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STEFAN PROBLEMS WITH THE UNILATERAL BOUNDARY CONDITION ON THE FIXED BOUNDARY II

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Contents

0. Introduction

In this paper we consider the following one-dimensional two-phase Stefan problem with the unilateral boundary condition on the fixed boundary: Given the initial data, l and ϕ, find a critical time T^*, and the two functions $s=s(t)$ and $u=u(x, t)$ defined on $[0, T^*]$ such that

$$\begin{cases} (0.1) \quad s(0)=l, \qquad 0<s(t)<1 \ (0\leqq t<T^*), \\ (0.2) \quad u_{xx}-c_0u_t=0 \qquad (0<x<s(t),\ 0<t<T^*), \\ (0.3) \quad u_{xx}-c_1u_t=0 \qquad (s(t)<x<1,\ 0<t<T^*), \end{cases}$$

$$
\text{(S)} \quad \begin{cases} (0.4) & \begin{aligned} &\text{(a)} \quad u_x(0, t) \in \gamma_0(u(0, t)) \qquad (0<t<T^*) \\ &\text{(b)} \quad -u_x(1, t) \in \gamma_1(u(1, t)) \qquad (0<t<T^*), \end{aligned} \\ (0.5) & \begin{aligned} &\text{(a)} \quad u(x, 0) = \phi(x) \qquad (0<x<l), \\ &\text{(b)} \quad u(x, 0) = \phi(x) \qquad (l<x<1), \end{aligned} \\ (0.6) & u(s(t), t) = 0 \qquad (0<t\leqq T^*), \\ (0.7) & b\dot{s}(t) = -u_x^-(s(t), t) + u_x^+(s(t), t) \qquad (0<t<T^*). \end{cases}
$$

The critical time T^*, $0<T^*<\infty$, is defined to be the first time that the free boundary $x=s(t)$ touches the fixed boundary $x=0$ or $x=1$. The quantities c_0, c_1 and b are positive physical parameters of the problem. The assumption for the boundary condition (0.4) at the fixed boundary is that γ_0 and γ_1 are maximal monotone graphs in $\boldsymbol{R}^2$ such that both $\gamma_0^{-1}(0)\cap[0, \infty[$ and $\gamma_1^{-1}(0)\cap]-\infty, 0]$ are not empty sets. We put this assumption from the physical reasoning, that is, there are a kind of heater at $x=0$ and a kind of freezer at $x=1$. (0.4) are the unilateral boundary conditions. (0.7) is the so-called Stefan's condition. The superscripts $^+$ and $^-$ indicate the limits from right and left respectively for the space variable x.

The system (0.1)–(0.7) is a simple model of a heat-conduction system consisting of two phases (e.g. liquid and solid) of the same substance which are in perfect thermal contact at an interfcae. $u(x, t)$ represents the temperature distribution in the system, and the curve $s(t)$ represents the position of the intreface which varies with time t as solid melts or liquid freezes. The unilateral boundary conditions (0.4) model several phyiscal situations, including the temperature control through the boundary [9, Ch. 1] and the heat flow subject to the nonlinear cooling by the radiation on the boundary [14, Ch. 7]. The boundary conditions at the interface ((0.6), (0.7)) reflect respectively the facts that the temperature at the interface must be equal to the melting temperature (taken to be zero) and that the rate of melting is proportional to the rate of absorption of the heat energy at the interface. In formulating (0.7), we have assumed, without loss of generality, that the thermal conductivity in both phases is 1.

The problem of this type with the linear boundary condition on the fixed boundary have been considered by many authors (Rubinstein [27], Kamenomostskaja [18], Friedman [13, 14], Brézis [2], Cannon & Primicerio [5, 6], Cannon—Henry—Kotlow [7], Nogi [25], Damlamian [8] e.t.c.). On the other hand Bénilan [1] has treated this type's Stefan problem of n-dimensional case using the theory of nonlinear contraction semigroups in Banach space L^1. He got an intgeral solution. However we do not know the differentiability of the Benilan's integral solution. Also Cannon & DiBenedetto [37], Visintin [38], and Niezgodka—Pawlow—Visintin [39] have considered the different type of weak solutions of the similar problems. One-phase problem of this type was recently studiesd by Yotsutani [34].

The purpose of this paper is to prove the global existence and uniqueness of the classical solution of the two-phase Stefan problem (S). We put some assumptions of signs of the data ϕ from the physical reasoning that it is positive in the liquid region and negative in the solid region. The following two points are the main difficulties of this problem (S). One is the fact $s(t)$ is unknown and the other is how we deal with the unilateral boundary condition.

We establish the existence of a local solution of (S) using the Schauder's fixed point theorem. For this we employ an approach used by Evans [10] to treat the flow of two immiscible fluids in one-dimensional porous medium. Then we show a global a priori estimate on $u(x, t)$, $s(t)$, and we get a global solution of (S). Uniqueness is based upon the maximum principle, its strong form [24], a parabolic version of Hopf's lemma [14] and the comparison theorem for the unilateral problem. Here we must note that our proof of the uniqueness is closely related to the existence of solutions of auxiliary Stefan problems.

The plan of this paper is as follows. In § 1 we state main theorems. § 2 collects some elementary results. § 3 introduces the moving boundary problem (M) which is auxiliary for the original one and useful in the proof of the main theorems. § 4–§ 7 are devoted to prove the existence of a solution of (M) using the finite difference method. In § 4 we introduce a difference scheme. In § 5 and § 6 we derive the estimates for solutions of the difference scheme. These are used in § 7 to prove the convergence of the difference scheme and the properties of the solutions of (M). § 8 gives several comparison theorems concerning the moving boundary problem (M). In § 9 we reform the Stefan's condition to an integral form. In § 10 we prove the existence of a local solution of (S) without assuming signs of $\phi(x)$. In § 11 we prepare propositions which we use in the study of the continuation of solutions. In § 12 we give global a priori estimates. In § 13 we show the global existence of a solution of (S) under the slightly stringent conditions on the data. In § 14 we prove the existence of a global solution of (S) for the general data. In § 15 we show a comparison theorem and the uniqueness of the solutions of the Stefan problem (S).

We will investigate the behavior of the solution in detail in [36].

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1. Statements of main results

As for the definition of maximal monotone graphs in $\boldsymbol{R}^2$, see Brézis [4] or Yotsutani [34, § 3]. The assumptions required on the Stefan data $\{l, \phi\}$ are following (A).

$$\text{(A)} \quad \begin{cases} 0<l<1\,. \quad \phi(x)\geqq 0\ (0<x<l),\ \phi(x)\leqq 0\ (l<x<1)\,. \\ \phi(x) \ \text{ is bounded, and continuous for a.e. } x\in[0,\,1]\,. \end{cases}$$

Remark 1.1. We put this assumption of signs for the data ϕ, because the interest of ours is in the two-phase problem.

We shall prepare some notations which will be used later.

$$(1.1) \qquad Q_T = \{(x,\,t);\ 0<x<1,\ 0<t<T\}\,,$$

$$(1.2) \qquad \bar{Q}_T = \{(x,\,t);\ 0\leqq x\leqq 1,\ 0\leqq t\leqq T\}\,,$$

$$(1.3) \qquad \bar{Q}_{\sigma,T} = \{(x,\,t);\ 0\leqq x\leqq 1,\ \sigma\leqq t\leqq T\}\,,$$

$$(1.4) \qquad D_T^i = \{(x,\,t)\in Q_T;\ (-1)^i(x-s(t))<0\} \qquad (i=0,\,1)\,,$$

$$(1.5) \qquad D_T = D_T^0\cup D_T^1\,,$$

$$(1.6) \qquad \bar{D}_T^i = \{(x,\,t)\in \bar{Q}_T;\ (-1)^i(x-s(t))\leqq 0\} \qquad (i=0,\,1)\,,$$

$$(1.7) \qquad S_T^i = \{(x,\,t)\in Q_T;\ (-1)^i(x-s(t))\leqq 0\} \qquad (i=0,\,1)\,,$$

$$(1.8) \qquad Z = \{(x,\,0);\ x\in[0,\,1] \text{ is a point of discontinuity of } \phi\}\,.$$

Let $I\subset \boldsymbol{R}^2$ or $I\subset \boldsymbol{R}$. We denote by $C(I)$, $C^{0,\alpha}(I)$ $(0<\alpha\leqq 1)$, $C^m(I)$ $(m=1, 2, \cdots)$ and $C^\infty(I)$ the space of continuous, Hölder continuous (exponent α), m times continuously differentiable, and infinitely differentiable functions on I respectively. Thus $C^{0,1}(I)$ denotes the space of Lipshitz continuous functions on I. We denote by $L^p(I)$ the usual Lebesgue space of measurable functions with the norm $\|\cdot\|_{L^p(I)}$ $(1\leqq p\leqq\infty)$. $H^p(I)$ denotse the usual Sobolev space. $L^p_{\text{loc}}(I)$ (resp. $H^p_{\text{loc}}(I)$) denotes the space of functions which belong to $L^p(E)$ (resp. $H^p(E)$) for any compact subset E of I.

Defintion 1.1. The pair $(s,\,u)$ is a solution of the Stefan problem (S) on $[0,\,T]$ if

(i) $s(0)=l$, $0<s(t)<1$ for $0\leqq t\leqq T$,

$$(1.9) \qquad s\in C([0,\,T])\cap C^\infty(]0,\,T])\,,$$

(ii) u is bounded on $\bar{Q}_T$, $u\in C(\bar{Q}_T-Z)\cap C^\infty(S_T^0)\cap C^\infty(S_T^1)$,

$$(1.10) \qquad \int_{T_1}^{T_2}\int_0^1 u_t(x,\,t)^2\,dxdt<\infty$$

for each T_1 and T_2 such that $0<T_1\leqq T_2\leqq T$,

(iii) (0.2), (0.3), (0.5), (0.6) and (0.7) hold on $[0,\,T]$,

(iv) for a.e. $t\in]0,\,T]$, $u_x(0,\,t)$ and $u_x(1,\,t)$ exist and satisfy (a), (b) of (0.4) on $[0,\,T]$ respectively.

In what follows, for example, when $T=\infty$, $[0,\,T]$ and $0\leqq t\leqq T$ denote $[0,\,\infty[$ and $0\leqq t<\infty$ respectively.

Definition 1.2. $(T^*,\,s,\,u)$ is a solution of the Stefan problem (S) if

(i) $0<T^*\leqq\infty$, a pair of function (s, u) denfied on $[0, T^*]$ is a solution of (S) on $[0, T]$ for any T with $0<T<T^*$,

(ii) if $T^*<\infty$, then $s\in C([0, T^*])$, $u\in C(\bar{Q}_{T^*}-Z)$, and $s(T^*)=0$ or 1.

Let θ_i $(i=0, 1)$ be a lower semicontinuous convex function $\boldsymbol{R}$ into $]-\infty, \infty]$ such that $\theta_i\geqq 0$, $\theta_i(H_i)=0$ and $\partial\theta_i=\gamma_i$. This is well-defined, since γ_i $(i=0, 1)$ is a maximal monotone graph in $\boldsymbol{R}^2$ with $\gamma_i(H_i)\ni 0$ (see [4, p. 43]). We can now state the existence and uniqueness theorems.

Theorem 1. *Let $\{l, \phi\}$ satisfy (A). Then there exists the unique solution (T^*, s, u) of (S). Further s and u have the following properties.*

$$(1.11)\qquad s\in C^{0,1/3}([0, T^*])\cap C^{0,2/3}(]0, T^*])\cap C^{\infty}(]0, T^*[)\,,$$

$$(1.12)\qquad 0\leqq u\leqq \max(\|\phi\|_{L^\infty(0,l)}, H_0)\qquad on\ \bar{D}^0_{T^*}\,,$$

$$(1.13)\qquad \min(-\|\phi\|_{L^\infty(l,1)}, H_1)\leqq u\leqq 0\qquad on\ \bar{D}^1_{T^*}\,,$$

$$(1.14)\qquad \sup_{\sigma\leqq t\leqq T^*}\left\{\int_0^1 u_x(x, t)^2dx+\theta_0(u(0, t))+\theta_1(u(1, t))\right\}$$
$$+\int_\sigma^{T^*}|\dot{s}|^3dt+\int_\sigma^{T^*}\int_0^{s(t)}u_{xx}{}^2dxdt+\int_\sigma^{T^*}\int_{s(t)}^1 u_{xx}{}^2dxdt\leqq C_\sigma\,,$$

$$(1.15)\qquad |u(x', t')-u(x, t)|\leqq C_\sigma(|x'-x|^{1/2}+|t'-t|^{1/4})\qquad on\ \bar{Q}_{\sigma,T^*}\,,$$

where

$$(1.16)\qquad H_0=\min\{H\geqq 0;\ H\in\gamma_0^{-1}(0)\}\,,$$

$$(1.17)\qquad H_1=\max\{H\leqq 0;\ H\in\gamma_1^{-1}(0)\}\,,$$

and C_σ is a constant depending on $\sigma\in]0, T^*[$.

We have the following regularity properties, if the data $\{l, \phi\}$ satisfies some additional conditions.

Theorem 2. *Let $\{l, \phi\}$ satisfy (A), $\phi\in H^1(0, 1)$, $\phi(0)\in D(\theta_0)$, $\phi(l)=0$ and $\phi(1)\in D(\theta_1)$. Then we have the following properties in addition to the conclusion of Theorem 1.*

$$(1.18)\qquad s\in C^{0,2/3}([0, T^*])\,,$$

$$(1.19)\qquad 2^{-1}\int_0^1 u_x(x, T)^2dx+\theta_0(u(0, T))+\theta_1(u(1, T))$$
$$+2^{-1}b^2\int_0^T|\dot{s}(t)|^3dt+c_2\int_0^T\int_0^1 u_t(x, t)^2dxdt$$
$$\leqq 2^{-1}\int_0^1\phi_x(x)^2dx+\theta_0(\phi(0))+\theta_1(\phi(1))\qquad for\ any\ T\in[0, T^*]\,,$$

$$(1.20)\qquad |u(x', t')-u(x, t)|\leqq C(|x'-x|^{1/2}+|t'-t|^{1/4})\qquad on\ \bar{Q}_{T^*}\,,$$

where $c_2=\min(c_0, c_1)$ and C is a constant.

Theorem 3. *Let $\{l, \phi\}$ satisfy (A). Suppose γ_i ($i=0, 1$) is a single valued maximal monotone function. Then it follows that $u_x \in C(\bar{D}_T^i - [0, 1] \times \{0\})$ for any T with $0 < T < T^*$, and*

$$(1.21) \qquad (-1)^i u_x(i, t) = \gamma_i(u(i, t)) \qquad (0 < t \leqq T^*).$$

We introduce conditions (A.1), (A.2) and (A.3) which will be used in the proof of the theorems above.

$$(\text{A.1}) \qquad 0 < l < 1 .$$

$$(\text{A.2}) \qquad \phi \in H^1(0, 1),\ \phi(0) \in D(\theta_0),\ \phi(1) \in D(\theta_1),\ \phi(l) = 0 .$$

$$(\text{A.3}) \qquad \phi(x) \geqq 0\ (0 \leqq x \leqq l),\ \phi(x) \leqq 0\ (l \leqq x \leqq 1) .$$

For simplicity (A′) denotes the conditions (A.1), (A.2) and (A.3).

2. Preliminaries

In this section we collect some elementary inequalities which will be useful later in obtaining necessary estimates.

We use the letter C throughuot this paper to denote various constants depending only on known quantities.

Lemma 2.1. *Let $]\alpha, \beta[$ be a finite open interval on the real line.*

(i) *There exists a constant C_1, depending on $\beta - \alpha$, such that*

$$(2.1) \qquad \|u_x\|^2_{L^\infty(\alpha,\beta)} \leqq C_1(1 + \|u\|^2_{L^\infty(\alpha,\beta)} + \|u_{xx}\|^2_{L^2(\alpha,\beta)}) \|u\|^{2/3}_{L^\infty(\alpha,\beta)}$$

for each $u \in H^2(\alpha, \beta)$.

(ii) *There exists a constant $C[\varepsilon]$, depending only on $\varepsilon > 0$ and $\beta - \alpha$, such that*

$$(2.2) \qquad \|u_x\|^2_{L^\infty(\alpha,\beta)} \leqq \varepsilon \|u_{xx}\|^2_{L^2(\alpha,\beta)} + C[\varepsilon] \|u_x\|^2_{L^2(\alpha,\beta)} ,$$

for each $u \in H^2(\alpha, \beta)$.

