpha,\beta}$ is a constant.

Now we define the semi-norms $|p|_{\ell_1,\ell_2}^{(\sigma)}$ of $p \in S_{\rho,\delta}^{\sigma}$ by

$$|p|_{\ell_1,\ell_2}^{(\sigma)} = \max_{|\alpha|\le \ell_1, |\beta|\le \ell_2} \sup_{(x,\xi)\in \boldsymbol{R}^d\times \boldsymbol{R}^d} \{|p_{(\beta)}^{(\alpha)}(x,\xi)|\langle\xi\rangle^{-(\sigma(x)-\rho|\alpha|+\delta|\beta|)}\}.$$

Then (3.2) shows that $|r_{\lambda,k}|_{\ell_1,\ell_2}^{(0)} \le C\lambda^{-k\varepsilon}$ for some constant C.

Next we estimete the semi-norms of the remainder term

$$Os - \int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^d} e^{-iy\eta}\left(N\int_0^1 \sum_{|\gamma|=N} \frac{(1-\theta)^{N-1}}{\gamma!} p_{\lambda}^{(\gamma)}(x,\xi+\theta\eta) q_{\lambda(\gamma)}^{0}(x+y,\xi)d\theta\right) dyd\eta.$$

Put

$$a_N(y,\eta;x,\xi) = N\int_0^1 \sum_{|\gamma|=N} \frac{(1-\theta)^{N-1}}{\gamma!} p_{\lambda}^{(\gamma)}(x,\xi+\theta\eta) q_{\lambda(\gamma)}^{0}(x+y,\xi)d\theta,$$

then we should only prove that there exists a constant M which is independent of λ such that

$$(3.3)\qquad \begin{aligned} &|\partial_\eta^\alpha \partial_y^\beta \partial_\xi^{\alpha'} \partial_x^{\beta'} a_N(y,\eta;x,\xi)| \\ &\quad \le M\lambda^{-\varepsilon}\langle\xi\rangle^{-\rho|\alpha'|+\delta|\beta'|}\langle\eta\rangle^{\sup|\sigma(x)|+N\rho+\rho|\alpha'|+\delta|\beta'|+\rho|\alpha|} \end{aligned}$$

for any multi-indices α, β, α', and β' (see Theorem 6.4 of Chapter 1 of [4]). From the definition of p_λ and q_λ^0, we write

$$\begin{aligned} &\partial_\eta^\alpha \partial_y^\beta \partial_\xi^{\alpha'} \partial_x^{\beta'} a_N(y,\eta;x,\xi) \\ &\quad = N\int_0^1 \sum_{|\gamma|=N} \frac{(1-\theta)^{N-1}}{\gamma!} \sum_{\alpha'=\alpha_1'+\alpha_2'} \sum_{\beta'=\beta_1'+\beta_2'} \binom{\alpha'}{\alpha_1'}\binom{\beta'}{\beta_1'} \\ &\qquad\qquad \theta^{|\alpha|} p_{\beta_1'}^{(\gamma+\alpha+\alpha_1')}(x,\xi+\theta\eta)\partial_\xi^{\alpha_2'}\partial_x^{\gamma+\beta+\beta_2'}\left(\frac{1}{\lambda+p(x+y,\xi)}\right)d\theta. \end{aligned}$$

Then, since $p(x,\xi)$ is strongly elliptic, we have

$$\begin{aligned} &|\partial_\eta^\alpha \partial_y^\beta \partial_\xi^{\alpha'} \partial_x^{\beta'} a_N(y,\eta;x,\xi)| \\ &\quad \le C_{N,\alpha,\beta,\alpha',\beta'} \int_0^1 \sum_{\alpha'=\alpha_1'+\alpha_2'} \sum_{\beta'=\beta_1'+\beta_2'} \langle\xi+\theta\eta\rangle^{\sigma(x)-\rho|\gamma+\alpha+\alpha_1'|+\delta|\beta_1'|} \\ &\qquad\qquad (\lambda+\langle\xi\rangle^{\sigma(x+y)})^{-1}\langle\xi\rangle^{-\rho|\alpha_2'|+\delta|\gamma+\beta+\beta_2'|}d\theta \end{aligned}$$

