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AN ISOLATED UMBILICAL POINT OF A WILLMORE SURFACE

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1. Introduction

Let S be a surface in $\mathbf{R}^3$. Then it is known that if S is a surface with constant mean curvature, then the index of an isolated umbilical point on S is negative ([16]). If S is special Weingarten, then the same result is obtained ([15]). In the present paper, we shall prove that the index of an isolated umbilical point on a Willmore surface does not exceed $1/2$.

We say that S is a *Willmore surface* if S is a stationary surface of the Willmore functional $\mathcal{W}$, where the *Willmore functional* is defined by the integral of the square of the mean curvature. It is known that S is a Willmore surface if and only if S satisfies the following partial differential equation ([12]):

$$\{\Delta + 2(H^2 - K)\}H = 0, \tag{1}$$

where Δ is the Laplace operator on S and K, H are the Gaussian and the mean curvatures of S, respectively. Equation (1) is the Euler-Lagrange equation for Willmore surfaces.

Willmore proved that $\mathcal{W} \geqq 4\pi$ for any compact surface in $\mathbf{R}^3$ and that the equality holds if and only if the surface is a round sphere ([36], [37]). In addition, he and Shiohama-Takagi proved that $\mathcal{W} \geqq 2\pi^2$ $(> 4\pi)$ for a torus represented as the boundary of a tubular neighborhood of a closed curve in $\mathbf{R}^3$ and that the equality holds if and only if the torus is a $\sqrt{2}$-anchor ring, i.e., the boundary of the tubular neighborhood with radius $a > 0$ of a circle with radius $\sqrt{2}a$ ([38], [27]). Willmore conjectured $\mathcal{W} \geqq 2\pi^2$ for any torus in $\mathbf{R}^3$ ([36]). Since White showed that if the surface is compact and orientable, then $\mathcal{W}$ is invariant under any conformal transformation of $\overline{\mathbf{R}}^3 := \mathbf{R}^3 \cup \{\infty\}$ ([35]), it has been expected that the equality in Willmore's conjecture holds if and only if the torus is conformally equivalent in $\overline{\mathbf{R}}^3$ to a $\sqrt{2}$-anchor ring. Li-Yau showed that Willmore's conjecture is true for tori with certain conformal structures close to the conformal structure of a $\sqrt{2}$-anchor ring ([21]); Montiel-Ros showed that Willmore's conjecture is also true for tori with more conformal structures ([22]). Simon proved that there exists an embedded torus in $\mathbf{R}^3$ at which $\mathcal{W}$ attains the infimum on all the immersed tori ([28], [29]). Recently, the author has had paper [26] by Schmidt the main theorem of which states that Willmore's conjecture is true for any torus immersed in $\mathbf{R}^3$.

Weiner proved that the image of any minimal surface in S^3 by a stereographic projection is a Willmore surface in $\mathbf{R}^3$ ([34]). Any compact two-dimensional manifold other than the projective plane may be realized in S^3 as a minimal surface ([20]), while the projective plane may not be realized in S^3 as any minimal surface ([1], [20]). Therefore we see that any compact two-dimensional manifold distinct from the projective plane may be realized in $\mathbf{R}^3$ as a Willmore surface. Pinkall showed that there exists a Hopf torus in S^3 which is not conformally equivalent in S^3 to any minimal surface and the image of which by a stereographic projection is a Willmore surface in $\mathbf{R}^3$ ([24]). In addition, Kusner found an example of a Willmore surface in $\mathbf{R}^3$ which is homeomorphic to the projective plane ([18], [19]). At this example, $\mathcal{W}$ attains 12π, the infimum on all the projective planes immersed in $\mathbf{R}^3$. Bryant described the moduli space of the Willmore projective planes in $\mathbf{R}^3$ for each of which $\mathcal{W}$ is equal to 12π ([11]).

By Hopf-Poincaré's theorem together with Kusner's example of a Willmore projective plane, we see that our estimate of the index of an isolated umbilical point on a Willmore surface is sharp.

