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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 41(4) P.865-P.876</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-12</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
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<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/12782">https://doi.org/10.18910/12782</a></td>
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<td>DOI</td>
<td>10.18910/12782</td>
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AN ISOLATED UMBILICAL POINT OF A WILLMORE SURFACE

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(Received April 9, 2003)

1. Introduction

Let $S$ be a surface in $\mathbb{R}^3$. Then it is known that if $S$ is a surface with constant mean curvature, then the index of an isolated umbilical point on $S$ is negative ([16]). If $S$ is special Weingarten, then the same result is obtained ([15]). In the present paper, we shall prove that the index of an isolated umbilical point on a Willmore surface does not exceed $1/2$.

We say that $S$ is a Willmore surface if $S$ is a stationary surface of the Willmore functional $\mathcal{W}$, where the Willmore functional is defined by the integral of the square of the mean curvature. It is known that $S$ is a Willmore surface if and only if $S$ satisfies the following partial differential equation ([12]):

$$\left\{ \Delta + 2(H^2 - K) \right\}H = 0,$$

where $\Delta$ is the Laplace operator on $S$ and $K, H$ are the Gaussian and the mean curvatures of $S$, respectively. Equation (1) is the Euler-Lagrange equation for Willmore surfaces.

Willmore proved that $\mathcal{W} \geq 4\pi$ for any compact surface in $\mathbb{R}^3$ and that the equality holds if and only if the surface is a round sphere ([36], [37]). In addition, he and Shiohama-Takagi proved that $\mathcal{W} \geq 2\pi^2 (> 4\pi)$ for a torus represented as the boundary of a tubular neighborhood of a closed curve in $\mathbb{R}^3$ and that the equality holds if and only if the torus is a $\sqrt{2}$-anchor ring, i.e., the boundary of the tubular neighborhood with radius $a > 0$ of a circle with radius $\sqrt{2}a$ ([38], [27]). Willmore conjectured $\mathcal{W} \geq 2\pi^2$ for any torus in $\mathbb{R}^3$ ([36]). Since White showed that if the surface is compact and orientable, then $\mathcal{W}$ is invariant under any conformal transformation of $\overline{\mathbb{R}^3} := \mathbb{R}^3 \cup \{ \infty \}$ ([35]), it has been expected that the equality in Willmore’s conjecture holds if and only if the torus is conformally equivalent in $\overline{\mathbb{R}^3}$ to a $\sqrt{2}$-anchor ring. Li-Yau showed that Willmore’s conjecture is true for tori with certain conformal structures close to the conformal structure of a $\sqrt{2}$-anchor ring ([21]); Montiel-Ros showed that Willmore’s conjecture is also true for tori with more conformal structures ([22]). Simon proved that there exists an embedded torus in $\mathbb{R}^3$ at which $\mathcal{W}$ attains the infimum on all the immersed tori ([28], [29]). Recently, the author has had paper [26] by Schmidt the main theorem of which states that Willmore’s conjecture is
true for any torus immersed in $\mathbb{R}^3$.

Weiner proved that the image of any minimal surface in $S^3$ by a stereographic projection is a Willmore surface in $\mathbb{R}^3$ ([34]). Any compact two-dimensional manifold other than the projective plane may be realized in $S^3$ as a minimal surface ([20]), while the projective plane may not be realized in $S^3$ as any minimal surface ([1], [20]). Therefore we see that any compact two-dimensional manifold distinct from the projective plane may be realized in $\mathbb{R}^3$ as a Willmore surface. Pinkall showed that there exists a Hopf torus in $S^3$ which is not conformally equivalent in $S^3$ to any minimal surface and the image of which by a stereographic projection is a Willmore surface in $\mathbb{R}^3$ ([24]). In addition, Kusner found an example of a Willmore surface in $\mathbb{R}^3$ which is homeomorphic to the projective plane ([18], [19]). At this example, $\mathcal{W}$ attains $12\pi$, the infimum on all the projective planes immersed in $\mathbb{R}^3$. Bryant described the moduli space of the Willmore projective planes in $\mathbb{R}^3$ for each of which $\mathcal{W}$ is equal to $12\pi$ ([11]).

