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<th>An isolated umbilical point of a Willmore surface</th>
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Osaka University
AN ISOLATED UMBILICAL POINT OF A WILLMORE SURFACE

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1. Introduction

Let $S$ be a surface in $\mathbb{R}^3$. Then it is known that if $S$ is a surface with constant mean curvature, then the index of an isolated umbilical point on $S$ is negative ([16]). If $S$ is special Weingarten, then the same result is obtained ([15]). In the present paper, we shall prove that the index of an isolated umbilical point on a Willmore surface does not exceed $1/2$.

We say that $S$ is a Willmore surface if $S$ is a stationary surface of the Willmore functional $\mathcal{W}$, where the Willmore functional is defined by the integral of the square of the mean curvature. It is known that $S$ is a Willmore surface if and only if $S$ satisfies the following partial differential equation ([12]):

$$\{\Delta + 2(\nabla^2 - K)\}H = 0,$$

where $\Delta$ is the Laplace operator on $S$ and $K, H$ are the Gaussian and the mean curvatures of $S$, respectively. Equation (1) is the Euler-Lagrange equation for Willmore surfaces.

Willmore proved that $\mathcal{W} \geq 4\pi$ for any compact surface in $\mathbb{R}^3$ and that the equality holds if and only if the surface is a round sphere ([36], [37]). In addition, he and Shiohama-Takagi proved that $\mathcal{W} \geq 2\pi^2$ ($> 4\pi$) for a torus represented as the boundary of a tubular neighborhood of a closed curve in $\mathbb{R}^3$ and that the equality holds if and only if the torus is a $\sqrt{2}$-anchor ring, i.e., the boundary of the tubular neighborhood with radius $a > 0$ of a circle with radius $\sqrt{2}a$ ([38], [27]). Willmore conjectured $\mathcal{W} \geq 2\pi^2$ for any torus in $\mathbb{R}^3$ ([36]). Since White showed that if the surface is compact and orientable, then $\mathcal{W}$ is invariant under any conformal transformation of $\overline{\mathbb{R}^3} := \mathbb{R}^3 \cup \{\infty\}$ ([35]), it has been expected that the equality in Willmore’s conjecture holds if and only if the torus is conformally equivalent in $\overline{\mathbb{R}^3}$ to a $\sqrt{2}$-anchor ring. Li-Yau showed that Willmore’s conjecture is true for tori with certain conformal structures close to the conformal structure of a $\sqrt{2}$-anchor ring ([21]); Montiel-Ros showed that Willmore’s conjecture is also true for tori with more conformal structures ([22]). Simon proved that there exists an embedded torus in $\mathbb{R}^3$ at which $\mathcal{W}$ attains the infimum on all the immersed tori ([28], [29]). Recently, the author has had paper [26] by Schmidt the main theorem of which states that Willmore’s conjecture is
true for any torus immersed in \( \mathbb{R}^3 \).

Weiner proved that the image of any minimal surface in \( S^3 \) by a stereographic projection is a Willmore surface in \( \mathbb{R}^3 \) ([34]). Any compact two-dimensional manifold other than the projective plane may be realized in \( S^3 \) as a minimal surface ([20]), while the projective plane may not be realized in \( S^3 \) as any minimal surface ([1], [20]). Therefore we see that any compact two-dimensional manifold distinct from the projective plane may be realized in \( \mathbb{R}^3 \) as a Willmore surface. Pinkall showed that there exists a Hopf torus in \( S^3 \) which is not conformally equivalent in \( S^3 \) to any minimal surface and the image of which by a stereographic projection is a Willmore surface in \( \mathbb{R}^3 \) ([24]). In addition, Kusner found an example of a Willmore surface in \( \mathbb{R}^3 \) which is homeomorphic to the projective plane ([18], [19]). At this example, \( W \) attains \( 12\pi \), the infimum on all the projective planes immersed in \( \mathbb{R}^3 \). Bryant described the moduli space of the Willmore projective planes in \( \mathbb{R}^3 \) for each of which \( W \) is equal to \( 12\pi \) ([11]).

By Hopf-Poincaré’s theorem together with Kusner’s example of a Willmore projective plane, we see that our estimate of the index of an isolated umbilical point on a Willmore surface is sharp.