It is expected that the index of an isolated umbilical point on a surface does not exceed one. We call this conjecture the *index conjecture*. In relation to the index conjecture, the following two conjectures are known: Carathéodory's conjecture and Loewner's conjecture. *Carathéodory's conjecture* asserts that there exist at least two umbilical points on a compact, strictly convex surface in $\mathbf{R}^3$. If the index conjecture is true, then we see from Hopf-Poincaré's theorem that there exist at least two umbilical points on a compact, orientable surface of genus zero, and this immediately gives the affirmative answer to Carathéodory's conjecture. Let F be a real-valued, smooth function of two real variables x, y, and set $\partial_{\bar{z}} := (\partial/\partial x + \sqrt{-1}\partial/\partial y)/2$. Then *Loewner's conjecture* for a positive integer $n \in \mathbf{N}$ asserts that if a vector field $\mathrm{Re}\left(\partial_{\bar{z}}^n F\right)(\partial/\partial x) + \mathrm{Im}\left(\partial_{\bar{z}}^n F\right)(\partial/\partial y)$ has an isolated zero point, then its index with respect to this vector field does not exceed n ([17], [33]). Loewner's conjecture for $n = 1$ is affirmatively solved; Loewner's conjecture for $n = 2$ is equivalent to the index conjecture. We may find [9], [13], [30], [31] and [32] as recent papers in relation to Carathéodory's and Loewner's conjectures. We discussed the index of an isolated umbilical point on a surface in [2]–[7], and in [8], we introduced and studied a conjecture in relation to Loewner's conjecture.

We see from our estimate of the index in the present paper that the index conjecture is true for any isolated umbilical point on a Willmore surface. In the proof of the main theorem, we shall encounter a situation on a surface with an isolated umbilical point which has not appeared in our previous studies.

2. Willmore surfaces

Let M be a connected, orientable two-dimensional manifold and $\iota\colon M \to \mathbf{R}^3$ an immersion of M into $\mathbf{R}^3$. Let H be the mean curvature of M with respect to ι and dA the area element of M with respect to the metric g induced by ι. Then the *Willmore functional* $\mathcal{W}$ is given by

$$\mathcal{W}(\iota) := \int_M H^2\,dA.$$

Let K be the Gaussian curvature of M with respect to the metric g and set

$$\widehat{\mathcal{W}}(\iota) := \int_M (H^2 - K)\,dA.$$

Then we obtain

$$\widehat{\mathcal{W}}(\iota) = \mathcal{W}(\iota) - \int_M K\,dA. \tag{2}$$

It is known that for any conformal transformation X of $\overline{\mathbf{R}}^3$ such that $X \circ \iota$ is an immersion, the following holds ([35]):

$$\widehat{\mathcal{W}}(X \circ \iota) = \widehat{\mathcal{W}}(\iota). \tag{3}$$

If M is compact, then by (2), (3) and Gauss-Bonnet's theorem, we obtain

$$\mathcal{W}(X \circ \iota) = \mathcal{W}(\iota).$$

Let M and ι be as above. Let ξ be a unit normal vector field on M with respect to ι and f a smooth function on M with compact support. Let ι_f be a smooth map from $M \times \mathbf{R}$ into $\mathbf{R}^3$ satisfying $\iota_f(p,0) = \iota(p)$, $(\partial\iota_f/\partial t)(p,0) = f(p)\xi(p)$ for $p \in M$ and the condition that $\iota_f(p,t) = \iota_f(p,0)$ for any $t \in \mathbf{R}$ and any point p of M outside the support of f. We set $\iota_{f,t}(p) := \iota_f(p,t)$ for $(p,t) \in M \times \mathbf{R}$. Then there exists an open interval I containing 0 such that for each $t \in I$, $\iota_{f,t}$ is an immersion of M into $\mathbf{R}^3$. We set

$$w_f(t) := \mathcal{W}(\iota_{f,t}), \;\; \widehat{w}_f(t) := \widehat{\mathcal{W}}(\iota_{f,t}).$$

An immersion ι is called *Willmore* if $(dw_f/dt)(0) = 0$ holds for any smooth function f on M with compact support; if ι is a Willmore immersion, then the pair (M,ι) or the image $\iota(M)$ of M by ι is called a *Willmore surface*. An immersion ι is Willmore if and only if (1) holds, where Δ is the Laplace operator on M with respect to the metric g ([12]). Let D be a domain in M which contains the support of f

and the boundary of which consists of a finite number of closed curves. Then for $t \in I$, $w_f(t) - \widehat{w}_f(t)$ is represented as follows:

$$w_f(t) - \widehat{w}_f(t) = \int_{M\setminus D} K_t\, dA_t + \int_D K_t\, dA_t, \tag{4}$$