(iii) *There exists a constant C' depending only on $\beta - \alpha$ such that*

$$(2.3) \qquad \|u_x\|^4_{L^\infty(\alpha,\beta)} \leqq C'(\|u_x\|^4_{L^2(\alpha,\beta)} + \|u_x\|^2_{L^2(\alpha,\beta)} \|u_{xx}\|^2_{L^2(\alpha,\beta)}) ,$$

for each $u \in H^2(\alpha, \beta)$.

The constants in (2.1), (2.2) *and* (2.3) *remain bounded as $\beta - \alpha$ ranges over any compact subset of* $]0, \infty[$.

Proof. (i). We see from [10, (2.4)] that

$$(2.4) \qquad \|u_x\|^2_{L^\infty(\alpha,\beta)} \leqq C(\|u\|^2_{H^2(\alpha,\beta)})^{2/3} \|u\|^{2/3}_{L^\infty(\alpha,\beta)} ,$$

where the constant C depends only on $\beta - \alpha$ and remains bounded for $\beta - \alpha$ in any compact subset of $]0, \infty[$. By Young's inequality

$$||u_x||^2_{L^\infty(\alpha,\beta)} \leqq C(1+||u||^2_{H^2(\alpha,\beta)})||u^{2/3}_{L^\infty(\alpha,\beta)}.$$

Thus we get (2.1) using the interpolation inequality.

(ii) and (iii). We see from [10, (2.5)] that

$$(2.5) \qquad ||u_x||_{L^\infty(\alpha,\beta)} \leqq C||u_x||^{1/2}_{H^1(\alpha,\beta)}||u_x||^{1/2}_{L^2(\alpha,\beta)}.$$

Thus we can get (2.2) and (2.3) easily. q.e.d.

Lemma 2.2. *Let $v(x, t)$ be a Lipshitz continuous function on $Q=[a_1, a_2]\times[b_1, b_2]$. Then we have*

$$(2.6) \qquad |v(x', t')-v(x, t)|$$
$$\leqq L[\sup_{b_1\leqq t\leqq b_2}||v_x(\cdot, t)||_{L^2(a_1,a_2)}+||v_t||_{L^2(Q)}]\times[|x'-x|^{1/2}+|t'-t|^{1/4}] \qquad \text{on } Q,$$

where $A=a_2-a_1$, $B=b_2-b_1$, $L=\max(2A^{1/2}B^{-1/4}, 2A^{-1/2}B^{1/4}, 1)$.

Proof. See [16, Lemma 3.1] or [34, Lemma 16.4]. q.e.d.

Lemma 2.3. *Let $\{F_n\}^N_{n=q}$, $\{K_n\}^N_{n=q+1}$, $\{R_n\}^{N-1}_{n=q}$ and $\{V_n\}^N_{n=q+1}$ be sequences of non-negative numbers such that*

$$(2.7) \qquad V_n+F_n\leqq(1+K_n)F_{n-1}+R_{n-1} \qquad (q<n\leqq N).$$

Then we have

$$(2.8) \qquad \max_{q\leqq p\leqq N} F_p+\sum^N_{p=q+1} V_p\leqq[F_q+\sum^{N-1}_{p=q} R_p][1+\exp(2\sum^N_{p=q+1} K_p)]$$

Moreover, if $R_n=0$ $(q\leqq n\leqq N-1)$, then

$$(2.9) \qquad F_n-F_q\leqq(\max_{q\leqq p\leqq N} F_p)(\sum^n_{p=q+1} K_p) \qquad (q<n\leqq N)$$

Proof. We have

$$(2.10) \qquad F_n\leqq(F_q+\sum^{n-1}_{p=q} R_p)\exp(\sum^n_{p=q+1} K_p) \qquad (q<n\leqq N)$$

by (2.7) and the induction (see [33, Lemma 4.1]). We see

$$(2.11) \qquad V_n\leqq(F_{n-1}-F_n)+R_{n-1}+F_{n-1}K_n$$

from (2.7). Hence we get by (2.10) and (2.11)

$$F_n+\sum^n_{p=q+1} V_p$$
$$\leqq(F_q+\sum^n_{p=q+1} R_{p-1})+(F_q+\sum^{n-2}_{p=q} R_p)\exp(\sum^{n-1}_{p=q+1} K_p)(\sum^n_{p=q+1}K_p)$$

Therefore we obtain (2.8) easily. We can get (2.9) using (2.11). q.e.d.

3. Moving boundary problem

Consider the following moving boundary problem: Given a time T, a data $\phi(x)$ and a function $s(t)\in C([0, T])\cap H^1_{loc}(]0, T])$ satisfying $0<s(t)<1$ $(0\leqq t\leqq T)$, find a function $u=u(x, t)$ such that

$$
(\mathrm{M})\quad \begin{cases}
(3.1)\quad u_{xx}-c_0u_t=0 \qquad (0<x<s(t),\ 0<t<T)\,, \\
(3.2)\quad u_{xx}-c_1u_t=0 \qquad (s(t)<x<1,\ 0<t<T)\,, \\
(3.3)\quad \begin{array}{l}(\mathrm{a})\quad u_x(0, t)\in\gamma_0(u(0, t)) \qquad (0<t\leqq T)\,, \\ (\mathrm{b})\quad -u_x(1, t)\in\gamma_1(u(1, t)) \qquad (0<t\leqq T),\end{array} \\
(3.4)\quad u(s(t), t)=0 \qquad (0<t\leqq T)\,, \\
(3.5)\quad \begin{array}{l}(\mathrm{a})\quad u(x, 0)=\phi(x) \qquad (0<x<l\equiv s(0)) \\ (\mathrm{b})\quad u(x, 0)=\phi(x) \qquad (l<x<1)\,.\end{array}
\end{cases}
$$

Here γ_0 and γ_1 are maximal monotone graphs in $\boldsymbol{R}^2$ with $\gamma_0(H_0)\ni 0$, $\gamma_1(H_1)\ni 0$ for some H_0 and H_1.

REMARK 3.1. We do not need any assumption of signs for $\phi(x)$, H_0 and H_1.

DEFINITION 3.1. $u=u(x, t)$ is a solution of the moving boundary problem (M) if

(i) u is bounded on $\bar{Q}_T$, $u\in C^\infty(D_T)\cap C(\bar{Q}_T-Z)$,

$$\int_{T_1}^{T_2}\int_0^1 u_t(x, t)^2\,dx\,dt\leqq C_{T_1,T_2}$$

for each T_1 and T_2, where D_T, $\bar{Q}_T$ and Z are sets defined by (1.5) (1.1) and (1.8) respectively, and C_{T_1,T_2} is a positive constant depending on T_1 and T_2 $(0<T_1\leqq T_2\leqq T)$.

(ii) (3.1), (3.2) (3.4), (3.5) hold,

(iii) for a.e. $t\in]0, T]$, $u_x(0, t)$ and $u_x(1, t)$ exist and satisfy (3.3).

Proposition 3.1. *If the data $\phi(x)$ is bounded, and continuous for a.e. $x\in[0, 1]$, then there exists the unique solution u of the moving boundary problem (M). Further u has the following properties.*

$$(3.6)\qquad |u(x, t)|\leqq\max(\|\phi\|_{L^\infty(0,1)},\ |H_0|,\ |H_1|) \qquad \text{on } \bar{Q}_T\,,$$

$$(3.7)\qquad \int_0^1 u_x(x, t)^2dx+\theta_0(u(0, t))+\theta_1(u(1, t))\leqq\bar{M}_\sigma \qquad (\sigma\leqq t\leqq T)\,,$$

$$(3.8)\qquad c_0^{-2}\int_\sigma^T\int_0^{s(t)}u_{xx}(x, t)^2\,dx\,dt+c_1^{-2}\int_\sigma^T\int_{s(t)}^1 u_{xx}(x, t)^2dxdt\leqq\bar{M}_\sigma\,,$$

$$(3.9)\qquad |u(x', t')-u(x, t)|\leqq\bar{M}_\sigma(|x'-x|^{1/2}+|t'-t|^{1/4}) \qquad \text{on } \bar{Q}_{\sigma,T}\,,$$

for any $\sigma\in]0, T[$, where $\bar{M}_\sigma$ is a constant bounded with $1/\sigma$, T, $\|\dot{s}\|_{L^2(\sigma/8,T)}$, $1/d^*$, $\|\phi\|_{L^2(0,1)}$. Here

$$d^*=\min\{\min(s(t),\ 1-s(t));\ 0\leqq t\leqq T\}\,.$$

Proposition 3.2. *Let $s(t)$ and $\phi(x)$ satisfy the assumptions of Proposition* 3.1. *Suppose that γ_i $(i=0, 1)$ is a single valued maximal monotone function. Then it follows that $u_x(x, t)\in C(D_T^i \cup \{i\} \times]0, T])$ and*

$$(-1)^i u_x(i, t) = \gamma_i(u(i, t)) \qquad (0<t\leqq T).$$

Proposition 3.3. *Let $s(t)\in H^1(0, T)$ and $\phi(x)$ satisfy* (A.1), (A.2). *Then u has the following properties in addition to the conclusion of Proposition* 3.1.

$$(3.10)\qquad \sup_{0\leqq t\leqq T} \Phi^t + c_2 \int_0^T\int_0^1 u_t(x, t)^2 dxdt$$
$$\leqq \Phi^0 \{1+\exp[2(2/d^*+c_3)(T+\int_0^T \dot{s}(t)^2 dt)]\},$$

$$(3.11)\qquad \Phi^{t_2}-\Phi^{t_1}\leqq K(t_2-t_1+\int_{t_1}^{t_2}\dot{s}(t)^2 dt) \qquad (0\leqq t_1\leqq t_2\leqq T),$$

$$(3.12)\qquad |u(x', t')-u(x, t)|\leqq K\{|x'-x|^{1/2}+|t'-t|^{1/4}\} \qquad on\ \overline{Q}_T,$$

where $K=K(T, ||\dot{s}||_{L^2(0,T)}, 1/d^*, \Phi^0)$ is a constant bounded with T, $||\dot{s}||_{L^2(0,T)}$, $1/d^*$, Φ^0. Here

$$(3.13)\qquad \Phi^t = 2^{-1}\int_0^1 u_x(x, t)^2 dx+\theta_0(u(0, t))+\theta_1(u(1, t)),$$

$$(3.14)\qquad \Phi^0 = 2^{-1}\int_0^1 \phi_x(x)^2 dx+\theta_0(\phi(0))+\theta_1(\phi(1)),$$

$$(3.15)\qquad c_2 = \min(c_0, c_1),\ c_3 = \max(c_0, c_1).$$

REMARK 3.2. θ_i $(i=0, 1)$ is a lower semicontinuous convex function θ_i from $\boldsymbol{R}$ into $]-\infty, \infty]$ such that $\theta_i\geqq 0$, $\theta_i(H_i)=0$ and $\partial\theta_i=\gamma_i$ (see § 1 and [4, p. 43]).

We shall state the results concerning the continuity of a family of solutions of the moving boundary problem (M) with respect to the moving boundary and the initial data.

Proposition 3.4. *Let $s^n(t)\in H^1(0, T)$ and $\phi(x)$ satisfy* (A.1), (A.2). *Suppose that*

$$(3.16)\qquad \int_0^T \dot{s}^n(t)^2 dt\leqq K,$$

$$(3.17)\qquad d\leqq s^n(t)\leqq 1-d \qquad (0\leqq t\leqq T),$$

$$(3.18)\qquad \lim_{n\to\infty} s^n(t) = s(t) \qquad (0\leqq t\leqq T),$$

where K and d are positive constants independent of n. Then we have

$$(3.19)\qquad \lim_{n\to\infty} u^n(x, t) = u(x, t) \qquad in\ C(\overline{Q}_T),$$

where u^n (resp. u) is the solution of (M) corresponding to the curve s^n (resp. s) and the initial data ϕ.

Proposition 3.5. *Let $s^n(t)\in H^1(0, T)$ and $\phi^n(x)$ satisfy* (A.1), (A.2). *Let $s(t)\in C(0, T])\cap H^1_{loc}(]0, T])$ and $\phi(x)$ be bounded, continuous for a.e. $x\in[0, 1]$. Suppose that*

$$(3.20)\qquad \int_\sigma^T \dot{s}^n(t)^2 dt \leqq K_\sigma ,$$

$$(3.21)\qquad d\leqq s^n(t)\leqq 1-d \qquad (0\leqq t\leqq T),$$

$$(3.22)\qquad \lim_{n\to\infty} s^n(t) = s(t) \qquad (0\leqq t\leqq T),$$

$$(3.23)\qquad |\phi^n(x)|\leqq K \qquad (0\leqq x\leqq 1),$$

$$(3.24)\qquad \lim_{n\to\infty} ||\phi^n(\cdot)-\phi(p)||_{C([p-\delta_p,p+\delta_p])}=0 \text{ for a.e. } p\in[0, 1]$$

(δ_p is a positive constant depending on p),

where K_σ (depending on $\sigma\in]0, T]$), d and K are constants. Then we have

$$(3.25)\qquad \lim_{n\to\infty} u^n(x, t) = u(x, t) \qquad \text{in } C(\overline{Q}_{\sigma,T})$$

for any $\sigma\in]0, T[$, where u^n (resp. u) is the solution of (M) corresponding to the curve s^n (resp. s) and the initial data ϕ^n (resp. ϕ).

REMARK 3.3. We can treat the problem (M) in a Hilbert space $L^2(0, 1)$ using the theory of the nonlinear semigroups. The related problems are shown in Damlamian [8], Kenmochi [20], Yamada [31] and Yotsutani [33].

REMARK 3.4. It is important to construct the solution of (M) by the finite difference method from the view point of the numerical analysis. A related work is shown in Jamet [17].

4. Difference scheme

In § 4–§ 7 we shall prove the existence of a solution of (M) introduced in the previous section using the finite difference method. In this section we introduce a difference scheme and state some simple lemmas.

First we extend $s(t)$ to the interval $[0, T+1]$ by defining $s(t)=s(T)$ for $T\leqq t\leqq T+1$ in order to clarify the following argument.

We use a net of rectangular meshes with uniform space width h and variable time step k_n ($n=1, 2, \cdots$). Here h varies in such a way that $1/h\equiv M$ is an integer. Let us introduce discrete coordinates.

$$x_j = jh \qquad (j = 0, 1, \cdots, M),$$
$$t_n = \textstyle\sum_{p=1}^n k_p \qquad (n = 1, 2, \cdots).$$

We shall give the definition of k_n and J_n. We put

$$J_0 = \max\{j \in N; jh < l + h/2\}, \qquad t_0 = 0.$$

For $n=1, 2, \cdots$ successively, we define

$$t'_n = \min\{t \geqq t_{n-1}; s(t) = (J_{n-1}-1)h \quad \text{or } (J_{n-1}+1)h\},$$

$$t_n = \begin{cases} t'_n & \text{if } t'_n - t_{n-1} \leqq 2h^{1/2}, \\ t_{n-1} + h^{1/2} & \text{if } t'_n - t_{n-1} > 2h^{1/2}, \end{cases}$$

$$J_n = \begin{cases} s(t'_n)/h & \text{if } t'_n - t_{n-1} \leqq 2h^{1/2}, \\ J_{n-1} & \text{if } t'_n - t_{n-1} > 2h^{1/2}, \end{cases}$$

$$k_n = t_n - t_{n-1} \qquad (>0).$$

REMARK 4.1. It follows from the definition above that

$$(J_n - 1)h < s(t_n) < (J_n + 1)h \qquad (n \geqq 0).$$

REMARK 4.2. We continue computing until $t_m \geqq T$ for some m. We put $N = \min\{n \in N; t_n \geqq T\}$.

Thus we get the following lemmas.

