UNIRATIONAL QUASI-ELLiptIC SURFACES  
IN CHARACTERISTIC 3

Dedicated to the memory of Taira Honda

MASAYOSHI MIYANISHI

(Received September, 1, 1975)

0. A non-singular projective surface $X$ is called a \textit{quasi-elliptic} surface if there exists a morphism $f : X \to C$, a curve, with almost all fibres irreducible singular rational curves $E$ with $p_a(E) = 1$ (cf. [4]). According to Tate [5], such surfaces can occur only in the case where the characteristic $p$ of the ground field $k$ is either 2 or 3, and almost all fibres $E$ have single ordinary cusps. Let $\mathfrak{f}$ be the function field of $C$. Then the generic fibre of $f$ with the unique singular point taken off is an elliptic $\mathfrak{f}$-form of the affine line $\mathbb{A}^1$ (cf. [2], [3]); if this form has a $\mathfrak{f}$-rational point(*) it is birational over $\mathfrak{f}$ to one of the following affine plane curves:

(i) If $p=3$, $t^2=x^3+\gamma$ with $\gamma \in \mathfrak{f}-\mathfrak{f}^*.$
(ii) If $p=2$, $t^2=x^3+\beta x+\gamma$ with $\beta, \gamma \in \mathfrak{f}$ and $\beta \in \mathfrak{f}^*$ or $\gamma \in \mathfrak{f}^*.$

On the other hand, if $X$ is unirational $C$ must be a rational curve. Conversely if $C$ is a rational curve $X$ is unirational. Indeed, $k(X) \otimes_{\mathfrak{f}} \mathfrak{f}^{1/\alpha}$ is rational over $k$ in the first case, and $k(X) \otimes_{\mathfrak{f}} \mathfrak{f}^{1/\alpha}$ is rational over $k$ in the second case. In this article we consider a unirational quasi-elliptic surface with a rational cross-section only in characteristic 3. Thus $X$ is birational to a hypersurface $t^2=x^3+\phi(y)$ in the affine 3-space $\mathbb{A}^3$, where $\phi(y) \in \mathfrak{f}=k(y)$. If $\phi(y)$ is not a polynomial, write $\phi(y)=a(y)/b(y)$ with $a(y), b(y) \in k[y]$. Substituting $t, x$ by $b(y)^{\alpha}t, b(y)^{\alpha}x$ respectively and replacing $\phi(y)$ with $b(y)^{\alpha}a(y)$ we may assume that $\phi(y) \in k[y]$. Moreover, after making suitable birational transformations we may assume that $\phi(y)$ has no monomial terms whose degree are congruent to 0 modulo 3; especially that $d=\deg \phi$ is prime to 3. It is easy to see that under this assumption $f(x, y)=x^3+\phi(y)$ is irreducible.

A main result of this article is:

\textbf{Theorem.} \textit{Let $k$ be an algebraically closed field of characteristic 3. Then}

(*) This is equivalent to saying that $f$ has a rational cross-section which is different from the section formed by the (movable) singular points of the fibres.
any unirational quasi-elliptic surface with a rational cross-section defined over k is birational to a hypersurface in \( A^3 : t^2 = x^3 + \phi(y) \) with \( \phi(y) \in k[y] \). Let \( K = k(t, x, y) \) be an algebraic function field of dimension 2 generated by \( t, x, y \) over \( k \) such that \( t^2 = x^3 + \phi(y) \) with \( \phi(y) \in k[y] \) and \( d = \deg_y \phi \) prime to 3. Let \( m \) be the quotient of \( d \) divided by 6, and let \( H_0 \) be the (non-singular) minimal model of \( K \) when \( K \) is not rational over \( k \). Moreover if \( d \geq 7 \) assume that the following conditions hold**:

1. For every root \( \alpha \) of \( \phi'(y) = 0 \), \( v_a(\phi(y) - \phi(\alpha)) \leq 5 \), where \( v_a \) is the \((y-\alpha)\)-adic valuation of \( k[y] \) with \( v_a(y - \alpha) = 1 \).
   
2. If, moreover, \( \phi(y) - \phi(\alpha) = a(y - \alpha)^3 \) plus terms of higher degree in \( y - \alpha \) for some root \( \alpha \) of \( \phi'(y) = 0 \) and \( a \in k - (0) \) then \( v_a(\phi(y) - \phi(\alpha) - a(y - \alpha)^3) \leq 5 \).

Then we have the following:

(i) If \( m = 0 \), i.e., \( d \leq 5 \), then \( K \) is rational over \( k \). If \( d \geq 7 \), \( K \) is not rational over \( k \), and the minimal model \( H_0 \) exists.

(ii) If \( m = 1 \), i.e., \( 7 \leq d \leq 11 \), then \( H_0 \) is a K3-surface.

(iii) If \( m > 1 \), i.e., \( d \geq 13 \), then \( p_a(H_0) = p_g(H_0) = m, q = \dim H^1(H_0, O_{H_0}) = 0, \) the \( r \)-genus \( P_r(H_0) = r(m - 1) + 1 \) for every positive integer \( r \), and \( \kappa(H_0) = 1 \).