where K_t and dA_t are the Gaussian curvature and the area element of M with respect to the metric induced by $\iota_{f,t}$, respectively. From Gauss-Bonnet's theorem, we see that the second term of the right hand side of (4) depends only on the boundary of D, which implies that this term does not depend on $t \in I$. In addition, since D contains the support of f, the first term of the right hand side of (4) does not depend on $t \in I$ either. Therefore we see that $w_f - \widehat{w}_f$ is constant on I. In particular, we obtain

$$\frac{d\widehat{w}_f}{dt}(0) = \frac{dw_f}{dt}(0). \tag{5}$$

By (3) together with (5), we obtain

Proposition 2.1. *Let ι be an immersion of M into $\mathbf{R}^3$ and X a conformal transformation of $\overline{\mathbf{R}}^3$ such that $X\circ\iota$ is an immersion. Then ι is Willmore if and only if $X\circ\iota$ is Willmore.*

3. The index of an isolated umbilical point

Let f be a smooth function of two variables x, y and G_f the graph of f. We set

$$p_f := \frac{\partial f}{\partial x},\ \ q_f := \frac{\partial f}{\partial y},\ \ r_f := \frac{\partial^2 f}{\partial x^2},\ \ s_f := \frac{\partial^2 f}{\partial x \partial y},\ \ t_f := \frac{\partial^2 f}{\partial y^2}.$$

Then the Gaussian curvature K_f and the mean curvature H_f of G_f are represented as follows:

$$K_f := \frac{r_f t_f - s_f^2}{(1+p_f^2+q_f^2)^2},\ \ H_f := \frac{r_f + t_f + p_f^2 t_f - 2p_f q_f s_f + q_f^2 r_f}{2(1+p_f^2+q_f^2)^{3/2}}. \tag{6}$$

Let D_f, N_f, PD_f be symmetric tensor fields on G_f of type $(0,2)$ represented in terms of the coordinates (x, y) as follows:

$$\begin{aligned}
\mathrm{D}_f &:= s_f\, dx^2 + (t_f - r_f)\, dx\, dy - s_f\, dy^2,\\
\mathrm{N}_f &:= (s_f p_f^2 - p_f q_f r_f)\, dx^2 + (t_f p_f^2 - r_f q_f^2)\, dx\, dy + (p_f q_f t_f - s_f q_f^2)\, dy^2,\\
\mathrm{PD}_f &:= \frac{1}{1+p_f^2+q_f^2}(\mathrm{D}_f + \mathrm{N}_f).
\end{aligned}$$

A tangent vector $\boldsymbol{v}_0$ to G_f at a point is in a principal direction if and only if $\mathrm{PD}_f(\boldsymbol{v}_0, \boldsymbol{v}_0) = 0$ holds ([5]). For a tangent vector $\boldsymbol{v}$, we set

$$\tilde{\mathrm{D}}_f(\boldsymbol{v}) := \mathrm{D}_f(\boldsymbol{v}, \boldsymbol{v}), \quad \tilde{\mathrm{N}}_f(\boldsymbol{v}) := \mathrm{N}_f(\boldsymbol{v}, \boldsymbol{v}), \quad \widetilde{\mathrm{PD}}_f(\boldsymbol{v}) := \mathrm{PD}_f(\boldsymbol{v}, \boldsymbol{v}).$$

For $\phi \in \mathbf{R}$, we set

$$u_\phi := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad \mathbf{U}_\phi := \cos\phi \frac{\partial}{\partial x} + \sin\phi \frac{\partial}{\partial y}.$$

We set

$$\mathrm{grad}_f := \begin{pmatrix} p_f \\ q_f \end{pmatrix}, \quad \mathrm{grad}_f^\perp := \begin{pmatrix} -q_f \\ p_f \end{pmatrix}, \quad \mathrm{Hess}_f := \begin{pmatrix} r_f & s_f \\ s_f & t_f \end{pmatrix}.$$

Let $\langle\ ,\ \rangle$ be the scalar product in $\mathbf{R}^2$. Then for any $\phi \in \mathbf{R}$, the following hold ([5]):

$$\begin{aligned} \tilde{\mathrm{D}}_f(\mathbf{U}_\phi) &= \langle \mathrm{Hess}_f\, u_\phi, u_{\phi+\pi/2} \rangle, \\ \tilde{\mathrm{N}}_f(\mathbf{U}_\phi) &= \langle \mathrm{grad}_f, u_\phi \rangle \langle \mathrm{grad}_f^\perp, \mathrm{Hess}_f\, u_\phi \rangle. \end{aligned}$$