Lemma 4.1. *We have*

$$(4.1) \qquad \sum_{p=q+1}^{n} h^2/k_p \leqq \int_{t_q}^{t_n} \dot{s}(t)^2 dt + t_n h \qquad (q \geqq 1).$$

Moreover, if $J_0 = l/h$ *and* $s \in H^1(0, T)$, *then we have*

$$(4.2) \qquad \sum_{p=1}^{n} h^2/k_p \leqq \int_0^{t_n} \dot{s}(t)^2 dt + t_n h.$$

Proof. Let $p \geqq 2$. We shall show that $|s(t_p) - s(t_{p-1})| = h$ if $k_p < h^{1/2}$. If $k_p < h^{1/2}$, then we see that $t'_p - t_{p-1} \leqq 2h^{1/2}$, $t_p = t'_p$, $s(t_p) = J_p h$ $(=(J_{p-1}-1)h$ or $(J_{p-1}+1)h)$ by the definition of k_p. Thus it must hold that $t'_{p-1} - t_{p-2} \leqq 2h^{1/2}$. In fact, suppose $t'_{p-1} - t_{p-2} > 2h^{1/2}$, then $k_{p-1} = h^{1/2}$ and $t'_{p-1} = t'_p$ $(=t_p)$. Hence $t'_{p-1} - t_{p-2} = t_p - t_{p-2} = k_p + k_{p-1} < 2h^{1/2}$, which is a contradiction. Therefore we have $t_{p-1} = t'_{p-1}$ and $s(t_{p-1}) = J_{p-1}h$. Hence $s(t_p) - s(t_{p-1}) = \pm h$. Consequently we see

$$(h/k_p)^2 k_p \begin{cases} = \left|\dfrac{s(t_p) - s(t_{p-1})}{t_p - t_{p-1}}\right|^2 k_p \leqq \displaystyle\int_{t_{p-1}}^{t_p} \dot{s}(t)^2 dt & (k_p < h^{1/2}), \\ \leqq (h/h^{1/2})^2 k_p = h k_p & (k_p \geqq h^{1/2}). \end{cases}$$

Therefore we obtain (4.1).

Let $J_0=l/h$. We get $|s(t_1)-s(t_0)|=h$ if $k_1<h^{1/2}$. Hence we gte (4.2) using (4.1). q.e.d.

Lemma 4.2. $\lim_{h\to 0} s_h(t) = s(t) \quad$ *in* $C([0, T])$,

where $s_h(t)$ *is a piecewise linear function such that* $s(t_p)=J_p h$.

Proof. We see that $|s_h(t)-s(t)|\leqq 2h$ noting Remark 4.1. q.e.d.

Let us introduce a net function u_j^n which corresponds to $u(x_j, t_n)$. Further we use usual divided differences.

$$\text{(4.3)} \qquad \begin{cases} u_{jx}^n = (u_{j+1}^n-u_j^n)/h\,, \quad u_{j\bar{x}}^n = (u_j^n-u_{j-1}^n)/h\,, \\ u_{jx\bar{x}}^n = (u_{j+1}^n-2u_j^n+u_{j-1}^n)/h^2\,, \quad u_{j\bar{t}}^n = (u_j^n-u_j^{n-1})/k_n\,, \quad \text{e.t.c.} \end{cases}$$

In our scheme the heat equations are replaced by pure implicit difference equations,

$$\text{(4.4)} \qquad u_{jx\bar{x}}^n-c_0u_{j\bar{t}}^n = 0 \qquad (1\leqq j\leqq J_n-1)\,,$$

$$\text{(4.5)} \qquad u_{jx\bar{x}}^n-c_1u_{j\bar{t}}^n = 0 \qquad (J_n+1\leqq j\leqq M-1)\,.$$

The boundary and initial conditions are put in the following forms,

$$\text{(4.6)} \qquad \begin{aligned} &\text{(a)} \quad u_{0x}^n\in\gamma_0(u_0^n) \\ &\text{(b)} \quad -u_{M\bar{x}}^n\in\gamma_1(u_M^n)\,, \end{aligned}$$

$$\text{(4.7)} \qquad \begin{aligned} &\text{(a)} \quad u_j^0 = \phi_j\equiv\phi(x_j) \qquad (0\leqq j\leqq J_0-1)\,, \\ &\text{(b)} \quad u_j^0 = \phi_j\equiv\phi(x_j) \qquad (J_0+1\leqq j\leqq M)\,, \end{aligned}$$

$$\text{(4.8)} \qquad u_{J_n}^n = 0\,.$$

Now we state the difference scheme.

0^0 Determine k_n $(1\leqq n\leqq N)$ and J_n $(0\leqq n\leqq N)$.

We determine u_j^n as follows.

1^0 $u_j^0 = \phi_j \qquad (0\leqq j\leqq M, j\neq J_0)$, $u_{J_0}^0=0$.

For $n=1, 2, \cdots, N$ successively,

2^0 solve the system of difference equations (4.4) and (4.5) for $\{u_j^n\}_j$ under the boundary conditions (4.6) and (4.8) with the initial condition $\{u_j^{n-1}\}_j$.

REMARK 4.3. Step 2^0 is well-defined by [34, Lemma 4.1].

REMARK 4.4. We define $u_{J_0}^0=0$ leaving the value of $\phi(x_{J_0})$ out of consideration. It holds that $\sum_{j=0}^{M-1} {u_{jx}^0}^2 h=\sum_{j=0}^{M-1} \phi(x_j)_x^2 h$ when $\phi(x_{J_0})=0$. We use this fact in the proof of Lemma 6.1.

5. L^2-estimates of the solutions

In this section we get the L^2-estimates of the differnece solutions. We employ the idea of the nonlinear semigroups (see Brézis [3,4] and Yotsutani [33,34] e.t.c.).

The following inequalities are so-called variational inequalities.

Lemma 5.1. *Let u_j^n satisfy* (4.4)–(4.8). *Then we have*

$$\begin{aligned}
(5.1)\quad & \textstyle\sum_{j=0}^{J_n-1} u_{jx}^n(w_{jx}^n-u_{jx}^n)h+\theta_0(w_0^n)-\theta_0(u_0^n)-u_{J_n\bar{x}}^n w_{J_n}^n \\
& \geqq -\textstyle\sum_{j=1}^{J_n-1} u_{jx\bar{x}}^n(w_j^n-u_j^n)h\,, \\
(5.2)\quad & 2^{-1}\textstyle\sum_{j=0}^{J_n-1}(w_{jx}^{n\,2}-u_{jx}^{n\,2})h+\theta_0(w_0^n)-\theta_0(u_0^n)-u_{J_n\bar{x}}^n w_{J_n}^n \\
& \geqq -\textstyle\sum_{j=1}^{J_n-1} u_{jx\bar{x}}^n(w_j^n-u_j^n)h\,, \\
(5.3)\quad & \textstyle\sum_{j=J_n}^{M-1} u_{jx}^n(w_{jx}^n-u_{jx}^n)h+\theta_1(w_M^n)-\theta_1(u_M^n)+u_{J_n x}^n w_{J_n}^n \\
& \geqq -\textstyle\sum_{j=J_n+1}^{M-1} u_{jx\bar{x}}^n(w_j^n-u_j^n)h\,, \\
(5.4)\quad & 2^{-1}\textstyle\sum_{j=J_n}^{M-1}(w_{jx}^{n\,2}-u_{jx}^{n\,2})h+\theta_1(w_M^n)-\theta_1(u_M^n)+u_{J_n x}^n w_{J_n}^n \\
& \geqq -\textstyle\sum_{j=J_n+1}^{M-1} u_{jx\bar{x}}^n(w_j^n-u_j^n)h\,, \\
(5.5)\quad & 2^{-1}\textstyle\sum_{j=0}^{M-1} w_{jx}^{n\,2}h+\theta_0(w_0^n)+\theta_1(w_M^n)+(-u_{J_n\bar{x}}^n+u_{n\,J_n x}^x)w_{J_n}^n \\
& -\{2^{-1}\textstyle\sum_{j=0}^{M-1} u_{jx}^{n\,2}h+\theta_0(u_0^n)+\theta_1(u_M^n)\} \\
& \geqq c_0\textstyle\sum_{j=1}^{J_n-1} u_{j\bar{t}}^n(u_j^n-w_j^n)h+c_1\textstyle\sum_{j=J_n+1}^{M-1} u_{j\bar{t}}^n(u_j^n-w_j^n)h\,,
\end{aligned}$$

for w_j^n such that $w_0^n\in D(\theta_0)$, $w_M^n\in D(\theta_1)$.

Proof. We get (5.1), (5.2), (5.3) and (5.4) by the proof analogous to that of [34, Lemma 7.1]. Adding (5.2) and (5.3), we have (5.5) by (4.4) and (4.5). q.e.d.

For simplicity we put

$$\Phi_n = 2^{-1}\textstyle\sum_{j=0}^{M-1} u_{jx}^{n\,2}h+\theta_0(u_0^n)+\theta_1(u_M^n)\,,$$
$$\hat{\Phi}_m = \max\{\Phi_p;\ m-1\leqq p\leqq N\}\,.$$

We give a simple lemma.

Lemma 5.2. *Let u_j^n satisfy* (4.4)–(4.8). *Then we have*

$$(5.6)\qquad \textstyle\sum_{p=m}^{n}\sum_{j=0}^{M} u_{j\bar{t}}^{p\,2}hk_p\leqq 4\sum_{p=m}^{n}\sum_{j=1}^{M-1} u_{j\bar{t}}^{p\,2}hk_p+12\hat{\Phi}_m\sum_{p=m}^{n}(h/k_p)^2k_p\,.$$

for any m and n with $m\leqq n$.

Proof. We see that

$$\begin{aligned}
(5.7)\qquad u_{0\bar{t}}^{p\,2} &= (-u_{0x}^p h+u_{1\bar{t}}^p k_p+u_{0\ x}^{p-1}h)^2k_p^{-2} \\
&\leqq 3u_{1\bar{t}}^{p\,2}+3(u_{0x}^{p\,2}+u_{0\ x}^{p-1\,2})h^2k_p^{-2}\,.
\end{aligned}$$

We get also

(5.8) $u_{M\bar{t}}^{p\ 2} \leqq 3 u_{M-1,\bar{t}}^{p\ 2} + 3(u_{M\bar{x}}^{p\ 2} + u_{M\ \bar{x}}^{p-1\ 2}) h^2 k_p^{-2}$.

Hence we have

$$\begin{aligned} &(u_{0\bar{t}}^{p\ 2} + u_{M\bar{t}}^{p\ 2}) hk_p \\ &\leqq 3(u_{1\bar{t}}^{p\ 2} + u_{M-1,\bar{t}}^{p\ 2}) hk_p + 3(2\Phi^p + 2\Phi^{p-1}) h^2 k_p^{-1} \\ &\leqq 3 \sum_{j=1}^{M-1} u_{j\bar{t}}^{p\ 2} hk_p + 12\hat{\Phi}_m (h/k_p)^2 k_p . \end{aligned}$$

Therefore we obtain (5.6) easily. q.e.d.

In what follows we assume that the uniform space width h is sufficiently small. We shall give the several estimates for u_j^n. We have the following lemma by the proof of [34, Lemma 5.1].

Lemma 5.3.

(5.9) $|u_j^n| \leqq \max, (\|\phi\|_{L^\infty(0,1)} |H_0|, |H_1|)$.

Let us fix the auxiliary function $v(x)$.

$$v(x) = \begin{cases} H_0(1-x/D_1) & (0 \leqq x \leqq D_1), \\ 0 & (D_1 \leqq x \leqq D_2) \\ H_1(1-(1-x)/(1-D_2)) & (D_2 \leqq x \leqq 1), \end{cases}$$

where $D_1 = d^*/2$ and $D_2 = 1 - d^*/2$. The next lemma is useful for further estimates.

Lemma 5.4. *There exists a constant* $\hat{M} = \hat{M}(1/d^*, T, \|\phi\|_{L^2(0,1)})$ *bounded with* $1/d^*$, T *and* $\|\phi\|_{L^2(0,1)}$ *such that*

(5.10) $\sum_{p=1}^{N} \Phi_p k_p \leqq \hat{M}$.

Proof. Substitute $v_j = v(x_j)$ for w_j^n in (5.5). Now we observe that

$$\begin{aligned} (5.11) \quad & u_{j\bar{t}}^n (u_j^n - v_j) k_n = [(u_j^n - v_j) - (u_j^{n-1} - v_j)](u_j^n - v_j) \\ & \geqq 2^{-1}(u_j^n - v_j)^2 - 2^{-1}(u_j^{n-1} - v_j)^2 . \end{aligned}$$

Multiplying (5.5) by k_n and noting (5.11), we have

$$\begin{aligned} &\sum_{j=0}^{M-1} v_{jx}^2 hk_n - 2\Phi_n k_n \\ &\geqq c_0 \sum_{j=1}^{J_n-1} (u_j^n - v_j)^2 h + c_1 \sum_{j=J_n+1}^{M-1} (u_j^n - v_j)^2 h \\ &\quad - [c_0 \sum_{j=1}^{J_n-1} (u_j^{n-1} - v_j)^2 h + c_1 \sum_{j=J_n+1}^{M-1} (u_j^{n-1} - v_j)^2 h] \\ &\geqq c_0 \sum_{j=1}^{J_n-1} (u_j^n - v_j)^2 h + c_1 \sum_{j=J_n+1}^{M-1} (u_j^n - v_j)^2 h \\ &\quad - [c_0 \sum_{j=1}^{J_{n-1}-1} (u_j^{n-1} - v_j)^2 h + c_1 \sum_{j=J_{n-1}+1}^{M-1} (u_j^{n-1} - v_j)^2 h] \end{aligned}$$

by $\theta_0(v_0)=\theta_1(v_M)=0$, $v_{J_n}=v_{J_{n-1}}=u^{n-1}_{J_{n-1}}=0$. Therefore we get

$$(5.12)\quad \sum_{p=1}^{n} \Phi_p k_p+2^{-1}c_2 \sum_{j=1}^{M-1} (u_j^n-v_j)^2 h \\ \leqq 2^{-1}\sum_{p=1}^{n}\sum_{j=0}^{M-1} v_{jx}{}^2 hk_p+2^{-1} c_3 \sum_{j=1}^{M-1} (u_j^0-v_j)^2 h$$

by summing up. Hence we obtain (5.10) easily. q.e.d.

Now we define a new net function $\tilde{u}_j^{n-1}$ from u_j^{n-1}. We give the definition according to the relation between J_{n-1} and J_n.

Case $J_n=J_{n-1}-1$.

$$\tilde{u}_j^{n-1}=\begin{cases} ((1-(j+1)r)u_j^{n-1}+jru_{j+1}^{n-1})/(1-r) & (0\leqq j\leqq J_n) \\ u_j^{n-1} \quad (J_n+1\leqq j\leqq M), \end{cases}$$

where $r=1/J_{n-1}$.

Case $J_n=J_{n-1}$.

$$\tilde{u}_j^{n-1}=u_j^{n-1} \qquad (0\leqq j\leqq M).$$

Case $J_n=J_{n-1}+1$.

$$\tilde{u}_j^{n-1}=\begin{cases} u_j^{n-1} \qquad (0\leqq j\leqq J_n-1) \\ ((M-j)r'u_{j-1}^{n-1}+(1-(M-j+1)r')u_j^{n-1})/(1-r') \\ \qquad\qquad (J_n\leqq j\leqq M) \end{cases}$$

where $r'=1/(M-J_{n-1})$.

We prepare some useful lemmas which are essential in obtaining necessary estimates.

Lemma 5.5.

$$(5.13)\quad \tilde{u}_0^{n-1}=u_0^{n-1}, \quad \tilde{u}_{J_n}^{n-1}=0, \quad \tilde{u}_M^{n-1}=u_M^{n-1},$$

$$(5.14)\quad 2^{-1}\sum_{j=0}^{M-1} \tilde{u}_j^{n-1}{}_x{}^2 h-2^{-1}\sum_{j=0}^{M-1} u_j^{n-1}{}_x{}^2 h \\ \leqq M_1(k_n+h^2/k_n)\Phi_{n-1},$$

$$(5.15)\quad 2^{-1}c_3[(\sum_{j=1}^{J_n-1}+\sum_{j=J_n+1}^{M-1})(\tilde{u}_j^{n-1}-u_j^{n-1})^2 hk_n^{-1}+u^n_{J_n\bar{t}}{}^2 hk_n] \\ \leqq c_3(h^2/k_n)\Phi_{n-1},$$

where $M_1=2/d_*$.