It is expected that the index of an isolated umbilical point on a surface does not exceed one. We call this conjecture the index conjecture. In relation to the index conjecture, the following two conjectures are known: Carathéodory’s conjecture and Loewner’s conjecture. Carathéodory’s conjecture asserts that there exist at least two umbilical points on a compact, strictly convex surface in \( \mathbb{R}^3 \). If the index conjecture is true, then we see from Hopf-Poincaré’s theorem that there exist at least two umbilical points on a compact, orientable surface of genus zero, and this immediately gives the affirmative answer to Carathéodory’s conjecture. Let \( F \) be a real-valued, smooth function of two real variables \( x, y \), and set \( \partial_\zeta := (\partial/\partial x + \sqrt{-1}\partial/\partial y)/2 \). Then Loewner’s conjecture for a positive integer \( n \in \mathbb{N} \) asserts that if a vector field \( \text{Re} (\partial_\zeta F) (\partial/\partial x) + \text{Im} (\partial_\zeta F) (\partial/\partial y) \) has an isolated zero point, then its index with respect to this vector field does not exceed \( n \) ([17], [33]). Loewner’s conjecture for \( n = 1 \) is affirmatively solved; Loewner’s conjecture for \( n = 2 \) is equivalent to the index conjecture. We may find [9], [13], [30], [31] and [32] as recent papers in relation to Carathéodory’s and Loewner’s conjectures. We discussed the index of an isolated umbilical point on a surface in [2]–[7], and in [8], we introduced and studied a conjecture in relation to Loewner’s conjecture.

We see from our estimate of the index in the present paper that the index conjecture is true for any isolated umbilical point on a Willmore surface. In the proof of the main theorem, we shall encounter a situation on a surface with an isolated umbilical point which has not appeared in our previous studies.
2. Willmore surfaces

Let $M$ be a connected, orientable two-dimensional manifold and $\iota: M \to \mathbb{R}^3$ an immersion of $M$ into $\mathbb{R}^3$. Let $H$ be the mean curvature of $M$ with respect to $\iota$ and $dA$ the area element of $M$ with respect to the metric $g$ induced by $\iota$. Then the Willmore functional $\mathcal{W}$ is given by

$$\mathcal{W}(\iota) := \int_M H^2 \, dA.$$

Let $K$ be the Gaussian curvature of $M$ with respect to the metric $g$ and set

$$\tilde{\mathcal{W}}(\iota) := \int_M (H^2 - K) \, dA.$$

Then we obtain

$$\tilde{\mathcal{W}}(\iota) = \mathcal{W}(\iota) - \int_M K \, dA. \quad (2)$$

It is known that for any conformal transformation $X$ of $\mathbb{R}^3$ such that $X \circ \iota$ is an immersion, the following holds ([35]):

$$\tilde{\mathcal{W}}(X \circ \iota) = \tilde{\mathcal{W}}(\iota). \quad (3)$$

If $M$ is compact, then by (2), (3) and Gauss-Bonnet’s theorem, we obtain

$$\mathcal{W}(X \circ \iota) = \mathcal{W}(\iota).$$

Let $M$ and $\iota$ be as above. Let $\xi$ be a unit normal vector field on $M$ with respect to $\iota$ and $f$ a smooth function on $M$ with compact support. Let $\iota_f$ be a smooth map from $M \times \mathbb{R}$ into $\mathbb{R}^3$ satisfying $\iota_f(p, 0) = \iota(p)$, $(\partial \iota_f/\partial t)(p, 0) = f(p)\xi(p)$ for $p \in M$ and the condition that $\iota_f(p, t) = \iota_f(p, 0)$ for any $t \in \mathbb{R}$ and any point $p$ of $M$ outside the support of $f$. We set $\iota_{f,t}(p) := \iota_f(p, t)$ for $(p, t) \in M \times \mathbb{R}$. Then there exists an open interval $I$ containing 0 such that for each $t \in I$, $\iota_{f,t}$ is an immersion of $M$ into $\mathbb{R}^3$. We set

$$w_f(t) := \mathcal{W}(\iota_{f,t}), \quad \tilde{w}_f(t) := \tilde{\mathcal{W}}(\iota_{f,t}).$$