and by the fact that $\langle \xi + \theta\eta \rangle \le \langle \xi \rangle \langle \eta \rangle$,

$$\begin{aligned} &\le C_{N,\alpha,\beta,\alpha',\beta'} \sum_{\alpha'=\alpha'_1+\alpha'_2} \sum_{\beta'=\beta'_1+\beta'_2} \langle \xi \rangle^{\sigma(x)-\rho|\gamma+\alpha+\alpha'_1|+\delta|\beta'_1|} \\ &\qquad \langle \eta \rangle^{|\sigma(x)|+\rho|\gamma+\alpha+\alpha'_1|+\delta|\beta'_1|} (\lambda + \langle \xi \rangle^{\sigma(x+y)})^{-\varepsilon} \\ &\qquad (\lambda + \langle \xi \rangle^{\sigma(x+y)})^{-(1-\varepsilon)} \langle \xi \rangle^{-\rho|\alpha'_2|+\delta|\gamma+\beta+\beta'_2|} \\ &\le C_{N,\alpha,\beta,\alpha',\beta'} \langle \xi \rangle^{\sigma(x)-\rho|\gamma+\alpha+\alpha'|+\delta|\beta'|} \\ &\qquad \langle \eta \rangle^{|\sigma(x)|+\rho|\gamma+\alpha+\alpha'|+\delta|\beta'|} \lambda^{-\varepsilon} \langle \xi \rangle^{-\sigma(x+y)(1-\varepsilon)} \end{aligned}$$

and then, when $N \ge (\overline{\sigma} - \underline{\sigma}(1-\varepsilon))/\rho$, we obtain

$$\le C_{N,\alpha,\beta,\alpha',\beta'} \lambda^{-\varepsilon} \langle \xi \rangle^{-\rho|\alpha'|+\delta|\beta'|} \langle \eta \rangle^{\sup|\sigma(x)|+N\rho+\rho|\alpha'|+\delta|\beta'|+\rho|\alpha|},$$

which implies (3.3).

Then the semi-norm $|r_\lambda|^{(0)}_{\ell_1,\ell_2}$ for $\ell_1 = 2[n/2+1]$ and $\ell_2 = 2[n/(2(1-\delta))+1]$ is less than $C\lambda^{-\varepsilon}$ for some constant C. Thus, for sufficiently large λ, there exists the inverse operator $(I + r_\lambda(X, D_x))^{-1}$ in $\mathcal{S}^{0}_{\rho,\delta}$ (Theorem I.1 of Appendix of [4]).

Let $v \in H^{k\sigma(\cdot)}(\boldsymbol{R}^d)$. Our purpose now is to prove that there exists $u \in H^{(k+1)\sigma(\cdot)}(\boldsymbol{R}^d)$ such that $v = (\lambda + p(X, D_x))u$. Now we put $u = q^0_\lambda(X, D_x)(I + r_\lambda(X, D_x))^{-1}v$. Then

$$p_\lambda(X, D_x)u = p_\lambda(X, D_x) q^0_\lambda(X, D_x)(I + r_\lambda(X, D_x))^{-1} v = v.$$

On the other hand, by the algebra (Theorem 2.1) and Theorem 2.7, we have

$$\langle D_x \rangle^{(k+1)\sigma(X)} u = \langle D_x \rangle^{(k+1)\sigma(X)} q^0_\lambda(X, D_x)(I + r_\lambda(X, D_x))^{-1} v \in L^2(\boldsymbol{R}^d).$$

This shows $u \in H^{(k+1)\sigma(\cdot)}(\boldsymbol{R}^d)$. □

Proof of Theorem 3.3. Let P be a strongly elliptic pseudodifferrential operator. Then, from Theorem 2.6, we have $\{u \in H^{-\infty}(\boldsymbol{R}^d); \langle D_x \rangle^{k\sigma(X)} Pu \in L^2(\boldsymbol{R}^d)\} = H^{(k+1)\sigma(\cdot)}(\boldsymbol{R}^d)$. Now let us consider the operator P as one on $C_\infty(\boldsymbol{R}^d)$ having the domain $\mathcal{D}(P) = H^{(k+1)\sigma}(\boldsymbol{R}^d)$ for sufficiently large integer k. Now we have only to check three conditions of Theorem 3.1. Condition i) follows from (3.1), condition ii) from Lemma 3.4, and condition iii) from Lemma 3.5 and (3.1). This completes the proof. □