Proof. We get (5.13) easily by the definition of $\tilde{u}_j^{n-1}$. We shall show (5.14) and (5.15). If $J_n=J_{n-1}$, they are obvious from the definition of $\tilde{u}_j^{n-1}$. Let us consider the case $J_n=J_{n-1}-1$. For simplicity $\tilde{u}_j$, u_j, J denote $\tilde{u}_j^{n-1}$, u_j^{n-1}, J_{n-1} respectively. We shall get (5.14). We have

$$\sum_{j=0}^{M-1} \tilde{u}_{jx}{}^2 h = \sum_{j=0}^{J_n} \tilde{u}_{jx}{}^2 h + \sum_{j=J_n+1}^{M-1} u_{jx}{}^2 h$$
$$= \sum_{j=0}^{J-2} \tilde{u}_{jx}{}^2 h + \sum_{j=J}^{M-1} u_{jx}{}^2 h$$
$$= (1-r)^{-2} h^{-1} \sum_{j=0}^{J-2} [(j+1) r(u_{j+2}-u_{j+1})$$
$$+(1-(j+1)r)(u_{j+1}-u_j)]^2 + \sum_{j=J}^{M-1} u_{jx}{}^2 h$$
$$\leqq (1-r)^{-2} \sum_{j=0}^{J-2} [(j+1) r u_{j+1,x}{}^2 + (1-(j+1)r) u_{jx}{}^2] h + \sum_{j=J}^{M-1} u_{jx}{}^2 h$$
$$= (1-r)^{-1} \sum_{j=0}^{M-1} u_{jx}{}^2 h \leqq (1+2r) \sum_{j=0}^{M-1} u_{jx}{}^2 h$$

by $r=1/J \geqq 1/2$. Hence we have

$$2^{-1} \sum_{j=0}^{M-1} \tilde{u}_{jx}{}^2 h - 2^{-1} \sum_{j=0}^{M-1} u_{jx}{}^2 h$$
$$\leqq 2(Jh)^{-1} h k_n^{-1/2} k_n^{1/2} (2^{-1} \sum_{j=0}^{M-1} u_{jx}{}^2 h)$$
$$\leqq M_1 (h^2/k_n + k_n) \Phi_{n-1}.$$

We shall get (5.15). It follows from $u^n_{J_n} = u^{n-1}_{J_{n-1}} = 0$ that

$$u^n_{J_n \bar{t}}{}^2 h k_n = (h^2/k_n) u_{J-1,x}{}^2 h\,.$$

Hence we obtain

$$(\sum_{j=1}^{J_n-1} + \sum_{j=J_n+1}^{M-1})(\tilde{u}_j^{n-1} - u_j^{n-1})^2 h k_n^{-1} + u^n_{J_n \bar{t}}{}^2 h k_n$$
$$= (1-r)^{-2} \sum_{j=1}^{J-2} [jr(u_{j+1}-u_j)]^2 h k_n^{-1} + u^n_{J_n \bar{t}}{}^2 h k_n$$
$$\leqq 2(h^2/k_n)(2^{-1} \sum_{j=1}^{J-1} u_{jx}{}^2 h) \leqq 2(h^2/k_n) \Phi_{n-1}\,.$$

We can treat the case $J_n = J_{n-1}+1$ in the same way. q.e.d.

Lemma 5.6.

$$(5.16)\quad 2^{-1} \sum_{j=0}^{M-1} (\tilde{u}^{n-1}_{j\,x}{}^2 - u^n_{jx}{}^2) h + \theta_0(u_0^{n-1}) + \theta_1(u_M^{n-1}) - \theta_0(u_0^n) - \theta_1(u_M^n)$$
$$\geqq 2^{-1} c_2 \sum_{j=1}^{M-1} u^n_{j\bar{t}}{}^2 h k_n$$
$$-2^{-1} c_3 [(\sum_{j=1}^{J_n-1} + \sum_{j=J_n+1}^{M-1})(u_j^{n-1} - \tilde{u}_j^{n-1})^2 h k_n^{-1} + u^n_{J_n \bar{t}}{}^2 h k_n]\,.$$

Proof. Taking $\tilde{u}_j^{n-1}$ as w_j^n in (5.5), we have

$$(5.17)\quad 2^{-1} \sum_{j=0}^{M-1} (\tilde{u}^{n-1}_{j\,x}{}^2 - u^n_{jx}{}^2) h + \theta_0(\tilde{u}_0^{n-1}) + \theta_1(\tilde{u}_M^{n-1}) - \theta_0(u_0^n) - \theta_1(u_M^n)$$
$$\geqq c_0 \sum_{j=1}^{J_n-1} u^n_{j\bar{t}} [(u_j^n - u_j^{n-1}) + (u_j^{n-1} - \tilde{u}_j^{n-1})] h$$
$$+ c_1 \sum_{j=J_n+1}^{M-1} u^n_{j\bar{t}} [(u_j^n - u_j^{n-1}) + (u_j^{n-1} - \tilde{u}_j^{n-1})] h$$
$$= c_0 \sum_{j=1}^{J_n-1} u^n_{j\bar{t}}{}^2 h k_n + c_0 \sum_{j=1}^{J_n-1} u^n_{j\bar{t}} (k_n h)^{1/2} (k_n^{-1} h)^{1/2} (u_j^{n-1} - \tilde{u}_j^{n-1})$$
$$+ c_1 \sum_{j=J_n+1}^{M-1} u^n_{j\bar{t}}{}^2 h k_n + c_1 \sum_{j=J_n+1}^{M-1} u^n_{j\bar{t}} (k_n h)^{1/2} (k_n^{-1} h)^{1/2} (u_j^{n-1} - \tilde{u}_j^{n-1})$$
$$\geqq 2^{-1} c_0 \sum_{j=1}^{J_n-1} u^n_{j\bar{t}}{}^2 h k_n - 2^{-1} c_0 \sum_{j=1}^{J_n-1} k_n^{-1} h (u_j^{n-1} - \tilde{u}_j^{n-1})^2$$
$$+ 2^{-1} c_1 \sum_{j=J_n+1}^{M-1} u^n_{j\bar{t}}{}^2 h k_n - 2^{-1} c_1 \sum_{j=J_n+1}^{M-1} k_n^{-1} h (u_j^{n-1} - \tilde{u}_j^{n-1})^2$$

by (5.13). Hence we obtain (5.16) easily. q.e.d.

We derive the recurrent estimates from Lemma 5.5 and 5.6.

Lemma 5.7.

$$(5.18)\quad 2^{-1}c_2\sum_{j=1}^{M-1}(t_{n-1}-t_{q-1})(u_{j\bar{t}}^{n})^2hk_n+(t_{n-1}-t_{q-1})\Phi_n$$
$$\leqq(1+M_2k_n+M_2h^2/k_n)(t_{n-2}-t_{q-1})\Phi_{n-1}$$
$$+(1+M_2k_n+M_2\|\dot{s}\|_{L^2(t_q,T)})(k_{n-2}\Phi_{n-1})\qquad(n>q\geqq 1)\,,$$

where $M_2=M_1+c_3$.

Proof. Using (5.14) and (5.15) in (5.16), we have

$$(5.19)\quad 2^{-1}c_2\sum_{j=1}^{M-1}u_{j\bar{t}}^{n\,2}hk_n+\Phi_n$$
$$\leqq(1+M_2k_n+M_2h^2/k_n)\Phi_{n-1}\,.$$

Multiplying (5.19) $\tilde{t}_{n-1}=t_{n-1}-t_{q-1}$, we have

$$2^{-1}c_3\sum_{j=1}^{M-1}\tilde{t}_{n-1}(u_{j\bar{t}}^{n})^2hk_n+\tilde{t}_{n-1}\Phi_n$$
$$\leqq(1+M_2k_n+M_2h^2/k_n)(\tilde{t}_{n-2}\Phi_{n-1})+(1+M_2k_n+M_2h^2/k_n)(k_{n-1}\Phi_{n-1})$$

by $\tilde{t}_{n-1}=\tilde{t}_{n-2}+k_{n-1}$. We obtain (5.17) using Lemma 4.1. q.e.d.

Now we state the most important L^2-estimates.

Lemma. 5.8. *Let* $q\geqq 1$. *Then we have*

$$(5.20)\quad (t_{n-1}-t_{q-1})\Phi_n\leqq M'(t_q)\qquad(q<n\leqq N)\,,$$
$$(5.21)\quad \sum_{p=q+1}^{N}\sum_{j=1}^{M-1}(t_{p-1}-t_{q-1})(u_{j\bar{t}}^{p})^2hk_p\leqq M'(t_q)\,,$$

where $M'(t_q)=\{2+M_2\|\dot{s}\|_{L^2(t_q,T)}\}\{1+\exp[2M_2(T+1+\|\dot{s}\|_{L^2(t_q,T)})]\}\hat{M}$.

Proof. We put

$$V_n=2^{-1}c_2\sum_{j=1}^{M-1}(t_{n-1}-t_{q-1})(u_{j\bar{t}}^{n})^2hk_n\,,\quad F_n=(t_{n-1}-t_{q-1})\Phi_n\,,$$
$$K_n=M_2(k_n+h^2/k_n)\,,\quad R_n=(2+M_2\|\dot{s}\|_{L^2(t_q,T)})k_n\Phi_n\,.$$

We note that $F_q=0$,

$$\sum_{p=q+1}^{N}K_p\leqq M_2(T+1+\|\dot{s}\|_{L^2(t_q,T)})\,,$$
$$\sum_{p=q}^{N-1}R_p\leqq(2+M_2\|\dot{s}\|_{L^2(t_q,T)})\hat{M}\,,$$

from Lemma 4.1 and 5.4. Hence we obtain the conclusion by Lemma 5.7 and 2.3. q.e.d.

Lemma 5.9. *Assume that* $\lim_{h\to\infty}t_m=\sigma$ *for arbitrary fixed* σ *with* $0<\sigma<T$. *Then we have*

$$(5.22)\quad \max_{m\leqq n\leqq N}\{\sum_{j=1}^{M-1}u_{jx}^{n\,2}h+\theta_0(u_0^n)+\theta_1(u_M^n)\}\leqq\tilde{M}_\sigma\,,$$
$$(5.23)\quad \sum_{p=m}^{N}\sum_{j=0}^{M}u_{j\bar{t}}^{p\,2}hk_p\leqq\tilde{M}_\sigma\,,$$

where $\tilde{M}_\sigma=\tilde{M}(1/\sigma, T, ||s||_{L^2(\sigma/8,T)}, 1/d^*, ||\phi||_{L^2(0,1)})$ *is a constant bounded with* $1/\sigma$, T, $||s||_{L^2(\sigma/8,T)}$, $1/d^*$, $||\phi|||_{L^2(0,1)}$.

Proof. We have frcm (5.20) and (5.21), if $m>q+1$,

$$(5.24)\qquad \hat{\Phi}_m=\max\{\Phi_p;\ m-1\leqq p\leqq N\}\leqq M'(t_q)/(t_{m-2}-t_{q-1}),$$

$$(5.25)\qquad \textstyle\sum_{p=m}^{N}\sum_{j=1}^{M-1} u_{j\bar{t}}^{p\ 2}hk_p\leqq M'(t_q)/(t_{m-1}-t_{p-1}).$$

Now we take $t_q=t_{q(m)}$ such that

$$(5.26)\qquad \sigma/8\leqq t_{q-1}\leqq\sigma/4 \qquad (\leqq\sigma/2\leqq t_{m-2}).$$

Hence we have by (5.24), (5.25) and (5.26)

$$(2.57)\qquad \hat{\Phi}_m\leqq M'(\sigma/8)/(\sigma/4),$$

$$(5.28)\qquad \textstyle\sum_{p=m}^{N}\sum_{j=1}^{M-1} u_{j\bar{t}}^{p\ 2}hk_p\leqq M'(\sigma/8)/(\sigma/4).$$

Therefore it follows from (5.27), (5.28), Lemma 4.1 and Lemma 5.2 that

$$\begin{aligned}&\textstyle\sum_{p=m}^{N}\sum_{j=0}^{M} u_{j\bar{t}}^{p\ 2}hk_p\leqq 4\sum_{p=m}^{N}\sum_{j=1}^{M-1} u_{j\bar{t}}^{p\ 2}hk_p+12\hat{\Phi}_m\sum_{p=m}^{N}(h^2/k_p)\\ &\leqq 16M'(\sigma/8)/\sigma+48M'(\sigma/8)(||\dot{s}||_{L^2(\sigma/8,T)}+1)/\sigma.\end{aligned}$$ q.e.d.

6. Estimates of the solutions under some additional conditions

In this section we give several estimates under some additional conditions in order to use these results in the proof of Proposition 3.2 and Theorem 2. We state an L^2-estimate.

Lemma 6.1. *Let* $s(t)\in H^1(0, T)$ *and* $\phi(x)\in H^1(0, 1)$ *with* $\phi(0)\in D(\theta_0)$, $\phi(1)\in D(\theta_1)$ *and* $\phi(x_{J_0})=0$. *Then*

$$(6.1)\qquad \max_{0\leqq n\leqq N}\Phi_n+c_2\textstyle\sum_{n=1}^{N}\sum_{j=1}^{M-1} u_{j\bar{t}}^{n\ 2}hk_n\leqq\Phi^0[1+\exp(2L_1)],$$

$$\begin{aligned}(6.2)\qquad &\Phi_n-\Phi^0\\ &\leqq\{\max_{0\leqq p\leqq N}\Phi_p\}\cdot\{M_2[t_n+\int_0^{t_n}|\dot{s}(t)|^2dt+t_nh]\}\qquad (0\leqq n\leqq N),\end{aligned}$$

where

$$L_1=M_2[T+\int_0^T|\dot{s}(t)|^2dt+(T+1)h]$$

Proof. It follows from (5.19) that

$$(6.3)\qquad V_n+F_n\leqq(1+K_n)F_{n-1}\qquad (0<n\leqq N),$$

where

$$V_n=2^{-1}c_2\textstyle\sum_{j=1}^{M-1} u_{j\bar{t}}^{n\ 2}hk_n,\quad F_n=\Phi_n,\quad K_n=M_2(k_n+h^2/k_n).$$

We see that

$$(6.4) \qquad \sum_{p=1}^{n} K_p \leqq M_2\left(t_n+\int_0^{t_n} |\dot{s}(t)|^2 dt+t_n h\right)$$

by Lemma 4.1. Hence we obtain the conclusion by Lemma 2.3, (6.3), (6.4) and the inequality $\Phi_0 \leqq \Phi^0$. q.e.d.

We shall use the next result in the proof of Proposition 3.2.

Lemma 6.2. *Let $\phi(x) \in C^{0,1}([0, 1])$ with $\phi(0) \in D(\gamma_0)$, $\phi(1) \in D(\gamma_1)$, then there exists a constant L_2 such that*

$$(6.5) \qquad |u_{jx}^n| \leqq L_2 \qquad (0 \leqq x_j \leqq d^*/2),$$

$$(6.5) \qquad |u_{j\bar{x}}^n| \leqq L_2 \qquad (1-d^*/2 \leqq x_j \leqq 1).$$

Proof. We shall show (6.5). It is easily seen from the assumption and the proof of [34, Lemma 6.1] that

$$(6.7) \qquad |u_{0x}^n| \leqq L_3,$$

where L_3 is a constant. Hence we get (6.5) by Lemma 5.3, (6.7), $\phi(x) \in C^{0,1}([0, 1])$ and the proof of [34, Lemma 16.1]. We can get (6.6) in the same way. q.e.d.

7. Convergence of the difference scheme

In this section we prove the convergence of the difference scheme under the assumption of Proposition 3.1. Further we give the proof of Proposition 3.1–3.5.

We shall show that the net functions u_j^n can be extended to the region $\bar{Q}_T$ in such a way that the family of the extended function $\{u_h(x, t)\}_h$ will be uniformly bounded and equicontinuous on $\bar{Q}_{\sigma,T}$. To begin with, we divide each rectangle $[x_j, x_{j+1}] \times [t_n, t_{n+1}]$ into triangles by a straight line connecting

$$(x_j, t_n) \text{ and } (x_{j+1}, t_{n+1}) \qquad \text{for } n \text{ s.t. } J_{n+1}=J_n \text{ or } J_n+1,$$

$$(x_{j+1}, t_n) \text{ nd } (x_j, t_{n+1}) \qquad \text{for } n \text{ s.t. } J_{n+1}=J_n-1.$$

We define $u_h(x, t)$ as a piecewise linear function which equals to the value of a net function u_j^n at the corner of triangles. It is easy to see that the function $u_h(x, t)$ constructed in this way is continuous on $\bar{Q}_T$, and it has the maximum at a mesh point. Hence we get the following result by Lemma 5.3, 2.2, 5.9 and the proof of [34, Lemma 8.1].

Lemma 7.1.

$$(7.1) \qquad |u_h(x, t)| \leqq \max(\|\phi\|_{L^\infty(0,1)}, |H_0|, |H_1|) \qquad \text{on } \bar{Q}_T,$$

$$(7.2) \qquad |u_h(x', t')-u_h(x, t)| \leqq K_\sigma(|x'-x|^{1/2}+|t'-t|^{1/4}) \qquad \text{on } \bar{Q}_{\sigma,T},$$

where K_σ is a constant depending on σ.

It follows from Lemma 7.1 and Ascoli-Arzelà's theorme that a subsequence of $\{u_h(x, t)\}$ convergse to a function $u(x, t)\in C(\bar{Q}_{\sigma,T})$ uniformly on $\bar{Q}_{\sigma,T}$, for any σ with $0<\sigma<T$. We denote by $\{s_h(t)\}$ and $\{u_h(x, t)\}$ the subsequence of $\{s_h(t)\}$ and $\{u_h(x, t)\}$ respectively again. We collect some properties of $s(t)$ and $u(x, t)$ in the next lemma.