An immersion $\iota$ is called Willmore if $(dW_f/dt)(0) = 0$ holds for any smooth function $f$ on $M$ with compact support; if $\iota$ is a Willmore immersion, then the pair $(M, \iota)$ or the image $\iota(M)$ of $M$ by $\iota$ is called a Willmore surface. An immersion $\iota$ is Willmore if and only if (1) holds, where $\Delta$ is the Laplace operator on $M$ with respect to the metric $g$ ([12]). Let $D$ be a domain in $M$ which contains the support of $f$.
and the boundary of which consists of a finite number of closed curves. Then for \( t \in I \), \( w_f(t) - \tilde{w}_f(t) \) is represented as follows:

\[
(4) \quad w_f(t) - \tilde{w}_f(t) = \int_{M \setminus D} K_t dA_t + \int_D K_t dA_t,
\]

where \( K_t \) and \( dA_t \) are the Gaussian curvature and the area element of \( M \) with respect to the metric induced by \( t_{f,t} \), respectively. From Gauss-Bonnet’s theorem, we see that the second term of the right-hand side of (4) depends only on the boundary of \( D \), which implies that this term does not depend on \( t \in I \). In addition, since \( D \) contains the support of \( f \), the first term of the right-hand side of (4) does not depend on \( t \in I \) either. Therefore we see that \( w_f - \tilde{w}_f \) is constant on \( I \). In particular, we obtain

\[
(5) \quad \frac{d\tilde{w}_f}{dt}(0) = \frac{dw_f}{dt}(0).
\]

By (3) together with (5), we obtain

**Proposition 2.1.** Let \( \iota \) be an immersion of \( M \) into \( \mathbb{R}^3 \) and \( X \) a conformal transformation of \( \mathbb{R}^3 \) such that \( X \circ \iota \) is an immersion. Then \( \iota \) is Willmore if and only if \( X \circ \iota \) is Willmore.

3. The index of an isolated umbilical point

Let \( f \) be a smooth function of two variables \( x, y \) and \( G_f \) the graph of \( f \). We set

\[
p_f := \frac{\partial f}{\partial x}, \quad q_f := \frac{\partial f}{\partial y}, \quad r_f := \frac{\partial^2 f}{\partial x^2}, \quad s_f := \frac{\partial^2 f}{\partial x \partial y}, \quad t_f := \frac{\partial^2 f}{\partial y^2}.
\]

Then the Gaussian curvature \( K_f \) and the mean curvature \( H_f \) of \( G_f \) are represented as follows:

\[
(6) \quad K_f := \frac{r_f t_f - s_f^2}{(1 + p_f^2 + q_f^2)^2}, \quad H_f := \frac{r_f + t_f + p_f^2 t_f - 2 p_f q_f s_f + q_f^2 r_f}{2(1 + p_f^2 + q_f^2)^{3/2}},
\]

Let \( D_f, N_f, PD_f \) be symmetric tensor fields on \( G_f \) of type \((0,2)\) represented in terms of the coordinates \((x, y)\) as follows:

\[
D_f := s_f dx^2 + (t_f - r_f) dx dy - s_f dy^2,
\]

\[
N_f := (s_f p_f^2 - p_f q_f r_f) dx^2 + (t_f p_f^2 - r_f q_f^2) dx dy + (p_f q_f t_f - s_f q_f^2) dy^2,
\]