By Hopf-Poincaré’s theorem together with Kusner’s example of a Willmore projective plane, we see that our estimate of the index of an isolated umbilical point on a Willmore surface is sharp.

It is expected that the index of an isolated umbilical point on a surface does not exceed one. We call this conjecture the index conjecture. In relation to the index conjecture, the following two conjectures are known: Carathéodory’s conjecture and Loewner’s conjecture. Carathéodory’s conjecture asserts that there exist at least two umbilical points on a compact, strictly convex surface in $\mathbb{R}^3$. If the index conjecture is true, then we see from Hopf-Poincaré’s theorem that there exist at least two umbilical points on a compact, orientable surface of genus zero, and this immediately gives the affirmative answer to Carathéodory’s conjecture. Let $F$ be a real-valued, smooth function of two real variables $x_1, y$, and set $\partial_x := (\partial/\partial x + \sqrt{-1}\partial/\partial y)/2$. Then Loewner’s conjecture for a positive integer $n \in \mathbb{N}$ asserts that if a vector field $\Re(\partial_x F) (\partial/\partial x) + \Im(\partial_x F) (\partial/\partial y)$ has an isolated zero point, then its index with respect to this vector field does not exceed $n$ ([17], [33]). Loewner’s conjecture for $n = 1$ is affirmatively solved; Loewner’s conjecture for $n = 2$ is equivalent to the index conjecture. We may find [9], [13], [30], [31] and [32] as recent papers in relation to Carathéodory’s and Loewner’s conjectures. We discussed the index of an isolated umbilical point on a surface in [2]–[7], and in [8], we introduced and studied a conjecture in relation to Loewner’s conjecture.

We see from our estimate of the index in the present paper that the index conjecture is true for any isolated umbilical point on a Willmore surface. In the proof of the main theorem, we shall encounter a situation on a surface with an isolated umbilical point which has not appeared in our previous studies.
2. Willmore surfaces

Let $M$ be a connected, orientable two-dimensional manifold and $\iota: M \to \mathbb{R}^3$ an immersion of $M$ into $\mathbb{R}^3$. Let $H$ be the mean curvature of $M$ with respect to $\iota$ and $dA$ the area element of $M$ with respect to the metric $g$ induced by $\iota$. Then the Willmore functional $\mathcal{W}$ is given by

$$ \mathcal{W}(\iota) := \int_M H^2 \, dA. $$

Let $K$ be the Gaussian curvature of $M$ with respect to the metric $g$ and set

$$ \mathcal{W}(\iota) := \int_M (H^2 - K) \, dA. $$

Then we obtain

$$ \mathcal{W}(\iota) = \mathcal{W}(\iota) - \int_M K \, dA. $$

(2)

It is known that for any conformal transformation $X$ of $\mathbb{R}^3$ such that $X \circ \iota$ is an immersion, the following holds ([35]):

$$ \mathcal{W}(X \circ \iota) = \mathcal{W}(\iota). $$

(3)

If $M$ is compact, then by (2), (3) and Gauss-Bonnet’s theorem, we obtain

$$ \mathcal{W}(X \circ \iota) = \mathcal{W}(\iota). $$

Let $M$ and $\iota$ be as above. Let $\xi$ be a unit normal vector field on $M$ with respect to $\iota$ and $f$ a smooth function on $M$ with compact support. Let $\iota_f$ be a smooth map from $M \times \mathbb{R}$ into $\mathbb{R}^3$ satisfying $\iota_f(p, 0) = \iota(p)$, $\left( \partial \iota_f / \partial t \right)(p, 0) = f(p) \xi(p)$ for $p \in M$ and the condition that $\iota_f(p, t) = \iota_f(p, 0)$ for any $t \in \mathbb{R}$ and any point $p$ of $M$ outside the support of $f$. We set $\iota_{f,t}(p) := \iota_f(p, t)$ for $(p, t) \in M \times \mathbb{R}$. Then there exists an open interval $I$ containing 0 such that for each $t \in I$, $\iota_{f,t}$ is an immersion of $M$ into $\mathbb{R}^3$. We set

$$ w_f(t) := \mathcal{W}(\iota_{f,t}), \quad \tilde{w}_f(t) := \mathcal{W}(\iota_{f,t}). $$