Lemma 7.2.
(i) *$u\in C(\bar{Q}_T-\{t=0\})\cap C^\infty(D_T)$, u satisfies* (3.1) *and* (3.2).

(ii)
$$|u(x, t)|\leqq\max(||\phi||_{L^\infty(0,1)}, |H_0|, |H_1|),$$
$$|u(x', t')-u(x, t)|\leqq K_\sigma(|x'-x|^{1/2}+|t'-t|^{1/4}) \qquad on\ \bar{Q}_{\sigma,T},$$
$$\int_0^1 u_x(x, t)^2dx+\theta_0(u(0, t))+\theta_1(u(1, t))\leqq\tilde{M}_\sigma \qquad (\sigma\leqq t\leqq T),$$
$$\int_\sigma^T\int_0^1 u_t(x, t)^2dx\leqq\tilde{M}_\sigma,$$

(iii) *For a.e. $t\in]0, T[$, $u_x(0, t)$ and $u_x(1, t)$ exists and $u_x(0, t)$, $u_x(1, t)\in L^2_{\mathrm{loc}}(]0, T])$.*
(iv) *u satisfies* (3.3).
(v) *u satisfies* (3.4) *and u is continuous at $(x, 0)\in[0, 1]\times\{0\}-Z$.*
(vi) *u satisfies* (3.5).

Proof. We have (i) using the proof of [34, Lemma 8.2 (i)]. We get (ii) using Lemma 7.1, (5.22) and (5.23) respectively. We get (iii) from (i), (ii) and the proof of [34, Lemma 8.3]. We have (iv) using the proof of [34, Lemma 8.4]. We have (v) using the Petrovskii's technique [26, p. 357–358]. We obtain (vi) using $u_h(s_h(t), t)=0$. q.e.d.

Now we give the proof of Proposition 3.1–3.5.

Proof of Proposition 3.1. We see the existence of the solution, (3.6), (3.7), (3.8) and (3.9) by Lemma 7.2. We get the uniqueness by Proposition 8.1 which we prove later. q.e.d.

REMARK 7.1. The full sequence of $\{u_h(x, t)\}_h$ converges to $u(x, t)$ in view of the uniqueness of the solution of (S).

Proof of Proposition 3.2. We note that u is constructed as the limit of the sequence $\{u_h\}_h$ of the solution of the difference equations. We may assume that ϕ satisfies the condition stated in Lemma 6.2, since u satisfies (3.3) and (3.8).

We shall show $u_x(0, t)=\gamma_0(u(0, t))$. Since γ_0 is a single valued maximal monotone graph in $\boldsymbol{R}^2$, $D(\gamma_0)$ is an open inver interval and $\gamma_0(\cdot)$ is a continuous function on $D(\gamma_0)$ by [34, Lemma 14.1]. Hence for any $t\in]0, T]$ and $\varepsilon>0$, there exists $\delta>0$ and $h_0>0$ such that

$$(7.3)\qquad |\gamma_0(u_h(0, s))-\gamma_0(u(0, t))|<\varepsilon$$

for $|s-t|<\delta$ and $h\leqq h_0$ in view of (7.2), and $u_h\to u$ in $C(\overline{Q}_T)$ as $h\to 0$. Set $z_j^n=u_{jx}^n$. It follow follows that $z_{jx\bar{x}}^n-c_0z_{j\bar{t}}^n=0$ and $z_0^n=\gamma_0(u_0^n)$. More over z_j^n is uniformly bounded near $x=0$ by Lemma 6.2. Combining these with (7.3) and applying the Petrovskii technique [26, p. 364–368] we observe that u_x is continuous to the boundary $x=0$ and $u_x(0, t)=\gamma_0(u(0, t))$. We can get $-u_x(1, t)=\gamma_1(u(1, t))$ and the continuity to the boundary $x=1$ in the same way. q.e.d.

Proof of Proposition 3.3. Consider the two cases: (i) l is a rational number, (ii) l is an irrational number.

(i). We can take a subsequence $\{h\}$ of the space widths such that $\phi(x_{J_0})=0$ for any h since l is a rational number. We get (3.10) and (3.12) using (6.1), Lemma 2.2 and the proof of Proposition 3.1. We shall show (3.11). It folfolws from (6.1) and (6.2) that

$$(7.4)\qquad \Phi^t-\Phi^0\leqq\Phi^0[1+\exp(2B)]M_2[t+\int_0^t|\dot{s}(t)|^2dt],$$

where $B=M_2[T+\int_0^T|\dot{s}(t)|^2dt]$. Therefore, taking t_1 as the initial time and $u(\cdot, t_1)$ as the initial data and repeating the same argument in § 5 and § 6, we obtain

$$(7.5)\qquad \begin{aligned}\Phi^{t_2}-\Phi^{t_1}&\leqq\Phi^{t_1}[1+\exp(2B)]M_2[t_2-t_1+\int_{t_1}^{t_2}|\dot{s}(t)|^2dt]\\ &\leqq\Phi^0[1+\exp(2B)]^2M_2[t_2-t_1+\int_{t_1}^{t_1}|\dot{s}(t)|^2dt]\end{aligned}$$

by noting (7.4) and (3.10). Thus we get (3.11).

(ii). We may repeat the argmuents used in the case (i) by defining $\phi_j=\phi^h(x_j)$ in (4.7), where $\phi^h(x)$ is a function introduced in the following lemma. q.e.d.

Lemma 7.3. *Let* $\{l, \phi(x)\}$ *satisfy* (A.1) *and* (A.2). *Then there exists* $\{l^h, \phi^h(x)\}_h$ *such that* $\{l^h, \phi^h\}$ *satisfies* (A.1), (A.2),

$$\lim_{h\to 0} l^h = l \qquad (l^h \text{ is a rational number}),$$

$$\phi^h(0) = \phi(0), \quad \phi^h(1) = \phi(1),$$

$$\lim_{h\to 0}\phi^h(x) = \phi(x) \qquad \text{in } C([0, 1]),$$

$$\lim_{h\to 0}\int_0^1\phi_x^h(x)^2dx = \int_0^1\phi_x(x)^2dx$$

Proof. Let $\alpha(x)$ be a cut-off function such that $\alpha(x)\in C^{\infty}([0, 1])$, $\alpha\equiv 1$ $(l/2\leqq x\leqq 1-l/2)$, $\alpha\equiv 0$ $(0\leqq x\leqq l/4,\ 1-l/4\leqq x\leqq 1)$, We define $\{l^h, \phi^h\}$ by

$$\phi^h(x) = \alpha(x)\phi(x-\varepsilon_h)+(1-\alpha(x))\phi(x)\,, \quad l^h = l+\varepsilon_h\,,$$

where $\{\varepsilon_h\}_h$ are numbers such that $l+\varepsilon_h$ is a rational number and $\varepsilon_h\to 0$ as $h\to 0$. q.e.d.

Proof of Proposition 3.4. It follows from (3.16) and (3.18) that $s^n(t)\to s(t)$ $\in C([0, T])$ uniformly on $[0, T]$ as $n\to\infty$. On the other hand we see from (3.10), (3.12), (3.16) and (3.17) that

$$(7.6)\qquad |u^n(x', t')-u^n(x, t)|\leqq K'(|x'-x|^{1/2}+|t'-t|^{1/4}) \qquad \text{on } \bar{Q}_T\,,$$

$$(7.7)\qquad \sup_{0\leqq t\leqq T}\{\int_0^1 u_x^n(x, t)^2dx\}+c_2\int_0^T\int_0^1 u_t^n(x, t)^2dxdt\leqq K'\,,$$

where K' is a constant. We get

$$(7.8)\qquad |u^n(x, t)|\leqq\max(|H_0|, |H_1|, ||\phi||_{L^\infty(0,1)}) \qquad \text{on } \bar{Q}_T\,,$$

by Proposition 8.2 which we prove later. Therefore it follows from (7.6), (7.8) and the Ascoli-Arzelà's theorem that there exist subsequence of $\{u^n\}$ (which we denote again by the same symbol), and a function $u(x, t)\in C(\bar{Q}_T)$ such that $u^n(x, t)\to u(x, t)$ in $C(\bar{Q}_T)$ as $n\to\infty$. We shall examine that u is the solution of (M) corresponding to the curve $s(t)$ and the data $\phi(x)$. We get (3.4) using $u^n(s^n(t), t)=0$ $(0\leqq t\leqq T)$ and $s^n\to s$ in $C([0, T])$ as $n\to\infty$. We note $u_{xx}^n-c_0u_t^n=0$ $(0<x<s^n(t),\ 0<t\leqq T)$ and $u_{xx}^n-c_1u_t^n=0$ $(s^n(t)<x<1,\ 0<t\leqq T)$. Thus it is easily seen that $u_{xx}-c_0u_t=0$ in D_T^0 and $u_{xx}-c_1u_t=0$ in D_T^1 in the distribution sense. Hence we have $u\in C^\infty(D_T)\cap C(\bar{Q}_T)$ and (3.1), (3.2) by the well-known result concerning the heat equation. We will show (3.3). It follows from Lemma 5.1 and the proof of [34, Lemma 8.4] that

$$\int_{t_1}^{t_2}\int_0^{s^n(t)} u_x^n(w-u^n)_x dxdt+\int_{t_1}^{t_2}\theta_0(\eta)dt-\int_{t_1}^{t_2}\theta_0(u^n(0, t))dt$$
$$\leqq -\int_{t_1}^{t_2}\int_0^{s^n(t)} u_{xx}^n(w-u^n)dxdt\,,$$

where $\eta\in D(\theta_0)$, $d'=d/2$,

$$w(x) = \begin{cases} \eta(1-x/d') & \text{for } 0\leqq x\leqq d'\,, \\ 0 & \text{for } x\geqq d'\,. \end{cases}$$

Hence it is easily seen from $u_{xx}^n=c_0u_t^n$, $u^n\to u$ in $C(\bar{Q}_T)$ as $n\to\infty$, $s^n\to s$ in $C([0, T])$ as $n\to\infty$, and (7.7) that

$$\int_{t_1}^{t_2}\int_0^{s(t)} u_x(w-u)_x dxdt+\int_{t_1}^{t_2}\theta_0(\eta)dt-\int_{t_1}^{t_2}\theta_0(u(0, t))dt$$
$$\geqq -\int_{t_1}^{t_2}\int_0^{s(t)} u_{xx}(w-u)dxdt\,.$$

Therefore we get (3.3) (a) by the arguments used in the last part of the proof of [34, Lemma 8.4]. We obtain (3.3) (b) in the same way. We get (3.5) using $u^n(x, 0)=\phi(x)$ and $u^n\to u$ in $C(\overline{Q}_T)$ as $n\to\infty$. Consequently we obtain (3.19), since the solution of (M) corresponding to the curve $s(t)$ and the data $\phi(x)$ is unique by Proposition 8.1 which we prove later. q.e.d.

Proof of Proposition 3.5. It follows from (3.20), (3.21), (3.22), (3.23), Proposition 3.1, Lemma 2.2 and the proof of Proposition 3.4 that there exists a subsequence of $\{u^n\}$ (which we denote again by the same symbol) and a function $u(x, t)\in C(\overline{Q}_T-\{t=0\})$ satisfying $u^n\to u$ in $C(\overline{Q}_{\sigma,T})$ as $n\to\infty$ for any $\sigma\in]0, T[$. We shall show that

(7.9) $$u(x, t)\to\phi(p) \qquad \text{as } (x, t)\to(p, 0)$$

for a.e. $p\in[0, 1]$. We use the tool of barriers (see [14, p. 70]). We introduce a barrier

$$w(x, t)=(x-p)^2+3c_2^{-1}t$$

at $(p, 0)$. We note that

(7.10) $$|u^n(x, t)|\leqq\max(|H_0|, |H_1|, K)\equiv K_1 \qquad \text{on } \overline{Q}_T$$

by (3.23) and Proposition 8.2 which we prove later. Let ε denote any arbitrary small positive number. We see from (3.24) that

(7.11) $$|\phi^n(x)-\phi(p)|<\varepsilon \qquad \text{on } [p-\delta_p, p+\delta_p]\equiv I_p$$

for sufficiently large n. Therefore it is easily seen from the maximum principle, (7.10) and (7.11) that

$$|u^n(x, t)-\phi(p)|\leqq K_2 w(x, t)+\varepsilon \qquad \text{on } I_p\times[0, T]\,,$$

where $K_2=2\delta_p^{-1}K_1$. Taking $n\to\infty$, we have

$$|u(x, t)-\phi(p)|\leqq K_2 w(x, t)+\varepsilon \qquad \text{on } I_p\times]0, T]\,.$$

Hence we obtain (7.9) easily. Consequently u is a solution of (M) corresponding to the curve $s(t)$ and the data $\phi(x)$. Thus we get (3.25) using the uniqueness of the solution of (M). q.e.d.

8. Comparison theorems for the moving boundary problem

We shall show the comparison theorems for the moving boundary problem

(M) stated in § 3. The following results are obtained using the proof of [34, Lemma 10.1, Proplsition 10.1 and Proposition 10.2].

Lemma 8.1. *For a given function* $r(t)\in C([0, T])\cap H^1_{\mathrm{loc}}(]0, T])$ *and* $i\in\{0, 1\}$, *let* $p(x, t)$ *and* $q(x, t)$ *be functions satisfying*

(8.1) $p, q\in C^\infty(D^i_T)\cap C(\bar{D}^i_T-Z^i_0)\cap L^\infty(\bar{D}^i_T)$

(8.2) $\iint_{D^i_T\cap\{t\geqq\sigma\}}(p_t^2+q_t^2)dxdt<\infty$ *for each* $\sigma\in]0, T]$,

(8.3) $p_{xx}-c_ip_t=0$ *in* D^i_T, $q_{xx}-c_iq_t=0$ *in* D^i_T ,

(8.4) $p(x, 0)\geqq q(x, 0)$ *for all* $x\in\{\xi\in[0, 1]; (-1)^i(\xi-\kappa)\leqq 0\}$,

(8.5) $p(r(t), t)\geqq q(r(t), t)$ *for all* $t\in]0, T]$,

(8.6) $(-1)^i(q_x(i, t)-p_x(i, t))(q(i, t)-p(i, t))^+\geqq 0$ *a.e.* $t\in]0, T]$,

where D^i_T, $\bar{D}^i_T$ *are sets defined by* (1.5) *and* (1.6) *with* $r(t)$ *instead of* $s(t)$ *respectively,*

$$\alpha^+=\max(\alpha, 0)\,,\quad 0<r(0)=\kappa<1\,,$$
$$Z^i_0=\tilde{Z}^i_0\times\{0\}\subset[0, 1]\times\{0\}\,,$$

and $\tilde{Z}^i_0$ *is a set of zero measure in* $\boldsymbol{R}^1$. *Then we have* $p(x, t)\geqq q(x, t)$ *on* $\bar{D}^i_T$.

Proposition 8.1. *Let* u_1 *and* u_2 *be solution of (M) under the assumptions of Proposition* 3.1 *corresponding, respectively, to the pairs of the moving boundary and the initial data* $\{s_1(t), \phi_1(x)\}$ *and* $\{s_2(t), \phi_2(x)\}$. *Suppose that*

$$s_1(t)\leqq s_2(t)\quad \textit{for all } t\in[0, T]\,,$$
$$\phi_1(x)\leqq\phi_2(x)\quad \textit{for all } x\in[0, s_1(0)]\cup[s_2(0), 1]\,,$$
$$u_1(s_1(t), t)\leqq u_2(s_1(t), t)\quad \textit{for all } t\in]0, T]\,,$$
$$u_1(s_2(t), t)\leqq u_2(s_2(t), t)\quad \textit{for all } t\in]0, T]\,.$$

Then we have

$$u_1(x, t)\leqq u_2(x, t)\quad \textit{on } \{(x, t)\in\bar{Q}_T;\ x\leqq s_1(t) \textit{ or } x\geqq s_2(t)\}\,.$$

REMARK 8.1. Proposition 8.1 implies the uniqueness of the solution of the moving boundary problem (M).