Now we will show that these Feller semigroups have transition density. For this aim, we state the theorem to construct a fundamental solution in the sense of pseudodifferential operators to the initial-value problem for the evolution equation with respect to an elliptic operator $P \in \mathcal{S}^{\sigma}_{\rho,\delta}$:

$$\text{(3.4)} \qquad \begin{cases} \{\partial_t + P\}u = f & \text{in} \quad (0, T) \\ \lim_{t\to 0} u(t) = u_0 & \text{in} \quad L^2(\boldsymbol{R}^d). \end{cases}$$

By virtue of Theorems 2.1, we can adapt the argument used in the proof of Theorem 2.1 in Section 2 of Chap. 8 in [4] to the proof of this theorem. In order to state this one, we introduce the following two definitions.

DEFINITION. We say that a sequence $\{p_j\}_{j=1}^{\infty}$ of $S^{\sigma}_{\rho,\delta}$ converges weakly to p in $S^{\sigma}_{\rho,\delta}$ if and only if $\{p_j\}_{j=1}^{\infty}$ is bounded in $S^{\sigma}_{\rho,\delta}$ and, for any $R > 0$ and for any multi-indices α, β, $(p_j)^{(\alpha)}_{(\beta)}(x,\xi)$ converges to $p^{(\alpha)}_{(\beta)}(x,\xi)$ uniformly for $(x,\xi) \in \boldsymbol{R}^n_x \times \{|\xi| < R\}$.

DEFINITION. Let I be an interval of $\boldsymbol{R}^1$ and V be a Fréchet space. For a mapping $\phi : I \to \phi(t) \in V$, we write $\phi \in \mathcal{B}^m(I; V)$ if ϕ is m times continuously differentiable in I in the topology of V and each derivative $D_t^l \phi$ is bounded ($l \leq m$).

Theorem 3.6. *There exists a fundamental solution E of the initial-value problem* (3.4) *such that it satisfies the following conditions: for each $T > 0$,*

(1) $$E(t) = e(t, X, D_x) \in \mathcal{B}^0((0,T]; \mathcal{S}^0_{\rho,\delta}) \cap \mathcal{B}^1((0,T]; \mathcal{S}^{\sigma}_{\rho,\delta})$$

and, for any $t_0 > 0$,

$$E(t) \in \mathcal{B}^1([t_0, T]; \mathcal{S}^{-\infty})$$

(2) *for any $t \in (0,T)$,*

$$(\partial_t + P(t))E(t) = 0$$

(3) $$e(t,x,\xi) \longrightarrow 1 \text{ in } S^0_{\rho,\delta} \quad \text{weakly as } t \searrow 0$$

(4) *put*

$$r_0(t,x,\xi) = e(t,x,\xi) - \exp(-tp(x,\xi)),$$

then $r_0(t,x,\xi)$ satisfies

$$r_0(t,x,\xi) \to 0 \qquad \text{in } S^{-(\rho-\delta)}_{\rho,\delta} \quad \text{weakly as } t \searrow 0$$

and

$$\frac{r_0(t,x,\xi)}{t} \in \mathcal{B}^0((0,T]; S^{\sigma-(\rho-\delta)}_{\rho,\delta}).$$

Proof. Let $e_0(t,x,\xi)=\exp(-tp(x,\xi))$. Then this function satisfies the equation:

$$\begin{cases} \{\partial_t + p(x,\xi)\}e_0(t,x,\xi) = 0 \\ e_0(0,x,\xi) = 1. \end{cases}$$