We use the following notations: Let \( X \) be a non-singular projective surface. Then \( K_X \) = the canonical divisor class on \( X \), \( p_a(X) = \dim H^0(X, K_X) \) = the geometric genus, \( q = \dim H^1(X, O_X) \) = the irregularity, \( p_a(X) - p_g(X) - q \) = the arithmetic genus, \( \kappa(X) \) = the Kodaira dimension of \( X \), and \( P_r(X) = \dim H^r(X, K_X^r) \) = the \( r \)-genus for a positive integer \( r \). For divisors \( D, D' \) etc. on \( X \), \( (D \cdot D') \) or \( (D^2) \) is the intersection number. We use sometimes the notation \( D \cdot D' \) or \( D^2 \) to indicate the intersection number if there is no fear of confusion.

1. Let \( k \) be an algebraically closed field of characteristic \( p = 3 \), let \( \phi(y) \) be a polynomial in \( y \) with coefficients in \( k \) of degree \( d > 0 \) and let \( f(x, y) = x^3 + \phi(y) \). Consider a hypersurface \( t^2 = x^3 + \phi(y) \) in the projective 3-space \( P^3 \), which is birational to a double covering** of \( F_0 = P^1 \times P^1 \). After a birational transformation of type \( (x, y, t) \rightarrow (x + \rho(y), y, t) \) with \( \rho(y) \in k[y] \) we may assume that \( (d, 3) = 1 \) and moreover that \( \phi(y) \) does not contain monomial terms whose degrees are congruent to zero modulo 3. Since \( K \) is apparently rational if \( d = 1 \) or \( 2 \) we may assume that \( d > 3 \).

The equation \( x^3 + \phi(y) = 0 \) defines a closed irreducible curve \( C \) in \( F_0 \). First of all, we shall look into singular points of \( C \) and the normalization \( \tilde{C} \) of \( C \). Let \( P : (x, y) = (\beta, \alpha) \) be a singular point of \( C \) lying on the affine part \( A^2 = F_0 - (x = 0) \).

(*) Note that if \( K \) is ruled and unirational then \( K \) is rational. Hence if \( K \) is not rational \( K \) has the minimal model.

(**) If either one of these conditions is violated we can drop the degree \( d \) by 6 by a suitable birational transformation.

(***) A morphism \( f : X' \rightarrow X \) of complete integral algebraic surfaces is called a double covering if \( f \) induces a separable quadratic extension of function fields \( k(X')/k(X) \).
oo) \cup (y=\infty). Then \( \phi'(\alpha)=0 \) and \( \beta^2+\phi(\alpha)=0 \). Conversely every root of \( \phi'(y)=0 \) gives rise to a singular point of \( C \) lying on \( A^2 \). Since \( \phi'(y)=0 \) has at least one root, \( C \) has at least one singular point on \( A^2 \subset F_0 \). The point \( Q \) of \( C \), which is situated outside of \( A^2 \), is given by \( (\xi, u)=(0, 0) \), where \( x=1/\xi, y=1/u \) and \( u^d+\xi^2\psi(u)=0 \) with \( \psi(u)=u^d(1/u) \) and \( \psi(0)=0 \). Hence \( Q \) is a cuspidal singular point with multiplicity \( (3, 3, \ldots, 3, 1, \ldots, 3) \) if \( d=3n+1 \) and \( (3, 3, \ldots, 3, 1, \ldots, 3, 2, 1, \ldots) \) if \( d=3n+2 \).

Here we introduce the following notations: Consider a fibration \( \mathcal{F}=\{l_0: l_0 \text{ is defined by } y=\alpha \} \) on \( F_0 \). We denote by \( l_0 \) the fibre \( y=\infty \), and by \( S_0 \) the cross-section \( x=\infty \). We denote by \( l \) a general fibre of \( \mathcal{F} \).

Let \( \sigma: F \to F_0 \) be the smallest blowings-up of \( F_0 \) with centers at all singular points of \( C \) and their infinitely near singular points, by which the proper transform \( \overline{C}=\sigma' C \) of \( C \) on \( F \) becomes non-singular. Let \( S_0=\sigma S_0 \), and let \( l_0=\sigma l_0 \).

The following figures will indicate the configuration of \( F \) in a neighbourhood of \( \sigma^{-1}(l_0 \cup C \cup S_0) \).

\[ \text{Fig. 1} \]

where \( d=3n+1 \) and \( (\overline{C} \cdot E_0)=3 \);

\[ \text{Fig. 2} \]

where \( d=3n+2 \) and \( (\overline{C} \cdot E_{n+1})=2 \).

Since \( (f)_{x=0}|_{F_0}=3S_0+dl_0 \), we have

\[
(f)|_{F}=\overline{C}+(3E_1+6E_2+\cdots+3nE_n)+D-3(S_0+E_1+2E_2+\cdots+nE_n)-d(l_0+E_1+\cdots+E_n)
\]

\[ (*) \text{ By this notation we mean that } Q \text{ is a point with multiplicity } 3, \text{ the infinitely near point of } C \text{ in the first neighborhood (which is a single point in this case) has multiplicity } 3, \text{ etc.} \]
if \( d=3n+1 \), where \( D \) is a positive divisor with support in the union \( \mathcal{E} \) of exceptional curves which arise from the blowings-up with centers at the singular points and their infinitely near singular points of \( C \) in the affine part \( \mathbb{A}^2 \subset F_0 \); and also