Propositoin 8.2. *Let* u *be a solution of (M) under the assumption of Proposition* 3.1. *Then we have*

$$|u(x, t)|\leqq\max(|H_0|, |H_1|, \|\phi\|_{L^\infty(0,1)})\quad \textit{on } \bar{Q}_T\,.$$

Proposition 8.3. *Let* u *be a solution of (M) under the assumption of Proposition* 3.1. *Suppose that*

$$H_0 \geqq 0, \quad H_1 \leqq 0,$$
$$\phi(x) \geqq 0 \quad (0 \leqq x < l), \qquad \phi(x) \leqq 0 \quad (l < x \leqq 1).$$

Then we have

$$0 \leqq u(x, t) \leqq \max(H_0, \|\phi\|_{L^\infty(0,l)}) \qquad on\ \bar{D}_T^0,$$
$$0 \geqq u(x, t) \geqq \min(H_1, -\|\phi\|_{L^\infty(l,1)}) \qquad on\ \bar{D}_T^1.$$

9. Reformation of the Stefan's condition

We shall state a useful result concerning the reformation of the Stefan's condition (0.7) in this section. To begin with, we refer to the following results concerning the behavior of the derivative of a solution of the heat equation which vanishes on a non-smooth boundary curve due to Cannon-Henry-Kotlow [7, Theorem 2.2] (see also [30, p. 10]).

Lemma 9.1. *Let $s(t)$ be such that $s(t) \geqq \delta > 0$ $(T_1 \leqq t \leqq T_2)$ and $s \in C^{0,\alpha}([T_1, T_2])$, where $\alpha > 1/2$. Let $v(x, t)$ be the solution of the moving boundary problem*

$$v_{xx} - v_t = 0 \qquad (0 < x < s(t),\ T_1 \leqq t \leqq T_2),$$
$$v(0, t) = f(t) \qquad (T_1 \leqq t \leqq T_2),$$
$$v(x, T_1) = \psi(x) \qquad (0 \leqq x \leqq s(T_1)),$$
$$v(s(t), t) = 0 \qquad (T_1 \leqq t \leqq T_2),$$

where $f(t) \in C([T_1, T_2])$, $\psi(x) \in C^{0,1}([0, s(T_1)])$ and $f(T_1) = \psi(0)$, $\psi(s(T_1)) = 0$. Then $v_x(x, t)$ converges to a limit $v_x(s(t)-0, t)$ uniformly on $[\tau, T_2]$ for any $\tau > 0$ as $x \to s(t)$, and $v_x(s(t)-0, t) \in L^\infty(]T_1, T_2]) \cap C(]T_1, T_2])$.

REMARK 9.1. If $\alpha = 1$, Geverey [15] has given this result.

Now we state the proposition.

Proposition 9.1. *Let $s(t) \in C([0, T]) \cap H^1_{\mathrm{loc}}(]0, T]) \cap C^{0,\alpha}(]0, T])$ with $\alpha > 1/2$, and $d \leqq (s(t) \leqq 1-d$ $(0 \leqq t \leqq T)$ for some $d \geqq 0$, and $u(x. t)$ be a solution of the moving boundary problem (M). Then the following two conditions are equivalent.*

(i) *$s(t) \in C^\infty([]0, T])$ and it satisfies the Stefan's condition*

$$b\dot{s}(t) = -u_x^-(s(t), t) + u_x^+(s(t), t) \qquad (0 < t \leqq T).$$

(ii) $bs(t_2) - bs(t_1)$

$$= \int_{t_1}^{t_2} [-u_x(d, t) + u_x(1-d, t)]dt - c_0 \int_d^{s(t_2)} u(x, t_2)dx$$
$$-c_1 \int_{s(t_2)}^{1-d} u(x, t_2)\,dx + c_0 \int_d^{s(t_1)} u(x, t_1)dx$$
$$+c_1 \int_{s(t_1)}^{1-d} u(x, t_1)dx \qquad for\ any\ 0 < t_1 < t_2 \leqq T.$$

Proof. We put

$$D^0=\{(x,\ t);\ d\leqq x\leqq s(t),\ t_1\leqq t\leqq t_2\}\ ,$$
$$D^1=\{(x,\ t);\ s(t)\leqq x\leqq 1-d,\ t_1\leqq t\leqq t_2\}\ .$$

It follows from the definition of the solution of (M) that

$$\int_{t_1/2}^{t_2}\int_0^{s(t)} u_{xx}{}^2 dxdt+\int_{t_1/2}^{t_2}\int_{s(t)}^1 u_{xx}{}^2 dxdt<\infty\,.$$

Therefore we see that there exists a time t_1' satisfying $t_1/2<t_1'<t_1$ and $u(\cdot,\ t_1')\in C^{0,1}([0,\ 1])$. Therefore we get $u_x(x,\ t)\in C(D^0)\cap C(D^1)$ by $s\in C^{0,\alpha}([t_1,\ t_2])$ and Lemma 9.1. Hence we can apply the Green's theorem to $u_{xx}-c_iu_t=0$ in D^i by virtue of $s\in H^1(t_1,\ t_2)$ (see e.g. [22, p. 144]). We get

$$(9.1)\qquad 0=\iint_{D^0}(u_{xx}-c_0u_t)\,dxdt=\iint_{\partial D^0}c_0udx+u_xdt\ ,$$

$$(9.2)\qquad 0=\iint_{D^1}(u_{xx}-c_1u_t)\,dxdt=\iint_{\partial D^1}c_1udx+u_xdt\,.$$

Hence we obtain by using $u(s(t),\ t)=0$

$$\begin{aligned}(9.3)\qquad &\int_{t_1}^{t_2}[-u_x^-(s(t),\ t)+u_x^+(s(t),\ t)]dt\\ &=\int_{t_1}^{t_2}[-u_x(d,\ t)+u_x(1-d,\ t)]dt-c_0\int_d^{s(t_2)}u(x,\ t_2)dx\\ &\quad -c_1\int_{s(t_2)}^{1-d}u(x,\ t_2)dx+c_0\int_d^{s(t_1)}u(x,\ t_1)dx+c_1\int_{s(t_1)}^{1-d}u(x,\ t_1)dx\,.\end{aligned}$$

Now it is obvious from (9.3) that (i) implies (ii). We shall show that (ii) implies (i). We have

$$b[s(t_2)-s(t_1)]=\int_{t_1}^{t_2}[-u_x^-(s(t),\ t)+u_x^+(s(t),\ t)]dt$$

by (ii) and (9.3). Hence we get

$$b\dot{s}(t)=-u_x^-(s(t),\ t)+u_x^+(s(t),\ t)\in C(]0,\ T])$$

using Lemma 9.1 and arbitrariness of t_1, t_2. Moreover it can be shown that $s(t)\in C^\infty(]0,\ T])$ by virtue of Schaeffer [28]. q.e.d.

10. Existence of a local solution of (S)

In this section we show the existence of a local solution of (S) under the conditions of (A.1) and (A.2).

Proposition 10.1. *Let $\{l,\ \phi\}$ satisfy* (A.1) *and* (A.2). *Then there exists a* *solution* (s, u) *of* (S) *on* $[0, T]$ *such that* $\dot{s}(t)\in L^4(0, T)$, *where*

$$(10.1)\quad \begin{aligned} &0<T=\min(\delta^2, 16^{-1}b^2C[\varepsilon_0]^{-1}C_2^{-1}),\\ &\delta=2^{-1}\min(l, 1-l),\\ &C[\varepsilon] \text{ is is the function of } \varepsilon \text{ from Lemma 2.1 (ii) (with } \delta\leqq\beta-\alpha\leqq 1),\\ &\varepsilon_0=4^{-1}b^2c_2(c_0{}^2+c_1{}^2)^{-1}C_2^{-1},\\ &C_2=\Phi^0\{1+\exp[4(2/\delta+c_3)]\}.\end{aligned}$$

REMARK 10.1. T is restricted to be small from the following two reasons. For one, we assure that the curve $s(t)$ starting at l does not hit the side $x=0$ or $x=1$ for $0\leqq t\leqq T$. For the other, we guarantee that the mapping H (defined below) preserves the unit ball in $L^2(0, T)$.

We shall prove the proposition above using several lemmas. We employ the method which is analogous to that of Evans [10, § 3]. We discover a curve $s(t)$ for which the function $u(x, t)$ provided by Proposition 3.3 satisfies not only (0.1)–(0.6) but also (0.7).

Let us denote by B the closed unit ball in the space $L^2(0, T)$. Notice that if $s(0)=l$ and $\dot{s}\in B$, then $0<\delta\leqq s(t)\leqq 1-\delta<1$. This fact follows immediately from the definition of δ and the estimate

$$|s(t)-l|\leqq t^{1/2}\|\dot{s}\|_{L^2(0,T)}\leqq T^{1/2}\leqq\delta.$$

DEFINITION 10.1. For $r(t)\in B$, define

$$H(r)(t)=b^{-1}(-u_x^-(s(t), t)+u_x^+(s(t), t))\qquad\text{for a.e. } t\in[0, T],$$

where $s(t)=l+\int_0^t r(\tau)d$ and u is the solution of the moving boundary problem (M) considered in Proposition 3.3.

Lemma 10.1.

(i) $H(r)(\cdot)$ is *measurable on* $[0, T]$.

(ii) $H(B)\subset B$.

Proof. (i) It follows from Proposition 3.3 that $u\in C^\infty(D_T)$ and (3.8) holds. Hence the map $t\rightarrow u_x(s(t)\pm 1/n, t)$ is continuous and $u_x(s(t)\pm 1/n, t)\rightarrow u_x^\pm(s(t), t)$ as $n\rightarrow\infty$ for a.e. $t\in[0, T]$. Therefore $H(r)(\cdot)$ is measurable.

(ii) Choose $r\in B$ and set $s(t)=l+\int_0^t r(\tau)d\tau$. Then

$$\int_0^T |H(r)(t)|^2dt\leqq 2b^{-2}\int_0^T(|u_x^-(s(t), t)|^2+|u_x^+(s(t), t|^2)dt$$

$$\leqq 2b^{-2}\int_0^T(\|u_x\|^2_{L^\infty(0,s(t))}+\|u_x\|^2_{L^\infty(s(t),1)})dt$$

$$\leqq 2b^{-2}(\varepsilon_0\int_0^T ||u_{xx}||^2_{L^\infty(0,s(t))}dt+TC[\varepsilon_0]\sup_{0\leqq t\leqq T}||u_x||^2_{L^2(0,s(t))}$$

$$+\varepsilon_0\int_0^T ||u_{xx}||^2_{L^2(s(t),1)}dt+TC[\varepsilon_0]\sup_{0\leqq t\leqq T}||u_x||^2_{L^2(s(t),1)})$$

$$\leqq 2b^{-2}\varepsilon_0(c_0{}^2+c_1{}^2)\int_0^T ||u_t||^2_{L^2(0,1)}dt+4b^{-2}TC[\varepsilon_0]\sup_{0\leqq t\leqq T}||u_x||^2_{L^2(0,1)}$$

$$\leqq 2b^{-2}\varepsilon_0(c_0{}^2+c_1{}^2)c_2^{-2}C_2+8b^{-2}TC[\varepsilon_0]C_2$$

$$\leqq 1/2+1/2\leqq 1\,,$$

by Lemma 2.1 (ii), (0.2), (0.3), Proposition 3.3 and (10.1). q.e.d.

We shall examine the hypotheses of Schauder's fixed point theorem for the mapping $H\colon B\to B$.

Lemma 10.2. *$H\colon B\to B$ is continuous and $H(B)\subset B$ is precompact in $L^2(0, T)$.*

Proof. We prove the continuity and compactness at the same time.

Suppose $r^n(t)\in B$ ($n=1, 2, \cdots$) and define $s^n(t)=l+\int_0^t r^n(\tau)d\tau$, then $0<\delta\leqq s^n(t)\leqq 1-\delta<1$ for $0\leqq t\leqq T$ and each n. Let u^n be the unique solutions of (M) associated with the curve $s^n(t)$. By Proposition 3.3 there are bounds, independent of n, on the following quantities,

(10.2) $\quad ||u^n||_{L^2(Q_T)}\,,$

(10.3) $\quad \int_0^T ||u_t^n||_{L^2(0,T)}dt\,.$

Since $L^2(0, T)$ is a Hilbert space, there exists a sequence (which we also denote by r^n) such that $r^n(t)$ converges weakly to some $r(t)\in B$. Set $s(t)=l+\int_0^t r(\tau)d\tau$. It holds that $s^n\to s$ unifomly on $[0, T]$ by Ascoli-Arzela's theorem. Thus we see that $u^n\to u$ uniformly on $\overline{Q}_T$ by Propositon 3.4, where u is the solution of (M) associated with the curve $s(t)$.

We now prove $H(r^n)\to H(r)$ in $L^2(0, T)$. For a.e. fixed $0\leqq t\leqq T$ there are following three cases.

(a) $s^n(t)<s(t)$. We get

$$|H(r^n)(t)-H(r)(t)|^2$$

$$\leqq b^{-2}|-u_x^{n-}(s^n(t), t)+u_x(s^n(t), t)-u_x^+(s(t), t)+u_x^n(s(t), t)$$

$$+u_x^-(s(t), t)-u_x(s^n(t), t)+u_x^{n+}(s^n(t), t)-u_x^n(s(t), t)|^2$$

$$\leqq 4b^{-2}(||u_x^n-u_x||^2_{L^\infty(0,s^n(t))}+||u_x^n-u_x||^2_{L^\infty(s(t),1)})$$

$$+4b^{-2}(|\int_{s^n(t)}^{s(t)}u_{xx}dx|^2+|\int_{s^n(t)}^{s(t)}u_{xx}^n dx|^2)$$

$$\leqq C(1+\|u^n-u\|^2_{L^\infty(0,s^n(t))}+\|u^n_{xx}-u_{xx}\|^2_{L^2(0,s^n(t))}$$
$$+\|u^n-u\|^2_{L^\infty(s(t),1)}+\|u^n_{xx}-u_{xx}\|^2_{L^2(s(t),1)})(\sup_{0\leqq t\leqq T}\|u^n-u\|^{2/3}_{L^\infty(0,1)})$$
$$+C(\sup_{0\leqq t\leqq T}|s^n(t)-s(t)|)(\int_{s^n(t)}^{s(t)}(u_{xx}{}^2+u^n_{xx}{}^2)dx)$$
$$\leqq C(1+\|u^n-u\|^2_{L^\infty(0,1)}+\|u^n_t-u_t\|^2_{L^2(0,1)})(\sup_{0\leqq t\leqq T}\|u^n-u\|^{2/3}_{L^\infty(0,1)})$$
$$+C(\sup_{0\leqq t\leqq T}|s^n(t)-s(t)|)(\|u_t\|^2_{L^2(0,1)}+\|u^n_t\|^2_{L^2(0,1)})$$

by Lemma 2.1 (i), (0.2) and (0.3). Hence we obtain

$$(10.4)\qquad |H(r^n)(t)-H(r)(t)|^2$$
$$\leqq C(1+\|u^n_t\|^2_{L^2(0,1)}+\|u_t\|^2_{L^2(0,1)})(\|u^n-u\|^{2/3}_{L^\infty(Q_T)}+\|s^n-s\|_{L^\infty(0,T)})$$

by (10.2).

(b) $s(t)<s^n(t)$. We can get (10.4) in the same way.

(c) $s(t)=s^n(t)$. We get

$$|H(r^n)(t)-H(r)(t)|^2$$
$$\leqq b^{-2}|-u^{n-}_x(s^n(t),t)+u^-_x(s(t),t)-u^+_x(s(t),t)+u^{n+}(s^n(t),t)|^2$$
$$\leqq 2b^{-2}(\|u^n_x-u_x\|^2_{L^\infty(0,s(t))}+\|u^n_x-u_x\|^2_{L^\infty(s(t),1)})\,.$$

Thus we get (10.4) by the argument used in the case (a).

Consequently it follows (10.3) and (10.4) that

$$\int_0^T|H(r^n)(t)-H(r)(t)|^2dt$$
$$\leqq C(\|u^n-u\|^{2/3}_{L^\infty(Q_T)}+\|s^n-s\|_{L^\infty(0,T)})\,.$$

Therefore we complete the proof, since $u^n\to u$ in $C(\overline{Q}_T)$, $s^n\to s$ in $C([0,T])$ as $n\to\infty$. q.e.d.

By Lemma 10.2 and Schauder's fixed point thoerem, $H\colon B\to B$ has at least one fixed point r. We shall show that $(s(t), u(x,t))$ is a solution of (S) on $[0,T]$, where $s(t)=l+\int_0^t r(\tau)d\tau$ and $u(x,t)$ is the solution of (M) associated with the curve $s(t)$. We may examine the Stefan's condition (0.7).