Furthermore, for any multi-indices α and β,

$$\partial_\xi^\alpha D_x^\beta e_0(t,x,\xi) = \sum_{k=1}^{|\alpha+\beta|} t^k p_{k,\alpha,\beta}(x,\xi)e_0(t,x,\xi),$$

where

$$p_{k,\alpha,\beta}(x,\xi) = \sum C_{\beta^1,\beta^2,\cdots,\beta^k}^{\alpha^1,\alpha^2,\cdots,\alpha^k} p_{(\beta^1)}^{(\alpha^1)}(x,\xi)p_{(\beta^2)}^{(\alpha^2)}(x,\xi)\cdots p_{(\beta^k)}^{(\alpha^k)}(x,\xi)$$

and the summation is taken over multi-indices α^j and β^j $(j=1,2,\cdots,k)$ such that $\sum_{j=1}^k \alpha^j = \alpha, \sum_{j=1}^k \beta^j = \beta$ and $C_{\beta^1,\beta^2,\cdots\beta^k}^{\alpha^1,\alpha^2,\cdots,\alpha^k}$ denotes a constant depending only on α^j and β^j $(j=1,2,\cdots,k)$. From (3), there exists a constant $C_1>0$ such that for any $(x,\xi)\in \boldsymbol{R}^d\times\boldsymbol{R}^d$

$$\Re p(x,\xi) > C_0\langle\xi\rangle^{\sigma(x)} - C_1$$

Therefore, putting $C=\exp(-TC_1)$, we have, for any $(t,x,\xi)\in(0,T]\times\boldsymbol{R}^d\times\boldsymbol{R}^d$,

$$|e_0(t,x,\xi)| \le C\exp(-tC_0\langle\xi\rangle^{\sigma(x)}).$$

Since $(t\langle\xi\rangle^{\sigma(x)})^k\exp(-tC_0\langle\xi\rangle^{\sigma(x)})$ is bounded in (t,x,ξ) of $(0,T]\times\boldsymbol{R}^d\times\boldsymbol{R}^d$, there exists a constant $C'_{\alpha,\beta}$ such that

$$\text{(3.5)} \qquad |\partial_\xi^\alpha D_x^\beta e_0(t,x,\xi)| \le C'_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|\rho+\delta|\beta|}$$

for any $(t,x,\xi)\in(0,T]\times\boldsymbol{R}^d\times\boldsymbol{R}^d$. Hence

$$\text{(3.6)} \qquad |\partial_\xi^\alpha D_x^\beta \partial_t e_0(t,x,\xi)| \le \sum_{k=0}^{|\alpha+\beta|} C_{0,\alpha,\beta,k}\, t^k\langle\xi\rangle^{(k+1)\sigma(x)-\rho|\alpha|+\delta|\beta|}\exp(-tC_0\langle\xi\rangle^{\sigma(x)})$$

for any $(t,x,\xi)\in(0,T]\times\boldsymbol{R}^d\times\boldsymbol{R}^d$, where $C_{0,\alpha,\beta,k}$ is a constant depending only on α, β, and k. These estimates (3.5) and (3.6) yield that

$$e_0 \in \mathcal{B}^0((0,T];S_{\rho,\delta}^0)\cap\mathcal{B}^1((0,T];S_{\rho,\delta}^\sigma),$$

and it is clear that $e_0 \longrightarrow 1$ weakly as $t \to 0$. We can define $\{e_j(t)\}_{j=1}^{\infty}$ and $\{q_j(t)\}_{j=1}^{\infty}$ ($0 \le t \le T$) inductively by

$$q_j(t) = \sum_{k=0}^{j-1} \sum_{|\alpha|+k=j} \frac{1}{\alpha!} p^{(\alpha)} e_{k\ (\alpha)}(t,x,\xi) \quad (j \ge 1)$$

and

$$\left\{ \begin{aligned} \{\partial_t + p\} e_j(t,x,\xi) &= -q_j(t,x,\xi) \\ e_j(0,x,\xi) &= 0 \qquad\qquad (j \ge 1). \end{aligned} \right. \tag{3.7}$$