An immersion $\iota$ is called Willmore if $(dw_f/dt)(0) = 0$ holds for any smooth function $f$ on $M$ with compact support; if $\iota$ is a Willmore immersion, then the pair $(M, \iota)$ or the image $\iota(M)$ of $M$ by $\iota$ is called a Willmore surface. An immersion $\iota$ is Willmore if and only if (1) holds, where $\Delta$ is the Laplace operator on $M$ with respect to the metric $g$ ([12]). Let $D$ be a domain in $M$ which contains the support of $f$.
and the boundary of which consists of a finite number of closed curves. Then for \( t \in I \), \( w_f(t) - \tilde{w}_f(t) \) is represented as follows:

\[
(4) \quad w_f(t) - \tilde{w}_f(t) = \int_{M \setminus D} K_t \, dA_t + \int_D K_t \, dA_t,
\]

where \( K_t \) and \( dA_t \) are the Gaussian curvature and the area element of \( M \) with respect to the metric induced by \( t_{f,t} \), respectively. From Gauss-Bonnet’s theorem, we see that the second term of the right hand side of (4) depends only on the boundary of \( D \), which implies that this term does not depend on \( t \in I \). In addition, since \( D \) contains the support of \( f \), the first term of the right hand side of (4) does not depend on \( t \in I \) either. Therefore we see that \( w_f - \tilde{w}_f \) is constant on \( I \). In particular, we obtain

\[
(5) \quad \frac{d\tilde{w}_f}{dt}(0) = \frac{dw_f}{dt}(0).
\]

By (3) together with (5), we obtain

**Proposition 2.1.** Let \( \iota \) be an immersion of \( M \) into \( \mathbb{R}^3 \) and \( X \) a conformal transformation of \( \mathbb{R}^3 \) such that \( X \circ \iota \) is an immersion. Then \( \iota \) is Willmore if and only if \( X \circ \iota \) is Willmore.

3. The index of an isolated umbilical point

Let \( f \) be a smooth function of two variables \( x, y \) and \( G_f \) the graph of \( f \). We set

\[
p_f := \frac{\partial f}{\partial x}, \quad q_f := \frac{\partial f}{\partial y}, \quad r_f := \frac{\partial^2 f}{\partial x^2}, \quad s_f := \frac{\partial^2 f}{\partial x \partial y}, \quad t_f := \frac{\partial^2 f}{\partial y^2}.
\]

Then the Gaussian curvature \( K_f \) and the mean curvature \( H_f \) of \( G_f \) are represented as follows:

\[
(6) \quad K_f := \frac{r_ft_f - s_f^2}{(1 + p_f^2 + q_f^2)^2}, \quad H_f := \frac{r_f + t_f + p_f^2 q_f - 2 p_f q_f s_f + q_f^2 r_f}{2(1 + p_f^2 + q_f^2)^{3/2}}.
\]

Let \( D_f, N_f, PD_f \) be symmetric tensor fields on \( G_f \) of type \((0, 2)\) represented in terms of the coordinates \((x, y)\) as follows:

\[
D_f := s_f dx^2 + (t_f - r_f) dx \, dy - s_f dy^2,
\]

\[
N_f := (s_f p_f^2 - p_f q_f r_f) dx^2 + (t_f p_f^2 - r_f q_f^2) dx \, dy + (p_f q_f t_f - s_f q_f^2) dy^2,
\]

\[
PD_f := \frac{1}{1 + p_f^2 + q_f^2} (D_f + N_f).
\]
A tangent vector $\mathbf{v}_0$ to $G_F$ at a point is in a principal direction if and only if $PD_f(\mathbf{v}_0, \mathbf{v}_0) = 0$ holds ([15]). For a tangent vector $\mathbf{v}$, we set

$$\tilde{D}_f(\mathbf{v}) := D_f(\mathbf{v}, \mathbf{v}), \quad \tilde{N}_f(\mathbf{v}) := N_f(\mathbf{v}, \mathbf{v}), \quad PD_f(\mathbf{v}) := PD_f(\mathbf{v}, \mathbf{v}).$$

For $\phi \in \mathbb{R}$, we set

$$u_\phi := \left( \begin{array}{c} \cos \phi \\ \sin \phi \end{array} \right), \quad U_\phi := \cos \phi \frac{\partial}{\partial x} + \sin \phi \frac{\partial}{\partial y}.$$ 