\[
(f) \mid_F = \mathcal{C}+(3\mathbb{E}_1+6\mathbb{E}_2+\cdots+3n\mathbb{E}_n+(3n+2)\mathbb{E}_{n+1})
\]
\[
+D-3(\mathbb{S}_0+\mathbb{E}_1+2\mathbb{E}_2+\cdots+n\mathbb{E}_n+(n+1)\mathbb{E}_{n+1})-d(\mathbb{I}_0+\mathbb{E}_1+\cdots
\]
\[
+\mathbb{E}_n+1) = \mathcal{C}-3\mathbb{S}_0-\mathbb{E}_{n+1}-d(\mathbb{I}_0+\mathbb{E}_1+\cdots+\mathbb{E}_{n+1})+D
\]

if \( d=3n+2 \).

On the other hand since \( K_F \sim -2\mathbb{S}_0-2\mathbb{I}_0 \), we have

\[
K_F \sim -2(\mathbb{S}_0+\mathbb{E}_1+2\mathbb{E}_2+\cdots+n\mathbb{E}_n)-2(\mathbb{I}_0+\mathbb{E}_1+\cdots+\mathbb{E}_n)
\]
\[
+\mathbb{E}_1+2\mathbb{E}_2+\cdots+n\mathbb{E}_n+D \quad \text{if } d=3n+1;
\]

and

\[
K_F \sim -2(\mathbb{S}_0+\mathbb{E}_1+\cdots+(n+1)\mathbb{E}_{n+1})-2(\mathbb{I}_0+\mathbb{E}_1+\cdots+\mathbb{E}_n+D)
\]
\[
+\mathbb{E}_1+2\mathbb{E}_2+\cdots+n\mathbb{E}_n+(n+1)\mathbb{E}_{n+1}+D_3 \quad \text{if } d=3n+2,
\]

where \( D_3 \) is a positive divisor with support in \( \mathcal{E} \).

We are now going to consider four cases separately.

(I) If \( d=6m+1 \) then \( d=3n+1 \) with \( n=2m \). Let \( B=\mathcal{C}+\mathbb{S}_0+(\mathbb{I}_0+\mathbb{E}_1+\cdots+\mathbb{E}_n)+D_1 \) and let \( Z=2\mathbb{S}_0+(3m+1)(\mathbb{I}_0+\mathbb{E}_1+\cdots+\mathbb{E}_n)-D_2 \), where \( D_1 \) and \( D_2 \) are the divisors uniquely determined by the conditions that \( D_1 \geq 0 \), every irreducible component of \( D_1 \) has multiplicity 1, \( D_2 \geq 0 \), \( D_1+D_2=D \), and \( \text{Supp}(D_1) \cup \text{Supp}(D_2) \subset \mathcal{E} \). Then \( f=B-2Z \), and \( K_F \sim Z \sim (3m-1)(\mathbb{I}_0+\mathbb{E}_1+\cdots+\mathbb{E}_n)
\]
\[
-(\mathbb{E}_1+2\mathbb{E}_2+\cdots+n\mathbb{E}_n)+(D_3-D_2) \sim (3m-1)\sigma^{-1}(1)-(\mathbb{E}_1+2\mathbb{E}_2+
\]
\[
\cdots+n\mathbb{E}_n+(D_3-D_2). \quad \text{Hence } Z \cdot (K_F+Z) \sim 2(3m-1)-2n+2D_2 \cdot (D_2-D_3), \quad \text{and } p_d(Z)=m+D_2 \cdot (D_2-D_3)/2.
\]

(II) If \( d=6m+2 \) then \( d=3n+2 \) with \( n=2m \). Let \( B=\mathcal{C}+\mathbb{S}_0+\mathbb{E}_{n+1}+D_1 \), and let \( Z=2\mathbb{S}_0+(3m+1)(\mathbb{I}_0+\mathbb{E}_1+\cdots+\mathbb{E}_n)+D_1 \), where \( D_1 \) and \( D_2 \) are divisors chosen as in the case (I). Then \( f=B-2Z \), and \( K_F \sim Z \sim (3m-1)(\mathbb{I}_0+\mathbb{E}_1+\cdots+\mathbb{E}_n)
\]
\[
-(\mathbb{E}_1+2\mathbb{E}_2+\cdots+n\mathbb{E}_n)+(D_3-D_2) \sim (3m-1)\sigma^{-1}(1)-(\mathbb{E}_1+2\mathbb{E}_2+
\]
\[
\cdots+n\mathbb{E}_n+(D_3-D_2). \quad \text{Hence } Z \cdot (K_F+Z) \sim 2(3m-1)-2n+n+D_2 \cdot (D_2-D_3)+2m-2D_2 \cdot (D_2-D_3), \quad \text{and } p_d(Z)=m+D_2 \cdot (D_2-D_3)/2.
\]

(III) If \( d=6m+4 \) then \( d=3n+1 \) with \( n=2m+1 \). Let \( B=\mathcal{C}+\mathbb{S}_0+D_1 \) and let \( Z=2\mathbb{S}_0+(3m+2)(\mathbb{I}_0+\mathbb{E}_1+\cdots+\mathbb{E}_n)-D_1 \), where \( D_1 \) and \( D_2 \) are divisors chosen as above. Then \( f=B-2Z \), and \( K_F \sim Z \sim 3m\sigma^{-1}(1)-(\mathbb{E}_1+\cdots+n\mathbb{E}_n)+D_2 \). \quad \text{Hence } Z \cdot (K_F+Z) \sim 6m-2n+2D_2 \cdot (D_2-D_3)+2m-2D_2 \cdot (D_2-D_3), \quad \text{and } p_d(Z)=m+D_2 \cdot (D_2-D_3)/2.