Then the solution $e_j(t,x,\xi)$ of (3.7) has the form:

$$e_j(t,x,\xi) = e_0(t,x,\xi) \int_0^t \frac{q_j(s,x,\xi)}{e_0(s,x,\xi)} ds.$$

Using the same argument as in the proof of Lemma 4.3 of Chapter 7 in [4], we have the following estimate:

$$|e_{j\ (\beta)}^{(\alpha)}(t,x,\xi)| \le \left\{ \begin{array}{l} C_{j,\alpha,\beta} \langle \xi \rangle^{-j(\rho-\delta)-|\alpha|+\delta|m|} \\ C'_{j,\alpha,\beta}\, t \, \langle \xi \rangle^{-j(\rho-\delta)-|\alpha|+\delta|\beta|+1} \end{array} \right.$$

for any $(t,x,\xi) \in (0,T] \times \boldsymbol{R}^d \times \boldsymbol{R}^d (j \ge 1)$, where $C_{j,\alpha,\beta}$ and $C'_{j,\alpha,\beta}$ are constants depending only on j, α and β. Therefore, we complete the proof of this theorem by the same way as in proof of Theorem 1.3 in [6]. □

From Theorem 2.7, we obtain that if σ and τ are in $\mathcal{B}^\infty(\boldsymbol{R}^d)$ and $p_j \to p$ in $S^{\sigma}_{\rho,\delta}$ weakly as $j \to \infty$, then

$$\begin{aligned} p_j(X,D_x)u &\longrightarrow p(X,D_x)u \text{ in } H^{\tau(\cdot)}(\boldsymbol{R}^d) \\ &\text{as } j \to \infty \quad \text{for } u \in H^{\sigma(\cdot)+\tau(\cdot)}(\boldsymbol{R}^d) \end{aligned} \tag{3.8}$$

(cf.[4] p.157). Therefore, from the above relation (3.8), Theorem 2.7, and Theorem 3.6, we get the following theorem.

Theorem 3.7. *Let $E(\cdot)$ be the same one as in Theorem 3.6 and let τ be any real-valued function in $\mathcal{B}^\infty(\boldsymbol{R}^n)$. Then, for $\phi \in H^{\tau(\cdot)}(\boldsymbol{R}^d)$, $u(\cdot) = E(\cdot)\phi$ belongs to $\mathcal{B}^0([0,T];\ H^{\tau(\cdot)}(\boldsymbol{R}^d)) \cap \mathcal{B}^1([0,T];\ H^{\tau(\cdot)-\sigma(\cdot)}(\boldsymbol{R}^d))$ for each $T > 0$ and is a solution to the initial-value problem for the evolution equation* (3.4).

Theorem 3.8. *Let $e(t,x,\xi)$ be the symbol of the fundamental solution $E(t)$ given by Theorem 3.6. Then, the function defined by*

$$K(t,x,y) = \int_{\boldsymbol{R}^d} \exp(i(x-y)\cdot\xi) e(t,x,\xi) đ\xi$$

($t \in (0,\infty)$, $x, y \in \boldsymbol{R}^d$) is a transition density with respect to the Feller semigroup $\{T_t\}_{t\geq 0}$ generated the operator P.

Let $X=(X(t), \boldsymbol{P}_x)$ be a Markov process whose generator is a pseudodifferential operator satisfying the conditions in Theorem 3.3. Then, we can investigate the behavior of sample paths of the Markov process $X=(X(t), \boldsymbol{P}_x)$ by the same way as in the cace of stable-like processes (cf. [6]). We state the result.

Theorem 3.9. *Let x be an arbitrarily fixed point.*
(1) *If $\sigma(x) < \gamma$, then*

$$\boldsymbol{P}_x(\lim_{t\to 0} |X(t) - x|/t^{\frac{1}{\gamma}} = 0) = 1.$$

(2) *If $\sigma(x) > \gamma > 0$, then*

$$\boldsymbol{P}_x(\limsup_{t\to 0} |X(t) - x|/t^{\frac{1}{\gamma}} = \infty) = 1.$$

Appendix

In this appendix we sum up some prorerties of Sobolev spaces of variable order. Here we consider L^p cases. Let p be a real number with $1 < p < \infty$. Then we define

(A.1) $$W^{\sigma(\cdot),p}(\boldsymbol{R}^d) = \{u \in W^{-\infty,p}(\boldsymbol{R}^d); \langle D_x\rangle^{\sigma(X)}u \in L^p(\boldsymbol{R}^d)\}.$$

We obtain the following facts in the same way as in the proofs of corresponding theorems in Section 2.