For $l \in \mathbf{N} \cup \{\infty\}$, let $\mathcal{C}_o^{(\infty,l)}$ be the set of smooth functions defined on a connected neighborhood of $(0,0)$ in $\mathbf{R}^2$ such that $(\partial^{m+n} F/\partial x^m \partial y^n)(0,0) = 0$ for each $F \in \mathcal{C}_o^{(\infty,l)}$ and non-negative integers m, n satisfying $0 \leqq m + n < l$. The following hold:

$$\mathcal{C}_o^{(\infty,l)} \supset \mathcal{C}_o^{(\infty,l+1)} \supset \mathcal{C}_o^{(\infty,\infty)} \neq \{0\},$$

where $l \in \mathbf{N}$. Let F be an element of $\mathcal{C}_o^{(\infty,2)}$ such that $o := (0,0,0)$ is an umbilical point of the graph of F, that is, there exists a real number a_F satisfying

$$F(x, y) = \frac{a_F(x^2 + y^2)}{2} + o(x^2 + y^2). \tag{7}$$

Let σ_F be an element of $\mathcal{C}_o^{(\infty,2)}$ defined by

$$\sigma_F := \begin{cases} 0 & \text{if } a_F = 0, \\ \dfrac{1}{a_F} - \dfrac{|a_F|}{a_F}\sqrt{\dfrac{1}{a_F^2} - (x^2 + y^2)} & \text{if } a_F \neq 0. \end{cases}$$

Then we obtain $F - \sigma_F \in \mathcal{C}_o^{(\infty,3)}$. For an integer $l \geqq 2$, let $\mathcal{C}_o^{\langle\infty,l\rangle}$ be the subset of $\mathcal{C}_o^{(\infty,l)}$ such that each $F \in \mathcal{C}_o^{\langle\infty,l\rangle}$ satisfies (7) for some $a_F \in \mathbf{R}$ and $F - \sigma_F \notin \mathcal{C}_o^{(\infty,\infty)}$. For an integer $k \geqq 3$, let $\mathcal{P}^k$ be the set of the homogeneous polynomials of degree k. Then for each $F \in \mathcal{C}_o^{\langle\infty,2\rangle}$, there exist an integer $k_F \geqq 3$ and a nonzero element g_F of $\mathcal{P}^{k_F}$ satisfying $F - \sigma_F - g_F \in \mathcal{C}_o^{(\infty,k_F+1)}$. Let g be an element of $\mathcal{P}^k$.

Then set $\mathrm{Hess}_g(\theta) := \mathrm{Hess}_g(\cos\theta, \sin\theta)$ for $\theta \in \mathbf{R}$ and let η_g be a continuous function on $\mathbf{R}$ such that for any $\theta \in \mathbf{R}$, $u_{\eta_g(\theta)}$ is an eigenvector of $\mathrm{Hess}_g(\theta)$, and let S_g denote the set of the numbers at each of which Hess_g is represented by the unit matrix up to a constant.

Let $\mathcal{C}_o^{\infty,2}$ be the subset of $\mathcal{C}_o^{\langle\infty,2\rangle}$ such that on the graph G_F of each $F \in \mathcal{C}_o^{\infty,2}$, o is an isolated umbilical point. For an element F of $\mathcal{C}_o^{\infty,2}$, let ρ_0 be a positive number such that there exists no umbilical point of G_F on $\{0 < x^2 + y^2 < \rho_0^2\}$ and ϕ_F a continuous function on $(0, \rho_0) \times \mathbf{R}$ such that for each $(\rho, \theta) \in (0, \rho_0) \times \mathbf{R}$, a tangent vector $\cos\phi_F(\rho, \theta)\partial/\partial x + \sin\phi_F(\rho, \theta)\partial/\partial y$ of G_F at $(\rho\cos\theta, \rho\sin\theta)$ is in a principal direction. Then the following (a)–(c) hold ([5], [6]):

(a) For any $\theta_0 \in \mathbf{R} \setminus S_{g_F}$, there exists a number $\phi_{F,o}(\theta_0)$ satisfying the following:
 (i) $\lim_{\rho\to 0} \phi_F(\rho, \theta_0) = \phi_{F,o}(\theta_0)$,
 (ii) $u_{\phi_{F,o}(\theta_0)}$ is an eigenvector of $\mathrm{Hess}_{g_F}(\theta_0)$;