We set

$$\text{grad}_f := \left( \begin{array}{c} p_f \\ q_f \end{array} \right), \quad \text{grad}_f := \left( \begin{array}{c} -q_f \\ p_f \end{array} \right), \quad \text{Hess}_f := \left( \begin{array}{cc} r_f & s_f \\ s_f & t_f \end{array} \right).$$

Let $\langle \ , \ \rangle$ be the scalar product in $\mathbb{R}^2$. Then for any $\phi \in \mathbb{R}$, the following hold ([15]):

$$\tilde{D}_f(U_\phi) = \langle \text{Hess}_f U_\phi, U_{\phi + \pi/2} \rangle,$$

$$\tilde{N}_f(U_\phi) = \langle \text{grad}_f, U_\phi \rangle \langle \text{grad}_f, \text{Hess}_f U_\phi \rangle.$$ 

For $l \in \mathbb{N} \cup \{\infty\}$, let $C_o^{(\infty, l)}$ be the set of smooth functions defined on a connected neighborhood of $(0, 0)$ in $\mathbb{R}^2$ such that $(\partial^m F / \partial x^m \partial y^n)(0, 0) = 0$ for each $F \in C_o^{(\infty, l)}$ and non-negative integers $m, n$ satisfying $0 \leq m + n < l$. The following hold:

$$C_o^{(\infty, l)} \supset C_o^{(\infty, l+1)} \supset C_o^{(\infty, \infty)} \neq \{0\},$$

where $l \in \mathbb{N}$. Let $F$ be an element of $C_o^{(\infty, 2)}$ such that $O := (0, 0, 0)$ is an umbilical point of the graph of $F$, that is, there exists a real number $a_F$ satisfying

$$F(x, y) = \frac{a_F(x^2 + y^2)}{2} + o(x^2 + y^2). \quad (7)$$

Let $\sigma_F$ be an element of $C_o^{(\infty, 2)}$ defined by

$$\sigma_F := \begin{cases} 
0 & \text{if } a_F = 0, \\
1 - \frac{|a_F|}{a_F} \sqrt{\frac{1}{a_F^2} - (x^2 + y^2)} & \text{if } a_F \neq 0.
\end{cases}$$

Then we obtain $F - \sigma_F \in C_o^{(\infty, 3)}$. For an integer $l \geq 2$, let $C_o^{(\infty, l)}$ be the subset of $C_o^{(\infty, l)}$ such that each $F \in C_o^{(\infty, l)}$ satisfies (7) for some $a_F \in \mathbb{R}$ and $F - \sigma_F \notin C_o^{(\infty, \infty)}$. For an integer $k \geq 3$, let $P^k$ be the set of the homogeneous polynomials of degree $k$. Then for each $F \in C_o^{(\infty, 2)}$, there exist an integer $k_F \geq 3$ and a nonzero element $g_F$ of $P^{k_F}$ satisfying $F - \sigma_F - g_F \in C_o^{(\infty, k_F+1)}$. Let $g$ be an element of $P^k$.
Then set $\text{Hess}_g(\theta) := \text{Hess}_g(\cos \theta, \sin \theta)$ for $\theta \in \mathbb{R}$ and let $\eta_g$ be a continuous function on $\mathbb{R}$ such that for any $\theta \in \mathbb{R}$, $u_{\eta_g(\theta)}$ is an eigenvector of $\text{Hess}_g(\theta)$, and let $S_g$ denote the set of the numbers at each of which $\text{Hess}_g$ is represented by the unit matrix up to a constant.

Let $C^{\infty, 2}_o$ be the subset of $C^{\infty, 2}_o$ such that on the graph $G_F$ of each $F \in C^{\infty, 2}_o$, $o$ is an isolated umbilical point. For an element $F$ of $C^{\infty, 2}_o$, let $\rho_0$ be a positive number such that there exists no umbilical point of $G_F$ on $\{0 < x^2 + y^2 < \rho_0^2\}$ and $\phi_F$ a continuous function on $(0, \rho_0) \times \mathbb{R}$ such that for each $(\rho, \theta) \in (0, \rho_0) \times \mathbb{R}$, a tangent vector $\cos \phi_F(\rho, \theta) \partial / \partial x + \sin \phi_F(\rho, \theta) \partial / \partial y$ of $G_F$ at $(\rho \cos \theta, \rho \sin \theta)$ is in a principal direction. Then the following (a)–(c) hold ([5], [6]):