\[
PD_f := \frac{1}{1 + p_f^2 + q_f^2}(D_f + N_f).
\]
A tangent vector $\mathbf{v}_0$ to $G$ at a point is in a principal direction if and only if $\text{PD}_f(\mathbf{v}_0, \mathbf{v}_0) = 0$ holds ([5]). For a tangent vector $\mathbf{v}$, we set
\[ \hat{D}_f(\mathbf{v}) := D_f(\mathbf{v}, \mathbf{v}), \quad \hat{N}_f(\mathbf{v}) := N_f(\mathbf{v}, \mathbf{v}), \quad \text{PD}_f(\mathbf{v}) := \text{PD}_f(\mathbf{v}, \mathbf{v}). \]
For $\phi \in \mathbb{R}$, we set
\[ u_\phi := \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \quad U_\phi := \cos \phi \frac{\partial}{\partial x} + \sin \phi \frac{\partial}{\partial y}. \]
We set
\[ \text{grad}_f := \begin{pmatrix} p_f \\ q_f \end{pmatrix}, \quad \text{grad}_f^\perp := \begin{pmatrix} -q_f \\ p_f \end{pmatrix}, \quad \text{Hess}_f := \begin{pmatrix} r_f & s_f \\ s_f & t_f \end{pmatrix}. \]
Let $\langle \ , \rangle$ be the scalar product in $\mathbb{R}^2$. Then for any $\phi \in \mathbb{R}$, the following hold ([5]):
\[ \hat{D}_f(U_\phi) = \langle \text{Hess}_f U_\phi, U_\phi \phi/2 \rangle, \]
\[ \hat{N}_f(U_\phi) = \langle \text{grad}_f, U_\phi \rangle \langle \text{grad}_f^\perp, \text{Hess}_f U_\phi \rangle. \]
For $l \in \mathbb{N} \cup \{\infty\}$, let $C^{(\infty)}_o$ be the set of smooth functions defined on a connected neighborhood of $(0,0)$ in $\mathbb{R}^2$ such that $((\partial^m F/\partial x^m \partial y^n)(0,0) = 0$ for each $F \in C^{(\infty)}_o$ and non-negative integers $m, n$ satisfying $0 \leq m + n < l$. The following hold:
\[ C^{(\infty)}_o \supset C^{(\infty+1)}_o \supset C^{(\infty,\infty)}_o \neq \{0\}, \]
where $l \in \mathbb{N}$. Let $F$ be an element of $C^{(\infty,2)}_o$ such that $O := (0,0,0)$ is an umbilical point of the graph of $F$, that is, there exists a real number $a_F$ satisfying
\[ F(x,y) = \frac{a_F(x^2 + y^2)}{2} + o(x^2 + y^2). \]
Let $\sigma_F$ be an element of $C^{(\infty,2)}_o$ defined by
\[ \sigma_F := \begin{cases} 0 & \text{if } a_F = 0, \\ \frac{1}{a_F} - \frac{|a_F|}{a_F} \sqrt{\frac{1}{a_F^2} - (x^2 + y^2)} & \text{if } a_F \neq 0. \end{cases} \]
Then we obtain $F - \sigma_F \in C^{(\infty,3)}_o$. For an integer $l \geq 2$, let $C^{(\infty,l)}_o$ be the subset of $C^{(\infty,l)}_o$ such that each $F \in C^{(\infty,l)}_o$ satisfies (7) for some $a_F \in \mathbb{R}$ and $F - \sigma_F \notin C^{(\infty,\infty)}_o$. For an integer $k \geq 3$, let $\mathcal{P}^k$ be the set of the homogeneous polynomials of degree $k$. Then for each $F \in C^{(\infty,2)}_o$, there exist an integer $k_F \geq 3$ and a nonzero element $g_F$ of $\mathcal{P}^{k_F}$ satisfying $F - \sigma_F - g_F \in C^{(\infty,k_F+1)}_o$. Let $g$ be an element of $\mathcal{P}^k$. 
Then set \( \text{Hess}_g(\theta) := \text{Hess}_g(\cos \theta, \sin \theta) \) for \( \theta \in \mathbb{R} \) and let \( \eta_g \) be a continuous function on \( \mathbb{R} \) such that for any \( \theta \in \mathbb{R} \), \( u_{\eta_g(\theta)} \) is an eigenvector of \( \text{Hess}_g(\theta) \), and let \( S_g \) denote the set of the numbers at each of which \( \text{Hess}_g \) is represented by the unit matrix up to a constant.

Let \( C_{\alpha}^{\infty,2} \) be the subset of \( C_{\alpha}^{\infty,2} \) such that on the graph \( G_F \) of each \( F \in C_{\alpha}^{\infty,2} \), \( \alpha \) is an isolated umbilical point. For an element \( F \) of \( C_{\alpha}^{\infty,2} \), let \( \rho_0 \) be a positive number such that there exists no umbilical point of \( G_F \) on \( \{ 0 < x^2 + y^2 < \rho_0^2 \} \) and \( \phi_F \) a continuous function on \( (0, \rho_0) \times \mathbb{R} \) such that for each \( (\rho, \theta) \in (0, \rho_0) \times \mathbb{R} \), a tangent vector \( \cos \phi_F(\rho, \theta) \partial_x + \sin \phi_F(\rho, \theta) \partial_y \) of \( G_F \) at \( (\rho \cos \theta, \rho \sin \theta) \) is in a principal direction. Then the following (a)–(c) hold ([5], [6]):

(a) For any \( \theta_0 \in \mathbb{R} \setminus S_{gf} \), there exists a number \( \phi_{F,\theta}(\theta_0) \) satisfying the following:

(i) \( \lim_{\rho \to 0} \phi_F(\rho, \theta_0) = \phi_{F,\theta}(\theta_0) \),

(ii) \( u_{\phi_{F,\theta}(\theta_0)} \) is an eigenvector of \( \text{Hess}_{gf}(\theta_0) \);