(IV) If \( d=6m+5 \) then \( d=3n+2 \) with \( n=2m+1 \). Let \( B=\mathcal{C}+\mathbb{S}_0+(\mathbb{I}_0+\mathbb{E}_1+\cdots+\mathbb{E}_n)+D_1 \) and let \( Z=2\mathbb{S}_0+(3m+3)(\mathbb{I}_0+\mathbb{E}_1+\cdots+\mathbb{E}_n)-D_1 \), where \( D_1 \) and \( D_2 \) are divisors chosen as above. Then \( f=B-2Z \), and \( K_F \sim Z \sim (3m+1)\sigma^{-1}(1)-(\mathbb{E}_1+\cdots+n\mathbb{E}_n+(n+1)\mathbb{E}_{n+1})+(D_3-D_2) \). \quad \text{Hence } Z \cdot (K_F+Z) \sim 2(3m+1)\sigma^{-1}(1)-(\mathbb{E}_1+\cdots+n\mathbb{E}_n+(n+1)\mathbb{E}_{n+1})+(D_3-D_2) \).
UNIRATIONAL QUASI-ELLIPTIC SURFACES

\[ -2(n+1)+D_z=(D_z-D_3)-2m-2+D_z,(D_z-D_3), \text{ and } p_a(Z)=m+D_z,(D_z-D_3)/2. \]

In each case, \( p_a(Z)=m+D_z,(D_z-D_3)/2 \). Let \( F \to F \) be the smallest blowings-up which make the branch locus of the double covering on \( F \) non-singular, let \( H \) be the normalization of \( F \) in the function field \( K=k(t, x, y) \) and let \( \pi: H \to F \) be the canonical morphism. Then \( H \) is a non-singular projective surface called the canonical model of \( K \), which is a double covering of \( F \) with branch locus \( B^{(*)} \) in each of the above four cases (cf. Artin [1]). Let \( K_H \) be the canonical divisor of \( H \). By Artin [1], we know that \( K_H \sim \pi^{-1}(K_F+Z) \) and \( p_a(H)=2p_a(F)+p_a(Z) \), since the singular points on the branch locus \( B \) on \( F \) are all negligible singularities(***) and since \( p_a(F)=0 \).

Thus we proved:

**Lemma 1.** Let \( m \) be the quotient of \( d \) divided by 6. Then \( p_a(H)=m+D_z,(D_z-D_3)/2 \).

Now we show:

**Lemma 2.** With the notations and assumptions as above, \( H \) is a rational surface if \( d \leq 5 \).

Proof. First of all, we may assume that \( d \leq 4 \). In effect, if \( d=5 \) we may assume that \( \phi(y) \) has no constant and degree 1 terms after a suitable change of variables \( x \) and \( y \). Then by a change of variables: \( t'=t/y^3, x'=x/y^2, y'=1/y \), we have

\[ t'^2 = x'^3 + \phi(y') \quad \text{with } \deg_{x'}\phi(y') \leq 4. \]

Now assuming that \( d \leq 4 \) and \( \phi(y) \) has no monomial terms whose degrees are congruent to zero modulo 3, we are going to compute \( D_z-D_3 \) and \( K_H \) explicitly. Let \( v \) be the number of distinct roots of \( \phi'(y)=0 \). If \( v=1 \), we may assume that \( \phi(y)=y^d \) after a suitable change of variables. Let \( P: (x, y)=(0, 0) \). \( P \) is a singular point of \( C \) with multiplicity \( (2, 1, \cdots) \) if \( d=2; (3,1,\cdots) \) if \( d=4 \). Then \( D=2E \) with \( E=\sigma^{-1}(P) \) if \( d=2; D=3E \) if \( d=4 \). Then \( D_z=0, D_z=D_3=E \) if \( d=2; D_z=D_3=D_3=E \) if \( d=4 \). In each case \( D_z-D_3=0 \). If \( v=2 \), let \( \alpha_1 \) and \( \alpha_2 \) be distinct roots. We have two possible cases: (i) Both \( \alpha_1 \) and \( \alpha_2 \) are simple roots; (ii) One of \( \alpha_1 \) and \( \alpha_2 \) is a double root and the other one is a simple root.