(a) For any $\theta_0 \in \mathbb{R} \setminus S_{Gr}$, there exists a number $\phi_{F, o}(\theta_0)$ satisfying the following:

(i) $\lim_{\rho \to 0} \phi_F(\rho, \theta_0) = \phi_{F, o}(\theta_0)$,

(ii) $u_{\phi_{F, o}(\theta_0)}$ is an eigenvector of $\text{Hess}_{gr}(\theta_0)$;

(b) For any $\theta_0 \in \mathbb{R}$, there exist numbers $\phi_{F, o}(\theta_0 + 0)$, $\phi_{F, o}(\theta_0 - 0)$ satisfying the following:

(i) $\lim_{\theta \to \theta_0} \phi_{F, o}(\theta) = \phi_{F, o}(\theta_0 \pm 0)$,

(ii) $\Gamma_{F, o}(\theta_0) := \phi_{F, o}(\theta_0 + 0) - \phi_{F, o}(\theta_0 - 0)$ is an element of $\{n\pi/2\}_{n \in \mathbb{Z}}$;

(c) The index $\text{ind}_o(G_F)$ of $o$ on $G_F$ is represented as follows:

$$\text{ind}_o(G_F) = \frac{\eta_{gr}(\theta + 2\pi) - \eta_{gr}(\theta)}{2\pi} + \frac{1}{2\pi} \sum_{\theta_0 \in S_{Gr} \cap \{\theta + n\pi/2\}} \Gamma_{F, o}(\theta_0).$$

For an integer $k \geq 3$, set $\mathcal{P}^k := \mathcal{P}^k \cap C^{\infty, 2}_o$. Then for any $g \in \mathcal{P}^k$, the following hold: $\Gamma_{g, o}(\theta_0) = -\pi/2$ for any $\theta_0 \in S_g$ ([4]); $\text{ind}_o(G_g) \in \{1 - k/2 + i\} \cap \mathbb{Z}$ ([2]). Let $C^{\infty, 2}_o$ be the subset of $C^{\infty, 2}_o$ such that for each $F \in C^{\infty, 2}_o$, $o$ is an isolated umbilical point on each of $G_F$ and $G_{gr}$. If $F$ is an element of $C^{\infty, 2}_o$ satisfying $S_{gr} = \emptyset$, then $F \in C^{\infty, 2}_o$ holds ([5], [6]). We see that if $F \in C^{\infty, 2}_o$ satisfies $S_{gr} = \emptyset$, then the following hold:

$$\text{ind}_o(G_F) = \text{ind}_o(G_{gr}) = \frac{\eta_{gr}(\theta + 2\pi) - \eta_{gr}(\theta)}{2\pi}.$$
we set \( \text{grad}_g(\theta) := \text{grad}_g(\cos \theta, \sin \theta) \). Then the following holds:

\[
(10) \quad (k - 1) \text{grad}_g(\theta) = \text{Hess}_g(\theta)u_{\theta^*}.
\]

From (10), we obtain

\[
(11) \quad \langle \text{Hess}_g(\theta)u_{\theta^*}, u_{\theta + \pi/2} \rangle = (k - 1) \frac{dg}{d\theta} (\theta^*).
\]

Therefore we obtain \( S_g \subset R_g \) and \( R(\text{Hess}_g) \subset R_g \). Suppose \( R_g = \mathbb{R} \). Then \( k \) is even and \( g \) is represented by \((x^2 + y^2)^{k/2}\) up to a constant. By direct computations, we obtain \( S_g = \emptyset \). Therefore \( \theta \) is an isolated umbilical point of \( G_g \). By (11), we see that \( R(\text{Hess}_g) = \mathbb{R} \), i.e., there exists a number \( \zeta_0 \in \{ n\pi/2 \}_{n \in \mathbb{Z}} \) satisfying \( \eta_g(\theta) = \theta + \zeta_0 \) for any \( \theta \in \mathbb{R} \). Therefore by (9), we obtain

\[
\text{ind}_g(G_g) = \frac{\eta_g(\theta + 2\pi) - \eta_g(\theta)}{2\pi} = 1.
\]