Lemma 10.3. *$s(t)\in C^{0,3/4}([0,T])\cap C^\infty(]0,T])$, $\dot{s}(t)\in L^4(0,T)$, $u(x,t)\in C(\overline{Q}_T)\cap C^\infty(S^0_T)\cap C^\infty(S^1_T)$ and the Stefan's condition* (0.7) *is satisfied.*

Proof. We get

$$\int_0^T|\dot{s}(t)|^4dt\leqq C(1+\int_0^T\int_0^1 u_t^2dxdt)<\infty\,,$$

by (0.7), Lemma 2.1 (iii), (0.2), (0.3) and (3.10). Thus we have $\dot{s}(t)\in L^4(0,T)$

and so $s(t)\in C^{0,3/4}([0, T])$. Therefore it follows from Lemma 9.1 that $u_x^-(s(t), t)$, $u_+^x(s(t), t)\in C([0, T])$, since there exists t_0 such that $u(\cdot, t_0)\in C^{0,1}([0, 1])$ and $0<t_0\leqq\varepsilon$ for any $\varepsilon\in]0, T]$ by (3.8). Hence we see from $H(r)=r$ that $s(t)\in C^1(]0, T])$ and $b\dot{s}(t)=-u_x^-(s(t), t)+u_x^+(s(t), t)$ $(0<t\leqq T)$. Consequently we see that $s(t)\in C^\infty(]0, T])$ using the result of Schaeffer [28]. q.e.d.

Thus we complete the proof of Proposition 10.1.

11. Continuation of solutions

We prepare fundamental propositions which are useful in the study of the continuation of solutions.

Proposition 11.1. *Let (s,u) be a solution of (M) on $[0, T]$ satisfying*

$$b\dot{s}(t) = -u_x^-(s(t), t)+u_x^+(s(t), t) \qquad a.e.\ t\in[0, T]\,.$$

Then (s, u) is a solution of (S) on $[0, T]$.

Proof. It is obvious from Propositoin 3.1 and the proof of Lemma 10.3. q.e.d.

Proposition 11.2. *Let (s_1, u_1) be a solution of (S) on $[0, T_1]$, and (s_2, u_2) be a solution of (S) on $[T_1, T_2]$ with the initial time T_1 and the initial data $\{s_1(T_1), u_1(\cdot, T_1)\}$. Assume that $s_1(t)\in H^1_{\mathrm{loc}}(]0, T_1])$ and $s(t)\in H^1(T_1, T_2)$. Then (s, u) is a solution of (S) on $[0, T_2]$, where*

$$s(t) = \begin{cases} s_1(t) & (0\leqq t\leqq T_1)\,, \\ s_2(t) & (T_1\leqq t\leqq T_2)\,, \end{cases}$$

$$u(x, t) = \begin{cases} u_1(x, t) & (0\leqq x\leqq 1,\ 0\leqq t\leqq T_1) \\ u_2(x, t) & (0\leqq x\leqq 1,\ T_1\leqq t\leqq T_2)\,. \end{cases}$$

Proof. We see from Proposition 3.1, 3.3 and the assumption $s(t)\in H^1_{\mathrm{loc}}(]0, T])$ that (s, u) is a solution of (M) on $[0, T_2]$. Hence (s,u) is a solution of (S) on $[0, T_2]$ by Proposition 11.1. q.e.d.

12. A priori estimates

We shall get a priori estimates to show the existence of a global solution of (S).

Proposition 12.1. *Let (s, u) be a solution of (S) on $[0, T]$ with $s(t)\in H^1(0, T)$ corresponding to the data $\{l, \phi\}$ satisfying the conditions (A'). Then we have*

$$(12.1) \qquad 2^{-1}\int_0^1 u_x(x, t)^2dx+\theta_0(u(0, T))+\theta_1(u(1, T))$$

$$+2^{-1}b^2\int_0^T|\dot{s}(t)|^3dt+c_2\int_0^T\int_0^1 u_t(x,t)^2dxdt$$
$$\leqq 2^{-1}\int_0^1\phi_x(x)^2dx+\theta_0(\phi(0))+\theta_1(\phi(1)).$$

Proposition 12.2. *Under the assumptions of Proposition* 12.1, *we have*

$$(12.2)\qquad 2^{-1}T\int_0^1 u_x(x,t)^2dx+T\theta_0(u(0,T))+T\theta_1(u(1,T))$$
$$+2^{-1}b^2\int_0^T t|\dot{s}(t)|^3dt+c_2\int_0^T\int_0^1 tu_t^2dxdt$$
$$\leqq 2^{-1}\int_0^T\int_0^1 u_x(x,t)^2dxdt+\int_0^T\theta_0(u(0,t))dt+\int_0^T\theta_1(u(1,t))dt.$$

REMARK 12.1. We shall use (12.1) and (12.2) in the proof of Theorem 2 and Theorem 1 respectively.

We prepare several lemmas to prove the propositions above. We suppose the assumptions in Proposition 12.1 throughout this section. We can regard u as the solution of the moving boundary problem (M) associated with the curve $s(t)$. We note that we can use Proposition 3.3.

Lemma 12.1. *Φ^t and $t\Phi^t$ are differentiable at a.e. $t\in[0,T]$, integrable on $[0,T]$ and*

$$(12.3)\qquad \Phi^T-\Phi^0\leqq\int_0^T d\Phi^t/dt\,dt$$

$$(12.4)\qquad T\Phi^T\leqq\int_0^T t\,d\Phi^t/dt\,dt+\int_0^T\Phi^t\,dt.$$

Proof. Three exists a constant K such that

$$(12.5)\qquad \Phi^{t_2}-\Phi^{t_1}\leqq K(t_2-t_1+\int_{t_1}^{t_2}\dot{s}(t)^2dt)\qquad(0\leqq t_1\leqq t_2\leqq T),$$

by (3.11). Hence we have

$$(12.6)\qquad t_2\Phi^{t_2}-t_1\Phi^{t_1}\leqq K'(t_2-t_1+\int_{t_1}^{t_2}\dot{s}(t)^2dt),$$

where $K'=TK+\sup_{0\leqq s\leqq T}\Phi^t\,(\leqq\Phi^0+K(2T+\int_0^T\dot{s}(t)^2dt))$. It follows from (12.5) and (12.6) that $\Phi^t-K\int_0^t(1+\dot{s}(\tau)^2)d\tau$ and $t\Phi^t-K'\int_0^t(1+\dot{s}(\tau))^2d\tau$ are bounded and nonincreasing on $[0,T]$. Consequently we have (12.3) and (12.4) using the wellknown theorems concerning Lebesgue's integral. q.e.d.

We shall calculate $d\Phi^t/dt$. We put $a=2^{-1}\min\{\min(s(t),1-s(t));\ 0\leqq t\leqq T\}$, (12.7) $\Phi_i^t = 2^{-1}\int_a^{1-a} u_x(x, t)^2 dx$

(12.8) $\Phi_b^t = 2^{-1}(\int_0^a u_x(x, t)^2 dx + \int_{1-a}^1 u_x(x, t)^2 dx) + \theta_0(u(0, t)) + \theta_1(u(1, t))$.

Lemma 12.2. $\Phi_i^t \in C^1(]0, T])$, Φ_b^t *is differentiable at a.e.* $t \in [0, T]$ *and it holds that*

(12.9) $d\Phi^t/dt = d\Phi_i^t/dt + d\Phi_b^t/dt \qquad$ *a.e.* $t \in [0, T]$.

Proof. We get $\Phi_i^t \in C^1(]0, T])$ by the following lemma. Hence we get the conclusion from Lemma 12.1. q.e.d.

Lemma 12.3. $\Phi_i^t \in C^1(]0, T])$ *and*

(12.10)
$$d\Phi_i^t/dt \leqq -2^{-1}b^2|\dot{s}(t)|^3 - (c_0\int_a^{s(t)} u_t^2 dx + c_1\int_{s(t)}^{1-a} u_t^2 dx) - u_x(a, t)u_t(a, t) + u_x(1-a, t)u_t(1-a, t).$$

Proof. We see that $u \in C^\infty(S_T^0) \cap C^\infty(S_T^1)$ and $s \in C^\infty(]0, T])$ by Definition 1.1. Thus we get $\Phi_i^t \in C^1(]0, T])$. We note that

(12.11) $u(s(t), t) = 0 \qquad (0 \leqq t \leqq T)$,

(12.12) $u_x^-(s(t), t),\ u_x^+(s(t), t) \leqq 0 \qquad (0 < t \leqq T)$

(12.13) $b\dot{s}(t) = -u_x^-(s(t). t) + u_x^+(s(t), t) \qquad (0 < t \leqq T)$,

where we used Proposition 8.3 to have (12.12). Thus we get

(12.14) $u_t^\pm(s(t), t) = -u_x^\pm(s(t), t)\dot{s}(t) \qquad (0 < t \leqq T)$,

(12.15) $-b|\dot{s}(t)| \geqq u_x^-(s(t), t) + u_x^+(s(t), t) \qquad (0 < t \leqq T)$.

We have

$$\begin{aligned} d\Phi_i^t/dt &= 2^{-1}d(\int_a^{s(t)} u_x(x, t)^2 dx + \int_{s(t)}^{1-a} u_x(x, t)^2 dx)/dt \\ &= 2^{-1}(u_x^-(s(t), t)^2\dot{s}(t) + 2\int_a^{s(t)} u_x u_{xt} dx - u_x^+(s(t), t)^2\dot{s}(t) + 2\int_{s(t)}^{1-a} u_x u_{xt} dx) \\ &= 2^{-1}(u_x^-(s(t), t)^2\dot{s}(t) - u_x^+(s(t), t)^2\dot{s}(t)) + [u_x u_t]_{x=a}^{x=s(t)-0} - \int_a^{s(t)} u_{xx}u_t dx \\ &\quad + [u_x u_t]_{x=s(t)+0}^{x=1-a} - \int_{s(t)}^{1-a} u_{xx}u_t dx \\ &= 2^{-1}(u_x^+(s(t), t)^2 - u_x^-(s(t), t)^2)\dot{s}(t) - c_0\int_a^{s(t)} u_t^2 dx - c_1\int_{s(t)}^{1-a} u_t^2 dx \\ &\quad - u_x(a, t)u_t(a, t) + u_x(1-a, t)u_t(1-a, t) \end{aligned}$$

by (0.2), (0.3) and (12.14). Consequently we obtain (12.10), since we have

$$(u_x^+(s(t), t)^2 - u_x^-(s(t), t)^2)\dot{s}(t)$$
$$= (u_x^+(s(t), t) - u_x^-(s(t), t))(u_x^+(s(t), t) + u_x^-(s(t), t))\dot{s}(t)$$
$$\leqq -b^2 |\dot{s}(t)|^3$$

by (0.7) and (12.15). q.e.d.

We shall calculate $d\Phi_b^t/dt$. We prepare a simple lemma.

Lemma 12.4. *It holds that for a.e.* $t \in [0, T]$

$$\lim_{h\to 0} \int_0^1 [(u(x, t) - u(x, t-h))/h - u_t(x, t)]^2 dx = 0 .$$

Proof. We see from Proposition 3.3 that

$$\int_0^T \int_0^1 u_t(x, t)^2 dx dt < \infty . \tag{12.16}$$

We define $u(t)$: $[0, 1] \to L^2(0, 1)$ by $u(t) = u(\cdot, t)$. It follows from [29, Proposition 8.3] and [4, Proposition A.7] that $u(t)$ is absolutely continuous as a $L^2(0, 1)$-valued function. Thus we get the conclusion easily. q.e.d.

Lemma 12.5. *It holds that for a.e.* $t \in [0, T]$

$$d\Phi_b^t/dt \leqq -c_0 \int_0^a u_t^2 dx - c_1 \int_{1-a}^1 u_t^2 dx \tag{12.17}$$
$$+ u_x(a, t)u_t(a, t) - u_x(1-a, t)u_t(1-a, t) .$$

Proof. For a.e. $t \in [0, T]$, we have

$$2^{-1} \int_0^a u_x(x, t-h)^2 dx + \theta_0(u(0, t-h)) - (2^{-1} \int_0^a u_x(x, t)^2 dx + \theta_0(u(0, t))) \tag{12.18}$$
$$\geqq \int_0^a u_x(x, t)(u_x(x, t-h) - u_x(x, t))dx + \theta_0(u(0, t-h)) - \theta_0(u(0, t))$$
$$= -\int_0^a u_{xx}(x, t)(u(x, t-h) - u(x, t))dx + u_x(a, t)(u(a, t-h) - u(a, t))$$
$$-u_x(0, t)(u(0, t-h) - u(0, t)) + \theta_0(u(0, t-h)) - \theta_0(u(0, t))$$
$$= c_0 \int_0^a u_t(x, t)(u(x, t) - u(x, t-h))dx - u_x(a, t)(u(a, t) - u(a, t-h))$$
$$-u_x(0, t)(u(0, t-h) - u(0, t)) + \theta_0(u(0, t-h)) - \theta_0(u(0, t))$$
$$\geqq c_0 \int_0^a u_t(x, t)(u(x, t) - u(x, t-h))dx - u_x(a, t)(u(a, t) - u(a, t-h))$$

by (0.2), (0.4) (a) and $\partial\theta_0 = \gamma_0$. We have also

$$2^{-1} \int_{1-a}^1 u_x(x, t-h)^2 dx + \theta_1(u(1, t-h)) - (2^{-1} \int_{1-a}^1 u_x(x, t)^2 dx + \theta_1(u(1, t))) \tag{12.19}$$

$$\geqq c_1 \int_{1-a}^{1} u_t(x, t)(u(x, t)-u(x, t-h))dx$$
$$+u_x(1-a, t)(u(1-a, t)-u(1-a, t-h))$$

for a.e. $t \in [0, T]$ in the same way. Consequently we obtain (12.17) using (12.18), (12.19), Lemma 12.2, Lemma 12.4 and $u \in C^\infty(S_T^0) \cap C^\infty(S_T^1)$ q.e.d.

We get the following result by Lemma 12.2, 12.3 and 12.5.

Lemma 12.6.

(12.20) $$d\Phi^t/dt \leqq -2^{-1}b^2 |\dot{s}(t)|^3 - c_2 \int_0^1 u_t(x, t)^2 dx .$$

Proof of Proposition 12.1. It is obvious from (12.3) and (12.20). q.e.d.

Proof of Proposition 12.2. It is obvious from (12.4) and (12.20). q.e.d.

13. Proof of Theorem 2

We show the global existence of a solution of (S) corresponding to the data $\{l, \phi\}$ satisfying (A′).

Proposition 13.1. *Let* $\{l, \phi\}$ *satisfy* (A'). *Then there exists a solution* (T^*, s, u) *of* (S) *satisfying* (1.18), (1.19) *and* (1.20).

Proof. According to Proposition 10.1 a solution (s, u) of (S) on $[0, T]$ exists for some $T>0$, and therefore on a maximal interval $[0, T^*[$. If $T^*=\infty$, there is no problem.

If $T^*<\infty$, it follows from the a priori estimate (12.1) that $s(t) \to s^*$ as $t \to T^*$ for some s^*. If $0<s^*<1$, then the a priori estimate (12.1) and Proposition 10.1 and 11.2 allow us to extend the solution still. This contradicts the maximality of the time interval $[0, T^*[$. Hence $\lim_{t \uparrow T^*} s(t)=0$ or 1. Moreover we get (1.18), (1.19) and (1.20) by (12.1) and Lemma 2.2. q.e.d.

Proof of Theorem 2. It is obvious from Proposition 13.1 and Theorem 1 which we prove later. q.e.d.

14. Proof of Theorem 1

In this section we shall show the local existence of the solution of (S) under the condition (A), and give the proof of Theorem 1 and Theorem 3.

We approximate the given data satisfying (A) by the data satisfying (A)′.

Lemma 14.1. *Let the data* $\{l, \phi\}$ *satisfy the condition* (A). *Then there exists a sequence of the data* $\{l, \phi^n\}_{n \geqq 1}$ *satisfying* (A'), (14.1) $0 \leqq \phi^n(x) \leqq \max(\|\phi\|_{L^\infty(0,l)}, H_0)$,

(14.2) $\min(-\|\phi\|_{L^\infty(l,1)}, H_1) \leqq \phi^n(x) \leqq 0$,

(14.3) $\lim_{n\to\infty} \|\phi^n(\cdot)-\phi(p)\|_{C([p-\delta_p, p+\delta_p])} = 0$ *for a.e.* $p \in [0, 1]$

(δ_p is a *positive constant depending on* p).