(b) For any $\theta_0 \in \mathbf{R}$, there exist numbers $\phi_{F,o}(\theta_0 + 0)$, $\phi_{F,o}(\theta_0 - 0)$ satisfying the following:
 (i) $\lim_{\theta\to\theta_0\pm 0} \phi_{F,o}(\theta) = \phi_{F,o}(\theta_0 \pm 0)$,
 (ii) $\Gamma_{F,o}(\theta_0) := \phi_{F,o}(\theta_0 + 0) - \phi_{F,o}(\theta_0 - 0)$ is an element of $\{n\pi/2\}_{n\in\mathbf{Z}}$;

(c) The *index* $\mathrm{ind}_o(\mathsf{G}_F)$ of o on G_F is represented as follows:

$$\mathrm{ind}_o(\mathsf{G}_F) = \frac{\eta_{g_F}(\theta + 2\pi) - \eta_{g_F}(\theta)}{2\pi} + \frac{1}{2\pi} \sum_{\theta_0 \in S_{g_F} \cap [\theta, \theta+2\pi)} \Gamma_{F,o}(\theta_0). \tag{8}$$

For an integer $k \geqq 3$, set $\mathcal{P}_o^k := \mathcal{P}^k \cap \mathcal{C}_o^{\infty,2}$. Then for any $g \in \mathcal{P}_o^k$, the following hold: $\Gamma_{g,o}(\theta_0) = -\pi/2$ for any $\theta_0 \in S_g$ ([4]); $\mathrm{ind}_o(\mathsf{G}_g) \in \{1 - k/2 + i\}_{i=0}^{[k/2]}$ ([2]). Let $\mathcal{C}_{oo}^{\infty,2}$ be the subset of $\mathcal{C}_o^{\infty,2}$ such that for each $F \in \mathcal{C}_{oo}^{\infty,2}$, o is an isolated umbilical point on each of G_F and G_{g_F}. If F is an element of $\mathcal{C}_o^{\langle\infty,2\rangle}$ satisfying $S_{g_F} = \emptyset$, then $F \in \mathcal{C}_{oo}^{\infty,2}$ holds ([5], [6]). We see that if $F \in \mathcal{C}_{oo}^{\infty,2}$ satisfies $S_{g_F} = \emptyset$, then the following hold:

$$\mathrm{ind}_o(\mathsf{G}_F) = \mathrm{ind}_o(\mathsf{G}_{g_F}) = \frac{\eta_{g_F}(\theta + 2\pi) - \eta_{g_F}(\theta)}{2\pi}. \tag{9}$$

For any $F \in \mathcal{C}_{oo}^{\infty,2}$, the following hold ([5], [6]):
(a) $-\pi/2 \leqq \Gamma_{F,o}(\theta_0) \leqq \pi/2$ for any $\theta_0 \in S_{g_F}$;
(b) $\mathrm{ind}_o(\mathsf{G}_{g_F}) \leqq \mathrm{ind}_o(\mathsf{G}_F) \leqq 1$.

If F is an element of $\mathcal{C}_o^{\infty,2}$ satisfying $\Gamma_{F,o}(\theta_0) \leqq \pi$ for any $\theta_0 \in S_{g_F}$, then $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1$ holds ([5], [6]). In general, it is expected that the index of an isolated umbilical point on a surface does not exceed one (which is called the *index conjecture* or the *local Carathéodory's conjecture*).

We presented one way of computing $\eta_g(\theta + 2\pi) - \eta_g(\theta)$ for any $g \in \mathcal{P}^k$ ([5]). For $\theta \in \mathbf{R}$, set $\tilde{g}(\theta) := g(\cos\theta, \sin\theta)$. A number $\theta_0 \in \mathbf{R}$ is called a *root* of g if $(d\tilde{g}/d\theta)(\theta_0) = 0$. The set of the roots of g is denoted by R_g. Let $R(\mathrm{Hess}_g)$ be the set of numbers such that each $\theta_0 \in R(\mathrm{Hess}_g)$ satisfies $\theta_0 \in \{\eta_g(\theta_0) + n\pi/2\}_{n\in\mathbf{Z}}$. For $\theta \in \mathbf{R}$, we set $\mathrm{grad}_g(\theta) := \mathrm{grad}_g(\cos\theta, \sin\theta)$. Then the following holds:

$$(k-1)\,\mathrm{grad}_g(\theta) = \mathrm{Hess}_g(\theta)u_\theta. \tag{10}$$

From (10), we obtain

$$\langle \mathrm{Hess}_g(\theta)u_\theta, u_{\theta+\pi/2}\rangle = (k-1)\frac{d\tilde{g}}{d\theta}(\theta). \tag{11}$$