Theorem A.1. *Let σ_1 and σ_2 be functions in $\mathcal{B}^\infty(\boldsymbol{R}^d)$ with $\sigma_1(x) \geq \sigma_2(x)$ for any $x \in \boldsymbol{R}^d$. Then we have $W^{\sigma_1(\cdot),p}(\boldsymbol{R}^d) \subset W^{\sigma_2(\cdot),p}(\boldsymbol{R}^d)$.*

Theorem A.2. *$W^{\sigma(\cdot),p}(\boldsymbol{R}^d)$ is a Banach space with the norm*

$$\| u|W^{\sigma(\cdot),p}(\boldsymbol{R}^d) \| = \left(\int_{\boldsymbol{R}^d} |(\langle D_x\rangle^{\sigma(X)}u)(x)|^p dx + \int_{\boldsymbol{R}^d} |(\langle D_x\rangle^{\underline{\sigma}}u)(x)|^p dx\right)^{\frac{1}{p}}.$$

Moreover $\mathcal{S}(\boldsymbol{R}^d)$ is dense in $W^{\sigma(\cdot),p}(\boldsymbol{R}^d)$.

Theorem A.3. *Let σ and τ be function in $\mathcal{B}^\infty(\boldsymbol{R}^d)$. Suppose that $P \in \mathcal{S}^{\sigma}_{1,\delta}$. Then there exist some constant $C > 0$ independent of P and some positive integer ℓ depending only on σ, τ, δ, and n such that*

$$\| u|W^{\tau(\cdot),p}(\boldsymbol{R}^d) \| \leq C|p|_\ell^{(\sigma)} \| u|W^{\sigma(\cdot)+\tau(\cdot),p}(\boldsymbol{R}^d) \|$$

for $u \in W^{\sigma(\cdot)+\tau(\cdot),p}(\boldsymbol{R}^d)$.

In this appendix we mention the duality relations of Sobolev spaces for variable case and remark that the "order" is a "local" property.

Theorem A.4. (1) *There exists a constant* $C = C(\sigma, p)$ *such that*

$$|(u,v)| \leq C \parallel u|W^{\sigma(\cdot),p}(\boldsymbol{R}^d) \parallel \cdot \parallel v|W^{-\sigma(\cdot),q}(\boldsymbol{R}^d) \parallel$$

for u, $v \in \mathcal{S}(\boldsymbol{R}^d)$, *where* q *is the number with* $p^{-1} + q^{-1} = 1$.
(2) $(W^{\sigma(\cdot),p}(\boldsymbol{R}^d))' = W^{-\sigma(\cdot),q}(\boldsymbol{R}^d)$.

Proof. (1) Note that

$$\begin{aligned}(u,v) &= ((Q_\sigma \langle D_x \rangle^{\sigma(X)} - R_L)u, v) \\ &= (\langle D_x \rangle^{\sigma(X)} u, Q_\sigma^* v) - (\langle D_x \rangle^{\underline{\sigma}} u, \langle D_x \rangle^{-\underline{\sigma}} R_L^* v).\end{aligned}$$

Therefore

$$\begin{aligned}|(u,v)| \leq &\parallel \langle D_x \rangle^{\sigma(X)} u \parallel_{L^p(\boldsymbol{R}^d)} \cdot \parallel Q_\sigma^* v \parallel_{L^q(\boldsymbol{R}^d)} \\ &+ \parallel \langle D_x \rangle^{\underline{\sigma}} u \parallel_{L^p(\boldsymbol{R}^d)} \cdot \parallel \langle D_x \rangle^{-\underline{\sigma}} R_L^* v \parallel_{L^q(\boldsymbol{R}^d)},\end{aligned}$$

and then, taking account of $Q_\sigma^* \in \mathcal{S}_{1,\delta}^{-\sigma}$, we obtain assertion (1).