Proof. Let $\{x_i^n\}_{0\leqq i\leqq 2n}$ $(=1, 2, \cdots)$ be a family of sets such that each x_i^n is a point of continuity of $\phi(x)$, $0=x_0^n<x_1^n<\cdots<x_{n-1}^n<x_n^n=l<x_{n+1}^n<\cdots<x_{2n-1}^n<x_{2n}^n=1$, $\lim_{n\to\infty}(\max_{1\leqq i\leqq 2^n} |x_i^1-x_{i-1}^n|)=0$, and $\{x_i^1\}_i \subset \{x_i^2\}_i \subset \cdots$. Define $\phi^n(x)$ as a piecewise linear function such that

$$(14.4)\quad \begin{cases} \phi^n(0) = H_0, \quad \phi^n(l) = 0, \quad \phi^n(1) = H_1, \\ \phi^n(x_i^n) = \phi(x_i^n) \qquad (i \neq 0, n, 2n). \end{cases}$$

q.e.d.

Let (T_n^*, s^n, n^n) be the unique solution of the Stefan problem (S) corresponding to the data $\{l, \phi^n\}$. This is well-defined, since $\{l, \phi^n\}$ satisfies (A') and we have Proposition 12.1. We shall show the several estimates which are independent of n.

The following lemma is useful for further estimates.

Lemma 14.2. *There exist positive constants T and d independent of n such that*

(14.5) $T_n^* \geqq T$,

(14.6) $d \leqq s^n(t) \leqq 1-d \qquad (0 \leqq t \leqq T)$.

Proof. We put $m=\max(\|\phi\|_{L^\infty(0,1)}, H_0, -H_1)$. Let (s_1, v) and (s_2, w) be the solutions of the following one-phase Stefan problem respectively,

$$\begin{cases} v_{xx}-c_0v_t=0 \qquad (0<x<s_1(t),\ t>0), \\ v_x(0, t) \in \gamma_0(v(0, t)) \qquad (t>0), \\ v(x, 0) = m \qquad (0 \leqq x \leqq (l+1)/2),\ s_1(0) = (l+1)/2, \\ v(s_1(t), t) = 0 \qquad (t>0) \\ b\dot{s}_1(t) = -v_x^-(s_1(t), t) \qquad (t>0), \end{cases}$$

$$\begin{cases} w_{xx}-c_1w_t = 0 \qquad (s_2(t)<x<1,\ t>0), \\ -w_x(1, t) \in \gamma_1(w(1, t)) \qquad (t>0), \\ w(x, 0) = -m \qquad (l/2 \leqq x \leqq 1),\ s_2(0) = l/2, \\ w(s_2(t), t) = 0 \qquad (t>0), \\ b\dot{s}_2(t) = w_x^+(s_2(t), t) \qquad (t>0). \end{cases}$$

These are well-defined by [34, Theorem 1]. It is easily seen from (14.1), (14.2) and the proof of Lemma 15.1 that

$$s_2(t)\leqq s^n(t)\leqq s_1(t) \qquad (0\leqq t\leqq T_n^*)$$

for all n. Hence we get the conclusion easily. q.e.d.

We shall show several estimates for $\{s^n\}a$ and $\{u^n\}$.

Lemma 14.3.

(i) $0\leqq u^n\leqq \max(\|\phi\|_{L^\infty(0,l)}, H_0)$ *on* $\bar{D}_T^0$,
$\min(-\|\phi\|_{L^\infty(l,1)}, H_1)\leqq u^n\leqq 0$ *on* $\bar{D}_T^1$,

(ii) *for any compact subset of* $\bigcap_{n=1}^{\infty}\{(x, t); 0<x<1, x\neq s_n(t), t>0\}$, *derivatives of* u^n *of all orders are uniformly bounded with respect to* n,

(iii) $\int_\sigma^T \dot{s}(t)^2 dt\leqq K_\sigma$,

(iv) $|s^n(t_2)-s^n(t_1)|\leqq K|t_2-t_1|^{1/3} \qquad (0\leqq t_1\leqq t_2\leqq T)$,

(v) $|s^n(t_2)-s^n(t_1)|\leqq K_\sigma|t_2-t_1|^{2/3} \qquad (\sigma\leqq t_1\leqq t_2\leqq T)$,

for any $\sigma\in]0, T[$, where K_σ (depending on $\sigma\in]0, T[$) and K are constants.

Proof. We have (i) using Proposition 8.3, (14.1) and (14.2). We get (ii) from (i) and the Bernstein's technique [21, p. 415]. It is easily seen from (14.1), (14.2), (14.6) and Lemma 5.4 that

$$\int_0^T \left\{\int_0^1 u_x^n(x, t)^2 dx+\theta_0(u^n(0, t))+\theta_1(u^n(1, t))\right\}dt\leqq \hat{M}, \tag{14.7}$$

where $\hat{M}$ is a constant independent of n. Hence we have

$$\int_0^T t|\dot{s}^n(t)|^3 dt\leqq 2b^{-2}\hat{M},$$

using Proposition 12.2 and (14.7). Consequently we obtain (iii), (iv) and (v) easily. q.e.d.

It follows from (14.6), Lemma 14.3 that there exist subsequence of $\{s^n\}$ (which we denote again by the same symbol) and a function $s\in C^{0,1/3}([0, T])\cap C^{0,2/3}(]0, T])\cap H^1_{loc}(]0, T])$ such that $s^n\to s$ in $C([0, T])$ as $n\to\infty$, and $d\leqq s(t)\leqq 1-d$ $(0\leqq t\leqq T)$. Hence we see from Proposition 3.5, (14.6) and Lemma 14.3 (iii) that $u^n\to u$ in $C(\bar{Q}_{\sigma,T})$ as $n\to\infty$ for any $0<\sigma<T$, where u is a solution of the moving boundary problem (M) corresponding to the curve $s(t)$ and the data $\phi(x)$.

We shall investigate the Stefan's condition.

Lemma 14.4. *$u\in C^\infty(S_T^0)\cap C^\infty(S_T^1)$, $s\in C^\infty(]0, T])$ and (0.7) is satisfied.*

Proof. Since each u^n satisfies (0.7), we have by Proposition 9.1

$$(14.8)\quad \begin{aligned} & bs^n(t_2)-bs^n(t_1) \\ &= \int_{t_1}^{t_2}(-u_x^n(a, t)+u_x^n(1-a, t))dt - c_0\int_a^{s^n(t_2)} u^n(x, t_2)dx \\ &-c_1\int_{s^n(t_2)}^{1-a} u^n(x, t_2)dx + c_0\int_a^{s^n(t_1)} u^n(x, t_1)dx \\ &+c^1\int_{s^n(t_1)}^{1-a} u^n(x, t_1)dx \qquad \text{for any } 0<t_1<t_2\leqq T, \end{aligned}$$

where $a=d/2$. Hence noting that $u_x^n(a, t)\to u_x(a, t)$, $u_x^n(1-a, t)\to u_x(1-a, t)$ uniformly on $[t_1, t_2]$ by Lemma 14.3 (ii), and letting $n\to\infty$ in (14.8), we have an equality similar to (14.8) as to (s, u). Hence we get the conclusion from Proposition 9.1, since $s(t)\in C^{0,1/3}([0, T])\cap C^{0,2/3}(]0, T])\cap H^1_{loc}(]0, T])$. q.e.d.

Consequently we have proved the existence of a solution (s, u) of (S) on $[0, T]$.

We shall show the global existence of a solution of (S). We note that $\{s(t), u(\cdot, t)\}$ satisfies the condition (A′) for all $t\in]0, T]$, since Proposition 3.1 and 8.3 hold. Hence $\{s(T), u(\cdot, T)\}$ satisfies the condition (A′). Let (T_1^*, s_1, u_1) be the solution of the Stefan problem (S) corresponding to the data $\{s(T), u(x, T)\}$. This is well-defined by Proposition 13.1. We put

$$s_2(t) = \begin{cases} s(t) & (0\leqq t\leqq T), \\ s_1(t-T) & (T\leqq t\leqq T_2^*) \end{cases}$$

$$u_2(x, t) = \begin{cases} u(x, t) & (0\leqq t\leqq T), \\ u_1(x, t-T) & (T\leqq t\leqq T_2^*), \end{cases}$$

where $T_2^*=T_1^*+T$. It follows from Proposition 11.2 that (T_2^*, s_2, u_2) is a solution of (S). We have (1.11) by Lemma 14.3 (iv) and Proposition 12.1. We get (1.12) and (1.13) by Proposition 8.3. We have (1.14) and (1.15) from Proposition 12.1, Lemma 2.2 and the fact that there exists a time σ' with $0<\sigma'<\sigma$ such that $u(x, \sigma')$ satisfies the condition (A′).

We can see the uniqueness of the solution of (S) by Proposition 15.1 which we prove later.

Thus we complete the proof of Theorem 1.

Proof of Theorem 3. It is obvious from Proposition 3.2 and Theorem 1. q.e.d.

15. Uniqueness of the solution of (S)

In this section we show the uniqueness of the solution of the Stefan problem (S) under the condition (A). We state a comparison theorem.

Proposition 15.1. *Let* (T_1^*, s_1, u_1) *and* (T_2^*, s_2, u_2) *be the solutions of* (*S*) *corresponding to the data* $\{l_1, \phi_1\}$ *and* $\{l_2, \phi_2\}$ *satisfying the condition* (*A*) *respectively. Suppose that* $l_1 \leqq l_2$ *and* $\phi_1 \leqq \phi_2$. *Then we have*

(15.1) $\quad s_1(t) \leqq s_2(t) \quad$ *on* $[0, T^*]$,

(15.2) $\quad u_1(x, t) \leqq u_2(x, t) \quad$ *on* $\bar{Q}_{T^*}$,

where $T^* = \min(T_1^*, T_2^*)$.

We prepare a lemma to prove the proposition above.

Lemma 15.1. *Under the assumptions of Proposition* 15.1, *let* $l_1 < l_2$ *and* $\phi_1 \leqq \phi_2$. *Then we have*

$$s_1(t) < s_2(t) \qquad on\ [0, T^*] .$$

Proof. Assuming the contrary, set $t_0 = \min\{t \in [0, T^*];\ s_1(t) = s_2(t)\}$. Clearly $\dot{s}_1(t_0) \geqq \dot{s}_2(t_0)$ and $t_0 > 0$. We may have that $u_2(s_1(t), t) > 0$ and $u_1(s_2(t), t) < 0$ $(0 < t < t_0)$ by virtue of Proposition 8.1, Proposition 8.3 and the strong maximum principle [24], ruling out the exceptional case $u_2 \equiv 0$ $(0 \leqq x \leqq s_2(t),\ 0 \leqq t \leqq t_0)$ or $u_1 \equiv 0$ $(s_1(t) \leqq x \leqq 1,\ 0 \leqq t \leqq t_0)$. In fact, (S) is reduced to a one-phase problem in the exceptional case, so we can derive a contradiction from the proof of [34, Lemma 12.1]. Now we have $u_2 - u_1 > 0$ in $\{(x, t);\ 0 < x < s_1(t),\ s_2(t) < x < 1,\ 0 < t \leqq t_0\}$ by Proposition 8.1 and the strong maximum principle. Since $u_2 - u_1$ vanishes at the point $(s_2(t_0), t_0)$, it follows from the parabolic version of Hopf's lemma [14] that

$$u_{2,x}^-(s_2(t_0), t_0) - u_{1,x}^-(s_2(t_0), t_0) < 0 ,$$
$$u_{2,x}^+(s_2(t_0), t_0) - u_{1,x}^+(s_2(t_0), t_0) > 0 .$$

Hence we have by $s_1(t_0) = s_2(t_0)$,

$$b\dot{s}_2(t_0) = u_{2,x}^+(s_2(t_0), t_0) - u_{2,x}^-(s_2(t_0), t_0)$$
$$> b\dot{s}_1(t_0) = u_{1,x}^+(s_1(t_0), t_0) - u_{1,x}^-(s_1(t_0), t_0) ,$$

which is a contradiction. q.e.d.

Proof of Proposition 15.1. We shall show (15.1). If $l_1 < l_2$, then we have (15.1) by Lemma 15.1. Hence we may treat the case $l_1 = l_2$. For simplicity we denote l_2, ϕ_2, s_2, u_2 by l, ϕ, s, u respectively. We put $l^\alpha = l + \alpha$,

$$(15.3)\qquad \phi^{\alpha}(x)=\begin{cases}\phi(x) & (0\leqq x<l,\ l^{\alpha}<x\leqq 1),\\ 0 & (l\leqq x\leqq l^{\alpha}),\end{cases}$$

for sufficiently small $\alpha>0$. Each $\{l^{\alpha}, \phi^{\alpha}\}$ satisfies (A). Therefore there exists a solution $(T^*_{\alpha}, s^{\alpha}, u^{\alpha})$ of (S) corresponding to the data $\{l^{\alpha}, \phi^{\alpha}\}$ by the result in § 14. We see from Lemma 15.1 that

$$(15.4)\qquad s_1(t)<s^{\alpha}(t) \qquad \text{on } [0, \min(T_1^*, T_{\alpha}^*)]\,.$$

Now (s, u) and (s^{α}, u^{α}), being the solutions of (S), must satisfy their versions of Proposition 9.1 (ii) respectively. Substracting them and noting that $u^{\alpha}-u\geqq 0$ by Proposition 8.1, we obtain

$$\begin{aligned}(15.5)\qquad & b(s^{\alpha}(t)-(s(t))\\ &= b(s^{\alpha}(\sigma)-s(\sigma))\\ &\quad -\int_0^t (u_x^{\alpha}(0, \tau)-u_x(0, \tau))d\tau+\int_{\sigma}^t (u_x^{\alpha}(1, \tau)-u_x(1, \tau))d\tau\\ &\quad -c_0\int_0^{s(t)} (u^{\alpha}(x, t)-u(x, t))dx-c_0\int_{s(t)}^{s^{\alpha}(t)} u^{\alpha}(x, t)dx\\ &\quad -c_1\int_{s^{\alpha}(t)}^1 (u^{\alpha}(x, t)-u(x, t))dx+c_1\int_{s(t)}^{s^{\alpha}(t)} u(x, t)dx\\ &\quad +c_0\int_0^{s^{\alpha}(\sigma)} u^{\alpha}(x, \sigma)dx-c_0\int_0^{s(\sigma)} u(x, \sigma)dx\\ &\quad +c_1\int_{s^{\alpha}(\sigma)}^1 u^{\alpha}(x, \sigma)dx-c_1\int_{s(\sigma)}^1 u(x, \sigma)dx\\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}+\mathrm{VII}+\mathrm{VIII}+\mathrm{IX}+\mathrm{X}+\mathrm{XI}\,,\end{aligned}$$

for any σ and t with $0<\sigma<t<\min(T_2^*, T_{\alpha}^*)$ for sufficiently small α. II and III are non-positive by the argument used in the proof of [34, Lemma 12.1], and moreover IV, V, VI and VII are non-positive since $u^{\alpha}-u\geqq 0$ and $s^{\alpha}\geqq s$. Hence letting σ tend to zero, we obtain

$$\begin{aligned}(15.6)\qquad & b(s^{\alpha}(t)-s(t))\leqq b(l^{\alpha}-l)\\ &\quad +c_0\Big(\int_0^{l^{\alpha}} \phi^{\alpha}dx-\int_0^l \phi dx\Big)+c_1\Big(\int_{l^{\alpha}}^1 \phi^{\alpha}dx-\int_l^1 \phi dx\Big)\\ &\leqq b\alpha+c_1\alpha\|\phi\|_{L^{\infty}(l,1)}\equiv I_{\alpha} \qquad \text{on } [0, \min(T_2^*, T_{\alpha}^*)]\,.\end{aligned}$$

Therefore we see from (15.4) and (15.6) that

$$s_1(t)<s^{\alpha}(t)<s_2(t)+b^{-1}I_{\alpha} \qquad \text{on } [0, \min(T_1^*, T_2^*, T_{\alpha}^*)]\,.$$

Letting $\alpha\to 0$, we have (15.1). We get (15.2) by Proposition 8.1. q.e.d.

Proof of Theorem 1 (Uniqueness). Let (T_1^*, s_1, u_1) and (T_2^*, s_2, u_2) be solutions of (S). Assume that $T_1^*\leqq T_2^*$. If $T_1^*<T_2^*$, then we can see from Pro-

position 15.1 that $s_1(t)=s_2(t)$ $(0\leqq t\leqq T_1^*)$. Hence $s_1(T_1^*)=s_2(T_1^*)$. This is a contradiction since $s_1(T_1^*)=0$ or 1, and $0<s_2(T_1^*)<1$ by $T_1^*<T_2^*$. Therefore $T_1^*=T_2^*$. Thus we have $s_1=s_2$ on $[0, T_1^*]$ and $u_1=u_2$ on $\bar{Q}_{T_1^*}$ by Proposition 15.1 and 8.1. q.e.d.

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