(b) For any \( \theta_0 \in \mathbb{R} \), there exist numbers \( \phi_{F,\theta}(\theta_0 + 0), \phi_{F,\theta}(\theta_0 - 0) \) satisfying the following:

(i) \( \lim_{\theta \to \theta_0 \pm 0} \phi_{F,\theta}(\theta) = \phi_{F,\theta}(\theta_0 \pm 0) \),

(ii) \( \Gamma_{F,\theta}(\theta_0) := \phi_{F,\theta}(\theta_0 + 0) - \phi_{F,\theta}(\theta_0 - 0) \) is an element of \( \{ n\pi/2 \}_{n \in \mathbb{Z}} \);

(c) The index \( \text{ind}_o(G_F) \) of \( o \) on \( G_F \) is represented as follows:

\[
(8) \quad \text{ind}_o(G_F) = \frac{\eta_{gf}(\theta + 2\pi) - \eta_{gf}(\theta)}{2\pi} + \frac{1}{2\pi} \sum_{\theta_0 \in S_{gf} \cap [\theta, \theta + 2\pi]} \Gamma_{F,\theta}(\theta_0).
\]

For an integer \( k \geq 3 \), set \( \mathcal{P}_k := \mathcal{P}^k \cap C_{\alpha}^{\infty,2} \). Then for any \( g \in \mathcal{P}_k \), the following hold: \( \Gamma_{g,\theta}(\theta_0) = -\pi/2 \) for any \( \theta_0 \in S_g \) ([4]); \( \text{ind}_o(G_g) \in \{ 1 - k/2 + i \}_{i \in \mathbb{Z}} \) ([2]). Let \( C_{\alpha}^{\infty,2} \) be the subset of \( C_{\alpha}^{\infty,2} \) such that for each \( F \in C_{\alpha}^{\infty,2} \), \( \alpha \) is an isolated umbilical point on each of \( G_F \) and \( G_{gf} \). If \( F \) is an element of \( C_{\alpha}^{\infty,2} \) satisfying \( S_{gf} = \emptyset \), then \( F \in C_{\alpha}^{\infty,2} \) holds ([5], [6]). We see that if \( F \in C_{\alpha}^{\infty,2} \) satisfies \( S_{gf} = \emptyset \), then the following hold:

\[
(9) \quad \text{ind}_o(G_F) = \text{ind}_o(G_{gf}) = \frac{\eta_{gf}(\theta + 2\pi) - \eta_{gf}(\theta)}{2\pi}.
\]

For any \( F \in C_{\alpha}^{\infty,2} \), the following hold ([5], [6]):

(a) \( -\pi/2 \leq \Gamma_{F,\theta}(\theta_0) \leq \pi/2 \) for any \( \theta_0 \in S_{gf} \);

(b) \( \text{ind}_o(G_{gf}) \leq \text{ind}_o(G_F) \leq 1 \).

If \( F \) is an element of \( C_{\alpha}^{\infty,2} \) satisfying \( \Gamma_{F,\theta}(\theta_0) \leq \pi \) for any \( \theta_0 \in S_{gf} \), then \( \text{ind}_o(G_F) \leq 1 \) holds ([5], [6]). In general, it is expected that the index of an isolated umbilical point on a surface does not exceed one (which is called the index conjecture or the local Carathéodory’s conjecture).

We presented one way of computing \( \eta_g(\theta + 2\pi) - \eta_g(\theta) \) for any \( g \in \mathcal{P}_k \) ([5]). For \( \theta \in \mathbb{R} \), set \( \bar{g}(\theta) := g(\cos \theta, \sin \theta) \). A number \( \theta_0 \in \mathbb{R} \) is called a root of \( g \) if \( (dg/d\theta)(\theta_0) = 0 \). The set of the roots of \( g \) is denoted by \( R_g \). Let \( R(\text{Hess}_g) \) be the set of numbers such that each \( \theta_0 \in R(\text{Hess}_g) \) satisfies \( \theta_0 \in \{ \eta_g(\theta_0) + i\pi/2 \}_{i \in \mathbb{Z}} \). For \( \theta \in \mathbb{R} \),
we set \( \text{grad}_g(\theta) := \text{grad}_g(\cos \theta, \sin \theta) \). Then the following holds:

(10) \( (k - 1) \text{grad}_g(\theta) = \text{Hess}_g(\theta)u_{\theta*} \).