(2) Assertion (1) implies $W^{-\sigma(\cdot),q}(\boldsymbol{R}^d) \subset (W^{\sigma(\cdot),p}(\boldsymbol{R}^d))'$. Here we prove the converse inclusion. Let f be a member of $(W^{\sigma(\cdot),p}(\boldsymbol{R}^d))'$. It is clear that $f \in H^{-\infty}(\boldsymbol{R}^d)$. Then we should only to prove $\langle D_x \rangle^{-\sigma(X)} f \in L^q(\boldsymbol{R}^d)$. The definition of pseudodifferential operator on $\mathcal{S}(\boldsymbol{R}^d)'$ means, for any $\phi \in \mathcal{S}(\boldsymbol{R}^d)$,

$$(\langle D_x \rangle^{-\sigma(X)} f, \phi) = (f, (\langle D_x \rangle^{-\sigma(X)})^* \phi).$$

Since f is a continuous linear functional on $W^{\sigma(\cdot),p}(\boldsymbol{R}^d)$, there exists a constant $C = C(f)$ such that

$$\begin{aligned}|(\langle D_x \rangle^{-\sigma(X)} f, \phi)| &= |(f, (\langle D_x \rangle^{-\sigma(X)})^* \phi)| \\ &\leq C \parallel (\langle D_x \rangle^{-\sigma(X)})^* \phi | W^{\sigma(\cdot),p}(\boldsymbol{R}^d) \parallel \leq C' \parallel \phi \parallel_{L^p(\boldsymbol{R}^d)} .\end{aligned}$$

Here last inequality follows from Theorem A.3. Thus $\langle D_x \rangle^{-\sigma(X)} f$ can be extended to a continuous linear functional on $L^p(\boldsymbol{R}^d)$ uniquely, and hence it is identified to a function in $L^q(\boldsymbol{R}^d)$. This completes the proof. □

REMARK. In [5] the Sobolev spaces of negative orders are defined indirectly by the use of parametrices. Theorem A.4 assures that the dual spaces of Sobolev spaces can be defined directly.

Theorem A.5. *Let ω be an open set in $\boldsymbol{R}^d$. If u belongs to $W^{\sigma(\cdot),p}(\boldsymbol{R}^d)$ and has its support in ω, then u belongs to $W^{s,p}(\boldsymbol{R}^d)$, where $s=\inf_{x\in\omega}\sigma(x)$.*

Proof. Let ϕ be a C^∞ function in $\boldsymbol{R}^d$ which satisfies $\phi=1$ on the support of u, and $=0$ on $\boldsymbol{R}^d\setminus\omega$. Then $\phi u=u$. Let $p(x,\xi)$ be the left simplified symbol (for the definition see [4]) of double symbol $\langle\xi\rangle^s\phi(x')$. That is,

$$\langle D_x\rangle^s u=\langle D_x\rangle^s(\phi u)=p(X,D_x)u. \tag{A.2}$$

By Theorem 3.1 of Chapter 2 of [4] we have the following representation

$$p(x,\xi)=p_0(x,\xi)+p_1(x,\xi)+\cdots+p_{N-1}(x,\xi)+r_N(x,\xi)$$

for any positive integer N, where

$$p_j(x,\xi)=\sum_{|\alpha|=j}D_x^\alpha\phi(x)\partial_\xi^\alpha(\langle\xi\rangle^s) \tag{A.3}$$

and $r_N(x,\xi)\in S^{s-N}_{1,0}$. This fact directly implies $p_j(x,\xi)\in S^{s-j}_{1,0}$ and $r_N\in S^{\sigma}_{1,0}$ when N is sufficiently large. Moreover we obtain from (A.3) that the support of $p_j(\cdot,\xi)$ is included in ω. Let χ_ω denote the characteristic function of ω. Then we have

$$|(p_j)^{(\alpha)}_{(\beta)}(x,\xi)|\langle\xi\rangle^{-\sigma(x)}\le C_{\alpha,\beta}\chi_\omega(x)\langle\xi\rangle^{s-j-|\alpha|-\sigma(x)}\le C_{\alpha,\beta}\chi_\omega(x)\langle\xi\rangle^{-j-|\alpha|},$$

for some constant $C_{\alpha,\beta}$. The second inequality follows from $s\le\sigma(x)$ for $x\in\omega$. Thus $p_j\in S^{\sigma-j}_{1,0}$ for $j=1,2,\cdots,N$. Hence $p\in S^{\sigma}_{1,0}$. Since $u\in W^{\sigma(\cdot),p}(\boldsymbol{R}^d)$, we have $p(X,D_x)u\in L^p(\boldsymbol{R}^d)$ by Theorem A.3. Thus the conclusion follows from (A.2). □

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