Therefore we obtain $S_g \subset R_g$ and $R(\mathrm{Hess}_g) \subset R_g$. Suppose $R_g = \mathbf{R}$. Then k is even and g is represented by $(x^2+y^2)^{k/2}$ up to a constant. By direct computations, we obtain $S_g = \emptyset$. Therefore o is an isolated umbilical point of G_g. By (11), we see that $R(\mathrm{Hess}_g) = \mathbf{R}$, i.e., there exists a number $z_0 \in \{n\pi/2\}_{n\in\mathbf{Z}}$ satisfying $\eta_g(\theta) = \theta + z_0$ for any $\theta \in \mathbf{R}$. Therefore by (9), we obtain

$$\mathrm{ind}_o(\mathsf{G}_g) = \frac{\eta_g(\theta+2\pi) - \eta_g(\theta)}{2\pi} = 1.$$

In the following, suppose $R_g \neq \mathbf{R}$. Then for each $\theta_0 \in R_g$, there exists a positive integer μ satisfying $(d^{\mu+1}\tilde{g}/d\theta^{\mu+1})(\theta_0) \neq 0$. The minimum of such integers is denoted by $\mu_g(\theta_0)$. A root $\theta_0 \in R_g$ is said to be
(a) *related* if θ_0 satisfies $\tilde{g}(\theta_0) = 0$ or if $\mu_g(\theta_0)$ is odd;
(b) *non-related* if θ_0 satisfies $\tilde{g}(\theta_0) \neq 0$ and if $\mu_g(\theta_0)$ is even.
Suppose that $\theta_0 \in R_g$ is related. Then it is said that the *critical sign* of θ_0 is positive (respectively, negative) if the following holds:

$$\tilde{g}(\theta_0)\frac{d^{\mu_g(\theta_0)+1}\tilde{g}}{d\theta^{\mu_g(\theta_0)+1}}(\theta_0) \leqq 0 \quad (\text{respectively}, \ > 0).$$

The critical sign of θ_0 is denoted by $\text{c-sign}_g(\theta_0)$. The set $R_g \setminus R(\mathrm{Hess}_g)$ consists of the numbers at each of which Hess_g is represented by the unit matrix up to a nonzero constant; in addition, an element $\theta_0 \in R_g \setminus R(\mathrm{Hess}_g)$ is related and satisfies $\text{c-sign}_g(\theta_0) = -$ ([5]). It is said that the *sign* of $\theta_0 \in R(\mathrm{Hess}_g)$ is positive (respectively, negative) if there exists a neighborhood U_{θ_0} of θ_0 in $\mathbf{R}$ satisfying

$$\{\theta - \eta_g(\theta) - (\theta_0 - \eta_g(\theta_0))\}(\theta - \theta_0) > 0 \quad (\text{respectively}, \ < 0)$$

for any $\theta \in U_{\theta_0} \setminus \{\theta_0\}$. For $\theta_0 \in R(\mathrm{Hess}_g)$, θ_0 is related if and only if the sign of θ_0 is positive or negative ([5]). If $\theta_0 \in R(\mathrm{Hess}_g)$ is related, then the sign of θ_0 is denoted by $\mathrm{sign}_g(\theta_0)$. For a related root θ_0 of g satisfying $\text{c-sign}_g(\theta_0) = +$, $\theta_0 \in R(\mathrm{Hess}_g)$ and $\mathrm{sign}_g(\theta_0) = +$ hold ([5]). Referring to [3], we see that if θ_0 is a related element of $R(\mathrm{Hess}_g)$ satisfying $\text{c-sign}_g(\theta_0) = -$, then the condition $\mathrm{sign}_g(\theta_0) = +$ (respectively, $-$) is equivalent to the following:

$$\frac{1}{\tilde{g}(\theta_0)}\frac{d^2\tilde{g}}{d\theta^2}(\theta_0) \in (k(k-2), \infty) \quad (\text{respectively}, \ [0, k(k-2))).$$

Let $n_{g,+}$ (respectively, $n_{g,-}$) denote the number of the related elements of $R(\mathrm{Hess}_g)$ in $[\theta, \theta+\pi)$ with positive (respectively, negative) sign. Then for any $\theta \in \mathbf{R}$, the following holds ([5]):

$$\frac{\eta_g(\theta + 2\pi) - \eta_g(\theta)}{2\pi} = 1 - \frac{n_{g,+} - n_{g,-}}{2}. \tag{12}$$