In the following, suppose \( R_g \neq \mathbb{R} \). Then for each \( \theta_0 \in R_g \), there exists a positive integer \( \mu \) satisfying \( (d^{\mu+1}g/d\theta^{\mu+1})(\theta_0) \neq 0 \). The minimum of such integers is denoted by \( \mu_g(\theta_0) \). A root \( \theta_0 \in R_g \) is said to be

(a) related if \( \theta_0 \) satisfies \( \tilde{g}(\theta_0) = 0 \) or if \( \mu_g(\theta_0) \) is odd;

(b) non-related if \( \theta_0 \) satisfies \( \tilde{g}(\theta_0) \neq 0 \) and if \( \mu_g(\theta_0) \) is even.

Suppose that \( \theta_0 \in R_g \) is related. Then it is said that the critical sign of \( \theta_0 \) is positive (respectively, negative) if the following holds:

\[
\tilde{g}(\theta_0) \frac{d^{\mu_g(\theta_0)+1} \tilde{g}}{d\theta^{\mu_g(\theta_0)+1}} (\theta_0) \leq 0 \quad (\text{respectively, } > 0).
\]

The critical sign of \( \theta_0 \) is denoted by \( c\text{-sign}_g(\theta_0) \). The set \( R_g \setminus R(\text{Hess}_g) \) consists of the numbers at each of which \( \text{Hess}_g \) is represented by the unit matrix up to a nonzero constant; in addition, an element \( \theta_0 \in R_g \setminus R(\text{Hess}_g) \) is related and satisfies \( c\text{-sign}_g(\theta_0) = - (5) \). It is said that the sign of \( \theta_0 \in R(\text{Hess}_g) \) is positive (respectively, negative) if there exists a neighborhood \( U_{\theta_0} \) of \( \theta_0 \) in \( \mathbb{R} \) satisfying

\[
\{ \theta - \eta_g(\theta) - (\theta_0 - \eta_g(\theta_0)) \mid \theta - \theta_0 > 0 \} \quad (\text{respectively, } < 0)
\]

for any \( \theta \in U_{\theta_0} \setminus \{ \theta_0 \} \). For \( \theta_0 \in R(\text{Hess}_g) \), \( \theta_0 \) is related if and only if the sign of \( \theta_0 \) is positive or negative \((5)\). If \( \theta_0 \in R(\text{Hess}_g) \) is related, then the sign of \( \theta_0 \) is denoted by \( \text{sign}_g(\theta_0) \). For a related root \( \theta_0 \) of \( g \) satisfying \( c\text{-sign}_g(\theta_0) = +, \theta_0 \in R(\text{Hess}_g) \) and \( \text{sign}_g(\theta_0) = + \) hold \((5)\). Referring to \([3]\), we see that if \( \theta_0 \) is a related element of \( R(\text{Hess}_g) \) satisfying \( c\text{-sign}_g(\theta_0) = - \), then the condition \( \text{sign}_g(\theta_0) = + \) (respectively, \(- \)) is equivalent to the following:

\[
\frac{1}{\tilde{g}(\theta_0)} \frac{d^2 \tilde{g}}{d\theta^2}(\theta_0) \in (k(k - 2), \infty) \quad (\text{respectively, } [0, k(k - 2))\).
\]
Let \( n_{g^+} \) (respectively, \( n_{g^−} \)) denote the number of the related elements of \( R(\text{Hess}_g) \) in \([θ, θ+ π)\) with positive (respectively, negative) sign. Then for any \( θ ∈ \mathbb{R} \), the following holds ([15]):

\[
\frac{η_{g}(θ + 2\pi) - η_{g}(θ)}{2\pi} = 1 - \frac{n_{g^+} - n_{g^−}}{2}.
\]

4. The main theorem

We shall prove

**Theorem 4.1.** Let \( F \) be an element of \( C_{\partial}^{(∞,2)} \) satisfying (7) for some \( a_F ∈ \mathbb{R} \) and suppose that the graph \( G_F \) of \( F \) is a Willmore surface such that there exists no totally umbilical neighborhood of \( θ \) in \( G_F \). Then the following hold:

(a) \( F ∈ C_{\partial}^{(∞,2)} \);
(b) If \( θ \) is an isolated umbilical point of \( G_F \), then \( \text{ind}_θ(G_F) ≤ 1/2 \).