From (10), we obtain

(11) \( \langle \text{Hess}_g(\theta)u_{\theta*}, u_{\theta + \pi/2} \rangle = (k - 1)\frac{d\hat{g}}{d\theta}(\theta) \).

Therefore we obtain \( S_g \subset R_g \) and \( R(\text{Hess}_g) \subset R_g \). Suppose \( R_g = R \). Then \( k \) is even and \( g \) is represented by \( (x^2 + y^2)^{k/2} \) up to a constant. By direct computations, we obtain \( S_g = \emptyset \). Therefore \( \mathbf{o} \) is an isolated umbilical point of \( G_g \). By (11), we see that \( R(\text{Hess}_g) = R \), i.e., there exists a number \( z_0 \in \{ n\pi/2 \}_{n \in \mathbb{Z}} \) satisfying \( \eta_g(\theta) = \theta + z_0 \) for any \( \theta \in R \). Therefore by (9), we obtain

\[
\text{ind}_\mu(G_g) = \frac{\eta_g(\theta + 2\pi) - \eta_g(\theta)}{2\pi} = 1.
\]

In the following, suppose \( R_g \neq R \). Then for each \( \theta_0 \in R_g \), there exists a positive integer \( \mu \) satisfying \( d^{\mu+1}\hat{g}/d\theta^{\mu+1}(\theta_0) \neq 0 \). The minimum of such integers is denoted by \( \mu_g(\theta_0) \). A root \( \theta_0 \in R_g \) is said to be

(a) related if \( \theta_0 \) satisfies \( \hat{g}(\theta_0) = 0 \) or if \( \mu_g(\theta_0) \) is odd;

(b) non-related if \( \theta_0 \) satisfies \( \hat{g}(\theta_0) \neq 0 \) and if \( \mu_g(\theta_0) \) is even.

Suppose that \( \theta_0 \in R_g \) is related. Then it is said that the critical sign of \( \theta_0 \) is positive (respectively, negative) if the following holds:

\[
\hat{g}(\theta_0) \frac{d^{\mu_g(\theta_0)+1}\hat{g}}{d\theta^{\mu_g(\theta_0)+1}(\theta_0)} \leq 0 \quad (\text{respectively, } > 0).
\]

The critical sign of \( \theta_0 \) is denoted by \( c\text{-sign}_g(\theta_0) \). The set \( R_g \setminus R(\text{Hess}_g) \) consists of the numbers at each of which \( \text{Hess}_g \) is represented by the unit matrix up to a nonzero constant; in addition, an element \( \theta_0 \in R_g \setminus R(\text{Hess}_g) \) is related and satisfies \( c\text{-sign}_g(\theta_0) = - \) ([15]). It is said that the sign of \( \theta_0 \in R(\text{Hess}_g) \) is positive (respectively, negative) if there exists a neighborhood \( U_{\theta_0} \) of \( \theta_0 \) in \( R \) satisfying

\[
\{ \theta - \eta_g(\theta) - (\theta_0 - \eta_g(\theta_0)) \} (\theta - \theta_0) > 0 \quad (\text{respectively, } < 0)
\]

for any \( \theta \in U_{\theta_0} \setminus \{ \theta_0 \} \). For \( \theta_0 \in R(\text{Hess}_g) \), \( \theta_0 \) is related if and only if the sign of \( \theta_0 \) is positive or negative ([5]). If \( \theta_0 \in R(\text{Hess}_g) \) is related, then the sign of \( \theta_0 \) is denoted by \( c\text{-sign}_g(\theta_0) \). For a related root \( \theta_0 \) of \( g \) satisfying \( c\text{-sign}_g(\theta_0) = + \), \( \theta_0 \in R(\text{Hess}_g) \) and \( c\text{-sign}_g(\theta_0) = + \) hold ([15]). Referring to [3], we see that if \( \theta_0 \) is a related element of \( R(\text{Hess}_g) \) satisfying \( c\text{-sign}_g(\theta_0) = - \), then the condition \( c\text{-sign}_g(\theta_0) = + \) (respectively, \( - \)) is equivalent to the following:

\[
\frac{1}{\hat{g}(\theta_0)} \frac{d^2\hat{g}}{d\theta^2}(\theta_0) \in (k(k - 2), \infty) \quad (\text{respectively, } [0, k(k - 2))).
\]
Let \( n_{g+} \) (respectively, \( n_{g-} \)) denote the number of the related elements of \( R(\text{Hess}_g) \) in \( [\theta, \theta + \pi) \) with positive (respectively, negative) sign. Then for any \( \theta \in \mathbb{R} \), the following holds ([15]):

\[
\frac{\eta_g(\theta + 2\pi) - \eta_g(\theta)}{2\pi} = 1 - \frac{n_{g+} - n_{g-}}{2}.
\]