4. The main theorem

We shall prove

Theorem 4.1. *Let F be an element of $\mathcal{C}_o^{(\infty,2)}$ satisfying* (7) *for some $a_F \in \mathbf{R}$ and suppose that the graph G_F of F is a Willmore surface such that there exists no totally umbilical neighborhood of o in G_F. Then the following hold*:
(a) $F \in \mathcal{C}_o^{\langle\infty,2\rangle}$;
(b) *If o is an isolated umbilical point of G_F, then* $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1/2$.

REMARK. Noticing Proposition 2.1 and that whether a one-dimensional subspace of the tangent plane at a point of a surface is a principal direction is invariant under any conformal transformation of $\overline{\mathbf{R}}^3$, we may suppose $F \in \mathcal{C}_o^{(\infty,3)}$ in Theorem 4.1.

REMARK. Although F is an element of $\mathcal{C}_o^{(\infty,2)}$ such that o is an isolated umbilical point of G_F, $F \in \mathcal{C}_o^{\langle\infty,2\rangle}$ does not always hold. Let f be a smooth function on a neighborhood of $(0,0)$ in $\mathbf{R}^2$ satisfying $f(0,0) = 0$ and $f > 0$ on a punctured neighborhood of $(0,0)$. Then $\exp(-1/f)$ is a smooth function defined on a punctured neighborhood of $(0,0)$ and smoothly extended to $(0,0)$ so that all the partial derivatives of $\exp(-1/f)$ at $(0,0)$ are equal to zero. Then we obtain $\exp(-1/f) \in \mathcal{C}_o^{(\infty,\infty)}$. Suppose that for each positive number $c > 0$, there exists a punctured neighborhood of $(0,0)$ on which the norm of the gradient vector field of $\log f$ is bounded from below by the number c. Then o is an isolated umbilical point on the graph of $\exp(-1/f)$ ([7]). However, since $\exp(-1/f) \in \mathcal{C}_o^{(\infty,\infty)}$, we obtain $\exp(-1/f) \notin \mathcal{C}_o^{\langle\infty,2\rangle}$. (a) of Theorem 4.1 is crucial to the proof of (b) of Theorem 4.1.

Proof of (a) of Theorem 4.1. Let Δ_F be the Laplace operator on G_F, and K_F, H_F the Gaussian and the mean curvatures of G_F, respectively. Then H_F satisfies the following elliptic partial differential equation:

$$\{\Delta_F + 2(H_F^2 - K_F)\}H_F = 0. \tag{13}$$

If $H_F \equiv 0$, then G_F is a minimal surface and F is real-analytic. Since G_F is not totally umbilical, we obtain $F \not\equiv 0$ and this implies $F \in \mathcal{C}_o^{\langle\infty,3\rangle}$. If $H_F \not\equiv 0$, then H_F is a non-trivial solution of (13) and referring to [14] as in [15], we see that not all the partial derivatives of H_F at $(0,0)$ are equal to zero. This implies $F \in \mathcal{C}_o^{\langle\infty,3\rangle}$.

Hence we obtain (a) of Theorem 4.1. □

Proof of (b) of Theorem 4.1. Let F be an element of $\mathcal{C}_o^{\langle\infty,3\rangle}$ such that the graph G_F of F is a Willmore surface. Then there exist an integer $k_F \geqq 3$ and a nonzero homogeneous polynomial $g_F \in \mathcal{P}^{k_F}$ satisfying $F - g_F \in \mathcal{C}_o^{\langle\infty,k_F+1\rangle}$, and noticing (6) and (13), we see that g_F satisfies $\Delta_0^2 g_F \equiv 0$, where $\Delta_0 := (\partial/\partial x)^2+(\partial/\partial y)^2$. Therefore there exist spherical harmonic functions h_{k_F}, h_{k_F-2} of degree k_F, k_F-2, respectively such that g_F is represented as

$$g_F = h_{k_F} + (x^2+y^2)h_{k_F-2}.$$

Suppose $S_{g_F} = \emptyset$. Then $F \in \mathcal{C}_{oo}^{\infty,2}$ holds. Noticing that the number of the zero points of $\tilde{g}_F$ in $[\theta, \theta+\pi)$ is more than or equal to $k_F - 2$, we obtain

$$k_F - 2 \leqq \sharp\{R_{g_F} \cap [\theta, \theta+\pi)\} \leqq k_F$$

and

$$(n_{g_F,+}, n_{g_F,-}) \in \{(k_F-2, 0), (k_F-1, 1), (k_F, 0)\}.$$

Therefore by (9), (12) and $k_F \geqq 3$, we obtain

$$\mathrm{ind}_o(\mathsf{G}_F) \leqq 1 - \frac{k_F-2}{2} = 2 - \frac{k_F}{2} \leqq \frac{1}{2}.$$