**Remark.** Noticing Proposition 2.1 and that whether a one-dimensional subspace of the tangent plane at a point of a surface is a principal direction is invariant under any conformal transformation of \( \mathbb{R}^3 \), we may suppose \( F ∈ C_{\partial}^{(∞,3)} \) in Theorem 4.1.

**Remark.** Although \( F \) is an element of \( C_{\partial}^{(∞,2)} \) such that \( θ \) is an isolated umbilical point of \( G_F \), \( F ∈ C_{\partial}^{(∞,2)} \) does not always hold. Let \( f \) be a smooth function on a neighborhood of \((0, 0)\) in \( \mathbb{R}^2 \) satisfying \( f(0, 0) = 0 \) and \( f > 0 \) on a punctured neighborhood of \((0, 0)\). Then \( \exp(-1/f) \) is a smooth function defined on a punctured neighborhood of \((0, 0)\) and smoothly extended to \((0, 0)\) so that all the partial derivatives of \( \exp(-1/f) \) at \((0, 0)\) are equal to zero. Then we obtain \( \exp(-1/f) ∈ C_{\partial}^{(∞,∞)} \).

Suppose that for each positive number \( c > 0 \), there exists a punctured neighborhood of \((0, 0)\) on which the norm of the gradient vector field of \( \log f \) is bounded from below by the number \( c \). Then \( \theta \) is an isolated umbilical point on the graph of \( \exp(-1/f) \) ([17]). However, since \( \exp(-1/f) ∈ C_{\partial}^{(∞,∞)} \), we obtain \( \exp(-1/f) \notin C_{\partial}^{(∞,2)} \). (a) of Theorem 4.1 is crucial to the proof of (b) of Theorem 4.1.

**Proof of (a) of Theorem 4.1.** Let \( Δ_F \) be the Laplace operator on \( G_F \), and \( K_F \), \( H_F \) the Gaussian and the mean curvatures of \( G_F \), respectively. Then \( H_F \) satisfies the following elliptic partial differential equation:

\[
\{Δ_F + 2(H_F^2 - K_F)\}H_F = 0.
\]

If \( H_F ≡ 0 \), then \( G_F \) is a minimal surface and \( F \) is real-analytic. Since \( G_F \) is not totally umbilical, we obtain \( F ≠ 0 \) and this implies \( F ∈ C_{\partial}^{(∞,3)} \). If \( H_F ≠ 0 \), then \( H_F \) is a non-trivial solution of (13) and referring to [14] as in [15], we see that not all the partial derivatives of \( H_F \) at \((0, 0)\) are equal to zero. This implies \( F ∈ C_{\partial}^{(∞,3)} \).
Hence we obtain (a) of Theorem 4.1.

Proof of (b) of Theorem 4.1. Let $F$ be an element of $C^{(\infty,3)}_o$ such that the graph $G_F$ of $F$ is a Willmore surface. Then there exist an integer $k_F \geq 3$ and a nonzero homogeneous polynomial $g_F \in \mathcal{T}_{k_F}$ satisfying $F - g_F \in C^{(\infty,k_F+1)}_o$, and noticing (6) and (13), we see that $g_F$ satisfies $\Delta^2_{\alpha}g_F \equiv 0$, where $\Delta_0 := (\partial/\partial x)^2 + (\partial/\partial y)^2$. Therefore there exist spherical harmonic functions $h_{k_F}, h_{k_F-2}$ of degree $k_F$, $k_F - 2$, respectively such that $g_F$ is represented as

$$g_F = h_{k_F} + (x^2 + y^2)h_{k_F-2}.$$

Suppose $S_{g_F} = \emptyset$. Then $F \in C^{\infty,2}_o$ holds. Noticing that the number of the zero points of $\tilde{g}_F$ in $[\theta, \theta + \pi)$ is more than or equal to $k_F - 2$, we obtain

$$k_F - 2 \leq \sharp\{S_{g_F} \cap [\theta, \theta + \pi)\} \leq k_F$$

and

$$(n_{g_F,+}, n_{g_F,-}) \in \{(k_F - 2, 0), (k_F - 1, 1), (k_F, 0)\}.$$

Therefore by (9), (12) and $k_F \geq 3$, we obtain

$$\text{ind}_o(G_F) \leq 1 - \frac{k_F - 2}{2} = 2 - \frac{k_F}{2} \leq \frac{1}{2}.$$