4. The main theorem

We shall prove

**Theorem 4.1.** Let \( F \) be an element of \( C_0^{(\infty, 2)} \) satisfying (7) for some \( \alpha_F \in \mathbb{R} \) and suppose that the graph \( G_F \) of \( F \) is a Willmore surface such that there exists no totally umbilical neighborhood of \( \alpha \) in \( G_F \). Then the following hold:

(a) \( F \in C_1^{(\infty, 2)} \);

(b) If \( \alpha \) is an isolated umbilical point of \( G_F \), then \( \text{ind}_\alpha(G_F) \leq 1/2 \).

**Remark.** Noticing Proposition 2.1 and that whether a one-dimensional subspace of the tangent plane at a point of a surface is a principal direction is invariant under any conformal transformation of \( \overline{\mathbb{R}}^3 \), we may suppose \( F \in C_0^{(\infty, 3)} \) in Theorem 4.1.

**Remark.** Although \( F \) is an element of \( C_0^{(\infty, 2)} \) such that \( \alpha \) is an isolated umbilical point of \( G_F \), \( F \in C_0^{(\infty, 2)} \) does not always hold. Let \( f \) be a smooth function on a neighborhood of \((0, 0)\) in \( \mathbb{R}^2 \) satisfying \( f(0, 0) = 0 \) and \( f > 0 \) on a punctured neighborhood of \((0, 0)\). Then \( \exp(-1/f) \) is a smooth function defined on a punctured neighborhood of \((0, 0)\) and smoothly extended to \((0, 0)\) so that all the partial derivatives of \( \exp(-1/f) \) at \((0, 0)\) are equal to zero. Then we obtain \( \exp(-1/f) \in C_0^{(\infty, \infty)} \).

Suppose that for each positive number \( c > 0 \), there exists a punctured neighborhood of \((0, 0)\) on which the norm of the gradient vector field of \( \log f \) is bounded from below by the number \( c \). Then \( \alpha \) is an isolated umbilical point on the graph of \( \exp(-1/f) \) ([7]). However, since \( \exp(-1/f) \in C_0^{(\infty, \infty)} \), we obtain \( \exp(-1/f) \notin C_0^{(\infty, 2)} \). (a) of Theorem 4.1 is crucial to the proof of (b) of Theorem 4.1.

**Proof of (a) of Theorem 4.1.** Let \( \Delta F \) be the Laplace operator on \( G_F \), and \( K_F \), \( H_F \) the Gaussian and the mean curvatures of \( G_F \), respectively. Then \( H_F \) satisfies the following elliptic partial differential equation:

\[
\{\Delta_F + 2(H_F^2 - K_F)\}H_F = 0.
\]

If \( H_F \equiv 0 \), then \( G_F \) is a minimal surface and \( F \) is real-analytic. Since \( G_F \) is not totally umbilical, we obtain \( F \not\equiv 0 \) and this implies \( F \in C_0^{(\infty, 3)} \). If \( H_F \not\equiv 0 \), then \( H_F \) is a non-trivial solution of (13) and referring to [14] as in [15], we see that not all the partial derivatives of \( H_F \) at \((0, 0)\) are equal to zero. This implies \( F \in C_0^{(\infty, 3)} \).
Hence we obtain (a) of Theorem 4.1.

Proof of (b) of Theorem 4.1. Let $F$ be an element of $C^3\infty,0)$ such that the graph $G_F$ of $F$ is a Willmore surface. Then there exist an integer $k_F \geq 3$ and a nonzero homogeneous polynomial $g_F \in \mathcal{P}^{k_F}$ satisfying $F - g_F \in C^\infty,0)$, and noticing (6) and (13), we see that $g_F$ satisfies $\Delta_0 g_F = 0$, where $\Delta_0 := (\partial/\partial x)^2 + (\partial/\partial y)^2$. Therefore there exist spherical harmonic functions $h_{k_F}$, $h_{k_F-2}$ of degree $k_F$, $k_F - 2$, respectively such that $g_F$ is represented as

$$g_F = h_{k_F} + (x^2 + y^2) h_{k_F-2}.$$

Suppose $S_{g_F} = \emptyset$. Then $F \in C^\infty,0)$ holds. Noticing that the number of the zero points of $\tilde{g}_F$ in $[\theta, \theta + \pi)$ is more than or equal to $k_F - 2$, we obtain