However neither case can occur. Indeed, \( d=3 \) in the first case, and the second case is impossible. If \( v=3 \), let \( \alpha_1, \alpha_2, \alpha_3 \) be distinct roots. Then \( d=4 \), and

\[ (*) \text{ A point } P \text{ of } F \text{ is a branch point, i.e., } P \in B \text{ if the normalization of } \mathcal{O}_{F,B} \text{ in } K \text{ is a local ring.} \]

\[ (**) \text{ A point } P \text{ of } B \text{ has negligible singularity if and only if it is of one of the following types: (i) a simple point of } B, \text{ (ii) a double point of } B, \text{ (iii) a triple point of } B \text{ with at most a double point (not necessarily ordinary) infinitely near (cf. Artin [1]). For the arithmetic genus formula, see also [B. Iversen: Numerical invariants and multiple planes, Amer. J. Math., 92 (1970), 968–996].} \]
\(\alpha_1, \alpha_2\) and \(\alpha_3\) are all simple roots. Let \(P_i (i=1, 2, 3)\) be the singular point of \(C\) with \(y\)-coordinate \(\alpha_i\). The multiplicity of \(P_i\) is \((2, 1, \ldots)\). Hence \(D=2\sigma^{-1}(P_1)+\sigma^{-1}(P_2)+\sigma^{-1}(P_3)\), \(D_1=0\) and \(D_2-D_3=\sigma^{-1}(P_1)+\sigma^{-1}(P_2)+\sigma^{-1}(P_3)\). Thus \(D_2-D_3=0\). Therefore \(p_a(H)=0\).

On the other hand, since \(K_H \sim \pi^{-1}(K_F+Z)\), we see from the above observations on \(K_F+Z\) that \(K_F+Z<0\) if \(d\leq 4\). Hence \(K_H<0\) and \(p_a(H)=0\). Therefore \(H\) is rational by virtue of Castelnuovo's criterion of rationality. Q.E.D.

2. Let us consider the following conditions on \(\phi(y)\):

1. For every root \(\alpha\) of \(\phi'(y)=0\), \(v_\alpha(\phi(y)-\phi(\alpha))\leq 5\), where \(v_\alpha\) is the \((y-\alpha)\)-adic valuation of \(k[y]\) with \(v_\alpha(y-\alpha)=1\).

2. If, moreover, \(\phi(y)-\phi(\alpha)\sim \alpha(y-\alpha)^3+\text{(terms of higher degree in } y-\alpha)\) for some root \(\alpha\) of \(\phi'(y)=0\) and \(\alpha(k-0)\) then \(v_\alpha(\phi(y)-\phi(\alpha)-\alpha(y-\alpha)^3)\leq 5\).

Assume that \(v_\alpha(\phi(y)-\phi(\alpha))\geq 6\) for some root \(\alpha\) of \(\phi'(y)=0\). Since \(d>0\), this assumption implies \(d \geq 6\). Then by a birational transformation \((t, x, y) \mapsto (t_1=t/(y-\alpha), x_1=(x+\phi(\alpha)^{1/3})/(y-\alpha)^2, y_1=y-\alpha)\), we have

\[
t=t_1^2+\phi_1(y_1) \quad \text{with} \quad \deg_\phi \phi_1 = \deg_\phi \phi - 6.
\]

Assume next that \(\phi(y)-\phi(\alpha)\sim \alpha(y-\alpha)^3+\text{(terms of higher degree in } y-\alpha)\) for some root \(\alpha\) of \(\phi'(y)=0\) and that \(v_\alpha(\phi(y)-\phi(\alpha)-\alpha(y-\alpha)^3)\geq 6\). Then by a birational transformation \((t, x, y) \mapsto (t_1=t, x_1=x+a^{1/3}(y-\alpha), y_1=y)\) we have

\[
t_1^2 = x_1^3+\phi_1(y_1) \quad \text{with} \quad \deg_\phi \phi_1 = d \quad \text{and} \quad v_\alpha(\phi_1(y_1)-\phi_i(\alpha))\geq 6.
\]

Therefore the argument in the former case applies, and we can drop the degree of \(\phi_i\) by 6. Therefore we may assume that \(d \geq 7\) and that the conditions (1) and (2) hold. Hereafter we assume these conditions for \(\phi(y)\). Then we have:

Lemma 3. With the notations as above, \(D_2=D_1\).

Proof. Let \(\alpha\) be a root of \(\phi'(y)=0\), and let \(P=(x, y)=(-\phi(\alpha)^{1/3}, \alpha)\) be the corresponding singular point of \(C\). Let \(e=v_\alpha(\phi(y)-\phi(\alpha))\). Since the conditions (1) and (2) hold, we may assume that \(e=2, 4\) or 5. In fact, the case where \(e=3\) can be reduced to the case where \(e=4\) or 5 by a birational transformation \((t, x, y) \mapsto (t, x+a^{1/3}(y-\alpha), y)\), which is biregular at \(P\). \(P\) is then a cuspidal singular point with multiplicity \((2, 1, \cdots)\) if \(e=2\); \((3, 1, \cdots)\) if \(e=4\); \((3, 2, 1, \cdots)\) if \(e=5\). Hence \(\sigma^{-1}(P)=E_1\) (irreducible) if \(e=2\) or \(4\); \(\sigma^{-1}(P)=E_1+E_2\) (\(E_1\) and \(E_2\) are irreducible) if \(e=5\). Then \(D_2=D_3=E_1\) if \(e=2\) or \(4\); \(D_2=D_3=E_1+2E_2\) if \(e=5\).

Corollary. Let \(m\) be the quotient of \(d\) divided by 6. If one assumes the conditions (1) and (2) on \(\phi(y)\), \(p_a(H)=m\).
The canonical model $H$ of $K$ might contain the exceptional curves of the first kind. When $p_a(H) = m > 0$ (i.e., $d \geq 7$), let $H_0$ be the minimal non-singular model of $K$, which is, needless to say, obtained from $H$ by contracting all exceptional curves of the first kind. We shall describe the canonical divisor $K_{H_0}$ of $H_0$.