Suppose $S_{g_F} \neq \emptyset$ and $F \in C^{\infty,2}_o$. Then we obtain $\sharp\{S_{g_F} \cap [\theta, \theta + \pi)\} = 1, (n_{g_F,+}, n_{g_F,-}) = (k_F - 1, 0)$ and $-\pi/2 \leq \Gamma_{F,o}(\theta_0) \leq \pi/2$ for any $\theta_0 \in S_{g_F}$. Therefore by (8), (12) and $k_F \geq 3$, we obtain

$$\text{ind}_o(G_F) \leq 1 - \frac{k_F - 1}{2} + \frac{1}{2} = 2 - \frac{k_F}{2} \leq \frac{1}{2}.$$

Suppose $S_{g_F} \neq \emptyset, F \in C^{\infty,2}_o$ and $F \notin C^{\infty,2}_o$. Then there exists an element $\theta_0 \in S_{g_F}$ satisfying $\tilde{g}_F(\theta_0) = 0$ and $\mu_{g_F}(\theta_0) = 2$. We obtain $\sharp\{S_{g_F} \cap [\theta, \theta + \pi)\} = 1$ and $(n_{g_F,+}, n_{g_F,-}) = (k_F - 1, 0)$. We shall prove $-\pi/2 \leq \Gamma_{F,o}(\theta_0) \leq \pi/2$, which implies $\text{ind}_o(G_F) \leq 1/2$. We may suppose $\theta_0 = 0$ and represent $g_F$ as

$$g_F(x, y) = g_0(x, y)y^3,$$

where $g_0$ is a homogeneous polynomial of degree $k_F - 3$ satisfying $g_0(x, 0) \neq 0$ for any $x \in \mathbb{R} \setminus \{0\}$. We set

$$a_F := s_F + s_Fp_F^2 - p_Fq_Fr_F,$$
$$2b_F := t_F - r_F + t_Fp_F^2 - r_Fq_F^2,$$
\[ c_F := -s_F - s_F q_F^2 + p_F q_F 1_F. \]

Then the following holds:
\[ (1 + p_F^2 + q_F^2) PD_F = a_F dx^2 + 2b_F dx dy + c_F dy^2. \]

We set
\[ \bar{b}_F(\rho, \theta) := b_F(\rho \cos \theta, \rho \sin \theta) \]
for \((\rho, \theta) \in (-\rho_0, \rho_0) \times \mathbb{R}\), where \(\rho_0 > 0\) is a positive number such that there exists no umbilical point of \(G_F\) on \(\{0 < x^2 + y^2 < \rho_0^2\}\). There exists a smooth function \(\bar{b}_F^{(k_F-2)}\) on \(\mathbb{R}\) satisfying
\[ \bar{b}_F(\rho, \theta) - \rho^{k_F-2} \bar{b}_F^{(k_F-2)}(\theta) = o(\rho^{k_F-2}), \]

From (14), we obtain \(d\bar{b}_F^{(k_F-2)}/d\theta(0) \neq 0\). Therefore by the implicit function theorem, we see that there exist a neighborhood \(V_0\) of \((0, 0)\) in \(\mathbb{R}^2\) and a curve \(C_0\) in \(V_0\) through \((0, 0)\) satisfying
(a) \( C_0 = \{(\rho, \theta) \in V_0; \bar{b}_F(\rho, \theta)/\rho^{k_F-2} = 0\}; \)
(b) \( C_0 \) is not tangent to the \(\theta\)-axis at \((0, 0)\).

Then noticing the behavior of the two continuous distributions around \(o\) defined by
\[ b_F dx^2 + (c_F - a_F) dx dy - b_F dy^2 = 0, \]
we obtain \(-\pi/2 \leq \Gamma_{F,0}(\theta_0) \leq \pi/2. \)

\[ \square \]

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\textbf{References}


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