Suppose $S_{g_F} \neq \emptyset$ and $F \in \mathcal{C}_{oo}^{\infty,2}$. Then we obtain $\sharp\{S_{g_F} \cap [\theta, \theta+\pi)\} = 1$, $(n_{g_F,+}, n_{g_F,-}) = (k_F-1, 0)$ and $-\pi/2 \leqq \Gamma_{F,o}(\theta_0) \leqq \pi/2$ for any $\theta_0 \in S_{g_F}$. Therefore by (8), (12) and $k_F \geqq 3$, we obtain

$$\mathrm{ind}_o(\mathsf{G}_F) \leqq 1 - \frac{k_F-1}{2} + \frac{1}{2} = 2 - \frac{k_F}{2} \leqq \frac{1}{2}.$$

Suppose $S_{g_F} \neq \emptyset$, $F \in \mathcal{C}_o^{\infty,2}$ and $F \notin \mathcal{C}_{oo}^{\infty,2}$. Then there exists an element $\theta_0 \in S_{g_F}$ satisfying $\tilde{g}_F(\theta_0) = 0$ and $\mu_{g_F}(\theta_0) = 2$. We obtain $\sharp\{S_{g_F} \cap [\theta, \theta+\pi)\} = 1$ and $(n_{g_F,+}, n_{g_F,-}) = (k_F-1, 0)$. We shall prove $-\pi/2 \leqq \Gamma_{F,o}(\theta_0) \leqq \pi/2$, which implies $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1/2$. We may suppose $\theta_0 = 0$ and represent g_F as

$$g_F(x, y) = g_0(x, y)y^3, \tag{14}$$

where g_0 is a homogeneous polynomial of degree $k_F - 3$ satisfying $g_0(x, 0) \neq 0$ for any $x \in \mathbf{R} \setminus \{0\}$. We set

$$a_F := s_F + s_F p_F^2 - p_F q_F r_F,$$
$$2b_F := t_F - r_F + t_F p_F^2 - r_F q_F^2,$$

$$c_F := -s_F - s_F q_F^2 + p_F q_F t_F.$$

Then the following holds:

$$(1 + p_F^2 + q_F^2)\,\mathrm{PD}_F = a_F\,dx^2 + 2b_F\,dx\,dy + c_F\,dy^2.$$

We set

$$\tilde{b}_F(\rho, \theta) := b_F(\rho\cos\theta, \rho\sin\theta)$$

for $(\rho, \theta) \in (-\rho_0, \rho_0) \times \mathbf{R}$, where $\rho_0 > 0$ is a positive number such that there exists no umbilical point of G_F on $\{0 < x^2 + y^2 < \rho_0^2\}$. There exists a smooth function $\tilde{b}_F^{(k_F-2)}$ on $\mathbf{R}$ satisfying

$$\tilde{b}_F(\rho, \theta) - \rho^{k_F-2}\tilde{b}_F^{(k_F-2)}(\theta) = o(\rho^{k_F-2}).$$

From (14), we obtain $(d\tilde{b}_F^{(k_F-2)}/d\theta)(0) \neq 0$. Therefore by the implicit function theorem, we see that there exist a neighborhood V_0 of $(0, 0)$ in $\mathbf{R}^2$ and a curve C_0 in V_0 through $(0, 0)$ satisfying
(a) $C_0 = \{(\rho, \theta) \in V_0\,;\, \tilde{b}_F(\rho, \theta)/\rho^{k_F-2} = 0\}$;
(b) C_0 is not tangent to the θ-axis at $(0, 0)$.
Then noticing the behavior of the two continuous distributions around o defined by

$$b_F\,dx^2 + (c_F - a_F)\,dx\,dy - b_F\,dy^2 = 0,$$

we obtain $-\pi/2 \leqq \Gamma_{F,o}(\theta_0) \leqq \pi/2$. $\square$

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