**Lemma 4.** Assume that $d = 6m + 1$ with $m > 0$. Then we have:

(i) $\pi^{-1}(I_\infty \cap E_i) = L_0^i$, $\pi^{-1}(E_i \cap E_i') = L_1^i$, $\pi^{-1}(E_i \cap E_{i-1}) = L_{i-1}^i$, where $L_i^i$ $(0 \leq i \leq n-1)$ is an irreducible non-singular rational curve with $(L_i^i)^2 = -2$ and $n = 2m$.

(ii) $\pi^{-1}(I_\infty) = 2L_0 + L_0'$, $\pi^{-1}(E_i) = L_{i-1}^i + 2L_i + L_i'$ $(1 \leq i \leq n-1)$, where $L_i$ $(0 \leq i \leq n-1)$ is an irreducible non-singular rational curve such that $(L_0^i) = -1$, $(L_1^i) = -2$ $(1 \leq i \leq n-1)$.

(iii) $K_H \sim \pi^{-1}(K_F + Z) - (m-1)\pi^{-1}(\sigma^{-1}(l)) + 4mL_0 + (4m-1)L_0 + (4m-2)L_1 + (4m-3)L_1 + \cdots + 3L_{2m-2} + 2L_{2m-1} + L_{2m-1}$.

(iv) $W := L_0 + L_0' + L_1 + \cdots + L_{2m-1}$ is contractible. Let $\tau: H \to H_0$ be the contraction of $W$. Then $H_0$ is a minimal model of $K$. Hence $K_{H_0} \sim (m-1)\pi^{-1}(\sigma^{-1}(l))$.

(v) For every positive integer $r$ the $r$-genus $P_r(H_{\sigma})$ of $H_0$ is $r(m-1)+1$. In particular, $p_g(H_0) = p_a(H_0) = m$ and $q = 0$.

(vi) If $m = 1$, i.e., $d = 7$, $H_0$ is a K3-surface. If $m > 1$, $\kappa(H_0) = 1$.

Proof. First of all note that $B = \{E_0 + E_1 + \cdots + E_n\} + D_1$ and $K_F + Z \sim (m-1)\pi^{-1}(l) + (2mL_0 + (2m-1)E_1 + \cdots + E_{2m-1})$. Let $\sigma_1: F_1 \to F$ be the blowings-up with centers at $E_0 \cup E_1 \cup \cdots \cup E_n$. Then $\pi: H \to F$ factors as $\pi = \pi_1 \circ \sigma_1$, $\pi = \pi_1 \circ \sigma_1$, i.e., $\pi = \pi_1 \circ \sigma_1$. Since the branch locus $B_1$ on $F_1$ is of the form $B_1 = \sigma_1(l_0) + \sigma_1(E_1) + \cdots + \sigma_1(E_{2m-1}) + B'$ with $B'$ having no intersections with $\sigma_1(E_0 + E_1 + \cdots + E_n)$, $\pi_1$ coincides with $\pi: H \to F$, which is the canonical normalization morphism, on a small open neighbourhood of $\sigma_1^{-1}(I_\infty \cup E_0 \cup \cdots \cup E_{2m-1})$. Now writing locally the equations of $\pi^{-1}(I_\infty \cap E_i) = \pi_1^{-1}(\sigma_1^{-1}(I_\infty \cap E_i))$, $\pi^{-1}(E_i \cap E_j)$, $\pi^{-1}(E_{i-1} \cap E_n)$, it is not hard to show that $L_0, \cdots, L_{2m-1}$ are irreducible non-singular rational curves. For $0 \leq i \leq n-1$, $(L_i^i)^2 = 2(\sigma_1^{-1}(E_i \cap E_{i+1}))^2 = -2$. This proves the assertion (i).

To show the assertion (ii), note that $I_\infty$, $E_0$, $\cdots$, $E_{2m-1}$ are components of the branch locus $B$. Therefore $\pi^{-1}(I_\infty) = 2L_0 + L_0'$ and $\pi^{-1}(E_0) = L_{i-1}^i + 2L_i + L_i'$ $(1 \leq i \leq n-1)$ with non-singular irreducible rational curves $L_i$ $(0 \leq i \leq n-1)$. Since $(\sigma_1(l_0)^2) = -2$ and $\pi^{-1}(\sigma_1(l_\infty)) = 2L_0$, we have $4(L_0^2) = -4$. Hence $(L_0^2) = -1$. 

(Fig. 3)
Similarly, \( (\sigma(E_i)^3) = -4 \) and \( \pi^{-1}(\sigma(E_i)) = iL_i \) for \( 1 \leq i \leq n-1 \). Hence \( (L^2_i) = -2 \) for \( 1 \leq i \leq n-1 \).