$$k_F - 2 \leq \sharp \{ S_{g_F} \cap [\theta, \theta + \pi) \} \leq k_F$$

and

$$(n_{g_F,+}, n_{g_F,-}) \in \{(k_F - 2, 0), (k_F - 1, 1), (k_F, 0)\}.$$ 

Therefore by (9), (12) and $k_F \geq 3$, we obtain

$$\text{ind}_o(G_F) \leq 1 - \frac{k_F - 2}{2} = 2 - \frac{k_F}{2} \leq \frac{1}{2}.$$ 

Suppose $S_{g_F} \neq \emptyset$ and $F \in C^\infty,0)$. Then we obtain $\sharp \{ S_{g_F} \cap [\theta, \theta + \pi) \} = 1$, $(n_{g_F,+}, n_{g_F,-}) = (k_F - 1, 0)$ and $-\pi/2 \leq \Gamma_{F,F_0}(\theta_0) \leq \pi/2$ for any $\theta_0 \in S_{g_F}$. Therefore by (8), (12) and $k_F \geq 3$, we obtain

$$\text{ind}_o(G_F) \leq 1 - \frac{k_F - 1}{2} + \frac{1}{2} = 2 - \frac{k_F}{2} \leq \frac{1}{2}.$$ 

Suppose $S_{g_F} \neq \emptyset$, $F \in C^\infty,0)$ and $F \notin C^\infty,0)$. Then there exists an element $\theta_0 \in S_{g_F}$ satisfying $\tilde{g}_F(\theta_0) = 0$ and $\mu_{g_F}(\theta_0) = 2$. We obtain $\sharp \{ S_{g_F} \cap [\theta, \theta + \pi) \} = 1$ and $(n_{g_F,+}, n_{g_F,-}) = (k_F - 1, 0)$. We shall prove $-\pi/2 \leq \Gamma_{F,F_0}(\theta_0) \leq \pi/2$, which implies $\text{ind}_o(G_F) \leq 1/2$. We may suppose $\theta_0 = 0$ and represent $g_F$ as

$$g_F(x, y) = g_0(x, y)y^3,$$

where $g_0$ is a homogeneous polynomial of degree $k_F - 3$ satisfying $g_0(x, 0) \neq 0$ for any $x \in \mathbb{R} \setminus \{0\}$. We set

$$a_F := s_F + s_F p_F^2 - p_F q_F r_F,$$
$$2b_F := t_F - r_F + t_F p_F^2 - r_F q_F^2,$$
Then the following holds:

\[(1 + p_F^2 + q_F^2)PD_F = a_F dx^2 + 2b_F dx dy + c_F dy^2.\]

We set

\[\tilde{b}_F(\rho, \theta) := b_F(\rho \cos \theta, \rho \sin \theta)\]

for \((\rho, \theta) \in (-\rho_0, \rho_0) \times \mathbb{R}\), where \(\rho_0 > 0\) is a positive number such that there exists no umbilical point of \(G_F\) on \(\{0 < x^2 + y^2 < \rho_0^2\}\). There exists a smooth function \(\tilde{b}_F^{(k_F-2)}\) on \(\mathbb{R}\) satisfying

\[\tilde{b}_F(\rho, \theta) - \rho^{k_F-2}b_F^{(k_F-2)}(\theta) = o(\rho^{k_F-2}),\]

From (14), we obtain \((d\tilde{b}_F^{(k_F-2)}/d\theta)(0) \neq 0\). Therefore by the implicit function theorem, we see that there exist a neighborhood \(V_0\) of \((0, 0)\) in \(\mathbb{R}^2\) and a curve \(C_0\) in \(V_0\) through \((0, 0)\) satisfying

(a) \(C_0 = \{(\rho, \theta) \in V_0; \tilde{b}_F(\rho, \theta)/\rho^{k_F-2} = 0\}\);

(b) \(C_0\) is not tangent to the \(\theta\)-axis at \((0, 0)\).

Then noticing the behavior of the two continuous distributions around \(o\) defined by

\[b_F dx^2 + (c_F - a_F) dx dy - b_F dy^2 = 0,\]

we obtain \(-\pi/2 \leq \Gamma_{F,o}(\theta_0) \leq \pi/2.\)

References

AN ISOLATED UMBILICAL POINT

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