By virtue of the assertions (i) and (ii), \( K_H \sim \pi^{-1}(K_F + Z) \sim (m-1)\pi^{-1}(l) + \pi^{-1}(2mL_0 + (2m-1)E_1 + \cdots + E_{2m-1}) = (m-1)\pi^{-1}(l) + 4mL_0 + (4m-1)L_0 + (4m-2)L_1 + \cdots + 3L_{2m-2} + 2L_{2m-1} + L_{2m-1}. \) Since \( L_i \)'s and \( L_i' \)'s \( (0 \leq i \leq 2m-1 \) have the configuration as indicated in the Fig. 3, it is easy to show that \( W \) is contractible, and \((m-1)\pi^{-1}(l) \) is the moving part of \( |K_H| \). Let \( \tau : H \rightarrow H_0 \) be the contraction of \( W \). Then \( K_{H_0} = \tau((m-1)\pi^{-1}(l)) \). Hence \( \dim |K_{H_0}| \geq 0 \) and \( |K_{H_0}| \) has no fixed components if \( m \geq 1 \). This implies that \( H_0 \) is a minimal model of \( K \). Thus the assertions (iii) and (iv) are proven.

Let us show that \( P_r(H_0) = (m-1)r + 1 \) for every positive integer \( r \). There exists a non-singular irreducible rational curve \( S_\omega \) on \( H \) such that \( \pi(S_\omega) = S_\omega, \pi^{-1}(S_\omega) > 2S_\omega \) and \( S_\omega \cap \text{Supp}(W) = \phi \). Let \( \hat{S}_\omega = \pi(S_\omega) \). Then \( \hat{S}_\omega \) is a non-singular irreducible rational curve. Since \( \dim |rK_{H_0}| = \dim Tr_{S_\omega} |rK_{H_0}| + \dim |rK_{H_0} - \hat{S}_\omega| + 1 \), we compute \( \dim Tr_{S_\omega} |rK_{H_0}| \) and \( \dim |rK_{H_0} - \hat{S}_\omega| \) as follows. Suppose that \( |rK_{H_0} - \hat{S}_\omega| \neq \phi \), and let \( M \in |rK_{H_0}| \) be such that \( M > \hat{S}_\omega \). Then \( \pi^{-1}M > \pi^{-1}\hat{S}_\omega = \hat{S}_\omega, \) so \( \pi^{-1}M \sim r(m-1)\pi^{-1}(l) \). Then \( \sigma \pi(\pi^{-1}M) > \sigma \pi\hat{S}_\omega = S_\omega, \) and \( \sigma \pi(\pi^{-1}M) \sim 2r(m-1)l \). This is a contradiction since no members of \( |2r(m-1)l| \) on \( F_0 = \mathbb{P}^1 \times \mathbb{P}^1 \) contain \( S_\omega \). Thus \( \dim |rK_{H_0} - \hat{S}_\omega| = -1 \). On the other hand, since \( \hat{S}_\omega \simeq \mathbb{P}^1 \) and \( \deg Tr_{\hat{S}_\omega} |rK_{H_0}| = r(m-1)(*) \) and \( Tr_{\hat{S}_\omega} |rK_{H_0}| \) is apparently complete we have \( \dim Tr_{\hat{S}_\omega} |rK_{H_0}| = r(m-1) \). Therefore \( P_r(H_0) = r(m-1) + 1 \). In particular, \( p_g(H_0) = p_g(H_0) = m = p_a(H_0) \). Hence \( q = \dim H^1(H_0, \mathcal{O}_{H_0}) = p_g(H_0) - p_a(H_0) = 0 \). Thus \( H_0 \) is a regular surface. If \( m = 1, H_0 \) is a K3-surface. If \( m > 1, \kappa(H_0) = 1 \) since \( P_r(H_0) \) is a linear polynomial in \( r \). This completes the proof of the assertions (v) and (vi).

Q.E.D.

In a similar fashion we can show:

**Lemma 5.** Assume that \( d = 6m + 2 \) with \( m > 0 \). Then we have:

1. \( \pi^{-1}(L_0) = L_0 + L_0, \pi^{-1}(E_i) = L_i + L_i \) (1 \( \leq i \leq 2m-1 \) where \( L_i \)'s and \( L_i' \)'s are irreducible non-singular rational curves such that \( (L^2_0) = (L^2_i) = -1 \) and \( (L^2_i) = -2 \) (1 \( \leq i \leq 2m-1 \). They have the following configuration:

```
  \[ \begin{array}{ccccccccc}
  L_0 & L_0 & L_0 & \cdots & L_{2m-1} \\
  -2 & -2 & -2 & \cdots & -2 \\
  \end{array} \]
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(Fig. 4)

(*) Cf. 2(\( \pi^{-1}(l) \)) = 2(\( \pi^{-1}(l) \)) = (\( 2\pi^{-1}(l) \)) = 2(\( \pi^{-1}(l) \)) = 2.
(ii) $K_H \sim \pi^{-1}(K_F + Z) \sim (m-1)\pi^{-1}q^{-1}(l) + 2mL_0 + (2m-1)L_1 + \cdots + L_{2m-1} + 2mL_0 + (2m-1)L_1 + \cdots + L_{2m-1}$.

(iii) Let $W := L_0 + L_1 + \cdots + L_{2m-1} + L_0' + L_1' + \cdots + L_{2m}'$. Then $W$ is contractible, and if $\tau : H \to H_0$ is the contraction of $W$, $H_0$ is a minimal model of $K$. Hence $K_{H_0} \sim (m-1)\pi^{-1}q^{-1}(l)$.

(iv) For every positive integer $r$, $P_r(H_0) = r(m-1) + 1$. In particular, $p_a(H_0) = p_a(H_0) = m$ and $q = 0$.

(v) If $m = 1$, i.e., $d = 8$, $H_0$ is a $K3$-surface. If $m > 1$, $\kappa(H_0) = 1$.

Lemma 6. Assume that $d = 6m + 4$ with $m > 0$. Then we have:

(i) $\pi^{-1}(\text{I}_0) = L_0 + L_0', \pi^{-1}(\text{I}_i) = L_i + L_i' (1 \leq i \leq 2m)$, where $L_i$'s and $L_i'$'s are irreducible non-singular rational curves such that $(L_0^2) = (L_0'^2) = -1$, $(L_i^2) = (L_i'^2) = -2$ ($1 \leq i \leq 2m$). They have the following configuration:

(ii) $K_H \sim \pi^{-1}(K_F + Z) \sim (m-1)\pi^{-1}q^{-1}(l) + (2m+1)L_0 + 2mL_1 + \cdots + L_{2m}$

(iii) Let $W := L_0 + L_1 + \cdots + L_{2m} + L_0' + L_1' + \cdots + L_{2m}'$. Then $W$ is contractible, and if $\tau : H \to H_0$ is the contraction of $W$, $H_0$ is a minimal model of $K$. Hence $K_{H_0} \sim (m-1)\pi^{-1}q^{-1}(l)$.

(iv) For every positive integer $r$, $P_r(H_0) = r(m-1) + 1$. In particular, $p_a(H_0) = p_a(H_0) = m$ and $q = 0$.

(v) If $m = 1$, i.e., $d = 10$, $H_0$ is a $K3$-surface. If $m > 1$, $\kappa(H_0) = 1$.

Lemma 7. Assume that $d = 6m + 5$ with $m > 0$. Then we have:

(i) $\pi^{-1}(\text{I}_0) = L_0 + L_0', \pi^{-1}(\text{E}_i) = L_i + L_i' (1 \leq i \leq 2m)$, where $n = 2m+1$ and $L_i' (0 \leq i \leq n)$ is an irreducible non-singular rational curve with $(L_i'^2) = -2$.

(ii) $\pi^{-1}(\text{I}_0) = 2L_0 + L_0'$ and $\pi^{-1}(\text{E}_i) = L_{i-1} + 2L_i + L_i' (1 \leq i \leq n)$, where $L_i$ $(0 \leq i \leq n)$ is an irreducible non-singular rational curve such that $(L_0^2) = -1$ and $(L_i^2) = -2$ $(0 < i \leq n)$. $L_i$'s and $L_i'$'s have the following configuration:

(Fig. 6)
(iii) \( K_H \sim \pi^{-1}(K_F + Z) \sim (m-1)\pi^{-1}\sigma^{-1}(l) + (4m+4)L_0 + (4m+3)L_0' + \cdots + 2L_{2m+1} + L'_{2m+1}. \)

(iv) Let \( W := L_0 + L_0' + \cdots + L_{2m+1} + L'_{2m+1}. \) Then \( W \) is contractible. If \( \tau: H \to H_0 \) is the contraction of \( W, H_0 \) is the minimal model of \( K. \) Hence \( K_{H_0} \sim \pi^{-1}(m-1)\tau\pi^{-1}\sigma^{-1}(l) \).

(v) For every positive integer \( r, P_r(H_0) = r(m-1) + 1. \) In particular, \( p_g(H_0) = p_g(H_0) - m \) and \( g = 0. \)

(vi) If \( m = 1, \) i.e., \( d = 11, \) \( H_0 \) is a \( K3 \)-surface. If \( m > 1, \) \( \kappa(H_0) = 1. \)

Combining the above results, we have our main theorem.

REMARK. If \( m > 1, \) \( H_0 \) is not birational to an elliptic surface. Assume the contrary, and let \( \rho: H' \to H_0 \) be a birational morphism with a non-singular projective surface \( H' \) endowed with an elliptic pencil \( \mathcal{L} = \{ C_{\alpha}; \alpha \in \mathbb{P} \}. \) Then \( K_{H'} \sim (m-1)\rho^{-1}\tau\pi^{-1}\sigma^{-1}(l) + E, \) where \( E \geq 0 \) with \( \text{Supp}(E) \) the union of exceptional curves arising from \( \rho. \) For a general member \( C \) of \( \mathcal{L} \) we have \( (C^2) = 0, \) and \( C \cdot K_{H'} \geq 0 \) because \( C \) is a non-singular irreducible curve distinct from components of \( E. \) Since \( 1 = p_a(C) = (C^2 + C \cdot K_{H'})/2 + 1 \) we have \( C \cdot K_{H'} = 0. \) Hence \( C \) coincides with a component of a member of \( |(m-1)\rho^{-1}\tau\pi^{-1}\sigma^{-1}(l)|, \) i.e., \( C = \rho^{-1}\tau\pi^{-1}\sigma^{-1}(l) = \tau\pi^{-1}\sigma^{-1}(l) \) for some \( l. \) This is absurd because \( \tau\pi^{-1}\sigma^{-1}(l) \) is rational.

OSAKA UNIVERSITY

References